Infinitesimal tensor rigidity of higher rank lattice actions.

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1 Introduction

Any C^∞ action of a group G on a manifold X induces a representation of G in the space $(\otimes_1^p S(T^*X)) \otimes (\otimes_1^q S(TX))$ of smooth (p,q)-tensor fields on X. Here $S(T^*X)$ and $S(TX)$ denote the spaces of smooth sections of, correspondingly, the cotangent and tangent bundles, i.e., covector and vector fields. We will denote the first cohomologies of that representation by

$$\mathrm{H}^1(G, X, (\otimes_1^p S(T^*X)) \otimes (\otimes_1^q S(TX)))$$

or, to shorten the notation, by $\mathrm{H}^1_{(p,q)}(G, X)$.

Definition 1.1 *A smooth action of a group G on a manifold X is called C^∞ infinitesimal (p,q)-tensor rigid if the C^∞ first cohomologies $\mathrm{H}^1_{(p,q)}(G, X)$ are trivial.*

Note that for $p = 0$, $q = 1$, this is a standard definition of infinitesimal rigidity [8].

Clearly, for any C^1 action, similar notions and definitions may be introduced in L^2, continuous or Hoelder categories.

For the most part we will work with the C^∞ actions and C^∞ cohomologies. We will mention some L^2, continuous and Hoelder results in Section 5.

The main result of this paper is the following rigidity

Theorem 1.1 *Let G be a semi-simple connected Lie group of $\mathbb{R}$-rank ≥ 3, with finite center and without either compact factors or factors of rank one. Let Γ be an irreducible cocompact lattice in G. Let N be a subgroup of G that* *contains the maximal connected split Cartan subgroup A of G. Let N_0 be the connected component of the unit in N.*

$(\times)$ Suppose that $N_0\backslash G$ is smoothly parallelizable.

Then every element of $\mathrm{H}^1_{(p,q)}(\Gamma, N\backslash G)$ uniquely extends to an element of $\mathrm{H}^1_{(p,q)}(G, N\backslash G)$.

Remark 1: In particular, Theorem 1.1 applies to the actions of Γ on the boundarues of the symmatric space associated with G (N is a parabolic subgroup).

Remark 2: Moreover, it will follow from our proof that

$$\mathrm{H}^1_{(p,q)}(\Gamma, N\backslash G) = \mathrm{H}^1_{(p,q)}(G, N\backslash G) \cong$$

$$\cong (\mathrm{H}^1)^{tr}(N, (\otimes_1^p Ad_N^*) \otimes (\otimes_1^q Ad_N)),$$

where Ad_N is the factor of the restriction of the adjoined representation of G to N by the subrepresentation in the tangent space $\mathcal{N}$ to N, Ad_N^* is its conjugate, and $(\mathrm{H}^1)^{tr}(N, (\otimes_1^p Ad_N^*) \otimes (\otimes_1^q Ad_N))$ is a certain subspace of $\mathrm{H}^1(N, (\otimes_1^p Ad_N^*) \otimes (\otimes_1^q Ad_N))$ (see Section 3 for the details). (Both here and elsewhere in the paper we shall denote the first cohomologies of a representation T of a group G by $\mathrm{H}^1(G, T)$.) Therefore, the infinite dimensional problem of the description of $\mathrm{H}^1_{(p,q)}(\Gamma, N\backslash G)$ reduces to a problem of calculating the first cohomologies of some finite dimensional representation of N. In particular, one can see that $H^1_{(p,q)}(\Gamma, N\backslash G)$ is finite dimensional and has dimension no greater then $(dim(G) - dim(N))^{(p+q)dim(N)}$.

In general the properties of $(\mathrm{H}^1)^{tr}(N, (\otimes_1^p Ad_N^*) \otimes (\otimes_1^q Ad_N))$ depend on the combinatorical relations between the roots of $\mathcal{G}$ – the Lie algebra of G – and $\mathcal{N}$. The different types of phenomena that play a crucial role in the study of $(\mathrm{H}^1)^{tr}(N, (\otimes_1^p Ad_N^*) \otimes (\otimes_1^q Ad_N))$ are discussed in Section 4, for the particular case when $G = SL(n, \mathbb{R})$ and N is a parabolic subgroup.

Remark 3: The $(\times)$ condition looks very unnatural. It seems reasonable to conjecture that Theorem 1.1 should be true without it, and that it is just a reflection of the limitations of our methods rather than of some real phenomena. Unfortunately, for the methods we use, it is essential for us to be able to work with twisted cocycles instead of the cohomologies in the tensor bundles, and for that we need the parallelization. One possible way to get rid of this condition is to use the L^2 analog of Theorem 1.1 (for the L^2 case $(\times)$ is, obviously, not a restriction at all) and then try to prove that the L^2 extensions of C^∞ or continuous elements of $\mathrm{H}^1_{(p,q)}(\Gamma, N\backslash G)$ are, correspondingly, C^∞ or continuous themselves. Thus far we have not been able to verify this, but we plan to pursue this direction in future work.

