Geometric Approach to Inverse Scattering for Hydrogen–like Systems in a Homogeneous Magnetic Field

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Abstract

We consider the Hamiltonian of one quantum particle in a homogeneous magnetic field B and a potential V in space dimensions $\nu = 3$. If the magnetic field B is given and V is of short range then the high velocity limit of the scattering operator uniquely determines the potential V.
If, in addition, long–range potentials V^l are present, some knowledge of (the far out tail of) V^l is needed to define a modified Dollard wave operator and a scattering operator S^D. Again its high velocity limit uniquely determines $V = V^s + V^l$, if B is given.
We generalize our results to a system of two interacting quantum particles with opposite electric charges.

1 Introduction

We study a system of one quantum particle in a homogeneous magnetic field in $\nu = 3$ dimensions. As an example of such a one–body system we mention

*This research was supported by a grant from Deutsche Forschungsgemeinschaft, hereby gratefully acknowledged, and is in partial fulfillment of the requirements for a Ph.D. degree at the RWTH Aachen.

a hydrogen atom with infinitely heavy proton. The interactions are given by a vector potential $\mathbf{A}$ with $\mathbf{rot}\,\mathbf{A} = \mathbf{B}$ and a scalar potential V. We chose the coordinates $(x, y, z) \in \mathbb{R}^3$ so that the magnetic field is pointed along the z–axis, i.e., $\mathbf{B} = (0, 0, b)$ for $b > 0$ constant. In the transversal gauge the corresponding vector potential is

$$\mathbf{A}(\mathbf{r}) = (-by/2, bx/2, 0) \text{ for a vector } \mathbf{r} = (x, y, z) \in \mathbb{R}^3 . \tag{1.1}$$

A classical particle with mass m, charge e and velocity $\mathbf{v}$ in a constant magnetic field $\mathbf{B}$ moves in helix with the Larmour radius

$$\mathbf{R}(t) = -\frac{m}{e}\frac{\mathbf{v}(t) \times \mathbf{B}}{\mathbf{B}^2}, \tag{1.2}$$

i.e., $|\mathbf{R}(t)| = (\mathbf{v}_x^2+\mathbf{v}_y^2)^{1/2} m\,|e|/b$ and the radius is proportional to the velocity in the (x, y)–direction. Then in the high velocity limit the radius tends to infinity, such that the particle moves in a straight line. The Larmour frequency is

$$\boldsymbol{\omega} = -\frac{e\,\mathbf{B}}{m}, \tag{1.3}$$

such that the minimal period of the particle $|T| = |2\pi/\boldsymbol{\omega}| = 2\pi\, m/(|e|\,b)$ is independent of the velocity. Thus for increasing velocity the particle moves in circles with increasing radius, but needs the same time for each circle. Therefore we have to assume that the particle moves parallel to the magnetic field with velocity $|\mathbf{v}_z| > 0$, such that the potential is hit only once.

Let $H_0 := \frac{1}{2m}\mathbf{p}^2$ be the free Hamiltonian. By

$$H_A := \frac{1}{2m}(\mathbf{p} - \mathbf{A}(\mathbf{r}))^2 \tag{1.4}$$

we denote the Hamiltonian with the magnetic field alone and by $H = H_A+V$ the interacting Hamiltonian with mass m, charge $e = 1$, momentum $\mathbf{p} = -i\nabla_{\mathbf{r}}$ and position $\mathbf{r}$. We abbreviate the first and second component of the position $\mathbf{r}$ with $\boldsymbol{\rho}$ and of the momentum $\mathbf{p}$ with $\boldsymbol{\pi}$:

$$\boldsymbol{\rho} := (x, y) \quad \text{and} \quad \boldsymbol{\pi} := (\mathbf{p}_x, \mathbf{p}_y). \tag{1.5}$$

It is useful to introduce the following representation (see [L 95] for the notation):

$$\begin{aligned} H_A &= H_0 + \frac{1}{2m}\left\{\frac{b^2}{4}\left(\frac{\mathbf{B}}{|\mathbf{B}|} \times \mathbf{r}\right)^2 - \mathbf{BL}\right\} && (1.6)\\ &= H_0 + \frac{1}{2m}\left\{\frac{b^2}{4}(x^2 + y^2) - b\,\mathbf{L}_z\right\} && (1.7)\\ &= \frac{1}{2m}\mathbf{p}_z^2 + H_{osc} - \frac{1}{2m}b\,\mathbf{L}_z && (1.8) \end{aligned}$$

with the z–component of the angular momentum $\mathbf{L}_z = x\mathbf{p}_y - y\mathbf{p}_x$ and

$$H_{osc} = \frac{1}{2m}\left\{(\mathbf{p}_x^2 + \mathbf{p}_y^2) + \frac{b^2}{4}(x^2 + y^2)\right\}. \tag{1.9}$$

We consider H acting on $\mathcal{H} = L^2(X)$ for $X = \mathbb{R}^3$. It is known (see [AHS 78a] for details) that $H_{xy} = H_{osc} - \frac{1}{2m} b\,\mathbf{L}_z$, acting on the Hilbert space $L^2(\mathbb{R}^2)$, has a complete set of eigenfunctions $\{\Theta_{ln}(x,y)\}_{l=0,\pm1,\ldots;n=0,1,\ldots}$ with corresponding eigenvalues

$$E_{ln} = (2m)^{-1} b\,(2n + |l| - l + 1), \tag{1.10}$$

where E_{ln} and Θ_{ln} are often referred to as Landau energy levels and Landau orbits. As H_{xy} does not act on the z–variable, we have $\exp[-iH_A t] = \exp[-iH_{xy}t]\exp[-i\mathbf{p}_z^2/(2m)]$.

The scalar potential can be split into a short–range and into a long–range part: $V = V^s + V^l$. V^s is assumed to be short–range: $V^s \in \mathcal{V}_{SR}$ with

$$\mathcal{V}_{SR} \;:=\; \left\{V^s \,|\, |V^s(\mathbf{r})| \le \text{const}\,(1+|z|)^{-(1+\varepsilon)}\right\}. \tag{1.11}$$

We mention that our results remain valid for unbounded short–range potentials, see Section 6.
V^l is assumed to be long–range: $V^l \in \mathcal{V}_{LR}$ with

$$\begin{aligned}\mathcal{V}_{LR} \;:=\; \big\{V^l \,|\, V^l \in C^1(\mathbb{R}^3),\ &V^l \text{ tends to zero for } |z| \to \infty \text{ and} \\ &|\nabla V^l(\mathbf{r})| \le \text{const}\,(1+|z|)^{-\gamma} \text{ with } \gamma > 3/2\big\}. \end{aligned}\tag{1.12}$$

We remark that we only assume decay in the z–direction. The splitting into short– and long–range parts is not unique. But without loss of generality we can make it in such a way that in addition

$$|D^\alpha V^l(\mathbf{r})| \le \text{const}\,(1+|z|)^{-1-|\alpha|(\delta+1/2)} \tag{1.13}$$

for $1 \le |\alpha| \le 4$, $0 < \delta < 1/2$ (see [Hö 76]).
Under these assumptions the interacting Hamiltonian is self–adjoint on its domain $\mathcal{D}(H) = \mathcal{D}(H_A) = W^{2,2}(X)$ in $\mathcal{H} = L^2(X)$.

In the special case, when V is of short range the following wave operators, which measure for a given $\mathbf{A}$ the additional effect on scattering due to V^s,

$$\Omega_\pm := \Omega_\pm(\mathbf{A}) := \operatorname*{s-lim}_{t\to\pm\infty} e^{iHt}e^{-iH_A t} \tag{1.14}$$

exist and are complete (see [AHS 78a, Chapter 4]). The scattering operator S is

$$S := S(\mathbf{A}; V^s) := (\Omega_+)^* \Omega_- . \tag{1.15}$$

As wave functions we chose asymptotic configurations $\Phi_0 \in \mathcal{H} = L^2(X)$ with compact momentum support in the ball around the origin with radius $m\eta$, $\eta > 0$, and momentum space wave function $\hat{\phi}_0 \in C_0^\infty(B_{m\eta}(0))$. These functions are dense in $\mathcal{H}$ and we can find a function $g \in C_0^\infty(\mathbb{R}^3)$ with $\Phi_0 = g(\mathbf{p})\Phi_0$ and especially a function $f \in C_0^\infty(\mathbb{R})$ with $\Phi_0 = f(\mathbf{p}_z)\Phi_0$.

