ON AN INTEGRAL EQUATION FOR CENTER MANIFOLDS: A DIRECT PROOF FOR NONAUTONOMOUS DIFFERENTIAL EQUATIONS ON BANACH SPACES

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Abstract. We study a nonlinear integral equation for a center manifold of a semilinear nonautonomous differential equation having mild solutions. We assume that the linear part of the equation admits, in a very general sense, a decomposition into center and hyperbolic parts. The center manifold is obtained directly as the graph of a fixed point for a Lyapunov-Perron type integral operator. We prove that this integral operator can be factorized as a composition of a nonlinear substitution operator and a linear integral operator Λ. The operator Λ is formed by the Green's function for the hyperbolic part and composition operators induced by a flow on the center part. We formulate the usual gap condition in spectral terms and show that this condition is, in fact, a condition of boundedness of Λ on corresponding spaces of differentiable functions. This gives a direct proof of the existence of a smooth global center manifold.

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1. Introduction

In this paper we give a direct proof of the existence of global smooth center manifolds of mild solutions of nonautonomous differential equations with sufficiently small nonlinearities defined on infinite dimensional Banach spaces. From its inception in now classic works, [4, 12, 18, 19, 27], the center manifold theory plays an important role in the modern theory of infinite dimensional dynamical systems. We refer the reader to [2, 3, 5, 9, 10, 13, 15, 22, 30, 37, 38, 42] and to the literature cited therein. Recently, several advances in the center manifold theory have been made using techniques related to scales of Banach spaces of exponentially growing functions, see [5, 11, 14, 39, 40, 41, 42, 43]. However, in this paper we will use the perhaps more direct traditional approach, see [5, 8, 12].

To explain our main innovation, consider an autonomous semilinear differential equation on a Banach space X:

$$x' = Ax + g(x), \quad t \in \mathbb{R}, \quad g \in C^{k,1}(X, X), \tag{1}$$

where A is the infinitesimal generator of a strongly continuous semigroup, $\{e^{tA}\}_{t\geq 0}$ such that the spectral mapping theorem holds, that is, $\sigma(e^{tA}) \setminus \{0\} = \exp t\sigma(A)$, $t > 0$. Further, suppose that the spectrum $\sigma(A)$ splits as

$$\sigma(A) = \sigma_c \cup \sigma_h, \quad \sigma_c, \sigma_h \neq \emptyset, \quad \omega_c < \beta_h, \tag{2}$$

where the numbers ω_c and β_h are computed as follows:

$$\omega_c = \sup\{|\operatorname{Re} z| : z \in \sigma_c\}, \quad \beta_h = \inf\{|\operatorname{Re} z| : z \in \sigma_h\}. \tag{3}$$

Under these assumptions there exists an A-invariant decomposition $X = X_c \oplus X_h$ such that, for the restrictions $A_c = A|X_c$, $A_h = A|X_h$ one has $\sigma_c = \{z : z \in \sigma(A_c)\}$ and $\sigma_h = \{z : z \in \sigma(A_h)\}$. Moreover,

$\{e^{tA_c}\}$ is a group on X_c, the semigroup $\{e^{tA_h}\}$ is hyperbolic on X_h, and there are complementary projections $P_\pm$ on X_h, restrictions of operators $A_h^\pm = A_h|\operatorname{Im} P_\pm$, and constants $M_c = M_c(\omega)$, $M_h = M_h(\beta)$ for which the following estimates hold

$$\|e^{tA_c}\| \le M_c e^{\omega|t|},\ t \in \mathbb{R}, \quad \|e^{tA_h^+}\| \le M_h e^{-\beta t},\ t > 0, \quad \|e^{tA_h^-}\| \le M_h e^{\beta t}, t < 0 \tag{4}$$

whenever $\omega > \omega_c$ and $\beta \in (0, \beta_h)$.

We assume that $k \in \{0, 1, 2, \dots\}$ and let $C^{k,1}(X, X)$ denote the space of k times continuously differentiable functions with Lipschitz kth derivative. As usual, we seek a center manifold for (1) of the form $\mathcal{M}_\psi = \{\xi + \psi(\xi) : \xi \in X_c\}$ where the function $\psi : X_c \to X_h$ is in $\psi \in B_\delta^{(k)}$, a δ-ball at the origin of $C^{k,1}(X, X)$. If we fix $\psi \in B_\delta^{(k)}$, project (1) on $X_c \oplus X_h$ to obtain "c and h equations", and use the Green's function $\mathcal{K}$, to be defined explicitly below, for the hyperbolic semigroup $\{e^{tA_h}\}_{t\ge 0}$, we see that $\mathcal{M}_\psi$ is invariant if and only if ψ is a fixed point of the operator $\mathcal{T}$ defined by

$$\mathcal{T}(\psi)(\xi) = \int_{-\infty}^{\infty} \mathcal{K}(-\tau) g_h(x_c(\tau, \xi) + \psi(x_c(\tau, \xi)))\, d\tau,$$

where $\xi \in X_c$, and $x_c(\cdot, \xi)$ is the solution of the c-equation $x_c' = A_c x_c + g_c(x_c + \psi(x_c))$ with the initial condition $x_c(0, \xi) = \xi$.

The solution $x_c(\cdot, \xi)$ defines a flow $S^t = S_\psi^t$ on X_c given by $S^t(\xi) = x_c(t, \xi)$. Using it, we introduce a group of *composition operators*, V^t, on $C^{k,1}(X_c, X_h)$ given by $(V^t\varphi)(\xi) = \varphi(S^t\xi)$. The operator $\mathcal{T}$ can be factored as a composition, $\mathcal{T}(\psi) = \Lambda_\psi \circ \mathcal{G}(\psi)$, where Λ_ψ is a *linear* operator and $\mathcal{G}$ is a nonlinear operator defined as follows:

$$\Lambda_\psi = \int_{-\infty}^{\infty} \mathcal{K}(-\tau) V^t\, d\tau, \quad \mathcal{G}(\varphi)(\xi) = g_h(\xi + \varphi(\xi)), \quad \xi \in X_c. \tag{5}$$

The ball $B_\delta^{(k)}$ is closed in the C^k-norm. To find a fixed point ψ for $\mathcal{T}$ one needs to show, first of all, that $\mathcal{T}$ preserves $B_\delta^{(k)}$ provided $\epsilon = \|g\|_{C^{k,1}}$ is sufficiently small. Using the factorization of $\mathcal{T}$, it is easy to see that $\mathcal{T}$ preserves such a small ball provided that Λ_ψ is a bounded operator on $C^{k,1}$.

A key feature of our analysis is the observation that the boundedness of Λ_ψ on $C^{k,1}$ is implied by the spectral gap condition

$$\beta_h - (k+1)\omega_c > 0.$$

Indeed, a direct (but long) calculation using the estimates (4) shows that the growth of $\|V^t\|$ on $C^{k,1}$ is bounded by $\exp[(k+1)(\omega + M_c\epsilon)|t|]$. Also, by (4), the growth of the norm of the Green's function, $\|\mathcal{K}(t)\|$, is bounded by $\exp[-\beta|t|]$. Using these facts together with the definition of Λ_ψ and the gap condition, it is easy to see that for ϵ small enough, Λ_ψ is bounded.

A similar argument, again using (4) and (5), shows that $\mathcal{T}$ is a contraction in the C^k-norm for sufficiently small ϵ provided the gap condition is satisfied. We remark that the C^k-contractivity of $\mathcal{T}$ for $k > 0$ is more than is needed to prove the desired result. In fact, by a lemma of D. Henry (see, e.g., [8]), the ball $B_\delta^{(k)}$ is closed in the C^0-norm. Thus, it suffices to prove that $\mathcal{T}$ is a C^0-contraction.

The arguments outlined above do not use the bounds ω_c and β_h that appear in (3) directly. Instead, they use the fact that β_h is the dichotomy bound and ω_h is the growth bound for the corresponding semigroups in the following sense: $\beta_h = \sup \beta$ and $\omega_h = \inf \omega$ for β and ω satisfying (4). We stress that the calculation (3) of $\beta_h = \sup \beta$ and $\omega_h = \inf \omega$ in terms of $\sigma(A)$ is possible because we assume the validity of the spectral mapping theorem for $\{e^{tA}\}$. These calculations are *invalid* for an arbitrary generator A. The spectral mapping theorem holds, for instance, for analytical semigroups, uniformly continuous semigroups, etc., see [28] for details. This point of view is helpful in understanding the corresponding spectral results in [3, 40].

In the present paper we will consider a generalization of (1) to nonautonomous differential equations of the form

$$x' = A(t)x + g(t, x(t)), \quad g \in C^{k,1}(\mathbb{R} \times X, X), \quad t \in \mathbb{R}. \tag{6}$$

We do not require that the linearization of (6) is well-posed, that is, that it has a differentiable propagator. Instead, our starting point is a strongly *continuous* evolutionary family (propagator) $\{U(t,s)\}_{t\geq s}$. Thus, we consider mild solutions of (6). The operators $A(t)$ that appear in (6) can be unbounded, can have variable domains, etc.

One of the difficulties encountered in our general setting is to formulate the correct assumptions on the decomposition of $\{U(t,s)\}_{t\geq s}$ into its "center" and "hyperbolic" parts. The spectra of the operators $A(t)$ do not give the appropriate information for this decomposition even when these operators are bounded. To remedy this, we introduce the so-called *evolutionary semigroup*, $\{T^t\}$, acting on a "super-space" $C_0(\mathbb{R}, X)$ of continuous X-valued functions that vanish at infinity, by the rule $(T^t\phi)(\tau) = U(\tau - t, \tau)\phi(\tau - t)$, $\tau \in \mathbb{R}$, $t > 0$. Let Γ denote the infinitesimal generator for $\{T^t\}$. For example, in the autonomous case above, Γ is just the closure of the operator $-d/d\tau + A$. It is known, see [23, 24, 33, 34] that the spectral mapping theorem holds for $\{T^t\}$, that $\sigma(\Gamma)$ is invariant with respect to translations along the imaginary axis, and that $\{U(t,s)\}_{t\geq s}$ has an exponential dichotomy on X if and only if Γ is invertible, or, equivalently, if T^t, for some $t > 0$, has no spectrum on the unit circle. Moreover, each spectral projection $\mathcal{P}$ for T^t

has the form $(\mathcal{P}\phi)(\tau) = P(\tau)\phi(\tau)$ for a bounded strongly continuous projection-valued function P defined on $\mathbb{R}$.

Our conditions on $\{U(t,s)\}_{t\geq s}$ are given in terms of the existence of a splitting of $\sigma(\Gamma)$ similar to (2). This splitting implies the existence of a spectral decomposition for T^t. Since spectral projections for T^t are operators of multiplication on projection-valued functions, we obtain the existence of evolutionary families $\{U_c(t,s)\}_{(t,s)\in\mathbb{R}^2}$ and $\{U_h(t,s)\}_{t\geq s}$, generalizing $\{e^{(t-s)A_c}\}_{(t,s)\in\mathbb{R}}$ and $\{e^{(t-s)A_h}\}_{t\geq s}$, considered above. We *define* ω_c and β_h that appear in the gap condition in terms of $\sigma(\Gamma)$, and *conclude* that ω_c and β_h are the growth and the dichotomy bounds for $\{U_c(t,s)\}_{(t,s)\in\mathbb{R}^2}$ and $\{U_h(t,s)\}_{t\geq s}$, respectively.

We construct an operator $\mathcal{T}$ and its factorization $\mathcal{T} = \Lambda \circ \mathcal{G}$ analogous to the construction above. Again, the gap condition is seen to be a condition on the operator Λ that controls the "race" between the exponential growth of the Green's function for $\{U_h(t,s)\}_{t\geq s}$ and composition operators V^t induced by $\{U_c(t,s)\}_{(t,s)\in\mathbb{R}^2}$. The gap condition again implies that Λ is bounded as long as the norm of g is sufficiently small. As a result, we prove the existence of a center manifold $\mathcal{M}_\psi$ for $\psi \in C^{k,1}$. We conjecture that, by a standard technique (see, e.g., [8]), this result in fact implies $\psi \in C^{k+1}$.

The main point of this paper is the presentation of a new direct proof of the existence of a center manifold based on a factorization analogous to (5). Our proof gives a clear understanding of the role of the gap condition that is imposed on the linear part in (6). Moreover, as far as we know, a proof of the existence of center manifolds in our general setting has not appeared in the literature.

The paper is organized as follows. In Section 2 we collect some facts about evolutionary families, evolutionary semigroups, and composition operators. In Section 3 we precisely describe our assumptions and prove the existence of center manifolds using certain norm estimates on the composition operator V^t. Section 4 contains the technical proofs of the required norm estimates. Finally, Section 5 contains an application of our result. In particular, we prove the existence of a Lipschitz invariant manifold for a semilinear skew-product flow. The existence of smooth invariant manifolds in the analogous finite dimensional case is considered in [11].

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2. Notations and Preliminaries

2.1. **Evolution families.** Let X denote a Banach space with norm $|\cdot|$ and let $\mathcal{L} = \mathcal{L}(X)$ denote the set of bounded linear operators on X. For a linear operator A, let $\sigma(A)$ denote the spectrum, $\rho(A)$ the resolvent set, $D(A)$ the domain, and $\mathrm{Im}(A)$ the image of the operator A. Also, let $J = \{(t,s) \in \mathbb{R}^2 : t \geq s\}$ or $J = \mathbb{R}^2$.

Definition 2.1. *An* evolution family *is a family of bounded operators* $\{U(t,s)\}_{(t,s)\in J}$ *on* X *that satisfy, for* $(t,s) \in J$*:*

i) $U(t,s) = U(t,r)U(r,s)$ *for all* $t \geq r \geq s$*;*

ii) $U(t,t) = I$*;*

iii) For each $x \in X$ *the function* $(t,s) \mapsto U(t,s)x$ *is continuous.*

iv) There exist positive constants β and c such that

$$\|U(t,s)\|_{\mathcal{L}} \leq ce^{\beta(t-s)}, \quad (t,s) \in J. \tag{7}$$

We remark that usually an evolution family is defined to satisfy just (i) and (ii) of the definition. If, in addition, it satisfies (iii) the family is called strongly continuous and if it satisfies (iv) it is called exponentially bounded. However, since we will only consider families satisfying all the properties of the definition, we will not make these distinctions below.

Definition 2.2. *An evolution family $\{U(t,s)\}_{t\geq s}$ is said to* solve *the abstract Cauchy problem*

$$x'(t) = A(t)x(t), \quad x(s) = x_s, \quad x_s \in D(A(s)), \quad t \geq s, \quad t,s \in \mathbb{R}, \tag{8}$$

if $x(\cdot) = U(\cdot,s)x_s$ is differentiable, $x(t) \in D(A(t))$ for $t \geq s$, and if x satisfies the differential equation defined in (8)*. The abstract Cauchy problem* (8) *is called* well-posed *if it is solved by an evolution family.*

Note that under our definition, the operators $A(t)$ in (8) are allowed to be unbounded. However, we will work in an even more general setting. In fact, we *will not require* that the evolution family $\{U(t,s)\}_{t\geq s}$ is differentiable. Thus, we do not assume that our evolution family solves an abstract Cauchy problem. We note that there are examples [32] of well-posed autonomous Cauchy problems $x' = A_0x$ such that (even for a bounded continuous function $B : \mathbb{R} \to \mathcal{L}(X)$) the abstract Cauchy problem (8) with $A(t) = A_0 + B(t)$ is not well-posed.

Definition 2.3. *Assume $J = \mathbb{R}^2$. The* growth bound ω_U *for an evolution family $\{U(t,s)\}_{(t,s)\in\mathbb{R}^2}$ on X is defined as the infimum over all*

$\omega > 0$ such that there exists a positive constant $M = M(\omega)$ so that the following estimate holds:

$$\|U(t,s)\|_{\mathcal{L}} \le Me^{\omega|t-s|}, \quad (t,s) \in \mathbb{R}^2. \tag{9}$$

If $A(t) \equiv A$ is a generator of a strongly continuous semigroup, $\{e^{tA}\}_{t\ge 0}$, on X, then $U(t,s) := e^{(t-s)A}$, $t \ge s$, defines an evolution family that solves (8). If A generates a strongly continuous group, then this evolution family is defined on $J = \mathbb{R}^2$. We stress that, in the last case, the growth bound for $U(t,s) := e^{(t-s)A}$, $t, s \in \mathbb{R}$ is, generally, strictly greater then the *spectral bound* for A defined as $\sup\{|\operatorname{Re} z| : z \in \sigma(A)\}$, see [28].

We recall the definition of exponential dichotomy for a strongly continuous evolution family, see [12, 18, 23, 27, 29]. For a projection-valued function $P_+ : \mathbb{R} \to \mathcal{L}(X)$, the complementary projection will be denoted by $P_-(t) = I - P_+(t)$, $t \in \mathbb{R}$. Suppose that $\{U(t,s)\}_{t\ge s}$ is an evolution family. If $P_+(t)U(t,s) = U(t,s)P_+(s)$ for all $t \ge s$, we define the restrictions $U_+(t,s) := P_+(t)U(t,s)P_+(s)$ and $U_-(t,s) := P_-(t)U(t,s)P_-(s)$. We stress that $U_+(t,s)$ is an operator from $\operatorname{Im} P_+(s)$ to $\operatorname{Im} P_+(t)$ and $U_-(t,s)$ acts from $\operatorname{Im} P_-(s)$ to $\operatorname{Im} P_-(t)$.