2 Some definitions, notations an preliminaries

In this section we recall standard definitions of cocycles of group actions and state some results from [4], [5] (also announced in [6]) that we will use in this article.

Definition 2.1 *If a group G acts on a space X then a cocycle of this action with coefficients in a group H is a map $\alpha : G \times X \to H$ such that*

$$\alpha(g_1 g_2, m) = \alpha(g_1, g_2(m))\alpha(g_2, m).$$

Two cocycles α and β are called cohomologous if there exists a function $P : X \to H$ such that

$$\beta(g, m) = P(gm)^{-1}\alpha(g, m)P(m).$$

We will denote a set of equivalence classes of cocycles by $\mathrm{H}(G, X, H)$.

Definition 2.2 *Let $\mathcal{R}$ be an H-module. Let α be a cocycle with coefficients in H. A map $\beta(g, m) : G \times X \to \mathcal{R}$ will be called a twisted cocycle if it satisfies*

$$\beta(g_1 g_2, m) = \beta(g_2, m) + \alpha(g_2, m)^{-1}\beta(g_1, g_2 m), \forall g_1, g_2 \in G, m \in X.$$

Two cocycles β and β_1 will be called cohomologous (or equivalent) if and only if there exists a function $f : X \to \mathcal{R}$ such that

$$\beta(g, m) - \beta_1(g, m) = f(m) - \alpha(g, m)^{-1}f(gm), \forall g \in G, m \in X.$$

We will denote the set of equivalence classes by $\mathrm{H}(G, X, \mathcal{R}, \alpha)$.

Definition 2.3 *We will call a cocycle or twisted cocycle constant if it does not depend on $m \in X$.*

Remark 1: One can easily see that, for cohomologous cocycles α_1 and α_2, $\mathrm{H}(G, X, \mathcal{R}, \alpha_\infty)$ and $\mathrm{H}(G, X, \mathcal{R}, \alpha_\in)$ are isomorphic. This justifies the notation $\mathrm{H}(G, X, \mathcal{R}, \alpha)$.

Remark 2: We define cocycles of different regularity, for example C^∞, L^2, continuous, Hoelder, imposing the corresponding regularity condition on the functions $\alpha(g, \cdot)$ and $\beta(g, \cdot)$.

The following definitions were introduced in [4] and [5].

Definition 2.4 *Suppose that G acts on X_1 and X_2 in such a way that the action on X_2 is a factor of the action on X_1. Then, for any group H, every element of $\mathrm{H}(G, X_2, H)$ can be uniquely lifted to an element of $\mathrm{H}(G, X_1, H)$. By $\mathrm{H}^{tr}(G, X_2, H)$ we will denote a set of those elements of $\mathrm{H}(G, X_2, H)$ that lift to a trivial element in $\mathrm{H}(G, X_1, H)$.*

We define $\mathrm{H}^{tr}(G, X_2, \mathcal{R}, \alpha)$, $\alpha \in \mathrm{H}(G, X_2, H)$ accordingly, where, as before, $\mathcal{R}$ is an H-module.

Remark: Sometimes, when it is not likely to cause any confusion, we will omit H and $\mathcal{R}$ in the notations introduced above.

Also, assume that there is a certain structure on X_1 and X_2 which is preserved by the factor map. This could be a structure of a topological space, measurable space, smooth manifold, etc. And suppose that we consider some class of cocycles with respect to this structure (for example, continuous, measurable, or smooth cocycles). The following duality theorems are true for all classes of cocycles which make sense for the spaces and factor maps involved. (For the proofs see [4].)

Theorem 2.1 *Let G_1 and G_2 be two groups acting on a space M. Consider cocycles with values in an arbitrary group H.*

If those actions commute, then:

$$\mathrm{H}^{tr}(G_1, M/G_2) \cong \mathrm{H}^{tr}(G_2, G_1 \backslash M).$$

To be more precise, there is a canonically defined map

$$K(G_1, G_2) : \mathrm{H}^{tr}(G_1, M/G_2) \to \mathrm{H}^{tr}(G_2, G_1 \backslash M)$$

such that $K(G_2, G_1) \circ K(G_1, G_2) = Id$ on $\mathrm{H}^{tr}(G_1, M/G_2)$.