By $\Phi_\mathbf{v}$ we denote the boosted configuration translated by $m\mathbf{v}$ in momentum space, where $\mathbf{v} = (\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z)$ is the velocity:

$$\Phi_\mathbf{v} = e^{im\mathbf{v}\mathbf{r}}\Phi_0 \quad \Leftrightarrow \quad \hat{\phi}_\mathbf{v}(\mathbf{p}) = \hat{\phi}_0(\mathbf{p} - m\mathbf{v}), \quad \mathbf{v}_z \neq 0, \tag{1.16}$$

where $\Phi_\mathbf{v}$ has compact velocity support in $B_\eta(\mathbf{v})$. Let $v := |\mathbf{v}|$. We will obtain asymptotics of S for the high velocity limit in an arbitrary direction $\hat{\mathbf{v}} = (\hat{\mathbf{v}}_x, \hat{\mathbf{v}}_y, \hat{\mathbf{v}}_z) := \mathbf{v}/v$, $v \to \infty$ with $\hat{\mathbf{v}}_z \neq 0$.

Regarding S as a mapping from the set of short–range potentials $\mathcal{V}_{SR}$ into the set of bounded operators $L(\mathcal{H})$ we have the following

Theorem 1 *Suppose that $V^s \in \mathcal{V}_{SR}$ and $V^l = 0$. Then the following reconstruction formula holds for all $\Phi_\mathbf{v}$, $\Psi_\mathbf{v}$ as in Eq. (1.16):*

$$\lim_{v\to\infty} iv((S - \mathbb{1})\Phi_\mathbf{v}, \Psi_\mathbf{v}) \;=\; \int_{-\infty}^{\infty} (V^s(\mathbf{r} + \tau\hat{\mathbf{v}})\Phi_0, \Psi_0)\, d\tau. \tag{1.17}$$

In particular, the scattering map

$$S(\mathbf{A}; \cdot) : \mathcal{V}_{SR} \to L(\mathcal{H})$$

is injective.

The modified Dollard wave operators

$$\begin{aligned}
\Omega_\pm^D := \Omega_\pm^D(\mathbf{A}) &:= \operatorname*{s-lim}_{t\to\pm\infty} e^{iHt} U^D(t) \quad \text{with} && (1.18)\\
U^D(t) &:= \exp[-iH_A t - i \int_0^t V^l(\hat{\boldsymbol{\rho}}_\mathbf{v}(s), s\mathbf{p}_z/m)\, ds] && (1.19)\\
\text{where} \quad \hat{x}_\mathbf{v}(s) &:= \omega^{-1}\{\mathbf{v}_x \sin(\omega\, s) + \mathbf{v}_y(1 - \cos(\omega\, s))\}, && (1.20)\\
\hat{y}_\mathbf{v}(s) &:= \omega^{-1}\{\mathbf{v}_y \sin(\omega\, s) - \mathbf{v}_x(1 - \cos(\omega\, s))\} \text{ and} && (1.21)\\
\boldsymbol{\rho}_\mathbf{v}(s) &:= (\hat{x}_\mathbf{v}(s), \hat{y}_\mathbf{v}(s)) && (1.22)
\end{aligned}$$

with $\omega := |\boldsymbol{\omega}| = b/m$ exist and are complete (see [L 95, Theorem 7.1]) for a Dollard correction with $\hat{\boldsymbol{\rho}}_{\mathbf{v}} = 0$).

$$S^D := S^D(\mathbf{A}, V^l; V^s) := (\Omega_+^D)^* \Omega_-^D \tag{1.23}$$

is the scattering operator S^D. Since the splitting of V into a short– and a long–range part is not unique the Dollard wave operators are not uniquely defined. Nevertheless, the scalar potential is uniquely obtained from *any* of the scattering operators S^D.

Theorem 2 *Suppose that $V^s \in \mathcal{V}_{SR}$ and $V^l \in \mathcal{V}_{LR}$. Then the following reconstruction formula holds for all $\Phi_{\mathbf{v}}$, $\Psi_{\mathbf{v}}$ as in Eq. (1.16):*

$$\begin{aligned} &\lim_{v\to\infty} iv((S^D - \mathbb{1})\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) - (\{V^l(\mathbf{r} + \tau\hat{\mathbf{v}}) - V^l(\tau\hat{\mathbf{v}})\}\, \Phi_0, \Psi_0) \\ = &\int_{-\infty}^{\infty} (V^s(\mathbf{r} + \tau\hat{\mathbf{v}})\Phi_0, \Psi_0)\, d\tau. \end{aligned} \tag{1.24}$$

In particular, for a given long–range potential V^l the scattering map

$$S^D(\mathbf{A}, V^l; \cdot) : \mathcal{V}_{SR} \to L(\mathcal{H})$$

is injective.
Moreover, the following formula holds for all $\Phi_{\mathbf{v}}$, $\Psi_{\mathbf{v}}$ as in Eq. (1.16):

$$\begin{aligned} &\lim_{v\to\infty} iv([S^D, \mathbf{p}_z]\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) \\ = &\int_{-\infty}^{\infty} \{(V^s(\mathbf{r} + \tau\hat{\mathbf{v}})\mathbf{p}_z\Phi_0, \Psi_0) - (V^s(\mathbf{r} + \tau\hat{\mathbf{v}})\Phi_0, \mathbf{p}_z\Psi_0) \\ &+ i((\partial_z V^l)(\mathbf{r} + \tau\hat{\mathbf{v}})\Phi_0, \Psi_0)\}\, d\tau. \end{aligned} \tag{1.25}$$

In particular, any of the scattering operators S^D uniquely determines the total potential V.

Remark 3 *We only need the knowledge of $\mathbf{B}$, in particular of b, and fixing a gauge $\mathbf{A}$ for the definition of the wave operators $\Omega_\pm^{(D)}$ and the scattering operator $S^{(D)}$. The reconstruction formulae (1.17), (1.24) and (1.25) are independent of $\mathbf{B}$.*

Remark 4 *The reconstruction formulae of Theorem 1 and 2 for $\mathbf{B} = 0$ are the same as in the case without magnetic field in [EW 95].*

Remark 5 *Our results are also valid for a system of two interacting quantum particles with opposite charges, see Section 5.*

Our proofs use a geometric, time–dependent method developed by Enss and Weder. This method has been used in [EW 95] to prove Theorems 1 and 2 in the case without magnetic field.
To our knowledge the case with decaying magnetic fields was previously treated by Novikov and Khenkin in [NK 87] where they proved that bounded, rapidly decreasing and sufficiently smooth potentials are uniquely obtained from the scattering operator at fixed energy. Eskin and Ralston proved in [ER 95] that the scattering operator at fixed energy uniquely determines the magnetic field and the scalar potential by stationary phase methods for exponentially decaying $\mathbf{A}$ and V^s. By stationary methods Nicoleau proves in [N 96] that the high energy limit uniquely determines the magnetic field and the scalar potential, where the potentials have to be C^∞ – functions with decay assumptions on the potentials and all derivatives. In [A 96a] and [A 96b] we prove a similar result with less decay assumptions on the scalar potential with the geometric, time–dependent method.
As far as we know, there are no rigorous results on inverse scattering for one–body and many–body (including two–body) systems in a homogeneous magnetic field.