Definition 2.4. *An evolution family $\{U(t,s)\}_{t\ge s}$ is said to have an exponential dichotomy with constant $\beta > 0$ if there exists a projection-valued function $P_+ : \mathbb{R} \to \mathcal{L}(X)$ such that the function $t \mapsto P_+(t)x$ is continuous and bounded for each $x \in X$, and if, for some constant $M = M(\beta) > 0$ and all $t \ge s$, the following hold:*

i) $P_+(t)U(t,s) = U(t,s)P_+(s)$;

ii) $U_-(t,s)$ is invertible as an operator from $\operatorname{Im} P_-(s)$ to $\operatorname{Im} P_-(t)$;

iii) $\|U_+(t,s)\|_{\mathcal{L}} \le Me^{-\beta(t-s)}$;

iv) $\|U_-(t,s)^{-1}\|_{\mathcal{L}} \le Me^{-\beta(t-s)}$.

If an evolution family $\{U(t,s)\}_{t\geq s}$ *has an exponential dichotomy, the* dichotomy bound β_U *for* $\{U(t,s)\}_{t\geq s}$ *is defined as* $\sup\{\beta > 0 : \{U(t,s)\}_{t\geq s}$ *has exponential dichotomy with constant* $\beta\}$. *The* Bohl spectrum, $\mathcal{B}$, *for* $\{U(t,s)\}_{t\geq s}$ *is defined as*

$$\mathcal{B} = \{\lambda \in \mathbb{R} : \{e^{\lambda(t-s)}U(t,s)\}_{t\geq s} \textit{ does not have exponential dichotomy } \}.$$

We note that Bohl spectrum, see [12], is in fact the same as the Sacker-Sell dynamical spectrum as defined in [35, 36], see [24] for more details.

2.2. **Evolution semigroups.** Consider an exponentially bounded evolution family $\{U(t,s)\}_{t\geq s}$ defined on the Banach space X and let $C_0(\mathbb{R}, X)$ denote the space of continuous functions from $\mathbb{R}$ to X that vanish at infinity with the uniform norm. There is a natural *evolution semigroup*, $\{T^t\}_{t\geq 0}$, defined on $C_0(\mathbb{R}, X)$ as follows:

$$(T^t\phi)(\tau) = U(\tau, \tau - t)\phi(\tau - t), \quad t \geq 0. \tag{10}$$

This semigroup is, by our definition of exponentially bounded evolution families, strongly continuous and therefore it has an infinitesimal generator Γ. For example, if $A : \mathbb{R} \to \mathcal{L}$ is bounded, then Γ is the closure of the operator $-d/d\tau + A(\tau)$, see [24].

The evolution semigroup defined using the propagator of a nonperiodic differential equation plays the same role as the monodromy operator does for the periodic case.

The following is a list of facts about evolution semigroups and their infinitesimal generators that we will use in our analysis, see [23, 24, 33, 34] for details and further references.

1. The spectrum of Γ is invariant with respect to translations along the imaginary axis; the spectrum of T^t is invariant with respect to rotations centered at origin.

2. The semigroup spectral mapping theorem holds; that is, $\sigma(T^t) \setminus \{0\} = e^{t\sigma(\Gamma)}$, $t > 0$.

3. If $\mathcal{P}$ is a spectral projection for T^1 that corresponds to a connected component in $\sigma(T^1)$, then there is a bounded strongly continuous projection-valued function $P : \mathbb{R} \to \mathcal{L}(X)$ such that $(\mathcal{P}\phi)(\tau) = P(\tau)\phi(\tau)$, $\tau \in \mathbb{R}$. Also, $P(t)U(t,s) = U(t,s)P(s)$ for all $t \geq s$.

4. The evolution family $\{U(t,s)\}_{t\geq s}$ has an exponential dichotomy if and only if $0 \in \rho(\Gamma)$ or, equivalently, $\sigma(T^t)$ does not intersect the unit circle. If this is the case, then the Riesz projection $\mathcal{P}_+$ that corresponds to the part of $\sigma(T^t)$ inside the unit disk is of the form $(\mathcal{P}_+\phi)(\tau) = P_+(\tau)\phi(\tau)$, where $P_+(\cdot)$ is the bounded projection-valued function mentioned in the definition of dichotomy.

5. The Bohl spectrum $\mathcal{B}$ for $\{U(t,s)\}_{t\geq s}$ and $\sigma(\Gamma)$ are related by the formula $\mathcal{B} = \sigma(\Gamma) \cap \mathbb{R}$. As a result, the growth bound ω_U for the evolution family $\{U(t,s)\}_{(t,s)\in\mathbb{R}^2}$, defined for $(t,s) \in \mathbb{R}^2$ and the dichotomy bound β_U for a dichotomic evolution family $\{U(t,s)\}_{t\geq s}$ can be computed as follows:

$$\omega_U = \sup\{|\lambda| : \lambda \in \sigma(\Gamma) \cap \mathbb{R}\}, \qquad \beta_U = \inf\{|\lambda| : \lambda \in \sigma(\Gamma) \cap \mathbb{R}\}.$$

2.3. **Spaces of multilinear operators.** For each integer $n \in \{1, 2, \cdots\}$, let $\mathcal{L}_n = \mathcal{L}_n(X)$ denote the set of n-multilinear operators on X. For a function $\varphi : \mathbb{R} \times X \to X$, let $D^k\varphi(t,x) \in \mathcal{L}_n$ denote its nth differential with respect to $x \in X$. Also, for each integer $k \in \{0, 2, \cdots\}$, let $C^{k,1} = C^{k,1}(\mathbb{R} \times X, X)$ denote the set of continuous functions $\varphi : \mathbb{R} \times X \to X$ that are k times differentiable with respect to $x \in X$

with each such derivative globally Lipschitz and with the uniform norm given by

$$\|\varphi\|_{C^{k,1}} = \sup_{t\in\mathbb{R}} \max\left\{ \max_{j=1,\ldots,k} \sup_{x\in X} \|D^j\varphi(t,x)\|_{\mathcal{L}_j},\ \sup_{x_1\neq x_2} \frac{\|D^k\varphi(t,x_1) - D^k\varphi(t,x_2)\|_{\mathcal{L}_k}}{|x_1 - x_2|} \right\}.$$

Also, for $\delta > 0$ and for $k \in \{0, 1, 2, \ldots\}$, let

$$B_\delta^{(k)} = \{\varphi \in C^{k,1} : \|\varphi\|_{C^{k,1}} \leq \delta\}$$

denote the δ-ball at the origin of $C^{k,1}$.

Recall the chain rule for the nth derivative of the composition of differentiable functions $G, F : X \to X$, (see, e.g., [1, p.97]):

$$D^n(G\circ F)(x)(e_1,\ldots,e_n) = \tag{11}$$

$$\sum_{i=1}^{k} \sum_{j_1+\cdots+j_i=k} \sum_{\{\ell\}} D^i G(F(x)) \left(D^{j_1}F(x)(e_{\ell_1},\ldots,e_{\ell_{j_1}}),\ldots, D^{j_i}F(x)(e_{\ell_{j_1+\cdots+j_{i-1}+1}},\ldots,e_{\ell_n}) \right),$$

where the last sum is taken over all $\ell_1 < \ldots \ell_{j_1}, \ldots,\ \ell_{j_1+\cdots+\ell_{i-1}+1} < \ell_n$, and $\mathbf{e} = (e_1,\ldots,e_n) \in X^n$. Also, for $r = 1,\ldots,i$, we define

$$u_r(x,F) = D^{j_r}F(x)\left(e_{\ell_{j_1+\cdots+\ell_{j_{r-1}}+1}},\ldots, e_{\ell_{j_1+\cdots+\ell_{j_r}}} \right). \tag{12}$$

If $\mathcal{A} \in \mathcal{L}_n$ and $\mathbf{e}^{(\nu)} = \left(e_1^{(\nu)},\ldots,e_n^{(\nu)}\right) \in X^n$, $\nu = 1, 2$, we have the following estimate:

$$\begin{aligned} &\left|\mathcal{A}(e_1^{(1)},\ldots,e_n^{(1)}) - \mathcal{A}(e_1^{(2)},\ldots,e_n^{(2)})\right|_X \\ \leq\ & \|\mathcal{A}\|_{\mathcal{L}_n} \sum_{j=1}^{n} \left(\prod_{p<j} |e_p^{(1)}| \right) \left| e_j^{(1)} - e_j^{(2)} \right| \left(\prod_{p>j} |e_p^{(2)}| \right). \end{aligned} \tag{13}$$

Also, for $\mathcal{A}$, $\mathcal{B} \in \mathcal{L}_n$ and $\mathbf{e} \in X^n$ we have

$$|\mathcal{A}(\mathbf{e})| \leq \|\mathcal{A}\|_{\mathcal{L}_n} \prod_{j=1}^{n} |e_j|, \quad |\mathcal{A}(\mathbf{e}) - \mathcal{B}(\mathbf{e})| \leq \|\mathcal{A}-\mathcal{B}\|_{\mathcal{L}_n} \prod_{j=1}^{n} |e_j|. \tag{14}$$

2.4. **Gronwall's inequality.** We will use the following simple consequence of Gronwall's inequality.

Proposition 2.5. *Suppose ω, a, b are all positive real numbers and n is a nonnegative integer. If, for some $t > 0$, we have*

$$v(t) \leq a \int_0^t e^{\omega(t-\tau)} \left[v(\tau) + b e^{n(a+\omega)\tau} \right] d\tau,$$

then

$$\begin{aligned} v(t) &\leq \frac{ab}{(n-1)(a+\omega)} e^{n(a+\omega)t} \quad \textit{for } n \geq 2, \\ v(t) &\leq abte^{(a+\omega)t} \quad \textit{for } n = 1, \quad \textit{and} \\ v(t) &\leq \frac{ab}{a+\omega} e^{(a+\omega)t} \quad \textit{for } n = 0. \end{aligned}$$

Proof. Define $u(t) := e^{-\omega t} v(t)$ and note that

$$u(t) \leq a \int_0^t u(\tau)\, d\tau + c(t), \quad c(t) = ab \int_0^t e^{[(n-1)\omega + na]\tau}\, d\tau.$$

The desired result follows from the version of Gronwall's inequality in [16, Lemma 4.1.2]. ■

2.5. **Composition operators.** Consider an evolution family $\{U(t,s)\}_{(t,s)\in\mathbb{R}^2}$ on X with finite growth bound. In particular, there are constants $M > 0$ and $\omega > 0$ such that the evolution family satisfies the inequality (9).

Proposition 2.6. *Suppose $f \in C^{k,1}$ and $(s, \xi) \in \mathbb{R} \times X$. If the evolution family $\{U(t,s)\}_{(t,s)\in\mathbb{R}^2}$ defined on a Banach space X has a finite growth bound, then there is a unique continuous function function $x = x(\cdot, \xi)$, defined for all $t \in \mathbb{R}$ with range in X, that satisfies the* *integral equation*

$$x(t,\xi) = U(t,s)\xi + \int_s^t U(t,\tau) f(\tau, x(\tau,\xi))\, d\tau. \tag{15}$$

Proof. The proof is similar to the proof given for the semigroup case in [31, p.184]. ■

Using the solution of (15), we define a flow S^t on $\mathbb{R} \times X$ by

$$S^t(s,\xi) = (t+s, x(t+s,\xi)),$$

and a family of *composition operators* V^t on $C^{k,1}$ given by

$$(V^t\varphi)(s,\xi) = \varphi(S^t(s,\xi)), \quad (s,\xi) \in \mathbb{R} \times X, \quad t \in \mathbb{R}. \tag{16}$$

In the next section we will need the following estimates for V^t.

Theorem 2.7. *Suppose $k \in \{0,1,2,\dots\}$ and $f \in C^{k,1}$ with $\epsilon := \|f\|_{C^{k,1}}$. If V^t is a family of composition operators defined, relative to an evolution family satisfying (9), by (16), then there exist positive constants $c_1 = c_1(\omega,k)$ and $c_2 = c_2(\omega,k)$ such that:*

$$\begin{aligned} \|V^t\|_{\mathcal{L}(C^{k,1})} &\leq c_1 e^{(k+1)(\omega+M\epsilon)|t|}, & (17)\\ \|V^t\|_{\mathcal{L}(C^{k})} &\leq c_2 e^{k(\omega+M\epsilon)|t|}, \quad t \in \mathbb{R}. & (18) \end{aligned}$$

Next, in addition to a given $f \in C^{k,1}$, fix $\delta > 0$ and let $\psi_1, \psi_2 \in B_\delta^{(k)}$. For each $(s,\xi) \in \mathbb{R}\times X$ let $x_\nu = x_\nu(\cdot\,,\xi)$, $\nu = 1,2$, denote the solutions of

$$x_\nu(t,\xi) = U(t,s)\xi + \int_s^t U(t,\tau) f(\tau, \psi_\nu(\tau, x_\nu(\tau,\xi)))\, d\tau. \tag{19}$$

Moreover, for $\nu = 1,2$, define the corresponding flows and composition operators as follows:

$$\begin{aligned} &S_\nu^t(s,\xi) = (t+s, x_\nu(t+s,\xi)), \quad t \in \mathbb{R},\\ &(V_\nu^t\varphi)(s,\xi) = \varphi(S_\nu^t(s,\xi)), \quad (s,\xi) \in \mathbb{R} \times X, \quad \varphi \in C^{k,1}. \quad (20) \end{aligned}$$

Theorem 2.8. *Suppose $k \in \{0, 1, 2, \dots, \infty\}$, the function $f \in C^{k,1}$, and $\delta > 0$. If V_ν^t, for $\nu = 1, 2$, are the composition operators defined by* (20) *relative to an evolution family satisfying* (9)*, then there are positive constants $c_3 = c_3(\omega, \delta, k)$ and $d = d(\omega, \delta, k)$ such that, for each $\psi_1, \psi_2 \in B_\delta^{(k)}$ and all $\varphi \in C^{k,1}$*

$$\|[V_1^t - V_2^t]\varphi\|_{C^k} \leq c_3 e^{(k+1)(\omega + d\epsilon)|t|} \|\psi_1 - \psi_2\|_{C^k} \|\varphi\|_{C^{k,1}}, \quad t \in \mathbb{R}. \tag{21}$$

The proofs of Theorems 2.7–2.8 are straightforward, and will be given in Section 4. They use only the chain rule (11) and the version of Gronwall's inequality given in Proposition 2.5.

2.6. **Substitution operator.** For $f \in C^{k,1}$, we define a (nonlinear) *substitution operator*, $\mathcal{F}$, by

$$\mathcal{F}(\varphi)(t, \xi) = f(t, \varphi(t, \xi)). \tag{22}$$

An application of the chain rule (11) yields the following result.

Proposition 2.9. *Suppose $k \in \{0, 1, 2, \dots\}$ and $f \in C^{k,1}$. If $\delta > 0$ and $\mathcal{F}$ is the composition operator defined by* (22)*, then there are positive constants $c_4 = c_4(\delta, k)$ and $c_5 = c_5(\delta, k)$ such that for all $\psi, \psi_1, \psi_2 \in B_\delta^{(k)}$*

$$\|\mathcal{F}(\psi)\|_{C^{k,1}} \leq c_4 \|f\|_{C^{k,1}}, \tag{23}$$

$$\|\mathcal{F}(\psi_1) - \mathcal{F}(\psi_2)\|_{C^k} \leq c_5 \|f\|_{C^{k,1}} \|\psi_1 - \psi_2\|_{C^k}. \tag{24}$$

We omit the proof. However, the proof is similar to the proofs of Lemmas 4.1 and 4.2 below.

3. Setting and Main Result

If we are given a strongly continuous exponentially bounded evolution family $\{U(t,s)\}_{t\geq s}$ and a function $g \in C^{k,1}$ where $k \in \{0,1,\dots\}$, then we will consider the following integral equation for functions $x : \mathbb{R} \to X$:

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\tau)g(\tau, x(\tau))\, d\tau, \quad t \geq s. \tag{25}$$

In fact, in this section we will formulate sufficient conditions on $\{U(t,s)\}_{t\geq s}$ and g, so that the integral equation has a center manifold. Note, that if $\{U(t,s)\}_{t\geq s}$ solves a well-posed nonautonomous abstract Cauchy problem (8), then each solution of (25), by definition, is a *mild solution* of (6).

3.1. **Linear setting.** We will consider an evolution family $\{U(t,s)\}_{t\geq s}$ that admits a splitting into "center" and "hyperbolic" parts. To be more precise, consider the evolution semigroup $T^t = e^{t\Gamma}$ on $C_0(\mathbb{R}, X)$ defined in (10).

Spectral assumptions (S): The spectrum $\sigma(\Gamma)$ splits as

$$\sigma(\Gamma) \cap \mathbb{R} = \Sigma_c \cup \Sigma_h, \quad \Sigma_c \cap \Sigma_h = \emptyset, \quad \Sigma_c, \Sigma_h \neq \emptyset. \tag{26}$$

Moreover, if

$$\omega_c := \sup\{|\lambda| : \lambda \in \Sigma_c\}, \quad \beta_h := \inf\{|\lambda| : \lambda \in \Sigma_h\}, \tag{27}$$

then $\omega_c < \beta_h$.