Here M/G_2 and $G_1 \backslash M$ are the factor spaces on which the other group acts in the obvious way; the H^{tr}-spaces are defined with respect to the lifts of the actions to the whole M.

The duality for twisted cocycles is "built over" the duality for non-twisted ones:

Theorem 2.2 *Let G_1, G_2, H and M be as above. Let $\mathcal{R}$ be any H-module. Let $\alpha \in \mathrm{H}^{tr}(G_1, M/G_2, H)$.*

Then

$$\mathrm{H}^{tr}(G_1, M/G_2, \alpha) \cong \mathrm{H}^{tr}(G_2, G_1\backslash M, K(G_1, G_2)(\alpha)).$$

To be more precise, there is a canonically defined group isomorphism (the spaces of twisted cocycles are naturally equipped with the abelian group structure induced from $\mathcal{R}$),

$$K_1(G_1, G_2) : \mathrm{H}^{tr}(G_1, M/G_2, \alpha) \to \mathrm{H}^{tr}(G_2, G_1\backslash M, K(G_1, G_2)(\alpha)),$$

such that $K_1(G_2, G_1) \circ K_1(G_1, G_2) = Id$ on $\mathrm{H}^{tr}(G_1, M/G_2, \alpha)$.

Remark: Whenever it is not likely to cause a misunderstanding we will denote the duality maps simply by K, instead of $K(G_1, G_2)$ and $K_1(G_1, G_2)$.

The following result is proved in [4].

Theorem 2.3 *Let G and Γ be as in Theorem 1.1. Let A be a maximal connected split Cartan subgroup of G. Let α be a constant real multiplicative cocycle, i.e. any representation of A in $\mathbb{R}^+$.*

Then for the C^∞ cocycles, every element of $\mathrm{H}(A, G/\Gamma, \mathbb{R}, \alpha)$ is C^∞ cohomologous to a constant twisted cocycle. (We think of $\mathbb{R}$ as being naturally equipped with the $\mathbb{R}^+$-module structure.)

Moreover, if α is non-trivial, i.e., not identically equal to 1, then

$$\mathrm{H}(A, G/\Gamma, \mathbb{R}, \alpha) = 0.$$

That is, its every element of $\mathrm{H}(A, G/\Gamma, \mathbb{R}, \alpha)$ is C^∞ cohomologous to a zero cocycle.

A slight modification of the arguments used to prove Theorem 4.1 in [5] yields the following

Theorem 2.4 *If $\mathcal{R}$, in addtition to being an H-module, also has a structure of an $\mathbb{R}^+$ module, and the actions of H and $\mathbb{R}^+$ on $\mathcal{R}$ commute, then, for any $\alpha \in \mathrm{H}(\Gamma, G, H)$,*

$$\mathrm{H}(\Gamma, G, \mathcal{R}, \alpha) = \prime,$$

where Γ acts on G by multiplication.

Remark 1: In fact, similarly to Theorem 4.1 in [5], the conclusion of Theorem 2.4 holds for a broad variety of totally discontinuous actions. For the details, see [5].

Remark 2: Another useful observation about trivializations of cocycles of subgroup actions on groups is the following: Assume G is a Lie group, P is its Lie subgroup and the orbit foliation for the left action of P on G admits a global transverse section T, which is a submanifold of G and every g in G decomposes as $p(g)t(g)$, where $p(g) \in P$, and $t(g) \in T$, and $p(g), t(g) : G \to G$ are C^∞. Let β be a cocycle or a twisted cocycle such that $\beta : P \times G \to H$ ($\beta : P \times G \to \mathcal{R}$) is C^∞ (which, of course, is much more than is required in the definition of a C^∞ cocycle). Then β is C^∞ cohomologous to a trivial cocycle or, correspondingly, trivial twisted cocycle.

Notice, that all elements of H^{tr}-subspaces have representatives that satisfy the above assumption.

Also, it is easy to see that all cocycles and twisted cocycles of G acting on G are trivial, regardless of the above mentioned regularity condition.

3 Proof of the main theorem.

Clearly it is enough to prove Theorem 1.1 for $N = N_0$.

Since $M = N \backslash P$ is parallelizable, $S(TM)$ can be identified with $F(M)$ which denotes the space of smooth functions on $P \backslash G$ with values in $\mathbb{R}^k$, where $k = dim M$.

Then the derivative of this action can be written as a cocycle $D(\gamma, m)$ with values in $GL(k, \mathbb{R})$: $D(\gamma, m)$ is equal to the derivative of the transformation $m \to m\gamma^{-1}$, evaluated at point $m \in P \backslash G$ in the coordinate system used to trivialize the tangent bundle over $P \backslash G$.