2 Proof in the Short–Range Case $V = V^s$

Since $S - \mathbb{1} = (\Omega_+ - \Omega_-)^*\Omega_-$ we obtain

$$\begin{aligned} & v((S-\mathbb{1})\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) \\ = \; & \underbrace{v((\Omega_- - \mathbb{1})\Phi_{\mathbf{v}}, (\Omega_+ - \Omega_-)\Psi_{\mathbf{v}})}_{=:R(\mathbf{v})} + \underbrace{v(\Phi_{\mathbf{v}}, (\Omega_+ - \Omega_-)\Psi_{\mathbf{v}})}_{=:l(\mathbf{v})}. \end{aligned} \tag{2.1}$$

In the following we will show that $R(\mathbf{v}) = \mathcal{O}(v^{-1})$ and that the high velocity limit $\lim_{v\to\infty} l(\mathbf{v})$ exists. For that purpose we need the following Lemma which is a propagation property expressing the fact that the solutions of the (one–dimensional) free Schrödinger equation have rapid decay away from the classically allowed region:

Lemma 6 *For any $f \in C_0^\infty(\mathbb{R}^1)$ with $\operatorname{supp} f \subset B_{m\eta}(0)$, for some $\eta > 0$ and any $l = 1, 2, 3, \ldots$, there is a constant C_l such that*

$$\left\| F(z \in \mathcal{M})\, e^{-i\frac{1}{2m}\mathbf{p}_z^2 t} f(\mathbf{p}_z - m\mathbf{v}_z) F(z \in \mathcal{M}') \right\| \le C_l\,(1 + s + |t|)^{-l} \tag{2.2}$$

for every $\mathbf{v}_z, t \in \mathbb{R}$ *and any measurable sets* $\mathcal{M}$ *and* $\mathcal{M}'$ *with* $s := \mathrm{dist}(\mathcal{M}, \mathcal{M}' + \mathbf{v}_z t) - \eta|t| \geq 0$.

Proof: see [E 83, Proposition 2.10]. ■

Proposition 7 *Assume that* $V^s \in \mathcal{V}_{SR}$. *Then for all* $\Phi_{\mathbf{v}}$ *as in Eq. (1.16)*

$$\|(\Omega_\pm - \mathbb{1})\Phi_{\mathbf{v}}\| = \mathcal{O}(v^{-1}). \tag{2.3}$$

Proof: We compute the difference

$$\begin{aligned}
&(\Omega_\pm - \mathbb{1})\Phi_{\mathbf{v}} \\
= &\int_0^{\pm\infty} dt\, \frac{d}{dt}\left(e^{iHt}e^{-iH_A t}\right)\Phi_{\mathbf{v}} \\
= &\int_0^{\pm\infty} dt\, e^{iHt} i(H - H_A)\, e^{-iH_A t}\Phi_{\mathbf{v}} \\
= &\; v^{-1}\int_0^{\pm\infty} d\tau\, e^{iH\tau/v} iV^s(\mathbf{r})\, e^{-iH_A\tau/v}\Phi_{\mathbf{v}} \qquad (2.4)
\end{aligned}$$

with the substitution $\tau = v\,t$. If we can show that the norm of the integrand in (2.4) is bounded uniformly in $\mathbf{v}$ by an integrable function $h \in L^1((\pm\infty, 0], d\tau)$, we finished the proof. To show integrability we will use $f(\mathbf{p}_z - m\mathbf{v}_z)\Phi_{\mathbf{v}} = \Phi_{\mathbf{v}}$ and $\|(1 + |z|)^2\Phi_{\mathbf{v}}\| \leq \text{const} < \infty$. In addition, we use that the particle moves in the direction parallel to the magnetic field with velocity $|\mathbf{v}_z| = v\,|\hat{\mathbf{v}}_z| > 0$. Then

$$\begin{aligned}
&\left\|V^s(\mathbf{r})e^{-iH_A\tau/v}\,\Phi_{\mathbf{v}}\right\| \\
\leq &\left\|V^s(\mathbf{r})e^{-iH_{xy}\tau/v}e^{-i\frac{1}{2m}\mathbf{p}_z^2\tau/v}f(\mathbf{p}_z - m\mathbf{v}_z)(1 + |z|)^{-2}\right\|\,\left\|(1 + |z|)^2\,\Phi_{\mathbf{v}}\right\| \\
\leq &\;\text{const}\,\Big\{\left\|F(|z| > |\tau\hat{\mathbf{v}}_z|/4)(1 + |z|)^{-2}\right\| \qquad (2.5)\\
&+\left\|F(|z - \tau\hat{\mathbf{v}}_z| \geq |\tau\hat{\mathbf{v}}_z|/2)\, e^{-i\frac{1}{2m}\mathbf{p}_z^2\tau/v}f(\mathbf{p}_z - m\mathbf{v}_z)F(|z| \leq |\tau\hat{\mathbf{v}}_z|/4)\right\| \qquad (2.6)\\
&+\|V^s(\mathbf{r})F(|z - \tau\hat{\mathbf{v}}_z| < |\tau\hat{\mathbf{v}}_z|/2)\|\Big\} \qquad (2.7)\\
\leq &\; h(|\tau\hat{\mathbf{v}}_z|) \qquad (2.8)
\end{aligned}$$

since (2.5) $\leq \text{const}\,(1+|\tau\hat{\mathbf{v}}_z|)^{-2} \leq h_1(|\tau\hat{\mathbf{v}}_z|)$, (2.6) $\leq h_2(|\tau\hat{\mathbf{v}}_z|)$ for $v \geq 8\eta/|\hat{\mathbf{v}}_z|$ because of Lemma 6 and (2.7) $\leq \|V^s(\mathbf{r})F(|z| \geq |\tau\hat{\mathbf{v}}_z|/2)\| \leq h_3(|\tau\hat{\mathbf{v}}_z|)$ because V^s is of short range. (2.8) follows with $h = h_1 + h_2 + h_3$. ■

The following Lemma expresses the translation in position and momentum space.

Lemma 8 *For all measurable bounded or polynomial functions f on suitable domains and all $s \in \mathbb{R}$ we have*

$$\begin{aligned} e^{i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}s} f(\mathbf{r})\, e^{-i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}s} &= f(\mathbf{r}+s\hat{\mathbf{v}}) \quad \textit{and} && (2.9)\\ e^{i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}s} f(\mathbf{p})\, e^{-i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}s} &= f(\mathbf{p}+s(\mathbf{B}\times\hat{\mathbf{v}})/2)). && (2.10)\end{aligned}$$

Proof: The translation in position space (2.9) can be obtained from

$$\begin{aligned} & e^{i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}s}\mathbf{r}\, e^{-i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}s} \\ = \; & e^{i\mathbf{p}\hat{\mathbf{v}}s} e^{-i\mathbf{A}\hat{\mathbf{v}}s}\mathbf{r}\, e^{-i\mathbf{A}\hat{\mathbf{v}}s} e^{-i\mathbf{p}\hat{\mathbf{v}}s} && (2.11)\\ = \; & e^{i\mathbf{p}\hat{\mathbf{v}}s}\mathbf{r}\, e^{-i\mathbf{p}\hat{\mathbf{v}}s} \\ = \; & \mathbf{r}+s\hat{\mathbf{v}}, \end{aligned}$$

where in (2.11) $i[\mathbf{p}\hat{\mathbf{v}}, \mathbf{A}\hat{\mathbf{v}}] = (b/2)\hat{\mathbf{v}}_x\hat{\mathbf{v}}_y + (-b/2)\hat{\mathbf{v}}_y\hat{\mathbf{v}}_x = 0$ is used.
The translation in momentum space (2.10) can be obtained from

$$\begin{aligned} e^{i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}s}\mathbf{p}\, e^{-i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}s} &= e^{-i\mathbf{A}\hat{\mathbf{v}}s}\mathbf{p}\, e^{i\mathbf{A}\hat{\mathbf{v}}s} && (2.12)\\ &= \mathbf{p}+s(\mathbf{B}\times\hat{\mathbf{v}})/2. && (2.13)\end{aligned}$$