Recall that the Bohl spectrum of $\{U(t,s)\}_{t\geq s}$ is given by $\mathcal{B} = \sigma(\Gamma) \cap \mathbb{R}$. Thus, the spectral assumption (S) has an equivalent reformulation in terms of the evolution family $\{U(t,s)\}_{t\geq s}$. To make this precise, we need the following assumptions. Assume there exists a bounded strongly continuous projection valued function $P : \mathbb{R} \to \mathcal{L}(X)$ such that $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$. For $Q(t) := I - P(t)$ define the subspaces $X_c(t) := \operatorname{Im} P(t)$ and $X_h(t) := \operatorname{Im} Q(t)$, and, for $t \geq s$ define the restricted operators

$$\begin{aligned} U_c(t,s) :=& U(t,s)|X_c(s) : X_c(s) \to X_c(t), \\ U_h(t,s) :=& U(t,s)|X_h(s) : X_h(s) \to X_h(t). \end{aligned}$$

Center part. Assume that, for all $t \geq s$, the operator $U_c(t,s)$ is invertible as an operator from $X_c(s)$ to $X_c(t)$, and define $U_c(s,t) := [U_c(t,s)]^{-1}$. Note that the elements of the family $\{U_c(t,s)\}$, for $(t,s) \in \mathbb{R}^2$, are operators on X that satisfy (i) and (iii) in Definition 2.1, and the replacement of condition (ii) given by the identity $U_c(t,t) = P(t)$, $t \in \mathbb{R}$. Assume there exists a positive constant ω and a constant $M_c = M_c(\omega) > 0$, such that for all $(t,s) \in \mathbb{R}$, we have the estimate

$$\|U_c(t,s)\|_{\mathcal{L}} \leq M_c e^{\omega|t-s|}. \tag{28}$$

Also, let $\omega_c \geq 0$ denote the growth bound for $\{U_c(t,s)\}_{(t,s)\in\mathbb{R}^2}$, that is, the infimum over all $\omega > 0$ such that (28) holds with some M_c.

Hyperbolic part. Assume that $\{U_h(t,s)\}_{t\geq s}$ has an exponential dichotomy. To be more precise, assume that there exist bounded strongly continuous operator-valued functions $P_\pm : \mathbb{R} \to \mathcal{L}$ such that $P_+(t) + P_-(t) = P(t)$, and for all $t \geq s$,

$$U_h(t,s)P_\pm(s) = P_\pm(t)U_h(t,s).$$

In addition, consider the restrictions

$$U_h^\pm(t,s) = U_h(t,s)|\operatorname{Im} P_\pm(s) : \operatorname{Im} P_\pm(s) \to \operatorname{Im} P_\pm(t), \quad t \geq s.$$

We assume that the operator $U_h^-(t,s)$ is invertible as an operator from $\operatorname{Im} P_-(s)$ to $\operatorname{Im} P_-(t)$ and that there exist positive constants β and

$M_h = M_h(\beta) > 0$ such that, for all $t \geq s$,

$$\begin{aligned} \|U_h^+(t,s)\|_{\mathcal{L}} &\leq M_h e^{-\beta(t-s)}, \\ \|[U_h^-(t,s)]^{-1}\|_{\mathcal{L}} &\leq M_h e^{-\beta(t-s)}. \end{aligned} \tag{29}$$

Let $\beta_h > 0$ denote the dichotomy bound for $\{U_h(t,s)\}_{t\geq s}$, that is, the supremum over all $\beta > 0$ such that (29) holds for some M_h.

With a slight abuse of terminology we summarize our assumptions on the evolution families $\{U_c(t,s)\}_{(t,s)\in\mathbb{R}^2}$ and $\{U_h(t,s)\}_{t\geq s}$ as follows:

Assumption (C). $\{U_c(t,s)\}$ is an evolution family, defined for all $(t,s) \in \mathbb{R}^2$, with the growth bound ω_c.

Assumption (H). $\{U_h(t,s)\}_{t\geq s}$ is a dichotomic evolution family with dichotomy bound β_h such that $\beta_h > \omega_c$.

We claim that (S) is equivalent to the assumptions (C) and (H). In fact, since $\sigma(\Gamma)$ is invariant with respect to translations along the imaginary axis, (26) is equivalent to the existence of a splitting for $\sigma(\Gamma)$. By the spectral mapping theorem for $\{T^t\}$, the existence of the splitting for $\sigma(\Gamma)$ is equivalent to the existence of a corresponding splitting for $\sigma(T^t)$ for each $t > 0$. Such a splitting exists if and only if T^t has complementary spectral projections $\mathcal{P}$ and $\mathcal{Q}$ on the space $C_0(\mathbb{R}, X)$ with the following properties: The projections are of the form $\mathcal{P} = P(\cdot)$ and $\mathcal{Q} = I - P(\cdot)$. The function $P(\cdot) : \mathbb{R} \to \mathcal{L}$ is the bounded strongly continuous projection valued function, mentioned in the definition of $U_c(t,s)$ and $U_h(t,s)$ above with $X_c(t) = \operatorname{Im} P(t)$ and $X_h(t) = \operatorname{Im} Q(t)$. With $C_{0,c} = \operatorname{Im}\mathcal{P}$ and $C_{0,h} = \operatorname{Im}\mathcal{Q}$, there is a T^t-invariant decomposition $C_0(\mathbb{R}; X) = C_{0,c} \oplus C_{0,h}$ such that $\Sigma_c = \sigma(\Gamma|C_{0,c}) \cap \mathbb{R}$ and $\Sigma_h = \sigma(\Gamma|C_{0,h}) \cap \mathbb{R}$. Thus, (S) is equivalent to (C) and (H).

In what follows we will need subspaces of "X_h-valued" functions on $\mathbb{R}$ and functions "from X_c to X_h". In fact, we define these subspaces as follows:

$$\begin{aligned} C_b(\mathbb{R}, X_h) &= \{\phi \in C_b(\mathbb{R}, X) : \phi(t) = Q(t)\phi(t), t \in \mathbb{R}\}; \\ C^0_{ch}(\mathbb{R} \times X, X) &= \{\varphi \in C^0 = C^0(\mathbb{R} \times X, X) : \\ &\qquad \varphi(t,x) = Q(t)\varphi(t, P(t)x) \text{ for all } (t,x) \in \mathbb{R} \times X\} \end{aligned} \tag{30}$$

where $C_b(\mathbb{R}, X)$ is the space of bounded continuous X-valued functions on $\mathbb{R}$.

We will need the following standard consequence of assumption (H) (see, e.g. [12] or [24]). For each $(t,s) \in \mathbb{R}^2$ with $t \neq s$, there is a bounded operator (Green's function) $\mathcal{K}(t,s) : X_h(s) \to X_h(t)$ defined as follows:

$$\mathcal{K}(t,s) = U_h^+(t,s) \text{ if } t > s, \quad \text{and} \quad \mathcal{K}(t,s) = -[U_h^-(t,s)]^{-1} \text{ if } t < s.$$

Proposition 3.1. *Assume hypothesis (H). If $\beta \in (0, \beta_h)$ then, with the constant $M_h = M_h(\beta)$ from (29), the following estimate holds:*

$$\|\mathcal{K}(t,s)\|_{\mathcal{L}} \leq M_h e^{-\beta|t-s|}, \quad (t,s) \in \mathbb{R}^2.$$

Moreover, for each $\phi \in C_b(\mathbb{R}, X_h)$, there exists a unique function $u \in C_b(\mathbb{R}, X_h)$ such that

$$u(t) = U_h(t,s)u(s) + \int_s^t U_h(t,\tau)\phi(\tau)\, d\tau, \quad t \geq s.$$

In fact, u is given by

$$u(t) = \int_{-\infty}^{\infty} \mathcal{K}(t,\tau)\phi(\tau)\, d\tau, \quad t \in \mathbb{R}.$$

We remark, see [26], that the existence of the unique $u \in C_b(\mathbb{R}, X)$ for each $\phi \in C_b(\mathbb{R}, X)$, as in the proposition, is, in fact, *equivalent* to the assumption (H).

3.2. **An Integral equation.** In this section, we consider an evolution family $U(t,s)$ that satisfies Assumption (S) (or, equivalently, (C) and (H)). Also, we fix $k \in \{0,1,\dots\}$ and $g \in C^{k,1}$.

Definition 3.2. *A set $\mathcal{M} \subset \mathbb{R} \times X$ is called an* invariant set *for* (25) *if, for each $(s,\xi) \in \mathcal{M}$, the solution $x(\cdot,\xi)$ of* (25) *with $x(s,\xi) = \xi$ is such that $(t, x(t,\xi)) \in \mathcal{M}$ for all $t \geq s$.*

For $\eta > 0$ and for $k \in \{0,1,2,\dots\}$, we define

$$L_\eta^{(k)} = \{\varphi \in C_{ch}^0(\mathbb{R} \times X, X) \cap C^{k,1}(\mathbb{R} \times X, X) : \|\varphi\|_{C^{k,1}} \leq \eta\}. \tag{31}$$

From the definitions in (30), we note that if $\varphi \in L_\eta^{(k)}$, then $\varphi(t,x) = Q(t)\varphi(t, P(t)x)$ for all $(t,x) \in \mathbb{R} \times X$. In other words, $\varphi(t,\cdot) : X_c(t) \to X_h(t)$ for each $t \in \mathbb{R}$.

We will look for an element $\psi \in L_\eta^{(k)}$ such that

$$\mathcal{M}_\psi = \{(t, P(t)x + \psi(t, P(t)x)) : (t,x) \in \mathbb{R} \times X\} \tag{32}$$

in an invariant set for (25). To this end we will construct an appropriate (nonlinear) integral operator $\mathcal{T}$ such that M_ψ is an invariant set for (25) whenever ψ is a fixed point of $\mathcal{T}$ in $L_\eta^{(k)}$.

To construct $\mathcal{T}$, we begin by defining

$$g_c(t,x) = P(t)g(t,x), \quad g_h(t,x) = Q(t)g(t,x), \quad (t,x) \in \mathbb{R} \times X,$$

as well as $x_c(t) = P(t)x(t)$ and $x_h(t) = Q(t)x(t)$. With these definitions, we rewrite (25) as a system:

$$x_c(t) = U_c(t,s)x_c(s) + \int_s^t U_c(t,\tau)g_c(\tau, x_c(\tau) + x_h(\tau))\, d\tau, \tag{33}$$

$$x_h(t) = U_h(t,s)x_h(s) + \int_s^t U_h(t,\tau)g_h(\tau, x_c(\tau) + x_h(\tau))\, d\tau, \quad t \geq s. \tag{34}$$

Moreover, for $\psi \in L_\eta^{(k)}$, define $f \in C^{k,1}$ by

$$f(t,x) = g_c(t, P(t)x + \psi(t, P(t)x)), \quad (t,x) \in \mathbb{R} \times X. \tag{35}$$

Since $\psi \in B_\eta^{(k)}$ and $P : \mathbb{R} \to \mathcal{L}$ is bounded, an application of Proposition 2.9 shows there is a constant $c > 0$ such that

$$\|f\|_{C^{k,1}} \leq c\|g\|_{C^{k,1}}, \quad c = c(\eta, k). \tag{36}$$

Fix $(s,\xi) \in \mathbb{R} \times X$. By Proposition 2.6, integral equation

$$x(t) = U_c(t,s)\xi + \int_s^t U_c(t,\tau)f(\tau, x(\tau))\, d\tau, \quad t \geq s,$$

has a unique solution $x_c(\cdot, \xi)$. Note that $U_c(t,s) = U_c(t,s)P(s)$ and that, without loss of generality, we may as well assume $\xi \in X_c(s)$. By assumption (C), the solution $x_c(\cdot,\xi)$ is defined for all $t \in \mathbb{R}$. Moreover, since $P(s)\xi \in X_c(s)$ and $f(t,x) \in X_c(t)$, one has $x_c(t,\xi) \in X_c(t)$ for all $t \in \mathbb{R}$. Define a flow $S^t = S_\psi^t$ on $\mathbb{R} \times X$ by

$$S^t(s,\xi) = (t+s, x_c(t+s,\xi)), \quad t \in \mathbb{R}. \tag{37}$$

Also, for $(s,\xi) \in \mathbb{R} \times X$, we let π denote the projection given by $\pi(s,\xi) = \xi$.

The set $\mathcal{M}_\psi$ is invariant for (25) if and only if $x_h(t) = \psi(t, x_c(t,\xi))$ satisfies

$$x_h(t) = U_h(t,s)x_h(s) + \int_s^t U_h(t,\tau)g_h(\tau, x_c(\tau,\xi) + \psi(\tau, x_c(\tau,\xi)))\,d\tau, \quad t \geq s. \tag{38}$$

Also, we have $(t, x_c(t,\xi)) = S^{t-s}(s,\xi)$, and $x_h(t) = \psi(S^{t-s}(s,\xi))$. Since the function $t \mapsto g_h(t, x_c(t,\xi) + \psi(t, x_c(t,\xi))$ is an element of $C_b(\mathbb{R}, X_h)$, we use Proposition 3.1, to conclude that (38) can be rewritten as

$$\psi(S^{t-s}(s,\xi)) = \int_{-\infty}^{\infty} \mathcal{K}(t, \tau+s)g_h(\tau+s, \pi(S^\tau(s,\xi)) + \psi(S^\tau(s,\xi)))\,d\tau.$$

Set $t = s$ in the last formula and conclude that $\mathcal{M}_\psi$ is an invariant set for (25) if and only if $\psi \in C^0_{ch}(\mathbb{R}\times X, X)$ is a fixed point of the nonlinear integral operator $\mathcal{T}$ defined as follows:

$$(\mathcal{T}\psi)(s,\xi) = \int_{-\infty}^{\infty} \mathcal{K}(s, \tau+s)g_h(\tau+s, \pi(S^\tau(s,\xi)) + \psi(S^\tau(s,\xi)))\,d\tau. \tag{39}$$

3.3. **Main theorem.** Our main idea is to express the operator $\mathcal{T}$ as a composition of a linear and a nonlinear operator that are then analyzed separately. To do this, we use the flow $S^t = S^t_\psi$ as in (37) and consider the composition operators V^t defined in (16). The linear operator Λ_ψ on $C^{k,1}$ is defined by the formula

$$(\Lambda_\psi\varphi)(s,\xi) = \int_{-\infty}^{\infty} \mathcal{K}(s,\tau+s)(V^\tau\varphi)(s,\xi)\,d\tau. \tag{40}$$

The nonlinear operator $\mathcal{G}$ is defined on $L^{(k)}_\eta$ by

$$\mathcal{G}(\psi)(s,\xi) = g_h(s, P(s)\xi + \psi(s, P(s)\xi)).$$

For $\psi \in L^{(k)}_\eta$, we clearly have

$$\mathcal{T}(\psi) = \Lambda_\psi \circ \mathcal{G}(\psi), \tag{41}$$

the desired decomposition of $\mathcal{T}$.

In addition to Assumption (S) (or, equivalently, Assumptions (C) and (H)), we also impose the following C^k *spectral gap condition*, $k \in \{0, 1, 2, \dots\}$:

Assumption (G). The spectral bounds ω_c and β_h, defined in (27), are such that

$$\beta_h - (k+1)\omega_c > 0. \tag{42}$$

The spectral gap condition has the following operator theoretic interpretation: If it holds, then the linear operator Λ_ψ is uniformly bounded as an operator on $C^{k,1}$ over $\psi \in L_\eta^{(k)}$ and all $g \in C^{k,1}$ with sufficiently small $C^{k,1}$-norm. Moreover, for all $\psi_1, \psi_2 \in L_\eta^{(k)}$, the norm of $\Lambda_{\psi_1} - \Lambda_{\psi_2}$, as an operator from $C^{k,1}$ to C^k, is $O(\|\psi_1 - \psi_2\|_{C^k})$. To be more precise, we have the following lemma.

Lemma 3.3. *Suppose $k \in \{0, 1, 2, \dots\}$ and $\eta > 0$. If the C^k spectral gap condition* (42) *holds, then there exist positive constants $\epsilon_0 = \epsilon_0(\eta, k, \omega_c, \beta_h)$ and $c_0 = c_0(\eta, k, \omega_c, \beta_h)$ such that for each $g \in C^{k,1}$ with $\|g\|_{C^{k,1}} \le \epsilon_0$ and all $\psi, \psi_1, \psi_2 \in L_\eta^{(k)}$,*

$$\|\Lambda_\psi\|_{\mathcal{L}(C^{k,1})} \le c_0, \tag{43}$$

$$\|\Lambda_\psi\|_{\mathcal{L}(C^k)} \le c_0, \tag{44}$$

$$\|[\Lambda_{\psi_1} - \Lambda_{\psi_2}]\varphi\|_{C^k} \le c_0 \|\psi_1 - \psi_2\|_{C^k} \|\varphi\|_{C^{k,1}},\ \varphi \in C^{k,1}. \tag{45}$$

Proof. Define $\gamma := \beta_h - (k+1)\omega_c$ and note that $\gamma > 0$. Fix $\omega = \omega_c + \gamma/(4(k+1))$ and $\beta = \beta_h - \gamma/4$ such that $\beta - (k+1)\omega = \gamma/2 > 0$. Find $M_h = M_h(\beta)$ and $M_c = M_c(\omega)$ such that estimates (28) and (29) hold. By (17) in Theorem 2.7, there is a constant $c_1 = c_1(\omega, k)$ such that

$$\|V^t\|_{\mathcal{L}(C^{k,1})} \le c_1 \exp[(k+1)(\omega + M_c\|f\|_{C^{k,1}})|t|],$$

where f is defined in (35). By (36), one has $\|f\|_{C^{k,1}} \leq c(\eta,k)\|g\|_{C^{k,1}}$ for all $\psi \in C^{k,1}$ with $\|\psi\|_{C^{k,1}} \leq \eta$. Choose $\epsilon_0 < \gamma[4(k+1)c(\eta,k)M_c]^{-1}$. Then, for each $g \in C^{k,1}$ with $\|g\|_{C^{k,1}} \leq \epsilon_0$ and $\psi \in L_\eta^{(k)}$ one has

$$\begin{aligned}\|\Lambda_\psi\|_{\mathcal{L}(C^{k,1})} &\leq \int_{-\infty}^{\infty} \|\mathcal{K}(s,\tau+s)\|_{\mathcal{L}}\|V^\tau\|_{\mathcal{L}(C^{k,1})}\,d\tau \\ &\leq c_1(\omega,k)M_h \int_{-\infty}^{\infty} \exp|\tau|[-\beta+(k+1)\omega+\gamma/4]\,d\tau = 8c_1(\omega,k)M_h/\gamma.\end{aligned}$$

This proves (43). Using (18) in Theorem 2.7, the proof of (44) is similar.