Then the action of Γ on $S(TM)$ translates into the following action on $F(M)$:

$$\gamma \cdot f(m) = D(\gamma, m\gamma) f(m\gamma) = D(\gamma^{-1}, m)^{-1} f(m\gamma), f(m) \in F(M).$$

Similarly the action of Γ on $S(T^*M)$ translates into the following action on $F(M)$:

$$\gamma \cdot f(m) = D(\gamma, m\gamma)^* f(m\gamma) = (D(\gamma^{-1}, m)^{-1})^* f(m\gamma), f(m) \in F(M),$$

where D^* denotes the inverse transpose of the matrix D.

We pass from the cohomologies to twisted cocycles by considering

$$\beta(\gamma, m) = \chi(\gamma^{-1}, m)$$

for $\chi(\gamma, m) \in \mathrm{H}^1_{(p,q)}(\Gamma, M)$. Then

$$\beta(\gamma, m) \in \mathrm{H}(\Gamma, N\backslash G, \mathbb{R}^{k^{(p+q)}}, (\otimes_1^p D^*) \otimes (\otimes_1^q D)),$$

and it is clear that $\mathrm{H}^1_{(p,q)}(\Gamma, M)$ is isomorphic to

$$\mathrm{H}(\Gamma, N\backslash G, \mathbb{R}^{k^{(p+q)}}, (\otimes_1^p D^*) \otimes (\otimes_1^q D)).$$

Lifting the action and the cocycle $(\otimes_1^p D^*) \otimes (\otimes_1^q D)$ to the whole G, and using Theorems 2.4, 2.1 and 2.2 (applied to the right action of Γ and the left action of N on G) we have

$$\mathrm{H}(\Gamma, N\backslash G, \mathbb{R}^{k^{(p+q)}}, (\otimes_1^p D^*)\otimes(\otimes_1^q D)) = \mathrm{H}^{tr}(\Gamma, N\backslash G, \mathbb{R}^{k^{(p+q)}}, (\otimes_1^p D^*)\otimes(\otimes_1^q D)) \cong$$

$$\cong \mathrm{H}^{tr}(N, G/\Gamma, \mathbb{R}^{k^{(p+q)}}, K((\otimes_1^p D^*) \otimes (\otimes_1^q D))).$$

It is clear from the construction of the duality map K (see [4] or [5]) that

$$K((\otimes_1^p D^*) \otimes (\otimes_1^q D)) = (\otimes_1^p K(D)^*) \otimes (\otimes_1^q K(D)).$$

It is proved in [4] that $K(D)$ is cohomologous to a constant cocycle given by the representation Ad_N of N which is the factor of the restriction of the adjoined representation of G to N by the subrepresentation in the tangent space $\mathcal{N}$ to N.

The exact matrix form of the cocycle $K(D) = Ad_N$ depends on the way we parallelize M. But different parallizations lead to equivalent representations, i.e., to cohomologous cocycles. Therefore, since the spaces of twised cocycles depend only on the cohomology class of the twisting, in order to study

$$\mathrm{H}(N, G/\Gamma, \mathbb{R}^{k^{(p+q)}}, (\otimes_1^p K(D)^*) \otimes (\otimes_1^q K(D)))$$

we may use whatever matrix representation of Ad_N we want.

It is easy to prove (see [4] or [5]) that $\mathcal{N}$ has a basis (as a linear space) that cosists of root vectors of G. Therefore, we may find a subspace $\mathcal{M}$ which is a complement of $\mathcal{N}$ in $\mathcal{G}$ and also has a basis v_i, $i = 1, \ldots, k$, consisting of root vectors. Let λ_i be the corresponding roots. Now, if $\mathfrak{a} \in \mathcal{A}$ and $a = exp(\mathfrak{a})$, $Ad_N(a)v_i = exp(\lambda_i(\mathfrak{a}))v_i$. We see that the matrix presentation of $Ad_N(a)$ is diagonal if $a \in A$. Therefore, the matrix $(\otimes_1^p K(D)^*(a)) \otimes (\otimes_1^q K(D)(a))$ is also diagonal, if $a \in A$.

Let $\beta(p,m) \in \mathrm{H}(N, G/\Gamma, \mathbb{R}^{k^{(p+q)}}, \alpha(p))$, $p \in N$, $m \in G/\Gamma$, where, both here and elsewhere, we denote the representation $(\otimes_1^p K(D)^*) \otimes (\otimes_1^q K(D)))$ of N by α. Consider the restriction of β to A. Then, the vector valued cocycle β splits into $k^{(p+q)}$ real cocycles. According to Theorem 2.3 each of the components is C^∞ cohomologous to a constant cocycle. Thus, the whole $\beta(a,m)$ is cohomologous to a constant cocycle. Therefore, from now on we may assume that $\beta(a,\cdot)$ is constant if $a \in A$.