The last equality can be proven by showing that the derivatives in s of (2.12) and (2.13) are equal. To show this we use $i[\mathbf{p}, \mathbf{A}\hat{\mathbf{v}}] = (b\hat{\mathbf{v}}_y/2, -b\hat{\mathbf{v}}_x/2, 0) = (\mathbf{B}\times\hat{\mathbf{v}})/2$. ■

To prove the next Proposition, where we compute the high velocity limit of the wave operators, we need the following Lemma:

Lemma 9 *Assume that $V^s \in \mathcal{V}_{SR}$. Then for fixed $\tau \in \mathbb{R}$*

$$\operatorname*{s-lim}_{v\to\infty} e^{i(H/v+(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}})\tau} = e^{i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}\tau} = \operatorname*{s-lim}_{v\to\infty} e^{i(H_A/v+(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}})\tau}. \qquad (2.14)$$

Proof: We show the first equality for H which implies the second with $V^s = 0$ and compute the difference

$$\begin{aligned} & e^{i(H/v+(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}})\tau} - e^{i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}\tau} \\ = \; & \left\{ e^{i(H/v+(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}})\tau} e^{-i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}\tau} - \mathbb{1} \right\} e^{-i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}\tau} \\ = \; & \int_0^1 ds\, e^{i(H/v+(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}})\tau\,(1-s)} (-iH/v) e^{i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}\tau\, s}. \end{aligned}$$

Then we obtain

$$\begin{aligned} & \left\| \left\{ e^{i(H/v+(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}})\tau} - e^{i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}\tau} \right\} \Psi_0 \right\| \\ \leq \; & v^{-1} \int_0^1 ds\, \left\| H\, e^{i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}\tau\, s}\Psi_0 \right\| && (2.15)\end{aligned}$$

and if we can show that the integrand in (2.15) is bounded by a constant, we finished the proof. Because of Lemma 8 the integrand in (2.15) is bounded by

$$\begin{aligned} & \left\| \left\{ [\mathbf{p} + \tau\, s\, (\mathbf{B} \times \hat{\mathbf{v}})/2 - \mathbf{A}(\mathbf{r} + \tau\, s\, \hat{\mathbf{v}})]^2/(2m) + V^s(\mathbf{r} + \tau\, s\, \hat{\mathbf{v}}) \right\} \Psi_0 \right\| \\ = \; & \| \{ [\mathbf{p} - \mathbf{A}(\mathbf{r})]^2/(2m) + V^s(\mathbf{r} + \tau\, s\, \hat{\mathbf{v}}) \} \Psi_0 \|, \end{aligned}$$

where both sides are equal because of the transversal gauge. The second term is bounded by a constant because our dense set of states Ψ_0 are localized in position and momentum space. ■

Proposition 10 *Assume that $V^s \in \mathcal{V}_{SR}$. Then for all $\Psi_{\mathbf{v}}$ as in Eq. (1.16)*

$$\|(\Omega_+ - \Omega_-)\Psi_{\mathbf{v}}\| = \mathcal{O}(v^{-1}) \quad \textit{and} \tag{2.16}$$

$$\lim_{v\to\infty} e^{im\mathbf{v}\mathbf{r}} v(\Omega_+ - \Omega_-)\Psi_{\mathbf{v}} = \int_{-\infty}^{\infty} d\tau\, iV^s(\mathbf{r} + \tau\hat{\mathbf{v}})\Psi_0. \tag{2.17}$$

Proof: Similarly to the proof of Proposition 7 we compute the difference

$$\begin{aligned} & (\Omega_+ - \Omega_-)\Psi_{\mathbf{v}} \\ = \; & v^{-1} \int_{-\infty}^{\infty} d\tau\, e^{iH\tau/v} iV^s(\mathbf{r})\, e^{-iH_A\tau/v}\Psi_{\mathbf{v}} \end{aligned} \tag{2.18}$$

and use that the norm of the integrand in (2.18) is bounded by an integrable function $h \in L^1((-\infty, \infty), d\tau)$, see (2.4) – (2.8). Hence (2.16) is proven. Now we calculate the high velocity limit (2.17). In the following we will use

$$e^{-im\mathbf{v}\mathbf{r}} f(\mathbf{p})\, e^{im\mathbf{v}\mathbf{r}} \;=\; f(\mathbf{p} + m\mathbf{v}) \tag{2.19}$$

for measurable and bounded functions f and Lemma 8. Because of the dominated convergence theorem we can interchange the limit and the integral:

$$\begin{aligned} & \lim_{v\to\infty} e^{im\mathbf{v}\mathbf{r}} v(\Omega_+ - \Omega_-)\Psi_{\mathbf{v}} \\ = \; & \lim_{v\to\infty} \int_{-\infty}^{\infty} d\tau\, e^{i(H + (\mathbf{p}-\mathbf{A})\mathbf{v})\tau/v} iV^s(\mathbf{r})\, e^{-i(H_A + (\mathbf{p}-\mathbf{A})\mathbf{v})\tau/v}\Psi_0 \\ = \; & \int_{-\infty}^{\infty} d\tau\, e^{i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}\tau} iV^s(\mathbf{r})\, e^{-i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}\tau}\Psi_0 && (2.20) \\ = \; & \int_{-\infty}^{\infty} d\tau\, iV^s(\mathbf{r} + \tau\hat{\mathbf{v}})\Psi_0, && (2.21) \end{aligned}$$

where we used Lemma 9 in (2.20). ■

With the help of Propositions 7 and 10 we calculate the following limit:

$$\begin{aligned}
& \lim_{v\to\infty} iv((S-\mathbb{1})\Phi_{\mathbf{v}},\Psi_{\mathbf{v}}) \\
= & \lim_{v\to\infty} iv((\Omega_- - \mathbb{1})\Phi_{\mathbf{v}},(\Omega_+ - \Omega_-)\Psi_{\mathbf{v}}) + iv(\Phi_{\mathbf{v}},(\Omega_+ - \Omega_-)\Psi_{\mathbf{v}}) \\
= & \int_{-\infty}^{\infty} d\tau\, (V^s(\mathbf{r}+\tau\hat{\mathbf{v}})\Phi_0,\Psi_0),
\end{aligned}$$

which is the desired reconstruction formula in Theorem 1.

We remark that we proved the reconstruction formula for all $\hat{\mathbf{v}} \in S^2$ with $\hat{\mathbf{v}}_z \neq 0$, i.e. for almost all $\hat{\mathbf{v}} \in S^2$, which is enough for the reconstruction of the scalar potential.

The injectivity of the scattering map results from the fact that the reconstruction of the potential V^s can be reduced to the inversion of the Radon transform of a bounded, continuous, square integrable function on $\mathbb{R}^2$ (see [EW 95, Proof of Theorem 1.1]) which uniquely determines the function (see [He 84, Chapter I, Theorem 2.17]).
Hence Theorem 1 is proved. ■

3 Proof in the Long–Range Case $V = V^s + V^l$

Similarly to (2.1) we obtain

$$\begin{aligned}
& v((S^D-\mathbb{1})\Phi_{\mathbf{v}},\Psi_{\mathbf{v}}) \\
= & \underbrace{v((\Omega_-^D - \mathbb{1})\Phi_{\mathbf{v}},(\Omega_+^D - \Omega_-^D)\Psi_{\mathbf{v}})}_{=:R^D(\mathbf{v})} + \underbrace{v(\Phi_{\mathbf{v}},(\Omega_+^D - \Omega_-^D)\Psi_{\mathbf{v}})}_{=:l^D(\mathbf{v})}.
\end{aligned} \tag{3.1}$$

In the following we will show that $R^D(\mathbf{v}) = \mathcal{O}(v^{-1})$ and that the high velocity limit $\lim_{v\to\infty} l^D(\mathbf{v})$ exists. For that purpose we need the following Lemma which is a propagation property for the Dollard correction $\tilde{U}(t)$ with

$$\tilde{U}(t) = \exp[-i\int_0^t V^l(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(s), s\mathbf{p}_z/m)\, ds] \tag{3.2}$$

and $\hat{\boldsymbol{\rho}}_{\mathbf{v}}(s)$ given in (1.20) – (1.22).