To prove (45), choose $\epsilon_0 < \gamma[4(k+1)c(\eta,k)d]^{-1}$, where $d = d(\eta,k,\omega_c,\beta_h)$ is given by Theorem 2.8. Then, with $c_3 = c_3(\eta,k,\omega_c,\beta_h)$ also from Theorem 2.8, we have

$$\|(\Lambda_{\psi_1}-\Lambda_{\psi_2})\varphi\|_{C^k} \leq M_h c_3 \int_{-\infty}^{\infty} \exp|\tau|[-\beta+(k+1)(\omega+d\|f\|_{C^{k,1}})]\,d\tau \leq 8c_3M_h/\gamma,$$

as required.

We now have all the ingredients necessary to prove our main theorem.

Theorem 3.4. *Suppose that $k \in \{0,1,2,\dots\}$ and $\eta > 0$. If the evolution family $\{U(t,s)\}_{t\geq s}$ satisfies assumption (S) (or, equivalently, (C) and (H)), and if its growth bound ω_c and dichotomy bound β_h satisfy the C^k spectral gap condition*

$$\beta_h - (k+1)\omega_c > 0,$$

then there is a positive constant $\epsilon = \epsilon(\eta,k,\omega_c,\beta_h)$ such that for each $g \in C^{k,1}$ with $\|g\|_{C^{k,1}} \leq \epsilon$ there exists a unique $\psi \in L_\eta^{(k)}$ such that the manifold $\mathcal{M}_\psi$, given by (32), is an invariant manifold for the integral equation (25).

Proof. If $\psi \in L_\eta^{(k)}$, we claim that $\mathcal{T}(\psi) \in L_\eta^{(k)}$ provided ϵ is sufficiently small. Indeed, if we choose $\epsilon \le \epsilon_0$ with ϵ_0 as in Lemma 3.3, then

$$\begin{aligned}\|\mathcal{T}(\psi)\|_{C^{k,1}} &= \|\Lambda_\psi \circ \mathcal{G}(\psi)\|_{C^{k,1}} \le \|\Lambda\|_{\mathcal{L}(C^{k,1})}\|\mathcal{G}(\psi)\|_{C^{k,1}} \\ &\le c_0 c_4 \|f\|_{C^{k,1}} \le c_0 c_4 c(\eta,k)\|g\|_{C^{k,1}},\end{aligned}$$

with c_0 from Lemma 3.3, the constant c_4 from Proposition 2.9, and $c(\eta,k)$ from (36). In particular, $\mathcal{T}(\psi) \in L_\eta^{(k)}$ provided $\epsilon < [c_0 c_4 c(\eta,k)]^{-1}\eta$.

We will show that $\mathcal{T}$ is a contraction in the C^k-norm, provided $\epsilon > 0$ is sufficiently small. Since $L_\eta^{(k)}$ in closed in the C^k-norm, an application of the contraction principle then implies the existence of a unique fixed point for $\mathcal{T}$ on $L_\eta^{(k)}$. We remark that, by a lemma due to D. Henry (see, e.g, [8]), $L_\eta^{(k)}$ is closed in the C^0-norm. Hence, it suffices to show that $\mathcal{T}$ is a C^k-contraction for $k = 0$. However, we will give the proof for an arbitrary choice of k.

To prove the contraction property, choose $\epsilon \le \epsilon_0$ for ϵ_0 as in Lemma 3.3. Then, with c_0 as in this lemma, we have

$$\begin{aligned}\|\mathcal{T}(\psi_1) - \mathcal{T}(\psi_2)\|_{C^k} &\le \|(\Lambda_{\psi_1} - \Lambda_{\psi_2})\mathcal{G}(\psi_1)\|_{C^k} + \|\Lambda_{\psi_2}\|_{\mathcal{L}(C^k)}\|\mathcal{G}(\psi_1) - \mathcal{G}(\psi_2)\|_{C^k} \\ &\le c_0\left(\|\psi_1 - \psi_2\|_{C^k}\|\mathcal{G}(\psi_1)\|_{C^{k,1}} + \|\mathcal{G}(\psi_1) - \mathcal{G}(\psi_2)\|_{C^k}\right).\end{aligned}$$

Define $\varphi_\nu(t,x) = P(t)x + \psi_\nu(t, P(t)x)$, $(t,x) \in \mathbb{R} \times X$, $\nu = 1,2$. Since $P : \mathbb{R} \to \mathcal{L}$ is bounded, $\varphi_\nu \in B_\delta^{(k)}$ for some $\delta = \delta(\eta)$. Also, there is some positive constant d such that $\|\varphi_1 - \varphi_2\|_{C^k} \le d\|\psi_1 - \psi_2\|_{C^k}$. Apply Proposition 2.9 with $f(t,x) = g_h(t,x)$ and $\psi_\nu = \varphi_\nu$. Then, for some positive constant $c(\eta,k)$, we have

$$\begin{aligned}\|\mathcal{G}(\psi_1)\|_{C^{k,1}} &= \|\mathcal{F}(\varphi_1)\|_{C^{k,1}} \le c(\eta,k)\|g\|_{C^{k,1}}, \\ \|\mathcal{G}(\psi_1) - \mathcal{G}(\psi_2)\|_{C^k} &\le c_5(\delta,k)\|g\|_{C^{k,1}}\|\varphi_1 - \varphi_2\|_{C^k} \le c(\eta,k)\|\psi_1 - \psi_2\|_{C^k}\|g\|_{C^{k,1}}.\end{aligned}$$

As a result, we obtain

$$\|\mathcal{T}(\psi_1) - \mathcal{T}(\psi_2)\|_{C^k} \leq c_0 c(\eta, k) \|\psi_1 - \psi_2\|_{C^k} \|g\|_{C^{k,1}}.$$

By choosing $\epsilon < [c_0 c(\eta, k)]^{-1}$, we have the desired result. ■

4. Composition Operators

In this section we give the proofs of Theorems 2.7 and 2.8. For this, consider a Banach space X, a function $f : \mathbb{R} \times X \to X$ in $C^{k,1}$, and an evolution family $\{U(t,s)\}_{(t,s)\in\mathbb{R}^2}$ on X that satisfies (9). Recall that, for a fixed $(s, \xi) \in \mathbb{R} \times X$, we let $x(\cdot, \xi)$ denote the solution of the equation (15). In particular, the identity

$$x(t+s, \xi) = U(t+s, s)\xi + \int_0^t U(t+s, \tau+s) f(\tau+s, x(\tau+s, \xi))\, d\tau \tag{46}$$

holds for all $(s, \xi) \in \mathbb{R}\times X$ and $t \in \mathbb{R}$. Since we will have to differentiate (46) with respect to ξ, it is convenient to introduce, for $(s, \xi) \in \mathbb{R} \times X$, the functional notations $G_s(\xi) := f(s, \xi)$ and $F_s(\xi) := x(s, \xi)$ so that s is viewed as a parameter. For example, in this notation, the identity (46) is expressed as follows:

$$F_{t+s}(\xi) = U(t+s, s)\xi + \int_0^t U(t+s, \tau+s) G_{t+s} \circ F_{t+s}(\xi)\, d\tau. \tag{47}$$

This notation together with several variants will be used throughout this section.

We start with two lemmas that give estimates for $x(\cdot, \xi)$ and its derivatives.

Lemma 4.1. *Suppose that $k \in \{0, 1, \dots\}$ and, as in (47), $F_s(\xi) = x(s, \xi)$. If $f \in C^{k,1}$ and $\epsilon = \|f\|_{C^{k,1}}$, then there exists a positive constant $c = c(\omega, k)$ such that, for all $(s, \xi) \in \mathbb{R} \times X$ and $t \in \mathbb{R}$,*

$$\|D^k F_{t+s}(\xi)\|_{\mathcal{L}_k} \leq c e^{k(\omega + M\epsilon)|t|}. \tag{48}$$

Proof. We claim that it suffices to prove (48) for $t \geq 0$. Indeed, for $t, s \in \mathbb{R}$ and $\xi \in X$, if we define

$$x'(t, \xi) = x(-t, \xi), \quad U'(t, s) = U(-t, -s), \quad f'(t, \xi) = -f(-t, \xi),$$

$$F'_t(\xi) = x'(t, \xi), \quad G'_t(\xi) = f'(t, \xi),$$

then, by replacing t by $-t$ and s by $-s$ in (46), we obtain

$$x'(t + s, \xi) = U'(t + s, s)\xi + \int_0^t U'(t + s, \tau + s) f'(t + s, x'(\tau + s, \xi))\, d\tau.$$

Since, $\{U'(t, s)\}_{(t,s)\in\mathbb{R}^2}$ satisfies (9), $\|f\|_{C^k} = \|f'\|_{C^k}$, and $D^k F'_{t+s}(\xi) = D^k F_{-t-s}(\xi)$, the estimate (48) for $x(\cdot, \xi)$ and $\{U(\cdot, \cdot)\}$ with $t < 0$ is exactly the same as the estimate for $x'(\cdot, \xi)$ and $\{U'(\cdot, \cdot)\}$, but with t replaced by $-t$. This proves the claim.

For the remainder of the proof we assume that $t \geq 0$.

To prove (48) for $k = 1$, we differentiate (47) and use (9) to obtain

$$\|DF_{t+s}(\xi)\|_{\mathcal{L}_1} \leq M e^{\omega t} + M\epsilon \int_0^t e^{\omega(t-\tau)} \|DF_{\tau+s}(\xi)\|_{\mathcal{L}_1}\, d\tau.$$

After multiplication by $e^{-\omega t}$, an application of Gronwall's inequality yields (48).

Proceeding by induction on k, assume that (48) holds for $1, 2, \dots, k-1$. After differentiation of both sides of (47), we find that

$$D^k F_{t+s}(\xi) = \int_0^t U(t + s, \tau + s) D^k (G_{\tau+s} \circ F_{\tau+s})(\xi)\, d\tau.$$

The chain rule (11) together with (9) yield the estimate

$$\|D^k F_{t+s}(\xi)\|_{\mathcal{L}_k} \leq M \int_0^t e^{\omega(t-\tau)} \sum_{i=1}^{k} \sum_{j_1+\cdots+j_i=k} \sum_{\{\ell\}} \|f\|_{C^i} \prod_{r=1}^{i} \|D^{j_r} F_{\tau+s}(\xi)\|_{\mathcal{L}_{j_r}} \, d\tau.$$

Note that, for $i = 2, \ldots, k$, the condition $j_1 + \cdots + j_i = k$, implies $j_r \leq k-1$, for $r = 1, \ldots, i$. Using the induction assumption, we have

$$\begin{aligned} \|D^k F_{t+s}(\xi)\|_{\mathcal{L}_k} \leq & M\epsilon \Bigg\{ \int_0^t e^{\omega(t-\tau)} \|D^k F_{\tau+s}(\xi)\|_{\mathcal{L}_k} \, d\tau \\ & + \sum_{i=2}^{k} \sum_{j_1+\cdots+j_i=k} \sum_{\{\ell\}} \int_0^t e^{\omega(t-\tau)} c \exp\Big(\sum_{r=1}^{i} j_r(\omega + M\epsilon)\tau\Big) \, d\tau \Bigg\} \\ & \leq M\epsilon \left\{ \int_0^t e^{\omega(t-\tau)} \|D^k F_{\tau+s}(\xi)\|_{\mathcal{L}_k} \, d\tau + c \int_0^t e^{\omega(t-\tau)+k(\omega+M\epsilon)\tau} \, d\tau \right\}. \end{aligned}$$

The estimate (48) follows from Proposition 2.5. ■

Lemma 4.2. *Suppose that $k \in \{1, 2, \ldots\}$ and, as in (47), $F_s(\xi) = x(s, \xi)$. If $f \in C^{k,1}$ and $\epsilon = \|f\|_{C^{k,1}}$, then there exists a positive constant $c = c(\omega, k)$ such that, for all $\xi_1 \neq \xi_2 \in X$ and $s, t \in \mathbb{R}$,*

$$\|D^k F_{t+s}(\xi_1) - D^k F_{t+s}(\xi_2)\|_{\mathcal{L}_k} \leq c e^{(k+1)(\omega+M\epsilon)|t|} |\xi_1 - \xi_2|_X. \tag{49}$$

Proof. As in Lemma 4.1, it suffices to give the proof for $t \geq 0$.

For $k = 0$ the estimate (49) is proved in [12, Lemma VII.5.2].

Proceeding by induction on k, we fix $k \geq 1$ and assume that (49) holds for $0, 1, 2, \ldots, k-1$.

Apply D^k in (47) and use the chain rule (11) with $\mathbf{e} = (e_1, \ldots, e_k)$ and $|e_i| = 1$ to obtain

$$[D^k F_{t+s}(\xi_1) - D^k F_{t+s}(\xi_2)](\mathbf{e}) = \int_0^t U(t+s, \tau+s) \sum_{i=1}^{k} \sum_{j_1+\cdots+j_i=k} \sum_{\{\ell\}} \Delta_i \, d\tau \tag{50}$$

where,

$$\Delta_i = D^i G_{\tau+s}(F_{\tau+s}(\xi_1))(\mathbf{u}^{(1)}) - D^i G_{\tau+s}(F_{\tau+s}(\xi_2))(\mathbf{u}^{(2)}),$$

and $\mathbf{u}^{(\nu)} = (u_1^{(\nu)}, \dots, u_i^{(\nu)})$, $\nu = 1, 2$ with

$$u_r^{(\nu)} = D^{j_r} F_{\tau+s}(\xi_\nu)(e_{\ell_{j_1}+\cdots+\ell_{j_{r-1}}+1}, \dots, e_{\ell_{j_1}+\cdots+\ell_{j_r}}), \tag{51}$$

for $r = 1, \dots, i$ and $i = 1, \dots, k$. Since $|e_i| = 1$, using Lemma 4.1 we have

$$|u_r^{(\nu)}| \le c e^{j_r(\omega+M\epsilon)|\tau|}, \quad r = 1, \dots, i, \quad i = 1, \dots, k, \quad \nu = 1, 2. \tag{52}$$

For $i = 1$, by (51) with $j_1 = k$, we obtain

$$|u_1^{(1)} - u_1^{(2)}| \le \|D^k F_{\tau+s}(\xi_1) - D^k F_{\tau+s}(\xi_2)\|_{\mathcal{L}_k}. \tag{53}$$

For $i = 2, \dots, k$, we find $j_r \le k - 1$, $r = 1, \dots, i$. Hence, by the induction assumption,

$$|u_r^{(1)} - u_r^{(2)}| \le c e^{(j_r+1)(\omega+M\epsilon)|\tau|} |\xi_1 - \xi_2|, \quad r = 1, \dots, i. \tag{54}$$

Also, for $k = 0$, (49) gives

$$|x(\tau + s, \xi_1) - x(\tau + s, \xi_2)| \le c e^{(\omega+M\epsilon)|\tau|} |\xi_1 - \xi_2|. \tag{55}$$

We will estimate $|\Delta_i|$, $i = 1, \dots, k$ in (50). For this, apply (13) with $n = i$, $\mathcal{A} = D^i G_{\tau+s}(F_{\tau+s}(\xi_2))$, and $\mathbf{e}^{(\nu)} = \mathbf{u}^{(\nu)} = (u_1^{(\nu)}, \dots, u_i^{(\nu)})$. Clearly,

$$\begin{aligned} |\Delta_i| &\le \left|[D^i G_{\tau+s}(F_{\tau+s}(\xi_1)) - D^i (G_{\tau+s}(F_{\tau+s}(\xi_2))](\mathbf{u}^{(1)})\right| + |\mathcal{A}(\mathbf{u}^{(1)}) - \mathcal{A}(\mathbf{u}^{(2)})| \\ &\le \|f\|_{C^{i,1}} |x(\tau + s, \xi_1) - x(\tau + s, \xi_2)| \prod_{r=1}^{i} |u_r^{(1)}| \\ &\quad + \|f\|_{C^i} \sum_{r=1}^{i} \left(\prod_{p<r} |u_p^{(1)}| \right) |u_r^{(1)} - u_r^{(2)}| \left(\prod_{p>r} |u_p^{(2)}| \right). \end{aligned}$$

For $i = 1$, the estimates (52), (53), (55), with $j_1 = k$, yield

$$|\Delta_1| \le \epsilon c e^{(\omega+M\epsilon)\tau}|\xi_1 - \xi_2| \cdot c e^{k(\omega+M\epsilon)\tau} + \epsilon \|D^k F_{\tau+s}(\xi_1) - D^k F_{\tau+s}(\xi_2)\|_{\mathcal{L}_k}. \tag{56}$$

For $i = 2, \ldots, k$, the estimates (52), (54), (55) give

$$\begin{aligned} |\Delta_i| &\le \epsilon c e^{(\omega+M\epsilon)\tau}|\xi_1 - \xi_2| \cdot c e^{(\omega+M\epsilon)\sum_{r=1}^{i} j_r \tau} \\ &+ \epsilon \sum_{r=1}^{i} c e^{(\omega+M\epsilon)\sum_{p<r} j_p \tau} \cdot c e^{(\omega+M\epsilon)(j_r+1)\tau}|\xi_1 - \xi_2| \cdot c e^{(\omega+M\epsilon)\sum_{p>r} j_p \tau}. \end{aligned}$$

Taking into account (see (50)) the fact that $\sum_{r=1}^{i} j_r = k$, we conclude that there exists a constant $c = c(\omega, k)$ such that, for $i = 2, \ldots, k$,

$$|\Delta_i| \le \epsilon c e^{(k+1)(\omega+M\epsilon)\tau}|\xi_1 - \xi_2|. \tag{57}$$

We substitute (56) and (57) in (50) and use (9) to obtain

$$\begin{aligned} &\|D^k F_{t+s}(\xi_1) - D^k F_{t+s}(\xi_2)\|_{\mathcal{L}_k} \\ &\le M\epsilon \int_0^t e^{\omega(t-\tau)} \left\{ \|D^k F_{\tau+s}(\xi_1) - D^k F_{\tau+s}(\xi_2)\|_{\mathcal{L}_k} + c e^{(k+1)(\omega+M\epsilon)\tau}|\xi_1 - \xi_2| \right\} d\tau. \end{aligned}$$

Gronwall's inequality (Proposition 2.5) yields the estimate (49) for the integer k. ■

Proof of Theorem 2.7. In the proofs of (17) and (18) we will give estimates only for the k-th derivative; lower order derivatives are estimated in a similar manner.