Let $\theta(p) = \int_M \beta(p,m)d\mu$, where μ is the normalized G invariant smooth measure on G/Γ. Since α is a constant cocycle, and, thus, the integration commutes with multiplication by α, we see that θ is a constant cocycle belonging to $\mathrm{H}(N, G/\Gamma, \alpha)$. We will prove that $\beta(p,m) = \theta(p)$.

Indeed, let $\Delta(p,m) = \beta(p,m) - \theta(p)$. Then, $\Delta(a,m) = 0$, i.e., $\Delta(a,m)$ is a zero vector, if $a \in A$.

Let p_λ be such an element of N that $p_\lambda = exp(\mathfrak{p}_\lambda)$, where λ is a root of G, and $\mathfrak{p}_\lambda$ is a vector from the corresponding root space. Let $a \in Ker(\lambda)$. Then, since a and p_λ commute we have

$$\Delta(ap_\lambda, m) = \Delta(p_\lambda, m) + \alpha(p_\lambda)^{-1}\Delta(a, p_\lambda m) = \Delta(p_\lambda, m) =$$

$$\Delta(p_\lambda a, m) = \Delta(a,m) + \alpha(a)^{-1}\Delta(p_\lambda, am).$$

Therefore,

$$\Delta(p_\lambda, m) = \alpha(a)^{-1}\Delta(p_\lambda, am).$$

Then for the j-th component $\Delta(p_\lambda, m)_j$ of $\Delta(p_\lambda, m)$, $j = 1, \ldots, k^{(p+q)}$, we have:

$$\Delta(p_\lambda, m)_j = \alpha(a)_j^{-1}\Delta(p_\lambda, am)_j,$$

where $\alpha(a)_j$ denotes the j-th component of α. But for a non-zero function $\Delta(p_\lambda, m)_j$ such an equality is possible only if $\alpha(a)_j = 1$. And even then, due to Moore's ergodicity theorem, $\Delta(p_\lambda, m)_j$ has to be a constant function. But since $\Delta(p,m) = \beta(p,m) - \theta(p)$, all components of Δ must be orthogonal to constants, for all $p \in N$. Therefore, we see that $\Delta(p_\lambda, m) = 0$.

Since A and the exponents of the root vectors from $\mathcal{N}$ generate the whole N we conclude that $\Delta(p,m) = 0$ for all $p \in N$ and $m \in M$. Therefore we have that $\beta(p,m)$ is cohomologous to a constant cocycle $\theta(p)$.

We denote by

$$(\mathrm{H}^1)^{tr}(N, (\otimes_1^p Ad_N^*) \otimes (\otimes_1^q Ad_N))$$

the subspace in

$$\mathrm{H}^1(N, (\otimes_1^p Ad_N^*) \otimes (\otimes_1^q Ad_N))$$

that consists of those elements of $\mathrm{H}^1(N, (\otimes_1^p Ad_N^*) \otimes (\otimes_1^q Ad_N))$ that correspond to trivial elements of $\mathrm{H}(N, G, \mathbb{R}^{k^{(p+q)}}, \alpha)$.

Then

$$\mathrm{H}^1_{(p,q)}(N, M) \cong (\mathrm{H}^1)^{tr}(N, (\otimes_1^p Ad_N^*) \otimes (\otimes_1^q Ad_N)).$$

On the other hand,

$$\mathrm{H}^1_{(p,q)}(G, M) = \mathrm{H}(G, N\backslash G, \mathbb{R}^{k^{(p+q)}}, (\otimes_1^p \overline{D}^*) \otimes (\otimes_1^q \overline{D})) =$$

$$= \mathrm{H}^{tr}(G, N\backslash G, \mathbb{R}^{k^{(p+q)}}, (\otimes_1^p \overline{D}^*) \otimes (\otimes_1^q \overline{D})) \cong$$

$$\cong \mathrm{H}^{tr}(N, G/G, \mathbb{R}^{k^{(p+q)}}, K(G,N)((\otimes_1^p \overline{D}^*) \otimes (\otimes_1^q \overline{D}))) =$$

$$= \mathrm{H}^{tr}(N, G/G, \mathbb{R}^{k^{(p+q)}}, (\otimes_1^p K(G,N)(\overline{D})^*) \otimes (\otimes_1^q K(G,N)(\overline{D}))),$$

where $\overline{D}$ is the derivative cocycle for the action of G on $G\backslash N$. It follows from the construction of the duality maps [4], [5] and the fact that $\overline{D}$ is an extension of D that $K(G,N)(\overline{D}) = K(\Gamma, N)(D)$.