Lemma 11 *For any $f \in C_0^\infty(\mathbb{R}^1)$ with* $\operatorname{supp} f \subset B_{m\eta}(0)$ *and* $\mathbf{v}_z \in \mathbb{R}^1$ *with* $|\mathbf{v}_z| \geq 2\eta$ *the following estimates are true:*

$$\begin{aligned} &\left\| z\, \tilde{U}(t) f(\mathbf{p}_z - m\mathbf{v}_z)(1+|z|^2)^{-1/2} \right\| \\ \leq\ & C\,(1 + |\mathbf{v}_z|^{-2} |\mathbf{v}_z t|^{2-\gamma}) \leq C\,(1+|\mathbf{v}_z t|)^{2-\gamma}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\left\| F(|z| \geq |\mathbf{v}_z t|/4)\, \tilde{U}(t) f(\mathbf{p}_z - m\mathbf{v}_z)(1+|z|^2)^{-2} \right\| \\ \leq\ & C\,(1+|\mathbf{v}_z t|)^{-(2+\varepsilon)} \end{aligned} \tag{3.4}$$

for some $\varepsilon > 0$.

Proof: The estimates can be proven in the same way as the estimates for $\hat{\boldsymbol{\rho}}_{\mathbf{v}} = 0$ in [EW 95, Proposition 3.1] because $\hat{\boldsymbol{\rho}}_{\mathbf{v}}$ commutes with $\mathbf{p}_z$. ■

To show the boundedness of the norm of $\{\boldsymbol{\rho} - \hat{\boldsymbol{\rho}}_{\mathbf{v}}(t)\} e^{-iH_{xy}t}\Phi_{\mathbf{v}}$ we need an explicit expression of $e^{-iH_{xy}t} \boldsymbol{\rho}\, e^{-iH_{xy}t}$:

Proposition 12 *With $\boldsymbol{\rho}^\perp := (y, -x)$ and $\boldsymbol{\pi}^\perp := (\mathbf{p}_y, -\mathbf{p}_x)$ and $\omega = b/m$ the following equality holds*

$$\begin{aligned} e^{-iH_{xy}t} \boldsymbol{\rho}\, e^{-iH_{xy}t} &= \boldsymbol{\rho}\,\{1+\cos(\omega\, t)\}/2 + \boldsymbol{\rho}^\perp \sin(\omega\, t)/2 \\ &\quad + \boldsymbol{\pi}\,\omega^{-1}\sin(\omega\, t) + \boldsymbol{\pi}^\perp \omega^{-1}\{1-\cos(\omega\, t)\} \end{aligned} \tag{3.5}$$

on the domain $(\boldsymbol{\rho}^2 + \boldsymbol{\pi}^2)^{-1/2}\,\mathcal{H}$.

Proof: Let us denote the left hand side of the equation (3.5) with $f(t)$ and the left hand side with $g(t)$. We calculate derivatives in t of $f(t)$ by computing a series of commutators $i[H_{xy}, \boldsymbol{\rho}]$ and of $g(t)$ and observe that $f''' = -\omega^2 f'$, $g''' = -\omega^2 g'$ and $f'(0) = g'(0)$, $f''(0) = g''(0)$, which implies $f' = g'$. In addition we know that $f(0) = g(0)$, which implies $f = g$. ■

Corollary 13 *For $\Phi_{\mathbf{v}}$ as in Eq. (1.16) and $\hat{\boldsymbol{\rho}}_{\mathbf{v}}(t)$ given in (1.20) – (1.22) the following expression is uniformly bounded in time t and velocity $\mathbf{v}$:*

$$\left\| \{\boldsymbol{\rho} - \hat{\boldsymbol{\rho}}_{\mathbf{v}}(t)\}\, e^{-iH_{xy}t}\Phi_{\mathbf{v}} \right\| \leq \text{const}\,. \tag{3.6}$$

Proof: Using Proposition 12 and the localization in position and momentum space we obtain the boundedness uniformly in time and velocity. ■

Proposition 14 *Assume that $V^s \in \mathcal{V}_{SR}$ and $V^l \in \mathcal{V}_{LR}$. Then for all $\Phi_{\mathbf{v}}$ as in Eq. (1.16)*

$$\|(\Omega_\pm^D - \mathbb{1})\Phi_{\mathbf{v}}\| = \mathcal{O}(v^{-1}). \tag{3.7}$$

Proof: We compute the difference

$$\begin{aligned}
&(\Omega_\pm^D - \mathbb{1})\Phi_{\mathbf{v}} \\
&= \int_0^{\pm\infty} dt \, \frac{d}{dt}\left(e^{iHt}U^D(t)\right)\Phi_{\mathbf{v}} \\
&= \int_0^{\pm\infty} dt \, e^{iHt} i(H - H_A - V^l(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(t), t\mathbf{p}_z/m)U^D(t)\Phi_{\mathbf{v}} \\
&= v^{-1}\int_0^{\pm\infty} d\tau \, e^{iH\tau/v} i\{V^s(\mathbf{r}) + V^l(\mathbf{r}) - V^l(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), \tau\mathbf{p}_z/(v\,m))\} \\
&\qquad \times U^D(\tau/v)\Phi_{\mathbf{v}}.
\end{aligned} \tag{3.8}$$

If we can show that the norm of the integrand in (3.8) is bounded by an integrable function $h \in L^1((\pm\infty, 0], d\tau)$, we finished the proof.
Since $i[\mathbf{p}_z, (\mathbf{p} - \mathbf{A})^2] = 0$, the components in U^D commute and $U^D(t) = \exp[-iH_A t]\,\tilde{U}(t)$. Let $\bar{f} \in C_0^\infty(\mathbb{R})$ with $\bar{f} \equiv 1$ on the support of f. We use that the particle moves in the direction parallel to the magnetic field with velocity $|\mathbf{v}_z| = v\,|\hat{\mathbf{v}}_z| > 0$. We show the integrability for the term with V^s and then for the term with V^l:

$$\left\| V^s(\mathbf{r})e^{-iH_A\tau/v}\tilde{U}(\tau/v)\Phi_{\mathbf{v}} \right\| \tag{3.9}$$

$$= \left\| V^s(\mathbf{r})e^{-i\frac{1}{2m}\mathbf{p}_z^2\tau/v}e^{-iH_{xy}\tau/v}\tilde{U}(\tau/v)\Phi_{\mathbf{v}} \right\|$$

$$\leq \text{const}\left\{ \left\| F(|z| > |\tau\hat{\mathbf{v}}_z|/4)\,\tilde{U}(\tau/v)f(\mathbf{p}_z - m\mathbf{v}_z)(1 + |z|^2)^{-2} \right\| \right. \tag{3.10}$$

$$+ \left\| F(|z - \tau\hat{\mathbf{v}}_z| \geq |\tau\hat{\mathbf{v}}_z|/2)\,e^{-i\frac{1}{2m}\mathbf{p}_z^2\tau/v}\bar{f}(\mathbf{p}_z - m\mathbf{v}_z)F(|z| \leq |\tau\hat{\mathbf{v}}_z|/4) \right\| \tag{3.11}$$

$$\left. + \|V^s(\mathbf{r})F(|z - \tau\hat{\mathbf{v}}_z| < |\tau\hat{\mathbf{v}}_z|/2)\| \right\}. \tag{3.12}$$

(3.10) $\leq h(|\tau\hat{\mathbf{v}}_z|)$ with Lemma 11 for $v \geq 2\eta/|\hat{\mathbf{v}}_z|$, (3.11) and (3.12) are the same terms as (2.6) and (2.7) respectively. Thus (3.9) is bounded by an integrable function.