Fix $\varphi \in C^{k,1}$. We will use the notation $\Phi_s(\xi) = \varphi(s, \xi)$ and, for a solution $x(\cdot, \xi)$ of (46), we will use $F_s(\xi) = x(s, \xi)$.

We will estimate $\|D^k(\varphi \circ S^t)(s, \xi)\|_{\mathcal{L}_k}$, that is, $\|D^k(\Phi_s \circ F_s)(\xi)\|_{\mathcal{L}_k}$ by using the chain rule (11). In fact, we have

$$[D^k(\Phi_{t+s} \circ F_{t+s})](\xi)(e_1, \ldots, e_k) = \sum_{i=1}^{k} \sum_{j_1+\cdots+j_i=k} \sum_{\{\ell\}} D^i \Phi_{t+s}(F_{t+s}(\xi))(u_1, \ldots, u_i).$$

Here, see (12), we define

$$u_r = D^{j_r} F_{t+s}(\xi)(e_{\ell_{j_1}+\cdots+\ell_{j_{r-1}}+1}, \ldots, e_{\ell_{j_1}+\cdots+\ell_{j_r}}), \quad r = 1, \ldots, i.$$

Using (48) from Lemma 4.1, we obtain the estimate

$$\begin{aligned}\|D^k(\varphi \circ S^t)(s,\xi)\|_{\mathcal{L}_k} &\leq \sum_{i=1}^{k} \sum_{j_1+\cdots+j_i=k} \sum_{\{\ell\}} \|\varphi\|_{C^i} \prod_{r=1}^{i} \|D^{j_r} F_{t+s}(\xi)\|_{\mathcal{L}_{j_r}} \\ &\leq c(\omega,k) e^{\sum_{r=1}^{i} j_r(\omega+M\epsilon)|t|} \|\varphi\|_{C^k} = c(\omega,k) e^{k(\omega+M\epsilon)|t|} \|\varphi\|_{C^k}.\end{aligned}$$

This proves (18).

To finish the proof of (17), fix $\xi_1 \neq \xi_2$. Again, apply the chain rule (11) with Φ and F as above. This time, see (12), we define, for $r = 1, \ldots, i$ and $i = 1, \ldots, k$:

$$u_r^{(\nu)} = D^{j_r} F_{t+s}(\xi_\nu)(e_{\ell_{j_1}+\cdots+\ell_{j_{r-1}}+1}, \ldots, e_{\ell_{j_1}+\cdots+\ell_{j_r}}), \quad \nu = 1, 2.$$

Then, for $\mathbf{e} = (e_1, \ldots, e_k)$ with $|e_i| = 1$ and $i = 1, \ldots, k$, we have

$$|D^k(\varphi \circ S^t)(s,\xi_1)(\mathbf{e}) - D^k(\varphi \circ S^t)(s,\xi_2)(\mathbf{e})| \leq \sum_{i=1}^{k} \sum_{j_1+\cdots+j_i=k} \sum_{\{\ell\}} |\Delta_i|, \tag{58}$$

where, for $i = 1, \ldots, k$,

$$\begin{aligned}|\Delta_i| \leq &\left| D^i \Phi_{t+s}(F_{t+s}(\xi_1))(u_1^{(1)}, \ldots, u_i^{(1)}) - D^i \Phi_{t+s}(F_{t+s}(\xi_1))(u_1^{(2)}, \ldots, u_i^{(2)}) \right| \\ &+ \left| [D^i \Phi_{t+s}(F_{t+s}(\xi_1)) - D^i \Phi_{t+s}(F_{t+s}(\xi_2))](u_1^{(2)}, \ldots, u_i^{(2)}) \right|.\end{aligned} \tag{59}$$

To estimate the first norm in (59), we apply (13) with $n = i$, $\mathcal{A} = D^i\Phi_{t+s}(F_{t+s}(\xi))$, and $\mathbf{e}^{(\nu)} = \mathbf{u}^{(\nu)} = (u_1^{(\nu)}, \ldots, u_i^{(\nu)})$, $\nu = 1, 2$, to obtain

$$\begin{aligned}
&|D^i\Phi_{t+s}(F_{t+s}(\xi_1))(\mathbf{u}^{(1)}) - D^i\Phi_{t+s}(F_{t+s}(\xi_1))(\mathbf{u}^{(2)})| \\
&\le \|\varphi\|_{C^i} \sum_{r=1}^{i} \left(\prod_{p<r} |u_p^{(1)}| \right) |u_r^{(1)} - u_r^{(2)}| \left(\prod_{p>r} |u_p^{(2)}| \right) \\
&\le \|\varphi\|_{C^i} \sum_{r=1}^{i} \left\{ ce^{(\omega+M\epsilon)\sum_{p<r} j_p|t|} \cdot ce^{(\omega+M\epsilon)(j_r+1)|t|}|\xi_1 - \xi_2| \cdot ce^{(\omega+M\epsilon)\sum_{p>r} j_p|t|} \right\} \\
&\le c(\omega, k)\|\varphi\|_{C^i} e^{(\omega+M\epsilon)(k+1)|t|}|\xi_1 - \xi_2|.
\end{aligned}$$

Here, to estimate $|u_r^{(\nu)}| \le \|D^{j_r}F_{t+s}(\xi_\nu)\|_{\mathcal{L}_{j_r}}$ and

$$|u_r^{(1)} - u_r^{(2)}| \le \|D^{j_r}F_{t+s}(\xi_1) - D^{j_r}F_{t+s}(\xi_2)\|_{\mathcal{L}_{j_r}},$$

we have used Lemma 4.1 and Lemma 4.2, respectively.

Returning to equation (59), we use Lemma 4.1 and Lemma 4.2 with $k = 0$ to estimate the second norm in (59) as follows (recall, that $\sum_{r=1}^{i} j_r = k$):

$$\begin{aligned}
&\left|[D^i\Phi_{t+s}(F_{t+s}(\xi_1)) - D^i\Phi_{t+s}(F_{t+s}(\xi_2))](u_1^{(2)}, \ldots, u_i^{(2)})\right| \\
&\le \|\varphi\|_{C^{i,1}}|x(t+s, \xi_1) - x(t+s, \xi_2)| \prod_{r=1}^{i} |u_r^{(2)}| \\
&\le \|\varphi\|_{C^{i,1}} ce^{(\omega+M\epsilon)|t|}|\xi_1 - \xi_2| \cdot e^{(\omega+M\epsilon)\sum_{r=1}^{i} j_r|t|} \\
&\le ce^{(k+1)(\omega+M\epsilon)|t|}|\xi_1 - \xi_2|\|\varphi\|_{C^{i,1}}.
\end{aligned}$$

Using this estimate in (58) yields

$$\|D^k(\varphi \circ S^t)(s, \xi_1) - D^k(\varphi \circ S^t)(s, \xi_2)\|_{\mathcal{L}_k} \le c(\omega, k)e^{(\omega+M\epsilon)(k+1)|t|}|\xi_1 - \xi_2|\|\varphi\|_{C^{k,1}},$$

and the proof of (17) is complete.■

Our next goal is to prove Theorem 2.8. Recall that the operators V_ν^t in (21) are constructed using the solutions $x_\nu(\cdot, \xi)$ of (19), where

$f, \psi_\nu \in C^{k,1}$ are given for $\nu = 1, 2$. We also define $\epsilon = \|f\|_{C^{k,1}}$ and $\delta = \max\{\|\psi_\nu\|_{C^{k,1}} : \nu = 1, 2\}$, and assume that $\{U(t, s)\}$ satisfies (9).

We know that $x_\nu(\cdot, \xi)$ satisfies

$$x_\nu(t+s, \xi) = U(t+s, s)\xi + \int_0^t U(t+s, \tau+s) f_\nu(\tau+s, x_\nu(\tau+s, \xi))\, d\tau, \tag{60}$$

where, naturally, we let $f_\nu(t, x) = f(t, \psi_\nu(t, x))$ for $\nu = 1, 2$. Since we will have to differentiate (60), for notational convenience we define as above

$$G_t(\xi) = f(t, \xi), \quad F_t^\nu(\xi) = x_\nu(t, \xi), \quad \Psi_t^\nu(\xi) = \psi_\nu(t, \xi). \tag{61}$$

By Proposition 2.9, if $k \in \{0, 1, \dots\}$ and $\delta > 0$, then there exists a positive constant $d = d(\delta, k)$ such that

$$\|f_\nu\|_{C^{k,1}} \le d\epsilon, \quad \nu = 1, 2. \tag{62}$$

The main step in the proof of Theorem 2.8 is contained in the following lemma.

Lemma 4.3. *Let $x_\nu(\cdot, \xi)$, for $\nu = 1, 2$, denote solutions of* (19) *and define $F_s^\nu(\xi) = x_\nu(s, \xi)$. If $k \in \{0, 1, \dots\}$ and if ω is the exponent in* (9)*, then there exist positive constants $d = d(\omega, k)$ and $c = c(\omega, k)$ so that, for all $(s, \xi) \in \mathbb{R} \times X$ and $t \in \mathbb{R}$,*

$$\|D^k F_{t+s}^1(\xi) - D^k F_{t+s}^2(\xi)\|_{\mathcal{L}_k} \le c e^{(k+1)(\omega + d\epsilon)|t|} \|\psi_1 - \psi_2\|_{C^k}. \tag{63}$$

Proof. If $k = 0$, then, using (60) and (9), we compute

$$|x_1(t+s,\xi) - x_2(t+s,\xi)| \le$$
$$M\|f\|_{C^{0,1}}\int_0^t e^{\omega(t-\tau)}\left\{|\psi_1(\tau+s,x_1(\tau+s,\xi)) - \psi_2(\tau+s,x_1(\tau+s,\xi))|\right.$$
$$\left. + |\psi_2(\tau+s,x_1(\tau+s,\xi)) - \psi_2(\tau+s,x_2(\tau+s,\xi))|\right\} d\tau$$
$$\le M\epsilon\int_0^t e^{\omega(t-\tau)}\left\{\|\psi_1-\psi_2\|_{C^0} + \|\psi_2\|_{C^{0,1}}|x_1(\tau+s,\xi) - x_2(\tau+s,\xi)|\right\} d\tau,$$

and note that (63) follows from Proposition 2.5 with $n = 0$.

Fix $k \ge 1$ and assume, by induction on k, that (63) holds for $0, 1, 2, \ldots, k-1$. Apply D^k in (60), and use the notation (61) to obtain

$$D^kF^1_{t+s}(\xi) - D^kF^2_{t+s}(\xi) =$$
$$\int_0^t U(t+s,\tau+s)\left\{D^k(G_{\tau+s}\circ\Psi^1_{\tau+s}\circ F^1_{\tau+s})(\xi) - D^k(G_{\tau+s}\circ\Psi^2_{\tau+s}\circ F^2_{\tau+s})(\xi)\right\} d\tau. \tag{64}$$

From the chain rule (11) we find that, for $\nu = 1, 2$,

$$D^k(G_{\tau+s}\circ(\Psi^\nu_{\tau+s}\circ F^\nu_{\tau+s}))(\xi)(\mathbf{e}) = \sum_{i=1}^k \sum_{j_1+\cdots+j_i=k}\sum_{\{\ell\}} D^iG_{\tau+s}(\Psi^\nu_{\tau+s}\circ F^\nu_{\tau+s})(\mathbf{u}^{(\nu)}), \tag{65}$$

where $\mathbf{e} = (e_1, \ldots, e_k) \in X^k$ with $|e_i| = 1$ for $i = 1, \ldots, k$, and (see the notations (12)), $\mathbf{u}^{(\nu)} = (u_1^{(\nu)}, \ldots, u_i^{(\nu)})$ where, for $r = 1, \ldots, i$,

$$u_r^{(\nu)} = D^{j_r}(\Psi^\nu_{\tau+s}\circ F^\nu_{\tau+s})(\xi)(e_{\ell_{j_1}+\cdots+\ell_{j_{r-1}}+1}, \ldots, e_{\ell_{j_1}+\cdots+\ell_{j_r}}).$$

To compute $u_r^{(\nu)}$, we again apply the chain rule (11). For $\nu = 1, 2$ we obtain

$$u_r^{(\nu)} = \sum_{j=1}^{j_r}\sum_{p_1+\cdots+p_j=j_r}\sum_{\{\ell\}} D^j\Psi^\nu_{\tau+s}(F^\nu_{\tau+s}(\xi))(\mathbf{v}^{(\nu)}), \tag{66}$$

where again, as in (12), for $j = 1, \dots, j_r$, $r = 1, \dots, i$, $i = 1, \dots, k$ we define

$$\mathbf{v}^{(\nu)} := (v_1^{(\nu)}, \dots, v_j^{(\nu)}), \quad v_\sigma^{(\nu)} = D^{p_\sigma} F^\nu_{\tau+s}(\xi)(\mathbf{e}), \quad \mathbf{e} \in X^{p_\sigma}, \sigma = 1, \dots, j.$$

To complete the induction step for (63), we will use the estimates for $|u_r^{(\nu)}|$ and $|u_r^{(1)} - u_r^{(2)}|$ given in Proposition 4.5 below. To prove this proposition, in its turn, we will need estimates for $|v_\sigma^{(\nu)}|$ and $|v_\sigma^{(1)} - v_\sigma^{(2)}|$, given in the next Proposition 4.4. We remark, that for $i = 1$ in (65) one has $j_1 = k$ and, for $r = 1$ and $j = 1$ in (66), also $p_1 = k$. For $i = 1$ in (65) and $r = 1$, $j = 2, \dots, k$ in (66) one has $p_\sigma \leq k - 1$, $\sigma = 1, \dots, j$. Finally, for $i = 2, \dots, k$ in (65) one has $j_r \leq k - 1$, $r = 1, \dots, i$, and, as a result, $p_\sigma \leq k - 1$ for $\sigma = 1, \dots, j$ and $j = 1, \dots, j_r$ in (66).

Proposition 4.4. *If $k \in \{0, 1, \dots\}$, $\delta > 0$, and $\omega > 0$ is as in* (9)*, then there exist positive constants $c = c(\omega, \delta, k)$ and $d = d(\omega, \delta, k)$ such that the following estimates hold:*

$$|v_\sigma^{(\nu)}| \leq c e^{p_\sigma(\omega + d\epsilon)|\tau|}, \quad \textit{for each } p_\sigma, \sigma = 1, \dots, j; \tag{67}$$

$$|v_\sigma^{(1)} - v_\sigma^{(2)}| \leq \|D^k F^1_{\tau+s}(\xi) - D^k F^2_{\tau+s}(\xi)\|_{\mathcal{L}_k} \tag{68}$$

$$\textit{for } p_1 = j_1 = k, i = 1, j = 1;$$

$$|v_\sigma^{(1)} - v_\sigma^{(2)}| \leq c e^{(p_\sigma + 1)(\omega + d\epsilon)|\tau|} \|\psi_1 - \psi_2\|_{C^{p_\sigma}} \quad \textit{for each } p_\sigma \leq k - 1. \tag{69}$$

Proof. Since $x_\nu(\cdot, \xi)$ satisfies (60), one can apply Lemma 4.1 with $f = f_\nu$ and $k = p_\sigma$ to obtain

$$|v_\sigma^{(\nu)}| \leq \|D^{p_\sigma} F^\nu_{\tau+s}(\xi)\|_{\mathcal{L}_{p_\sigma}} \leq c e^{p_\sigma(\omega + M\|f_\nu\|_{C^{p_\sigma,1}})|\tau|}.$$

Now (62) implies (67).

For $i = 1$, $j = 1$, and $p_1 = j_1 = k$, the estimate (68) follows from the definition of the quantity $v_{1,2}^{(\nu)}$.