Therefore, we have (since G/G consists of a single point)

$$\mathrm{H}^1_{(p,q)}(G, M) = (\mathrm{H}^1)^{tr}(N, (\otimes_1^p Ad_N^*) \otimes (\otimes_1^q Ad_N)).$$

And, moreover, it follows from the constructions in [4] that for any element $\xi \in (\mathrm{H}^1)^{tr}(N, (\otimes_1^p Ad_N^*) \otimes (\otimes_1^q Ad_N))$ the element

$$K_1(N, G)(\xi) \in \mathrm{H}^1_{(p,q)}(G, M)$$

is an extension of the element

$$K_1(N, \Gamma)(\xi) \in \mathrm{H}^1_{(p,q)}(\Gamma, M).$$

Theorem 1.1 is proved.

4 Projective actions.

In this section we consider the case $G = SL(n, \mathbb{R})$, $n \geq 4$. Let P be the minimal parabolic subgroup of G consisting of the upper triangular martices with positive diagonal entries and the determinant equal to 1. Then the following is true:

Proposition 4.1 *The action of* Γ *on* $P\backslash G$

1. *is not* C^∞ *infinitesimal* (p, q)*-tensor rigid if* $p \leq (n-1)q$ *and* $q \leq (n-1)p$;
2. *is* C^∞ *infinitesimal* (p, q)*-tensor rigid if* $p > (n-1)(q+1)$, *or if* $q > (n-1)(p+1)$;
3. *may or may not be* C^∞ *infinitesimal* (p, q)*-tensor rigid if* $(n-1)q < p \leq (n-1)(q+1)$, *or if* $(n-1)p < q \leq (n-1)(p+1)$.

Many important points in the proof of Proposition 4.1 rely on the arguments similar to the ones used in [4], [5] and [7]. Therefore, considering the fact that, on one hand, these arguments are explained in complete detail in the works cited above, and, on the other hand, that they are quite tedious, we will only outline the proofs of Propositions 4.1 and 4.2.

Outline of the proof of Proposition 4.1: First of all we notice that the action of P on G has a smooth transversal cross-section, as required in Remark 2 following the statement of Theorem 2.4 at the end of Section 2. Therefore

$$(\mathrm{H}^1)^{tr}(P, (\otimes_1^p Ad_P^*) \otimes (\otimes_1^q Ad_P)) = \mathrm{H}^1_{C^\infty}(P, (\otimes_1^p Ad_P^*) \otimes (\otimes_1^q Ad_P)),$$

where

$$\mathrm{H}^1_{C^\infty}(P, (\otimes_1^p Ad_P^*) \otimes (\otimes_1^q Ad_P))$$

denotes the set of those elements in

$$\mathrm{H}^1(P, (\otimes_1^p Ad_P^*) \otimes (\otimes_1^q Ad_P))$$

that satisfy the additional smoothness requirement with respect to $p \in P$.

Proof of 1): Under the conditions on p and q we can find some root vectors v_l, $l = 1, \ldots, p$, and v_m, $m = 1, \ldots, q$, from the basis of $\mathcal{M}$ such that for the corresponding roots λ_l, $l = 1, \ldots, p$, and λ_m, $m = 1, \ldots, q$,

$$\sum_{l=1}^{p} \lambda_l - \sum_{m=1}^{q} \lambda_m = 0 \tag{1}$$

(This follows from the observation the the root λ_{n1} may be represented as a sum of any number, from 1 to $n-1$ of negative roots of G, where λ_{n1} is the root corresponding to the root vector E_{1n}, where E_{ij}, $i,j=1,\dots,n$ are the elementary matrices.)

Then the diagonal entry $\alpha(a)_j$ of the matrix $\alpha(a)$ corresponding to the basis vector $(\otimes_{l=1}^p v_l^*)\otimes(\otimes_{m=1}^q v_m)$ in $(\otimes_{l=1}^p \mathcal{M}^*)\otimes(\otimes_{\Updownarrow=\infty}^{\amalg}\mathcal{M})$ is identically equal to 1.

If we let $\beta(a)$ be equal to a vector with zero components at all places except the j-th one, and with a non-trivial additive one-dimensional real representation of A at the j-th place, we get a non-trivial element of $\mathrm{H}^1_{C^\infty}(A,\alpha)$. Now we can extend β to a non-trivial element of $\mathrm{H}^1_{C^\infty}(P,\alpha)$ defining $\beta(p,m)=0$ for any unipotent $p\in P$.

This means that there are non-trivial elements of $\mathrm{H}^1_{(p,q)}(\Gamma,P\backslash G)$.