To show integrability for the term with the difference of V^l, we introduce a cut–off function $\chi \in C^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$, $\chi(\lambda) = 1$ for $\lambda \geq 1/2$, and $\chi(\lambda) = 0$ for $\lambda \leq 1/4$; and define $V^l_\tau(\boldsymbol{\rho}, z) := V^l(\boldsymbol{\rho}, z)\,\chi(|z|/|\tau\hat{\mathbf{v}}_z|)$. Then it follows from (1.13) that

$$\begin{aligned}
\|\nabla V^l_\tau\| &= \sup|\nabla V^l_\tau(\mathbf{r})| \leq \text{const}\ |\tau\hat{\mathbf{v}}_z|^{-\gamma}, \quad \gamma > 3/2, && (3.13)\\
\|D^{(0,0,2)}V^l_\tau\| &\leq \text{const}\ |\tau\hat{\mathbf{v}}_z|^{-(2+\delta)}, \quad \delta > 0. && (3.14)
\end{aligned}$$

We remark that $V^l(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), \tau\mathbf{p}_z/(v\,m)) = V^l_\tau(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), \tau\mathbf{p}_z/(v\,m))$ on the support of $f(\mathbf{p}_z - m\mathbf{v}_z)$. Then

$$\begin{aligned}
& \left\| \{V^l(\mathbf{r}) - V^l_\tau(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), \tau\mathbf{p}_z/(v\,m))\} e^{-iH_A\tau/v}\tilde{U}(\tau/v)\Phi_{\mathbf{v}} \right\| \qquad (3.15)\\
= & \left\| \{V^l(\mathbf{r}) - V^l_\tau(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), \tau\mathbf{p}_z/(v\,m))\}\, e^{-i\frac{1}{2m}\mathbf{p}_z^2\tau/v} e^{-iH_{xy}\tau/v}\tilde{U}(\tau/v)\Phi_{\mathbf{v}} \right\| \\
\le & \ \text{const} \left\{ \left\| F(|z| > |\tau\hat{\mathbf{v}}_z|/4)\, \tilde{U}(\tau/v) f(\mathbf{p}_z - m\mathbf{v}_z)(1+|z|^2)^{-2} \right\| \right. \qquad (3.16)\\
& + \left\| \{V^l(\mathbf{r}) - V^l_\tau(\mathbf{r})\} e^{-iH_{xy}\tau/v} e^{-i\frac{1}{2m}\mathbf{p}_z^2\tau/v} \bar{f}(\mathbf{p}_z - m\mathbf{v}_z) F(|z| \le |\tau\hat{\mathbf{v}}_z|/4) \right\| \\
& \left. + \left\| \{V^l_\tau(\mathbf{r}) - V^l_\tau(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), \tau\mathbf{p}_z/(v\,m))\} e^{-i\frac{1}{2m}\mathbf{p}_z^2\tau/v} e^{-iH_{xy}\tau/v}\tilde{U}(\tau/v)\,\Phi_{\mathbf{v}} \right\| \right\} \\
\le & \ h_1(|\tau\hat{\mathbf{v}}_z|) \\
& + \text{const} \left\{ \left\| F(|z - \tau\hat{\mathbf{v}}_z| \ge |\tau\hat{\mathbf{v}}_z|/2)\, e^{-i\frac{1}{2m}\mathbf{p}_z^2\tau/v} \bar{f}(\mathbf{p}_z - m\mathbf{v}_z) \right. \right. \\
& \qquad \times F(|z| \le |\tau\hat{\mathbf{v}}_z|/4) \| \qquad (3.17)\\
& \qquad + \left\| \{V^l_\tau(\boldsymbol{\rho}, z + \tau\mathbf{p}_z/(v\,m)) - V^l_\tau(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), \tau\mathbf{p}_z/(v\,m))\} \right. \\
& \qquad \left. \left. \times\, e^{-iH_{xy}\tau/v}\tilde{U}(\tau/v)\Phi_{\mathbf{v}} \right\| \right\}, \qquad (3.18)
\end{aligned}$$

where (3.16) $\le h_1(|\tau\hat{\mathbf{v}}_z|)$ with Lemma 11 for $v \ge 2\eta/|\hat{\mathbf{v}}_z|$ and (3.17) $\le h_2(|\tau\hat{\mathbf{v}}_z|)$ with Lemma 6 for $v \ge 8\eta/|\hat{\mathbf{v}}_z|$. We used $\exp[i\mathbf{p}_z^2 t/(2m)] f(\boldsymbol{\rho}, z) \exp[-i\mathbf{p}_z^2 t/(2m)] = f(\boldsymbol{\rho}, z + t\mathbf{p}_z/m)$ to obtain (3.18). Finally (3.18) can be calculated using the Baker–Campbell–Hausdorff formula

$$\begin{aligned}
& \exp[-i\mathbf{q}(\boldsymbol{\rho}, z + t\mathbf{p}_z/m)] \exp[i\mathbf{q}(s\boldsymbol{\rho} + (1-s)\hat{\boldsymbol{\rho}}_{\mathbf{v}}(t), sz + t\mathbf{p}_z/m)] \\
= & \exp[-i\mathbf{q}(\boldsymbol{\rho} - \hat{\boldsymbol{\rho}}_{\mathbf{v}}(t), z)(1-s)] \exp[it\mathbf{q}_z^2(1-s)/(2m)]
\end{aligned}$$

for all $\mathbf{q} \in \mathbb{R}^3$. Then for the difference of V^l_τ in (3.18) we obtain

$$\begin{aligned}
& V^l_\tau(\boldsymbol{\rho}, z + \tau\mathbf{p}_z/(v\,m)) - V^l_\tau(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), \tau\mathbf{p}_z/(v\,m)) \\
= & \int_0^1 ds \left\{ -(\nabla V^l_\tau)(s\boldsymbol{\rho} + (1-s)\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), sz + \tau\mathbf{p}_z/(v\,m))\{\mathbf{r} - (\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), z)\} \right. \\
& \left. + \frac{i}{2m}(D^{(0,0,2)}V^l_\tau)(s\boldsymbol{\rho} + (1-s)\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), sz + \tau\mathbf{p}_z/(v\,m))\tau/v \right\}.
\end{aligned}$$

With (3.13), (3.14), Corollary 13 and Lemma 11 integrability of (3.18) is proven. ■

Lemma 15 *For $\hat{\boldsymbol{\rho}}_{\mathbf{v}}(t)$ given in (1.20)–(1.22) and all fixed $\tau \in \mathbb{R}$ the following high velocity limit holds:*

$$\operatorname*{s-lim}_{v\to\infty} V^l\left(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), \tau(\mathbf{p}_z + m\mathbf{v}_z)/(m\,v)\right) = V^l(\tau\hat{\mathbf{v}}). \qquad (3.19)$$

Proof: We compute the difference

$$
\begin{aligned}
& V^l\left(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), \tau(\mathbf{p}_z + m\mathbf{v}_z)/(m\,v)\right) - V^l(\tau\hat{\mathbf{v}}) \\
= & \int_0^1 ds\, (-\nabla V^l)\left(\hat{x}_{\mathbf{v}}(\tau/v) + s\tau\hat{\mathbf{v}}_x, \hat{y}_{\mathbf{v}}(\tau/v) + s\tau\hat{\mathbf{v}}_y, s\tau(\mathbf{p}_z + m\mathbf{v}_z)/(m\,v)\right) \\
& \qquad \times \left(\hat{x}_{\mathbf{v}}(\tau/v) - \tau\hat{\mathbf{v}}_x, \hat{y}_{\mathbf{v}}(\tau/v) - \tau\hat{\mathbf{v}}_y, \mathbf{p}_z\tau/(m\,v)\right). \qquad (3.20)
\end{aligned}
$$