For $i = 1$, $j = 2, \ldots, k$, or for $i = 2, \ldots, k$ one has $p_\sigma \le k-1$. Hence, one can apply the induction assumption:

$$\begin{aligned} |v_\sigma^{(1)} - v_\sigma^{(2)}| &\le \|D^{p_\sigma} F^1_{\tau+s}(\xi) - D^{p_\sigma} F^2_{\tau+s}(\xi)\|_{\mathcal{L}_{p_\sigma}} \\ &\le c e^{(p_\sigma+1)(\omega+d\epsilon)|\tau|} \|\psi_1 - \psi_2\|_{C^{p_\sigma}}, \end{aligned}$$

and (69) is proved.■.

Proposition 4.5. *With the same assumptions as in Proposition 4.4, there exist positive constants $c = c(\omega, \delta, k)$ and $d = d(\omega, \delta, k)$ such that the following estimates hold:*

$$|u_r^{(\nu)}| \le c e^{j_r(\omega+d\epsilon)|\tau|} \quad \textit{for } r = 1, \ldots, i; \quad i = 1, \ldots, k; \quad \nu = 1, 2; \tag{70}$$

$$\begin{aligned} |u_1^{(1)} - u_1^{(2)}| &\le c\Big\{\|D^k F^1_{\tau+s}(\xi) - D^k F^2_{\tau+s}(\xi)\|_{\mathcal{L}_k} \\ &+ e^{(k+1)(\omega+d\epsilon)|\tau|}\|\psi_1 - \psi_2\|_{C^k}\Big\} \quad \textit{for } i = 1; \end{aligned} \tag{71}$$

$$|u_r^{(1)} - u_r^{(2)}| \le c e^{(j_r+1)(\omega+d\epsilon)|\tau|}\|\psi_1 - \psi_2\|_{C^k} \quad \textit{for } r = 1, \ldots, i, i = 2, \ldots, k. \tag{72}$$

Proof. To prove (70), we use (66) and apply (67) from Proposition 4.4 to obtain, for $r = 1, \ldots, i$, $i = 1, \ldots, k$, and $\nu = 1, 2$,

$$\begin{aligned} |u_r^{(\nu)}| &\le \sum_{j=1}^{j_r} \sum_{p_1+\cdots+p_j=j_r} \sum_{\{\ell\}} \|\psi_\nu\|_{C^j} \prod_{s=1}^{j} |v_s^{(\nu)}| \\ &\le \sum_{j=1}^{j_r} \sum_{p_1+\cdots+p_j=j_r} \sum_{\{\ell\}} c\delta e^{\sum_{s=1}^j p_s(\omega+d\epsilon)|\tau|} \le c e^{j_r(\omega+d\epsilon)|\tau|}. \end{aligned}$$

Similarly, for $|u_r^{(1)} - u_r^{(2)}|$, we use (66) to obtain

$$\begin{aligned} &|u_r^{(1)} - u_r^{(2)}| \\ &\le \sum_{j=1}^{j_r} \sum_{p_1+\cdots+p_j=j_r} \sum_{\{\ell\}} \Big\{|D^j \Psi^1_{\tau+s}(F^1_{\tau+s}(\xi))(\mathbf{v}^{(1)}) - D^j(\Psi^2_{\tau+s}(F^2_{\tau+s}(\xi))(\mathbf{v}^{(2)})|\Big\} \\ &\le \sum_{j=1}^{j_r} \sum_{p_1+\cdots+p_j=j_r} \sum_{\{\ell\}} \Big\{|[D^j \Psi^1_{\tau+s}(F^1_{\tau+s}(\xi)) - D^j \Psi^2_{\tau+s}(F^2_{\tau+s}(\xi))](\mathbf{v}^{(1)})| \\ &\quad + |D^j \Psi^2_{\tau+s}(F^2_{\tau+s}(\xi))(\mathbf{v}^{(1)}) - D^j \Psi^2_{\tau+s}(F^2_{\tau+s}(\xi))(\mathbf{v}^{(2)})|\Big\}. \end{aligned}$$

Using (13)–(14) with $\mathbf{e}^{(\nu)} = \mathbf{v}^{(\nu)}$ and $\mathcal{A} = D^j\Psi^2_{\tau+s}(F^2_{\tau+s}(\xi))$, we find that $|u_r^{(1)} - u_r^{(2)}|$ is bounded by

$$\sum_{j=1}^{j_r} \sum_{p_1+\cdots+p_j=j_r} \sum_{\{\ell\}} \Big\{ \Big\{ \|D^j\Psi^1_{\tau+s}(F^1_{\tau+s}(\xi)) - D^j\Psi^1_{\tau+s}(F^2_{\tau+s}(\xi))\|_{\mathcal{L}_j}$$

$$+ \|D^j\Psi^1_{\tau+s}(F^2_{\tau+s}(\xi)) - D^j\Psi^2_{\tau+s}(F^2_{\tau+s}(\xi))\|_{\mathcal{L}_j} \Big\} \prod_{s=1}^{j} |v_s^{(1)}|$$

$$+ \|\psi_2\|_{C^j} \sum_{s=1}^{j} \left(\prod_{l<s} |v_l^{(1)}| \right) |v_s^{(1)} - v_s^{(2)}| \left(\prod_{l>s} |v_l^{(2)}| \right) \Big\}. \tag{73}$$

Note that, in (73) by the induction assumption (63) with $k = 0$,

$$\|D^j\Psi^1_{\tau+s}(F^1_{\tau+s}(\xi)) - D^j\Psi^1_{\tau+s}(F^2_{\tau+s}(\xi))\|_{\mathcal{L}_j}$$
$$\leq \|\psi_1\|_{C^{j,1}} |x_1(\tau+s,\xi) - x_2(\tau+s,\xi)| \leq ce^{(\omega+d\epsilon)|\tau|}\|\psi_1 - \psi_2\|_{C^k}.$$

Also, we have

$$\|D^j(\Psi^1_{\tau+s} \circ F^2_{\tau+s})(\xi) - D^j(\Psi^2_{\tau+s} \circ F^2_{\tau+s})(\xi)\|_{\mathcal{L}_j} \leq \|\psi_1 - \psi_2\|_{C^0}.$$

Finally, we apply (67) from Proposition 4.4 in (73) to obtain

$$|u_r^{(1)} - u_r^{(2)}| \leq \sum_{j=1}^{j_r} \sum_{p_1+\cdots+p_j=j_r} \sum_{\{\ell\}} \Big\{ ce^{(j_r+1)(\omega+d\epsilon)|\tau|}\|\psi_1 - \psi_2\|_{C^k}$$
$$+ \delta \sum_{s=1}^{j} \left(\prod_{l<s} |v_l^{(1)}| \right) |v_s^{(1)} - v_s^{(2)}| \left(\prod_{l>s} |v_l^{(2)}| \right) \Big\} \tag{74}$$

To finish the proof of (71) and (72), we consider two cases: $i = 1$ and $2 \leq i \leq k$.

For $i = 1$, we apply the estimates (68), (69) in (74) to get

$$|u_1^{(1)} - u_1^{(2)}| \leq c \Bigg\{ e^{(k+1)(\omega+d\epsilon)|\tau|}\|\psi_1 - \psi_2\|_{C^k} + \|D^kF^1_{\tau+s}(\xi) - D^kF^2_{\tau+s}(\xi)\|_{\mathcal{L}_j}$$

$$+ \sum_{j=2}^{j_r} \sum_{p_1+\cdots+p_s} \sum_{\{\ell\}} e^{\sum_{l<s} p_s(\omega+d\epsilon)|\tau|} \cdot e^{(p_s+1)(\omega+d\epsilon)|\tau|}\|\psi_1 - \psi_2\|_{C^{p_s}} \cdot e^{\sum_{l>s} p_s(\omega+d\epsilon)|\tau|} \Bigg\}.$$

This proves (71).

For $i = 2, \dots, k$ and $r = 1, \dots, i$, we apply the estimates (69) and (67) in (74) to get

$$\begin{aligned}|u_1^{(1)} - u_1^{(2)}| \;\leq\; & \sum_{j=1}^{j_r} \sum_{p_1+\cdots+p_j=j_r} \sum_{\{\ell\}} \Big\{ c^{(j_r+1)(\omega+d\epsilon)|\tau|} \|\psi_1 - \psi_2\|_{C^k} \\ + \;& c e^{\sum_{l<s} p_s(\omega+d\epsilon)|\tau|} \cdot e^{(p_s+1)(\omega+d\epsilon)|\tau|} \|\psi_1 - \psi_2\|_{C^{p_s}} \cdot e^{\sum_{l>s} p_s(\omega+d\epsilon)|\tau|} \Big\}.\end{aligned}$$

This proves (72).■

We have all the ingredients to finish the proof of Lemma 4.3. To this end, we will make an estimate in (64) using (65). Recall that, for $i = 1$ in (65), one has $j_1 = k$, while, for $i = 2, \dots, k$, one has $j_r \leq k - 1$ for $r = 1, \dots, i$. We use (9) together with (64) to obtain

$$|[D^k F^1_{t+s}(\xi) - D^k F^2_{t+s}(\xi)](\mathbf{e})| \leq M \int_0^t e^{\omega(t-\tau)}$$

$$\Big\{ |DG_{\tau+s}(\Psi^1_{\tau+s} \circ F^1_{\tau+s}(\xi))(u_1^{(1)}) - DG_{\tau+s}(\Psi^2_{\tau+s} \circ F^2_{\tau+s}(\xi))(u_1^{(2)})| \tag{75}$$

$$+ \sum_{i=2}^{k} \sum_{j_1+\cdots+j_i=k} \sum_{\{\ell\}} |D^i G_{\tau+s}(\Psi^1_{\tau+s} \circ F^1_{\tau+s}(\xi))(\mathbf{u}^{(1)}) \tag{76}$$

$$- D^i G_{\tau+s}(\Psi^2_{\tau+s} \circ F^2_{\tau+s}(\xi))(\mathbf{u}^{(2)})| \Big\}\, d\tau.$$

First, we see that the expression in (75) is bounded by

$$\begin{aligned}&|[DG_{\tau+s}(\Psi^1_{\tau+s} \circ F^1_{\tau+s}(\xi)) - DG_{\tau+s}(\Psi^2_{\tau+s} \circ F^2_{\tau+s}(\xi))](u_1^{(1)})| \\ &\quad + \|DG_{\tau+s}(\Psi^2_{\tau+s} \circ F^2_{\tau+s}(\xi))\|_{\mathcal{L}_1} |u_1^{(1)} - u_1^{(2)}| \\ &\leq \|f\|_{C^{1,1}} \left\{ |\psi_1(\tau+s, x_1(\tau+s,\xi)) - \psi_1(\tau+s, x_2(\tau+s,\xi))| \right. \\ &\quad \left. + |\psi_1(\tau+s, x_2(\tau+s,\xi)) - \psi_2(\tau+s, x_2(\tau+s,\xi))| \right\} |u_1^{(1)}| + \|f\|_{C^1} |u_1^{(1)} - u_1^{(2)}| \\ &\leq \|f\|_{C^{1,1}} \left\{ \|\psi_1\|_{C^{0,1}} |x_1(\tau+s,\xi) - x_2(\tau+s,\xi)| + \|\psi_1 - \psi_2\|_{C^0} \right\} |u_1^{(1)}| \\ &\quad + \|f\|_{C^1} |u_1^{(1)} - u_1^{(2)}|.\end{aligned}$$

We will apply Proposition 4.5. Since $i = 1$ in (75), we have $j_1 = k$. We use (70) and (71), together with the induction assumption (63) with $k = 0$ to see that in fact the expression in (75) is bounded by

$$c\epsilon \left\{ \|D^k F^1_{\tau+s}(\xi) - D^k F^2_{t+s}(\xi)\|_{\mathcal{L}_k} + e^{(k+1)(\omega+d\epsilon)|\tau|}\|\psi_1 - \psi_2\|_{C^k} \right\}. \tag{77}$$

Next, we use (13)–(14) to observe that the expression(76) is bounded above by

$$\begin{aligned}
&\sum_{i=2}^{k} \sum_{j_1+\cdots+j_i=k} \sum_{\{\ell\}} \Big\{ |[D^i G_{\tau+s}(\Psi^1_{\tau+s} \circ F^1_{\tau+s}(\xi)) - D^i G_{\tau+s}(\Psi^2_{\tau+s} \circ F^2_{\tau+s}(\xi))](\mathbf{u}^{(1)})| \\
&\quad + |D^i G_{\tau+s}(\Psi^2_{\tau+s} \circ F^2_{\tau+s}(\xi))(\mathbf{u}^{(1)}) - D^i G_{\tau+s}(\Psi^2_{\tau+s} \circ F^2_{\tau+s}(\xi))](\mathbf{u}^{(2)})| \Big\} \\
&\leq \sum_{i=2}^{k} \sum_{j_1+\cdots+j_i=k} \sum_{\{\ell\}} \Big\{ \Big\{ \|[D^i G_{\tau+s}(\Psi^1_{\tau+s} \circ F^1_{\tau+s}(\xi)) - D^i G_{\tau+s}(\Psi^2_{\tau+s} \circ F^2_{\tau+s}(\xi))\|_{\mathcal{L}_i} \\
&\quad + \|[D^i G_{\tau+s}(\Psi^1_{\tau+s} \circ F^2_{\tau+s}(\xi)) - D^i G_{\tau+s}(\Psi^2_{\tau+s} \circ F^2_{\tau+s}(\xi))\|_{\mathcal{L}_i} \Big\} \prod_{r=1}^{i} |u^{(1)}_r| \\
&\quad + \|D^i G_{\tau+s}(\Psi^2_{\tau+s} \circ F^2_{\tau+s}(\xi))\|_{\mathcal{L}_i} \sum_{r=1}^{i} \left(\prod_{s<r} |u^{(1)}_s| \right) |u^{(1)}_r - u^{(2)}_r| \left(\prod_{s>r} |u^{(2)}_s| \right) \Big\} \\
&\leq \sum_{i=2}^{k} \sum_{j_1+\cdots+j_i=k} \sum_{\{\ell\}} \{ \{ \|f\|_{C^{i,1}} \|\psi_1\|_{C^{0,1}} |x_1(\tau+s,\xi) - x_2(\tau+s,\xi)| \\
&\quad + \|f\|_{C^{i,1}} \|\psi_1 - \psi_2\|_{C^0} \} \prod |u^{(1)}_r| \\
&\quad + \|f\|_{C^i} \sum_{r=1}^{i} \left(\prod_{s<r} |u^{(1)}_s| \right) |u^{(1)}_r - u^{(2)}_r| \left(\prod_{s>r} |u^{(2)}_s| \right) \Big\}.
\end{aligned}$$

As before, we will apply the induction assumption (63) with $k = 0$, and the estimates (70) and (72) to conclude that the expression (76) is bounded by

$$c\epsilon e^{(k+1)(\omega+d\epsilon)|\tau|}\|\psi_1 - \psi_2\|_{C^k}. \tag{78}$$

The estimates (77) and (78) for (75) and (76) yield the inequality

$$\|D^k F^1_{t+s}(\xi) - D^k F^2_{t+s}(\xi)\|_{\mathcal{L}_k} \le d\epsilon \int_0^t e^{\omega(t-\tau)}$$

$$\left\{ \|D^k x_1(\tau+s,\xi) - D^k x_2(\tau+s,\xi)\|_{\mathcal{L}_k} + \|\psi_1 - \psi_2\|_{C^k} e^{(k+1)(\omega+d\epsilon)|\tau|} \right\} d\tau$$

with a positive constant $d = d(\omega, \delta, k)$. Now Proposition 2.5 with $n = k+1 \ge 2$, $a = d\epsilon$, and $b = \|\psi_1 - \psi_2\|_{C^k}$ gives the inequality

$$\|D^k x_1(t+s,\xi) - D^k x_2(t+s,\xi)\|_{\mathcal{L}_k} \le R e^{(k+1)(\omega+d\epsilon)|t|},$$

where

$$R = \frac{d\epsilon}{k(\omega + d\epsilon)} \|\psi_1 - \psi_2\|_{C^k} \le c(\omega,\delta,k)\|\psi_1 - \psi_2\|_{C^k}. \blacksquare$$

Proof of Theorem 2.8. For a fixed $t \in \mathbb{R}$, we will estimate

$$\sup \left\{ \|D^k(\Phi_{t+s} \circ F^1_{t+s}(\xi) - D^k(\Phi_{t+s} \circ F^2_{t+s})(\xi)\|_{\mathcal{L}_k} : s \in \mathbb{R}, \quad \xi \in X \right\}$$

where, as above, $\Phi_t(\xi) = \phi(t,\xi)$. The estimates for the lower order derivatives follow similarly. Recall that $x_\nu(\cdot,\xi)$ for $\nu = 1,2$ are the solutions to (19) and that $F^\nu_t(\xi) = x_\nu(t,\xi)$.