Proof of 2): For p and q as in 2) we see that the equality(1) is impossible. Therefore, every diagonal entry of $\alpha(a)$ is a non-trivial representation of A. And by Theorem 2.3 we may assume that for any $\beta\in\mathrm{H}^1(P,\alpha)$, $\beta(a)=0$ for all $a\in A$. Then for a fixed j there are two possibilities.

The first is that $log(\alpha(a)_j)$ is not proportional to a root λ of G that corresponds to a vector $\mathfrak{p}_\lambda$ from $\mathcal{P}$, i.e., a positive root since P is the minimal parabolic. Then similarly to the proof of Theorem 1.1 we may prove that for $p_\lambda=exp(\mathfrak{p}_\lambda)$ the j-th component of $\beta(p_\lambda)$ is equal to zero.

The second possibility is that $log(\alpha(a)_j)$ is proportional to λ. But due to the restrictions on p and q the proportionality coefficient can not be equal to 1. Then, if it is negative, we may use the fact that $\alpha(a)_j$ decreases when λ increases and vice versa to prove that $\beta(p_\lambda)_j=0$ if β belongs to the H^{tr}-space. (for detailed similar argument see [4]). If the proportionality coefficient is positive, we can use the fact that $\alpha(a)_j$ grows at least two times faster then λ to prove that still $\beta(p_\lambda)_j=0$ (a detailed argument of this kind is carried out in [7]).

This proves that, for β from the H^{tr}-space, $\beta(p)=0$ for all $p\in P$ and, thus, that $\mathrm{H}^1_{(p,q)}(\Gamma,P\backslash G)=0$.

Proof of 3): It is hard to give a complete answer to the question about the infinitesimal (p,q)-tensor rigidity for p and q as in 3). All we can say is that for different p and q satisfying 3) the action may or may not be rigid.

For example, it is $(0,q)$-rigid, for any q. This follows from the arguments similar to the ones used for a second possibility in 2) (for the details see [4]). In particular, it is infinitesimally rigid in the usual sense [8].

On the other hand, it is not $(1,0)$-rigid. Indeed, let us order the basis E_{ij}, i,j in $\mathcal{M}$ so that E_{ij} goes before E_{kl} if either $i<k$, or $i=k$ but $j<l$. Then,

$$\beta(p)=(p_{11}^{-1}p_{12},p_{11}^{-1}p_{13},p_{11}^{-1}p_{14},\ldots,p_{11}^{-1}p_{1n},0,0,0,\ldots,0)^{\top}$$

defines a constant non-trivial element of $\mathrm{H}_{C^\infty}(P,G/\Gamma,Ad_P^*)$. (Here, we assume that p is given by a matrix $\{p_{ij}\}$.) Moreover, similar to the end of the proof of Proposition 4.2, one can deduce that there is no rigidity for any q and $p=q+1$.

This finishes the proof of Proposition 4.1.

Remark: We can easily extend Proposition 4.1 to include finite extensions of P; for example the subgroup of all upper triangular matrices with the determinant equal to 1, which corresponds to the Γ action on the complete flags manifold.

Our study of the tensor infinitesimal rigidity was prompted by attempts to generalize the methods used in [7] to prove the local C^3 rigidity of projective actions of cocompact lattices in $PSL(2,\mathbb{R})$ to a higher dimensional situation. Namely, we wanted to prove some sort of local rigidity for the action of cocompact lattices Γ in $SL(n,\mathbb{R})$ on the $n-1$ dimensional projective space $\mathbb{P}R^{n-1}$. It turns out that for C^3 actions on projective manifolds the existence of a C^3 conjugacy of the action to a projective action is equivalent to a cohomological triviality of a certain continuous $(2,1)$-tensor cocycle of the action which is the milti-dimensional analog of the Schwarzian cocycle. In particular, continuous infinitesimal $(2,1)$-tensor rigidity of a perturbation of the Γ action on S^{n-1} would imply that the perturbed action is C^3 conjugate to the original one. Unfortunately, even the original action is not infinitesimal $(2,1)$-tensor rigid. (Of course, it does not mean that there is no local rigidity. It only means that more accurate information about the Schwarzian cocycle is necessary to prove it.)

Because of the $(\times)$ condition in Theorem 1.1 our methods are applicable to the actions on spheres only of dimension 3 and 7, i.e. $n=4$ and $n=8$. In these cases we have:

Proposition 4.2 *The action of Γ on S^{n-1}*

1. *is not C^∞ infinitesimal (p,q)-tensor rigid if $p=q$ or $p-1=q$;*
2. *is C^∞ infinitesimal (p,q)-tensor rigid for all other p and q.*

Remark: Proposition 4.2 describes the C^∞ tensor cohomologies, so strictly speaking it does not answer the question related to the Schwarzian cocycle. But, in fact, a similar result holds for the continuous cocycles as well (see Section 5).