As ∇V^l is bounded, $\lim_{v\to\infty} \hat{x}_{\mathbf{v}}(\tau/v) - \tau\hat{\mathbf{v}}_x = 0$, $\lim_{v\to\infty} \hat{y}_{\mathbf{v}}(\tau/v) - \tau\hat{\mathbf{v}}_y = 0$ and $\lim_{v\to\infty} \mathbf{p}_z\tau/(m\,v) = 0$ for fixed τ we obtain (3.19). ■

Proposition 16 *Assume that $V^s \in \mathcal{V}_{SR}$ and $V^l \in \mathcal{V}_{LR}$. Then for all $\Psi_{\mathbf{v}}$ as in Eq. (1.16)*

$$
\|(\Omega_+^D - \Omega_-^D)\Psi_{\mathbf{v}}\| = \mathcal{O}(v^{-1}) \quad \textit{and} \qquad (3.21)
$$

$$
\begin{aligned}
\lim_{v\to\infty} e^{im\mathbf{v}\mathbf{r}} v(\Omega_+^D - \Omega_-^D)\Psi_{\mathbf{v}} = & \int_{-\infty}^{\infty} d\tau\, i\{V^s(\mathbf{r} + \tau\hat{\mathbf{v}}) \\
& + V^l(\mathbf{r} + \tau\hat{\mathbf{v}}) - V^l(\tau\hat{\mathbf{v}})\}\Psi_0. \qquad (3.22)
\end{aligned}
$$

Proof: Similarly to the proof of Proposition 14 we compute the difference

$$
\begin{aligned}
& (\Omega_+^D - \Omega_-^D)\Psi_{\mathbf{v}} \\
= & \; v^{-1} \int_{-\infty}^{\infty} d\tau\, e^{iH\tau/v} i\{V^s(\mathbf{r}) + V^l(\mathbf{r}) - V^l(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), \tau\mathbf{p}_z/(v\,m))\} \\
& \qquad \times U^D(\tau/v)\Psi_{\mathbf{v}} \qquad (3.23)
\end{aligned}
$$

and can show that the norm of the integrand in (3.23) is bounded by an integrable function $h \in L^1((-\infty,\infty), d\tau)$. Hence (3.21) is proven.
Now we calculate the high velocity limit (3.22). Because of the dominated convergence theorem we can interchange the limit and the integral and obtain similarly to (2.21):

$$
\begin{aligned}
& \lim_{v\to\infty} e^{im\mathbf{v}\mathbf{r}} v(\Omega_+^D - \Omega_-^D)\Psi_{\mathbf{v}} \\
= & \lim_{v\to\infty} \int_{-\infty}^{\infty} d\tau\, e^{i(H + (\mathbf{p}-\mathbf{A})\mathbf{v})\tau/v} i\{V^s(\mathbf{r}) + V^l(\mathbf{r}) \\
& \qquad - V^l(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(\tau/v), \tau(\mathbf{p}_z + m\mathbf{v}_z)/(v\,m))\} \\
& \qquad \times e^{-i(H_A + (\mathbf{p}-\mathbf{A})\mathbf{v})\tau/v - i\int_0^{\tau/v} V^l(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(s), s(\mathbf{p}_z + m\mathbf{v}_z)/m)\, ds}\Psi_0 \\
= & \int_{-\infty}^{\infty} d\tau\, e^{i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}\tau} i\{V^s(\mathbf{r}) + V^l(\mathbf{r}) - V^l(\tau\hat{\mathbf{v}})\}\, e^{-i(\mathbf{p}-\mathbf{A})\hat{\mathbf{v}}\tau}\Psi_0 \qquad (3.24) \\
= & \int_{-\infty}^{\infty} d\tau\, i\{V^s(\mathbf{r} + \tau\hat{\mathbf{v}}) + V^l(\mathbf{r} + \tau\hat{\mathbf{v}}) - V^l(\tau\hat{\mathbf{v}})\}\Psi_0,
\end{aligned}
$$

where in (3.24) s-$\lim_{v\to\infty} \exp[i(H/v + (\mathbf{p} - \mathbf{A})\hat{\mathbf{v}})\tau] = \exp[i(\mathbf{p} - \mathbf{A})\hat{\mathbf{v}}\tau] =$ s-$\lim_{v\to\infty} \exp[i(H_A/v + (\mathbf{p} - \mathbf{A})\hat{\mathbf{v}})\tau + i\int_0^{\tau/v} V^l(\hat{\boldsymbol{\rho}}_{\mathbf{v}}(s), s(\mathbf{p}_z + m\mathbf{v}_z)/m)\, ds]$ for fixed τ is used. This can similarly be shown as the equality in the short–range case in Lemma 9. ■

With the help of Propositions 14 and 16 we calculate the following limit:

$$
\begin{aligned}
& \lim_{v\to\infty} iv((S^D - \mathbb{1})\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) \\
= & \lim_{v\to\infty} iv((\Omega_-^D - \mathbb{1})\Phi_{\mathbf{v}}, (\Omega_+^D - \Omega_-^D)\Psi_{\mathbf{v}}) + iv(\Phi_{\mathbf{v}}, (\Omega_+^D - \Omega_-^D)\Psi_{\mathbf{v}}) \\
= & \int_{-\infty}^{\infty} d\tau\, (V^s(\mathbf{r} + \tau\hat{\mathbf{v}})\Phi_0, \Psi_0) \\
& + \int_{-\infty}^{\infty} d\tau\, (\{V^l(\mathbf{r} + \tau\hat{\mathbf{v}}) - V^l(\tau\hat{\mathbf{v}})\}\Phi_0, \Psi_0)
\end{aligned}
\tag{3.25}
$$

which is the reconstruction formula (1.24) in Theorem 2. The injectivity already has been shown in the short–range case above.
A short computation for the commutator of S^D with the third component of the momentum $\mathbf{p}_z$ gives (1.25). Then the total potential V can be computed from (1.25) (see [EW 95, Proof of Theorem 1.2]).
Hence Theorem 2 is proved. ■

4 Error Bounds

All formulae hold in the high velocity limit. For large, but fixed, v we obtain an error term of order $\mathcal{O}(|\mathbf{v}_z|^{-1})$ if the scalar L^1 and compactly supported: In this case we only have to know the Radon transform for infinitely many directions $\hat{\mathbf{v}}$, but not for all, to uniquely determine the potential from its X–ray transform (see [K 89, Theorem 5.2]). Then the error terms $|R(\mathbf{v})|$ and $|R^D(\mathbf{v})|$ can be calculated from (2.1) and (3.1) respectively and are of order $\mathcal{O}(|\mathbf{v}_z|^{-1})$.

5 The Two–Body Case

Now we study a system of two interacting quantum particles with opposite charges in a homogeneous magnetic field. As the most important example of such a system we note the hydrogen atom (see [L 95] and references there).