As usual, for $\nu = 1,2$ and for a fixed $(s,\xi) \in \mathbb{R} \times X$, we will apply the chain rule (11). In fact, we find that

$$D^k(\Phi_{t+s} \circ F^1_{t+s})(\xi)(\mathbf{e}) = \sum_{i=1}^k \sum_{j_1+\cdots+j_i=k} \sum_{\{\ell\}} D^i \Phi_{t+s}(F^1_{t+s}(\xi))(\mathbf{u}^{(\nu)}),$$

where, see the notation of (12), $\mathbf{e} = (e_1, \ldots, e_k) \in X^k$, with $|e_i| = 1$, for $i = 1, \ldots, k$, and for $r = 1, \ldots, i$,

$$\mathbf{u}^{(\nu)} = (u_1^{(\nu)}, \ldots, u_i^{(\nu)}), u_r^{(\nu)} = D^{j_r} F^\nu_{t+s}(\xi)(e_{\ell_{j_1}+\cdots+\ell_{j_{r-1}}+1}, \ldots, e_{\ell_{j_1}+\cdots+\ell_{j_r}}).$$

We have

$$|[D^k(\Phi_{t+s}\circ F^1_{t+s}(\xi) - D^k(\Phi_{t+s}\circ F^2_{t+s}(\xi)](\mathbf{e})| \le \sum_{i=1}^{k}\sum_{j_1+\cdots+j_i=k}\sum_{\{\ell\}}$$

$$\Big\{|[D^i\Phi_{t+s}(F^1_{t+s}(\xi)) - D^i\Phi_{t+s}(F^2_{t+s}(\xi))](\mathbf{u}^{(1)})| \tag{79}$$

$$+ |D^i\Phi_{t+s}(F^2_{t+s}(\xi))(\mathbf{u}^{(1)}) - D^i\Phi_{t+s}(F^2_{t+s}(\xi))(\mathbf{u}^{(2)})|\Big\}. \tag{80}$$

We use (14) to show that the expression in (79) is bounded by

$$\|\varphi\|_{C^{i,1}}|x_1(t+s,\xi) - x_2(t+s,\xi)|\prod_{r=1}^{i}|u^{(1)}_r|.$$

To estimate $|x_1(t+s,\xi) - x_2(t+s,\xi)|$, we use Lemma 4.3 with $k=0$. To estimate $|u^{(1)}_r|$ we use Lemma 4.1 with $k=j_r$ and $f=f_\nu$ in this lemma. Then, using (62), we find that the expression (79) is bounded by

$$\|\varphi\|_{C^{i,1}}ce^{(\omega+d\epsilon)|t|}\|\psi_1-\psi_2\|_{C^0}\cdot ce^{\sum_{r=1}^{i}j_r(\omega+M\|f_\nu\|_{C^{j_r,1}})|t|}$$
$$\le ce^{(k+1)(\omega+d\epsilon)|t|}\|\psi_1-\psi_2\|_{C^k}\|\varphi\|_{C^{k,1}}. \tag{81}$$

To estimate (80), we use (13)–(14) with $n=i$, $\mathcal{A} = D^i\Phi_{t+s}(F^2_{t+s}(\xi))$, and $\mathbf{e}^{(\nu)} = \mathbf{u}^{(\nu)}$, $\nu=1,2$, to show that the expression (80) is bounded by

$$\|\varphi\|_{C^i}\sum_{r=1}^{i}\left(\prod_{p<r}|u^{(1)}_p|\right)|u^{(1)}_r - u^{(2)}_r|\left(\prod_{p>r}|u^{(2)}_p|\right).$$

Now, for $|u^{(\nu)}_p|$ and $|u^{(1)}_r - u^{(2)}_r|$, we apply Lemma 4.1 with $k=j_p$, $f=f_\nu$ and Lemma 4.3 with $k=j_r$, respectively, to show that the expression (80) is bounded by

$$\|\varphi\|_{C^i}ce^{\sum_{p<r}j_p(\omega+M\|f_1\|_{C^{j_p,1}})|t|}\cdot e^{(j_r+1)(\omega+d\epsilon)|t|}\|\psi_1-\psi_2\|_{C^{j_r}}\cdot e^{\sum_{p>r}j_p(\omega+M\|f_2\|_{C^{j_p,1}})|t|}.$$

By (62), we find that the expression in (80) is bounded by

$$ce^{(k+1)(\omega+d\epsilon)|t|}\|\psi_1-\psi_2\|_{C^k}\|\varphi\|_{C^{k,1}}. \tag{82}$$

Finally, we apply (81)–(82) in (79)–(80) to obtain

$$\|D^k(\Phi_{t+s}\circ F^1_{t+s})(\xi)-D^k(\Phi_{t+s}\circ F^2_{t+s})(\xi)\|_{\mathcal{L}_k}\le ce^{(k+1)(\omega+d\epsilon)|t|}\|\psi_1-\psi_2\|_{C^k}\|\varphi\|_{C^{k,1}},$$

and Theorem 2.8 is proved.■

5. Center manifolds for linear skew-product flows

In this section we will give an application of Theorem 3.4. Actually, we will apply this theorem in an autonomous situation where the evolution family $\{U(t,s)\}$ is, in fact, a specific semigroup, the evolution semigroup generated by a linear skew-product flow. As a result, we will prove the existence of a Lipschitz invariant manifold for a semilinear skew-product flow. The existence of a *differentiable* invariant manifold for this situation in the finite dimensional setting was proved by S.-N. Chow and Y. Yi in [11]. We do not consider the question of the smoothness of the invariant manifold in this paper, but we believe that our approach could be used to analyze this question as well.

Let ϕ^t denote a flow on a compact metric space Θ, let Z denote a Banach space, and let $\{\Phi^t\}_{t\ge 0}$ denote a linear cocycle over ϕ^t; that is, for each $\theta\in\Theta$, we have $\Phi^t(\theta)\in\mathcal{L}(Z)$ and, for all $t,s\ge 0$, we have the following identities:

$$\Phi^{t+s}(\theta)=\Phi^t(\phi^s\theta)\Phi^s(\theta),\qquad \Phi^0(\theta)=I.$$

Here, we assume that the cocycle $\{\Phi^t\}$ is strongly continuous, that is $(t,\theta)\mapsto\Phi^t(\theta)z$ is continuous from $\mathbb{R}_+\times\Theta$ to Z for each $z\in Z$.

For $H : \Theta \times Z \to Z$, a globally Lipschitz function, let us consider the semilinear skew-product flow Υ^t on $\Theta \times Z$ given by

$$\Upsilon^t(\theta, \zeta) = (\phi^t\theta, z(t; \theta, \zeta)), \tag{83}$$

where, for $\theta \in \Theta$ and $\zeta \in Z$, the function $t \mapsto z(t; \theta, \zeta)$ is the solution of the following integral equation:

$$z(t; \theta, \zeta) = \Phi^t(\theta)\zeta + \int_0^t \Phi^{t-s}(\phi^s\theta) H(\phi^s\theta, z(s; \theta, \zeta))\, ds. \tag{84}$$

If $z(\cdot; \theta, \zeta)$ and the cocycle Φ^t are differentiable, and if

$$A(\theta) := \frac{d}{dt}\bigg|_{t=0} \Phi^t(\theta),$$

then $t \mapsto z(t; \theta, \zeta)$ is the solution of the abstract Cauchy problem

$$\dot{z} = A(\phi^t\theta)z + H(\phi^t\theta, z), \quad z(0) = \zeta. \tag{85}$$

The special case where the Banach space Z is finite dimensional and where z is the solution of an abstract Cauchy problem is considered in [11].

Returning to the general equation (84) for a strongly continuous cocycle, let Σ denote the Sacker-Sell spectrum for the linear skew-product flow

$$\hat{\phi}^t : \Theta \times Z \to \Theta \times Z : (\theta, \zeta) \mapsto (\phi^t(\theta),\, \Phi^t(\theta)\zeta).$$

In this section we will show that the semilinear skew-product flow Υ^t has a Lipschitz invariant manifold $M \subset \Theta \times Z$ provided that $\|H\|_{C^{0,1}}$ is sufficiently small and the Sacker-Sell spectrum Σ satisfies a gap condition. Our strategy is to reduce the existence of an invariant manifold for the nonautonomous skew-product flow Υ^t over Θ to a corresponding existence problem in an *autonomous* setting on the space X of continuous vector functions $x : \Theta \to Z$. A similar idea was used by

R. Johnson in [20] in the finite dimensional setting, see Remark 1 below. The philosophy of "lifting" the flow to a space of functions was exploited in [24] (see also [6, 21]) for the case of linear skew-product flows by the introduction of a special semigroup of operators on the space of continuous vector-functions, the *evolution semigroup*. This concept will play a fundamental role in the rest of this section.

The evolution semigroup $\{E^t\}_{t\geq 0}$ associated with a linear skew-product flow is a semigroup on the space $X = C(\Theta; Z)$ of continuous functions $x : \Theta \to Z$ with sup-norm—a space that is isomorphic to the space of continuous sections of the trivial bundle $\Theta \times Z \to \Theta$—that is defined by the following rule:

$$(E^t x)(\theta) = \Phi^t(\phi^{-t}\theta)x(\phi^{-t}\theta). \tag{86}$$

We will formulate below three facts about these evolution semigroups. In order to do this, let us recall that a semigroup $\{E^t\}$ is called hyperbolic provided that $\sigma(E^1) \cap \{z : |z| = 1\} = \emptyset$. Also, the linear skew-product flow $\hat{\phi}^t$ is said to have an exponential dichotomy (see [35, 36]) if there exists a strongly continuous projection-valued function $P : \Theta \to \mathcal{L}(Z)$ and a complementary projection Q defined by $Q(\theta) := I - P(\theta)$ such that $\Phi^t(\theta)P(\theta) = P(\phi^t\theta)\Phi^t(\theta)$, the restriction $\Phi^t_Q(\theta) = \Phi^t(\theta)|\mathrm{Im}\, Q(\theta)$ is invertible as an operator from $\mathrm{Im}\, Q(\theta)$ to $\mathrm{Im}\, Q(\phi^t\theta)$, and if there are constants $\beta > 0$ and $C > 0$ such that, for all $\theta \in \Theta$ and $t \geq 0$, the following estimates hold:

$$\|\Phi^t_P(\theta)\| \leq Ce^{-\beta t}, \qquad \|[\Phi^t_Q(\theta)]^{-1}\| \leq Ce^{-\beta t}.$$

We will need the following facts (see [24]).

Proposition 5.1. *The evolution semigroup $\{E^t\}$ is hyperbolic on X if and only if the linear skew-product flow $\hat{\phi}^t$ on $\Theta \times Z$ has exponential dichotomy.*

Proposition 5.2. *Suppose that the evolution semigroup $\{E^t\}$ is hyperbolic. If $\mathcal{P}$ denotes the Riesz projection for E^1 on $X = C(\Theta, Z)$ corresponding to the part of the spectrum of E^1 that lies inside of the open unit disk, then the projection $\mathcal{P}$ has a form $(\mathcal{P}x)(\theta) = P(\theta)x(\theta)$, where P is the projection-valued function given in the definition of the exponential dichotomy for $\{\hat{\phi}^t\}$. Conversely, if P is the projection-valued function given in the definition of the exponential dichotomy for $\{\hat{\phi}^t\}$, then $(\mathcal{P}x)(\theta) = P(\theta)x(\theta)$ defines the Riesz projection $\mathcal{P}$ for E^1 on $X = C(\Theta, Z)$.*

The following formula relates the spectrum of the evolution semigroup on X and the Sacker-Sell spectrum of the linear skew-product flow.

Proposition 5.3. $\Sigma = \ln|\sigma(E^1)\backslash\{0\}|$.

As a result, the spectral projections for E^1 give the Sacker-Sell spectral decomposition for $\{\hat{\phi}^t\}$ that corresponds to the components of Σ.

We formulate the gap condition for the linear skew-product flow $\{\hat{\phi}^t\}$ as follows: Assume that $\Sigma = \Sigma_c \cup \Sigma_h$ with $\Sigma_c \cap \Sigma_h = \emptyset$ and define $\beta_h = \inf\{|\lambda| : \lambda \in \Sigma_h\}$ and $\omega_c = \sup\{|\lambda| : \lambda \in \Sigma_c\}$. The skew-product flow satisfies the gap condition provided

$$\beta_h - \omega_c > 0. \tag{87}$$

By Proposition 5.3, the gap condition is, in fact, the spectral condition (S) imposed on the evolution semigroup $\{E^t\}$. Also, by Propositions 5.1–5.2, the existence of the decomposition $\Sigma = \Sigma_c \cup \Sigma_h$ implies the existence of the corresponding spectral (cf. [35, 36]) decomposition of the trivial bundle $\Theta \times Z$ into a direct sum of subbundles Z_c and Z_h where, in its turn, $Z_h = Z_h^+ \oplus Z_h^-$. The spectral subbundles Z_c, Z_h, and $Z_h^\pm$ are given by the corresponding spectral projections $\mathcal{P}_c$, $\mathcal{P}_h$, $\mathcal{P}_h^\pm$ for the operator E^1. We use the notation $Z_{c,h}(\theta) = \operatorname{Im} P_{c,h}(\theta)$ to specify the fibers of the spectral subbundles.

Theorem 5.4. *Consider the bundle $\Theta \times Z$, a function $H : \Theta \times Z \to Z$, and the corresponding (nonlinear) skew-product flow Υ^t defined by equations (83) and (84). For each $\eta > 0$ there exists $\epsilon > 0$ such that if $\|H\|_{C^{0,1}} \le \epsilon$, then Υ^t has an invariant manifold, given by the graph of a function $\psi : \Theta \times Z \to Z$. In fact, the invariant manifold has the form*

$$M_\psi = \{(\theta, \zeta_c + \psi(\theta, \zeta_c)) : \theta \in \Theta, \zeta_c \in Z_c(\theta)\} \tag{88}$$

where ψ is an element of the space

$$L_\eta := \{\psi \in C^0(\Theta \times Z, Z) : \|\psi\|_{C^{0,1}} \le \eta, \quad \psi(\theta, \cdot) : Z_c(\theta) \to Z_h(\theta)\}. \tag{89}$$

Proof. Consider a semigroup $\{\mathcal{E}^t\}_{t\ge 0}$ of (nonlinear) operators on $X = C(\Theta; Z)$ defined by

$$\mathcal{E}^t(x)(\theta) = z(t; \phi^{-t}\theta, x(\phi^{-t}\theta)). \tag{90}$$

Using (84), we see that for each fixed $x \in X$ and all $\theta \in \Theta$,

$$\mathcal{E}^t(x)(\theta) = \Phi^t(\phi^{-t}\theta)x(\phi^{-t}\theta) + \int_0^t \Phi^{t-s}(\phi^{s-t}\theta) \cdot H(\phi^{s-t}\theta, \mathcal{E}^s(x)(\phi^{s-t}\theta))\, ds. \tag{91}$$

Define $g : X \to X$ by $g(x)(\theta) = H(\theta, x(\theta))$. From the definition (86) of the evolution semigroup $\{E^t\}_{t\ge 0}$, equation (91) becomes

$$\mathcal{E}^t(x) = E^t x + \int_0^t E^{t-\tau} \circ g(\mathcal{E}^\tau(x))\, d\tau. \tag{92}$$

Thus, if we fix $x_0 \in X$ and define $U(t,\tau) = E^{t-\tau}$, then the function $\mathbb{R} \to X$ given by $t \mapsto \mathcal{E}^t(x_0)$ satisfies equation (25). Let $X = X_c \oplus X_h$ be the spectral decomposition for the linear semigroup $\{E^t\}_{t\geq 0}$ that satisfies

$$\ln|\sigma(E_c^1)\backslash\{0\}| = \Sigma_c, \quad \ln|\sigma(E_h^1)\backslash\{0\}| = \Sigma_h.$$

By Proposition 5.2, the corresponding spectral projections $\mathcal{P}_c$ and $\mathcal{P}_h$ for E^1 on X are multiplication operators whose multipliers are strongly continuous projection-valued functions $P_c, P_h : \Theta \to \mathcal{L}(Z)$; that is, $(\mathcal{P}_{c,h}x)(\theta) = P_{c,h}(\theta)x(\theta)$. Thus, for j equal to c or h, we have that

$$X_j = \{x \in C(\Theta; Z) : x(\theta) \in Z_j(\theta), \quad \theta \in \Theta\}.$$

Our strategy to complete the proof of the theorem is to first use Theorem 3.4 to prove the existence of a function $\Psi : X_c \to X_h$ whose graph is invariant for the flow $\mathcal{E}^t$ on X. This Ψ will be found as a fixed point of a Lyapunov-Perron operator on $C^0(X_c, X_h)$. Next, we will "push" the graph of Ψ back to $\Theta \times Z$. To be more precise, we will show that Ψ can be "localized in the variable θ"; that is, there exists a function $\psi : \Theta \times Z \to Z$ such that $\psi(\theta, \cdot) : Z_c(\theta) \to Z_h(\theta)$, and the following condition holds: For each fixed $\theta \in \Theta$ and $\zeta_c \in Z_c(\theta)$ we have $\psi(\theta, \zeta_c) = \Psi(x_c)(\theta)$ for every function $x_c \in X_c$ with the property $x_c(\theta) = \zeta_c$. The graph of this function ψ will be the desired invariant manifold for Υ^t.

To start, we define

$$\mathcal{L}_\eta = \{\Psi \in C^0(X, X) : \|\Psi\|_{C^{0,1}(X,X)} \leq \eta \quad \text{and} \quad \Psi : X_c \to X_h\}.$$

Thus, for $\Psi \in \mathcal{L}_\eta$ and $\theta \in \Theta$, if $x_c \in X_c$, we have $x_c(\theta) \in Z_c(\theta)$ and $\Psi(x_c)(\theta) \in Z_h(\theta)$. The graph $\mathcal{M}_\Psi = \{x_c + \Psi(x_c) : x_c \in X_c\}$ of the function $\Psi \in \mathcal{L}_\eta$ is an invariant set for the nonlinear evolution semigroup $\mathcal{E}^t$ on X if and only if Ψ is a fixed point of the Lyapunov-Perron integral operator $\mathcal{T}$ on $C^0(X_c, X_h)$, see formula (39) above, given by

$$\mathcal{T}(\Psi)(x_c^0) = \int_{-\infty}^{\infty} \mathcal{K}(-s) g_h(x_c(s, x_c^0) + \Psi(x_c(s, x_c^0)))\, ds. \tag{93}$$

Here, $\mathcal{K}$ denotes the Green's function for the hyperbolic linear semigroup $E_h^t = E^t|X_h$, and we have defined $g_h(x) := \mathcal{P}_h g(x)$ where the projection $\mathcal{P}_h : X \to X_h$, and $t \mapsto x_c(t, x_c^0)$ denotes the solution of the X_c-equation

$$x_c(t, x_c^0) = E_c^t x_c^0 + \int_0^t E_c^{t-s} g_c(x_c(s, x_c^0) + \Psi(x_c(s, x_c^0))\, ds. \tag{94}$$

In the course of the proof of Theorem 3.4, we showed that $\mathcal{T}$ is a strict contraction of $\mathcal{L}_\eta$ provided the gap condition (87) holds and $\|g\|_{C^{0,1}(X,X)}$ is sufficiently small.