Sketch of the proof. The proof of this proposition is very similar to the proof of Proposition 4.1. The reason for the fact that the proof gives a positive answer to the question about the infinitesimal (p,q)-tensor rigidity for a larger set of p, q than for the minimal parabolic case is that in this case $\mathcal{M}$ is spanned by only $n-1$ root vectors E_{1k}, $k=1,\dots,n$. This leaves us considerably "less space" for finding "bad" components of $\alpha(a)$, which are the source of the non-trivial elments in $\mathrm{H}_{C^\infty}(P,G/\Gamma,\alpha)$.

Namely, the only way to satisfy equation 1 is to have $p=q$, and then as before we see that $\mathrm{H}_{C^\infty}(P,G/\Gamma,\alpha)$ is non-trivial. And the only way to have the $log(\alpha(a)_j)$ be equal to a positive root is to to have $p=q-1$. In this case we can construct non-trivial cocycles for any q. Namely, similar to the minimal parabolic case

$$\beta(p)=(p_{11}^{-1}p_{12},p_{11}^{-1}p_{13},p_{11}^{-1}p_{14},\dots,p_{11}^{-1}p_{1n})^\top$$

defines a non-trivial constant element of $\mathrm{H}_{C^\infty}(P,M,Ad_P^*)$. Then we notice that $\mathcal{M}^*\otimes\mathcal{M}$ is canonically isomorphic to the space of linear transformations of $\mathcal{M}$. Therefore, the identity transformation $Id_{\mathcal{M}}$ of $\mathcal{M}$ is an element of $\mathcal{M}^*\otimes\mathcal{M}$ invariant with respect to $Ad_P^*\otimes Ad_P$. Thus $(\otimes_1^q Id_{\mathcal{M}})\otimes\beta(p)$ is a non-trivial constant element of $\mathrm{H}_{C^\infty}(P,M,(\otimes_1^q(Ad_P\otimes Ad_P^*)\otimes Ad_P^*)$.

Therefore, $\mathrm{H}^1_{(p,p)}(\Gamma,M)$ and $\mathrm{H}^1_{(p+1,p)}(\Gamma,M)$ are non-trivial for every p.

But, if $p\neq q$ and $p\neq q-1$, then we can prove the triviality of $\mathrm{H}^1_{(p,p)}(\Gamma,M)$ by methods similar to the ones used in the proof of Proposition 4.1 and in [4], [5], [7].

5 Continuous, L^2 and Hoelder cohomologies.

In this section we mention some continuous, L^2 and Hoelder results analogous to the C^∞ results of this paper.

Continuous and L^2 cases. The continuous and L^2 analogs of Theorem 1.1 are true even for a larger class of G. Namely, using the continuous results of [4] and [5] and their L^2 analogs (their proofs are straightforward compilations of the proofs for the continuous case), we may get rid of the condition that

G does not have local factors of rank one, and relax the rank condition to $\mathbb{R}$-rank being bigger than one.

The analog of Proposition 4.1 is true for $n \geq 3$, and Proposition 4.2 remains unchanged.

Hoelder case. Hoelder results are the same as continuous ones but we have to impose the additional condition of G being split and without factors locally isomorphic to $PSL(2, \mathbb{R})$. (The main reason for this restriction is that if G is split then the action of A on G/Γ is Anosov. For a detailed account of why this fact is so important for the Hoelder cocycles rigidity see [1] and [2].)

Perturbed actions. For a C^3 perturbation of the projective Γ actions, small enough in the C^1 topology on a set of finitely many generators, the affirmative parts of Propositions 4.1 and 4.2 (i.e., the second statement in 4.1 and 4.2) still hold in continuous and L^2 cases.

The proof relies on the fact that, due to the presence of a continuous conjugacy (see Remark at the end of the paper), the tensor cohomologies are isomorphic to twisted cocycles with the twistings close to the ones for the original Γ action. And we can prove their triviality using the continuity of the duality maps in C^0 topology, and the fact that the methods for the trivialization of cocycles used in [4] and [7] were actually designed to deal with the twistings close to constant ones, not just the constant ones as we did in the proofs of Propositions 4.1 and 4.2.

Remark: The existence of continuous conjugacy is a well known, but apparently unpublished result. The suspension construction that provides an easy proof for the maximal boundary case can be found, for example, in [9]. And the reduction to the minimal case may be obtained by the Katok-Spatzier method that they use for the smooth reduction in [3].

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