Let $H_0 := \sum_{i=1,2} \frac{1}{2m_i}\mathbf{p}_i^2$ be the free Hamiltonian. By

$$H_A := \sum_{i=1,2} \frac{1}{2m_i}(\mathbf{p}_i - e_i\mathbf{A}(\mathbf{r}_i))^2 \tag{5.1}$$

we denote the Hamiltonian with the magnetic field alone and by $H = H_A + V$ the interacting Hamiltonian with mass m_i, momentum $\mathbf{p}_i = -i\nabla_{\mathbf{r}_i}$, electric charge e_i and position $\mathbf{r}_i$ of the i–th particle, where we will assume that $e_1 = -e_2 = 1$. As in the one–body case it is useful to introduce the following representation (see [L 95] for the notation):

$$\begin{aligned} H_A &= H_0 + \sum_{i=1,2} \frac{1}{2m_i}\left\{\frac{b^2}{4}\left(\frac{\mathbf{B}}{|\mathbf{B}|}\times \mathbf{r}_i\right)^2 - e_i\mathbf{B}\mathbf{L}_i\right\} && (5.2)\\ &= H_0 + \sum_{i=1,2} \frac{1}{2m_i}\left\{\frac{b^2}{4}(x_i^2+y_i^2) - e_i b\,\mathbf{L}_{z,i}\right\} && (5.3)\\ &= \sum_{i=1,2}\left\{\frac{1}{2m_i}\mathbf{p}_{z,i}^2 + H_{osc,i} - \frac{1}{2m_i}e_i b\,\mathbf{L}_{z,i}\right\} && (5.4)\end{aligned}$$

with the z–component of the angular momentum $\mathbf{L}_{z,i} = x_i\mathbf{p}_{y,i} - y_i\mathbf{p}_{x,i}$ of the i–th particle and

$$H_{osc,i} = \frac{1}{2m_i}\left\{(\mathbf{p}_{x,i}^2 + \mathbf{p}_{y,i}^2) + \frac{b^2}{4}(x_i^2+y_i^2)\right\}. \tag{5.5}$$

As $H_{osc,i}$ and $\mathbf{L}_{z,i}$ do not act on the z–variable, we can separate the center of mass motion in the z–direction as usual. For that purpose we consider H acting on $\mathcal{H} = L^2(X)$ with

$$X = \{(\mathbf{r}_1,\mathbf{r}_2) \subset \mathbb{R}^6 \mid m_1 z_1 + m_2 z_2 = 0\}$$

instead of $L^2(\mathbb{R}^6)$, and write $z = z_1 - z_2$. The momentum space is

$$\hat{X} = \{(\mathbf{u}_1,\mathbf{u}_2) \subset \mathbb{R}^6 \mid \mathbf{u}_{z,1} + \mathbf{u}_{z,2} = 0\}.$$

It is known (see [AHS 78a] for details) that $H_{osc,i} - \frac{1}{2m_i}e_i b\,\mathbf{L}_{z,i}$, acting on the Hilbert space $L^2(\mathbb{R}^2)$, has a complete set of eigenfunctions $\{\Theta_{ln}(x_i,y_i)\}_{l=0,\pm1,\dots;\,n=0,1,\dots}$ with corresponding eigenvalues

$$E_{ln,i} = (2m_i)^{-1} b\,(2n + |l| - e_i l + 1), \tag{5.6}$$

where $E_{ln,i}$ and Θ_{ln} are often referred to as Landau energy levels and Landau orbits. Therefore the spectrum of $H_{xy} = \sum_{i=1,2} H_{osc,i} - \frac{1}{2m_i}e_i b\,\mathbf{L}_{z,i}$ acting on

$L^2(\mathbb{R}^4)$ consists of infinitely degenerated eigenvalues $\{E_\beta\}_{\beta=(l,n,l',n')}$, $l, l' = 0, \pm 1, \dots$; $n, n' = 0, 1, \dots$ with

$$E_\beta = \frac{b}{2m_1}(2n + |l| - l + 1) + \frac{b}{2m_2}(2n' + |l'| - l' + 1) \tag{5.7}$$

and corresponding eigenfunctions $\theta_\beta((x_1, y_1), (x_2, y_2)) = \theta_{ln}(x_1, y_1) \times \theta_{l'n'}(x_2, y_2)$.

We will use some notation similar to that commonly used in many–body scattering theory:

$$\mathbf{r} := \mathbf{r}_1 - \mathbf{r}_2, \qquad \mathbf{p} := (\mathbf{p}_1/m_1 - \mathbf{p}_2/m_2)m \tag{5.8}$$

with reduced mass $m = m_1 m_2/(m_1 + m_2)$, where $\mathbf{r}$ denotes the relative coordinate of $\mathbf{r}_1$ and $\mathbf{r}_2$ and $\mathbf{p}$ its conjugate momentum. As again H_{xy} does not act on the z–variable, we have $\exp[-iH_A t] = \exp[-iH_{xy}t]\exp[-i\mathbf{p}_z^2/(2m)]$.

Then the Theorems for the two–body case of Theorems 1 and 2 can be proven in the same way as in the one–body case where we use that the propagation properties of Lemma 6 and 11 are also true in the two–body case and that all other statements can be proven as before now using the new relative coordinates $\mathbf{r}$ and $\mathbf{p}$, see (5.8). Finally we obtain the following Theorems for the two–body case:

Theorem 17 (two–body case) *Suppose that $V^s \in \mathcal{V}_{SR}$ and $V^l = 0$. Then the following reconstruction formula holds for all $\Phi_\mathbf{v}$, $\Psi_\mathbf{v}$ as in Eq. (1.16):*

$$\lim_{v\to\infty} iv((S - \mathbb{1})\Phi_\mathbf{v}, \Psi_\mathbf{v}) \;=\; \int_{-\infty}^{\infty} (V^s(\mathbf{r} + \tau\hat{\mathbf{v}})\Phi_0, \Psi_0)\, d\tau. \tag{5.9}$$

In particular, the scattering map

$$S(\mathbf{A}; \cdot) : \mathcal{V}_{SR} \to L(\mathcal{H})$$

is injective.

Theorem 18 (two–body case) *Suppose that $V^s \in \mathcal{V}_{SR}$ and $V^l \in \mathcal{V}_{LR}$. Then the following reconstruction formula holds for all $\Phi_\mathbf{v}$, $\Psi_\mathbf{v}$ as in Eq. (1.16):*

$$\begin{aligned} &\lim_{v\to\infty} iv((S^D - \mathbb{1})\Phi_\mathbf{v}, \Psi_\mathbf{v}) - (\{V^l(\mathbf{r} + \tau\hat{\mathbf{v}}) - V^l(\tau\hat{\mathbf{v}})\}\,\Phi_0, \Psi_0) \\ = &\int_{-\infty}^{\infty} (V^s(\mathbf{r} + \tau\hat{\mathbf{v}})\Phi_0, \Psi_0)\, d\tau. \end{aligned} \tag{5.10}$$

In particular, for a given long–range potential V^l the scattering map

$$S^D(\mathbf{A}, V^l; \cdot) : \mathcal{V}_{SR} \to L(\mathcal{H})$$

is injective.
Moreover, the following formula holds for all $\Phi_{\mathbf{v}}$, $\Psi_{\mathbf{v}}$ as in Eq. (1.16):

$$\begin{aligned} & \lim_{v\to\infty} iv([S^D, \mathbf{p}_z]\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) \\ = \; & \int_{-\infty}^{\infty} \{(V^s(\mathbf{r} + \tau\hat{\mathbf{v}})\mathbf{p}_z\Phi_0, \Psi_0) - (V^s(\mathbf{r} + \tau\hat{\mathbf{v}})\Phi_0, \mathbf{p}_z\Psi_0) \\ & + i((\partial_z V^l)(\mathbf{r} + \tau\hat{\mathbf{v}})\Phi_0, \Psi_0)\}\, d\tau. \end{aligned} \tag{5.11}$$

In particular, any of the scattering operators S^D uniquely determines the total potential V.

6 Generalization

It is possible to generalize the results such that unbounded short–range V^s (i.e., V^s is only Kato–small and unbounded in the direction parallel to the field) are included. The decay assumption (1.11) can be weakened such that

$$\|V^s(\mathbf{r})\, (\mathbf{p}_z^2 + \mathbb{1}_z)^{-1} F(|z| \geq R)\| \in L^1([0, \infty), dR) \tag{6.1}$$

is sufficient for the reconstruction of V^s in Theorem 1 and of $V = V^s + V^l$ in Theorem 2 because V^s can be regularized with $f(\mathbf{p}_z - m\mathbf{v}_z)$ which commutes with $e^{-iH_A t}$ and $U^D(t)$.

Acknowledgement

I thank Professor V. Enss and Professor R. Weder for their encouragement and helpful conversations and Deutsche Forschungsgemeinschaft and the Graduiertenkolleg of the RWTH Aachen for their financial support. As this paper has been written during my visit of the Institute of Advanced Study in Princeton I like to thank the Institute for their hospitality.

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