First, we note that

$$\begin{aligned} \|g(x_1) - g(x_2)\|_X &= \sup_{\theta \in \Theta} |H(\theta, x_1(\theta)) - H(\theta, x_2(\theta))|_Z \\ &\leq \sup_{\theta \in \Theta} \|H\|_{C^{0,1}(\Theta \times Z, Z)} |x_1(\theta) - x_2(\theta)|_Z \\ &\leq \|H\|_{C^{0,1}(\Theta \times Z, Z)} \|x_1 - x_2\|_X \end{aligned}$$

where the Lipschitz norm of H is defined by

$$\|H\|_{C^{0,1}(\Theta \times Z, Z)} = \sup_{\theta \in \Theta} \max\{\sup_{z \in Z} |H(\theta, z)|, \quad \sup_{z_1 \neq z_2} |H(\theta, z_1) - H(\theta_1, z_2)|/|z_1 - z_2|\}. \tag{95}$$

This implies $\|g\|_{C^{0,1}(X,X)} \leq \|H\|_{C^{0,1}(\theta \times Z, Z)}$.

To prove the theorem, we will show that $\mathcal{T}$ preserves a closed subset $\mathcal{L}_{\eta,\text{loc}} \subset \mathcal{L}_\eta$, and, hence, has a fixed point $\Psi \in \mathcal{L}_{\eta,\text{loc}}$. We define $\mathcal{L}_{\eta,\text{loc}}$ to be the set of all functions $\Psi \in \mathcal{L}_\eta$ with the following property: For each point $\theta_0 \in \Theta$ and for all $x_1, x_2 \in X_c$, if $x_1(\theta_0) = x_2(\theta_0)$, then $\Psi(x_1)(\theta_0) = \Psi(x_2)(\theta_0)$.

Lemma 5.5. *$\mathcal{T}$ preserves the closed subset $\mathcal{L}_{\eta,\mathrm{loc}} \subset \mathcal{L}_\eta$.*

We will postpone the proof of the lemma and complete the proof of the theorem. To this end, let $\Psi \in \mathcal{L}_{\eta,\mathrm{loc}}$ denote the unique fixed point of the contraction $\mathcal{T}$. We define the function $\psi \in L_\eta$, where L_η is the space defined in the statement of the theorem, by the following procedure: For each $(\theta, \zeta_c) \in \Theta \times Z_c(\theta)$ choose a function $x_c \in X_c$ such that $x_c(\theta) = \zeta_c$ and then define

$$\psi(\theta, \zeta_c) := \Psi(x_c)(\theta). \tag{96}$$

Since $\Psi \in \mathcal{L}_{\eta,\mathrm{loc}}$, the function ψ is well defined.

Consider the graph M_ψ of the function ψ. We want to see that M_ψ is invariant under Υ^t in (83)–(84). Fix $\theta_0 \in \Theta$, $\zeta_c \in Z_c(\theta_0)$ and let $\zeta = \zeta_c + \psi(\theta_0, \zeta_c)$. In fact, we want to show the following: If $z_c(t) := P_c(\phi^t\theta_0) z(t, \theta_0, \zeta)$ satisfies

$$\begin{aligned} z_c(t) &= P_c(\phi^t\theta_0)\Phi^t(\theta_0)\zeta_c \\ &\quad + \int_0^t P_c(\phi^t\theta_0)\Phi^{t-s}(\phi^s\theta_0) H_c(\phi^s\theta_0, z_c(s) + \psi(\phi^s\theta_0, z_c(s)))\, ds, \end{aligned} \tag{97}$$

then $z_h(t) = \psi(\phi^t\theta_0, z_c(t))$ satisfies

$$\begin{aligned} z_h(t) &= P_h(\phi^t\theta_0)\Phi^t(\theta_0)\psi(\theta_0, \zeta_c) \\ &\quad + \int_0^t P_h(\phi^t\theta_0)\Phi^{t-s}(\phi^s\theta_0) H_h(\phi^s\theta_0, z_c(s) + \psi(\phi^s\theta_0, z_c(s)))\, ds. \end{aligned} \tag{98}$$

Here, $P_{c,h}(\theta) : Z \to Z_{c,h}(\theta)$ are the projections associated with the Riesz projections $\mathcal{P}_{c,h} : X \to X_{c,h}$ for E^1. In particular, we have

$$(\mathcal{P}_{c,h}x)(\theta) = P_{c,h}(\theta)x(\theta), \quad \text{and} \quad H_{c,h}(\theta, z) = P_{c,h}(\theta)H(\theta, z), \quad \theta \in \Theta, z \in Z.$$

Recall that $\mathcal{M}_\Psi$, the graph of Ψ, is invariant for $\mathcal{E}^t$ on $X = C(\Theta; Z)$. This means that, for each $x_c^0 \in X_c$, if the function $x_c(\cdot; x_c^0) : \mathbb{R}_+ \to X_c$ solves equation (94), then the function $\Psi(x_c(\cdot, x_c^0)) : \mathbb{R}_+ \to X_h$ satisfies the equation

$$\Psi(x_c(t; x_c^0)) = E_h^t \Psi(x_c^0) + \int_0^t E_h^{t-s} g_h(x_c(s, x_c^0) + \Psi(x_c(s, x_c^0)))\, ds. \tag{99}$$

Choose $x_c^0 \in X_c$ such that $x_c^0(\theta_0) = \zeta_c$. If equation (94) holds, then for every $\theta \in \Theta$ we have

$$\begin{aligned} x_c(t; x_c^0)(\theta) &= P_c(\theta)\Phi^t(\phi^{-t}\theta)x_c^0(\phi^{-t}\theta) \\ &\quad + \int_0^t P_c(\theta)\Phi^{t-s}(\phi^{s-t}\theta)P_c(\phi^{s-t}\theta)H(\phi^{s-t}\theta, x_c(s; x_c^0)(\phi^{s-t}\theta) \\ &\qquad + \Psi(x_c(s; x_c^0))(\phi^{s-t}\theta))\, ds. \end{aligned} \tag{100}$$

Recall that $\Psi \in \mathcal{L}_{\eta,\mathrm{loc}}$. In particular, for $\theta \in \Theta$ and for ψ as defined in (96), we have

$$\Psi(x_c(s; x_c^0))(\phi^s\theta) = \psi(\phi^s\theta, x_c(s; x_c^0)(\phi^s\theta)).$$

Using this fact, the evaluation of equation (100) at the point $\theta = \phi^t\theta_0$ has the following form:

$$\begin{aligned} x_c(t, x_c^0)(\phi^t\theta_0) &= P_c(\phi^t\theta_0)\Phi^t(\theta_0)x_c^0(\theta_0) \\ &\quad + \int_0^t P_c(\phi^t\theta_0)\Phi^{t-s}(\phi^s\theta))P_c(\phi^s\theta_0)H(\phi^s\theta_0, x_c(s; x_c^0)(\phi^s\theta_0) \\ &\qquad + \psi(\phi^s\theta_0, x_c(s; x_c^0)(\phi^s\theta_0)))\, ds. \end{aligned}$$

Thus, $z_c(t) := x_c(t; x_c^0)(\phi^t\theta_0)$ satisfies (97).

Similarly, in view of equation (99), for $\theta \in \Theta$ we have

$$\begin{aligned} \Psi(x_c(t; x_c^0)(\theta) &= P_h(\theta)\Phi^t(\phi^{-t}\theta)\Psi(x_c^0)(\phi^{-t}\theta) \\ &\quad + \int_0^t P_h(\theta)\Phi^{t-s}(\phi^{s-t}\theta)P_h(\phi^{s-t}\theta)H(\phi^{s-t}\theta, x_c(s; x_c^0)(\phi^{s-t}\theta) \\ &\qquad + \Psi(x_c(s; x_c^0))(\phi^{s-t}\theta))\, ds. \end{aligned} \tag{101}$$

Moreover, since $\Psi \in \mathcal{L}_{\eta,\mathrm{loc}}$, we have

$$\Psi(x_c(t;x_c^0))(\phi^t\theta_0) = \psi(\phi^t\theta_0, x_c(t,x_c^0)(\phi^t\theta_0)) = \psi(\phi^t\theta_0, z_c(t)) = z_h(t).$$

Using equation (101) evaluated at $\theta = \phi^t\theta_0$, we have that $z_h(t)$ indeed satisfies equation (98). $\square$

Proof of Lemma 5.5. Fix $\Psi \in \mathcal{L}_{\eta,\mathrm{loc}}$ and let the operator $\mathcal{T}$ be defined as in equation (93). We will first show the following proposition: For each $\theta_0 \in \Theta$ and every $x_1, x_2 \in C(\Theta, X)$, if $x_1(\theta_0) = x_2(\theta_0)$, then $\mathcal{T}(\Psi)(x_1)(\theta_0) = \mathcal{T}(\Psi)(x_2)(\theta_0)$.

Since $\Psi : X_c \to X_h$ is in $\mathcal{L}_{\eta,\mathrm{loc}}$, we can define the function ψ as in (96). Then, $\psi(\theta,\cdot) : Z_c(\theta) \to Z_h(\theta)$ and, for each $x_c \in X_c$, we have $\Psi(x_c)(\theta) = \psi(\theta, x_c(\theta))$.

Fix $\theta_0 \in \Theta$ and $x_i^0 \in X_c$, $i = 1,2$ such that $x_1^0(\theta_0) = x_2^0(\theta_0)$. Let $x_c(\cdot; x_i^0) : \mathbb{R} \to X_c$ denote the solutions of the equation

$$x_c(t;x_i^0) = E_c^t x_i^0 + \int_0^t E_c^{t-s} g_c(x_c(s;x_i^0) + \Psi(x_c(s;x_i^0)))\, ds. \tag{102}$$

Claim. *If $t \in \mathbb{R}$, then $x_c(t;x_1^0)(\phi^t\theta_0) = x_c(t;x_1^0)(\phi^t\theta_0)$.*

To prove the claim, we remark that using equation (102), if $\theta \in \Theta$ and $i = 1,2$, then

$$\begin{aligned} x_c(t,x_i^0)(\theta) &= P_c(\theta)\Phi^t(\phi^{-t}\theta)x_i^0(\phi^{-t}\theta) \\ &\quad + \int_0^t P_c(\theta)\Phi^{t-s}(\phi^{s-t}\theta)H(\phi^{s-t}\theta, x_c(s;x_i^0)(\phi^{s-t}\theta) \\ &\qquad + \psi(\phi^{s-t}\theta, x_c(s;x_i^0)(\phi^{s-t}\theta)))\, ds, \end{aligned} \tag{103}$$

where we have used the fact that $\Psi(x_c(s;x_i^0))(\phi^{s-t}\theta) = \psi(\phi^{s-t}\theta, x_c(s,x_i^0)(\phi^{s-t}\theta))$ for our function ψ defined by equation (96). For $i = 1,2$, if we define $z_i(t) = x_c(t;x_i^0)(\phi^t\theta_0)$ and evaluate equation (103) at $\theta = \phi^t\theta_0$, then we see that $z_i(\cdot)$ satisfies the equation

$$z_i(t) = \Phi_c^t(\theta_0)x_i^0(\theta_0) + \int_0^t \Phi_c^{t-s}(\phi^s\theta_0)H(\phi^s\theta_0, z_i(s) + \psi(\phi^s\theta_0, z_i(s)))\,ds.$$

Consider

$$h(t,\zeta) = H(\phi^t\theta_0, \zeta + \psi(\phi^t\theta_0, \zeta)), \quad t \in \mathbb{R}, \quad \zeta \in Z_c(\phi^t\theta_0),$$

and $U(t,\tau) = \Phi_c^{t-\tau}(\phi^\tau\theta_0)$. Clearly, h is globally Lipschitz and, by the cocycle property, $\{U(t,\tau)\}_{t\geq\tau}$ is an evolution family. By the assumption, $x_1^0(\theta_0) = x_2^0(\theta_0) =: \zeta$, and z_i are the (mild) solutions to the following nonautonomous equation

$$z(t) = U(t,0)\zeta + \int_0^t U(t,s)h(s,z(s))\,ds.$$

But, if h is Lipschitz, then this equation has a unique solution. Thus, we conclude that $z_1(t) = z_2(t)$ for all $t \in \mathbb{R}$, as required.

To finish the proof of the lemma, recall that

$$\mathcal{T}(\Psi)(x_i^0) = \int_{-\infty}^{\infty} \mathcal{K}(-s)g_h(x_c(s;x_i^0) + \Psi(x_c(s;x_i^0)))\,ds, \quad i = 1,2, \tag{104}$$

where the Green's function $\mathcal{K}$ for the hyperbolic evolution semigroup $\{E_h^t\}_{t\geq 0}$ is given (see, e.g., [25, Definition 4.1, sec 2], [24, formula (5.2)],) as follows: $\mathcal{K}(s) = E_+^s$ for $s > 0$ and $\mathcal{K}(s) = -E_-^s$ for $s < 0$. Here $E_\pm^s = E^s|\mathrm{Im}\,\mathcal{P}_\pm$ and $\mathcal{P}_+$, resp. $\mathcal{P}_-$, is the Riesz projection for E_h^1 that corresponds to $\sigma(E_h^1)\cap\{z : |z| < e^{-\beta_h}\}$, resp. to $\sigma(E_h^1)\cap\{z : |z| > e^{\beta_h}\}$. In other words, for the projections $(\mathcal{P}_\pm x)(\theta) = P_\pm(\theta)x(\theta)$ one has:

$$\begin{aligned}
\mathcal{K}(s)(x_h)(\theta) &= P_+(\theta)\Phi_h^s(\phi^{-s}\theta)x_h(\phi^{-s}\theta) \quad \text{for} \quad s > 0,\\
\mathcal{K}(s)(x_h)(\theta) &= -P_-(\theta)\Phi_h^s(\phi^{-s}\theta)x_h(\phi^{-s}\theta) \quad \text{for} \quad s < 0, \quad \theta \in \Theta, \quad x_h \in X_h.
\end{aligned}$$

By the claim proved above, we have $x_c(s;x_1^0)(\phi^s\theta_0) = x_c(s;x_2^0)(\phi^s\theta_0)$, $s \in \mathbb{R}$. If we evaluate equation (104) at $\theta = \theta_0$, we obtain

$$\begin{aligned}\mathcal{T}(\Psi)(x_i^0)(\theta_0) &= -\int_0^\infty P_-(\theta_0)\Phi_h^{-s}(\phi^s\theta_0)H(\phi^s\theta_0, x_c(s;x_i^0)(\phi^s\theta_0) \\ &\qquad +\psi(\phi^s\theta_0, x_c(s;x_i^0)(\phi^s\theta_0)))\,ds \\ &\quad +\int_{-\infty}^0 P_+(\theta_0)\Phi_h^{-s}(\phi^s\theta_0)H(\phi^s\theta_0, x_c(s;x_i^0)(\phi^s\theta_0) \\ &\qquad +\psi(\phi^s\theta_0, x_c(s;x_i^0)(\phi^s\theta_0)))\,ds.\end{aligned}$$

The right-hand side of the last formula remains the same for $i = 1$ and $i = 2$. Thus, $\mathcal{T}(\Psi)(x_i^0)(\theta_0) = \mathcal{T}(\Psi)(x_2^0)(\theta_0)$, as required. □

Remark 1. We thank Y. Yi for mentioning to us the paper [20] by R. Johnson where a proof is given for the existence of invariant manifolds for skew-product flows as in (85) in the finite dimensional setting with the base metric space Θ taken to be the hull of a certain matrix-function. In this paper, Johnson also used the idea of "lifting" the skew-product flow to a space X of functions $f : \Theta \to Z$. Instead of $X = C(\Theta, Z)$, as we have used, Johnson uses $X = B(\Theta, Z)$, the space of bounded functions. This allows him to use results from [19] concerning the existence and smoothness of invariant manifolds for the iterates of a *map* to prove the existence of invariant manifolds for the *flows* that he considers. We note that for *maps* one can obtain $\Psi \in B(\Theta, Z)$ and then compute ψ as in (96) for the function $x_c \in B(\Theta, Z)$ defined so that $x_c(\theta) = \zeta_c$ and so that x_c vanishes elsewhere. Thus, for $X = B(\Theta, Z)$ our "localization" argument used to show that $\Psi \in \mathcal{L}_{\eta,loc}$ is not needed. However, the advantage of our choice of $X = C(\Theta, Z)$ for the *flow* case is related to the fact that we obtain a space X where $\{E^t\}$ is a *strongly continuous* semigroup. Thus, our Theorem 3.4 is directly applicable, and the transition from maps to flows is not needed.

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