FLOWS, RANDOM PERTURBATIONS AND RATE OF MIXING

CARLANGELO LIVERANI

University of Rome *Tor Vergata*

Dicember 6, 1996

ABSTRACT. A new approach to the study of the rate of mixing in Anosov flows, recently proposed by N. Chernov, is simplified and generalized to the higher dimensional case.

CONTENT

§0 Introduction.

The problem of studying the rate of mixing in dynamical systems has been the subject of many investigations in the last decades. Many different techniques have been developed allowing to gain a good understanding of the situation as far as uniformly hyperbolic maps are concerned. For this systems it is possible to show that Hölder continuous observables enjoy an exponential decay of correlation [Bo].

I whish to thank Viviane Baladi, Nikolai Chernov and Thomas Spencer for stimulating my interest in the subject. I am indebted to Francois Ledrappier for many extremely helpful and enjoyable discussions and for his continuing encouragement during this project. Finally, I acknowledge the hospitality of the Ervin Shrödinger Institute, Vien, of IMPA, Rio de Janeiro, the support of the grant CHRX-CT94-0460 of the Commission of the European Community and the agreement CNPq-CNR.

This turns out to be true even if the system is only piecewise smooth, provided the singularities are not too wild [Li1], [Yo] or when the mechanism that produces the hyperbolicity is not a straightforward one [BY].

On the contrary, very little it is known in the non-uniformly hyperbolic case; this still stands as a challenge. Nevertheless, recently Chernov [Ch] has made a decisive progress concerning Anosov flows. On the one hand, hyperbolic flows can be considered to be an intermediate situation between uniform hyperbolicity and non-uniform hyperbolicity due to the zero Lyapunov exponent in the flow direction. On the other hand, the study of flows bears a clear interest both in itself and for its physical implications. The study of correlation for flows is open since the seventies [BR] but very little progress has been made since; apart from few results for the geodesic flows on constant negative curvature [CEG], [Mo], [Po1] and [Ra].

Chernov has been able to show that for Anosov flows on a three dimensional manifold satisfying some uniform non-integrability condition[1] and, for Hölder continuous observables, the correlations decay at least as $e^{-\sqrt{t}}$. In this paper I present Chernov idea in a nutshell, avoiding any reference to Markov partitions and therefore greatly simplifying Chernov's approach. As a byproduct of the above mentioned simplification I am able to extend the applicability of Chernov's method to the higher dimensional situation.

In order to further simplify the presentation of the technique I deal only with contact flows but the generalization to Anosov should present no difficulties. The results of the paper consist in the following.

Theorem A. *Given a uniformly hyperbolic contact flow ϕ_t, with Lipschitz stable and unstable foliation, on a $2d+1$-dimensional manifold $\mathtt{M}$ there exists a constant $\gamma_0 > 0$ such that for each two Hölder continuous functions f, g*

$$|\mu(fg\circ\phi_t) - \mu(f)\mu(g)| \le Ke^{-\gamma_0\sqrt{t}}$$

for some constant K.

Theorem B. *Given a uniformly hyperbolic contact flow ϕ_t on a $2d+1$-dimensional manifold $\mathtt{M}$ there exists a constant $\gamma_0 > 0$ such that for each two Hölder continuous functions f, g and each $\theta < \frac{2\alpha-1}{1-\alpha}$*

$$|\mu(fg\circ\phi_t) - \mu(f)\mu(g)| \le \frac{K}{1+(1+|t|)^\theta}$$

where $\alpha < 1$ is the order of Hölder continuity of the stable and unstable foliations.

Theorem B yields a polynomial decay of correlations for $\alpha > \frac{1}{2}$, with the degree of the polynomial going to infinity as $\alpha \to 1$. In particular, it follows that, for $\alpha > \frac{2}{3}$,

$$\int_0^\infty dt \int_{\mathtt{M}} d\mu g\circ\phi_t g < \infty,$$

[1]For contact flows Chernov conditions turns out to be essentially equivalent to requiring that the stable and unstable foliations are Lipschitz [H], see appendix III for more details.

for each $g \in C^{(1)}(\mathtt{M})$, $\int_{\mathtt{M}} g = 0$. The relevance of the above fact is that the integrability of the self-correlation is a basic ingredient to obtain the Central Limit Theorem for the function g [Li2].

Theorem B shows that Chernov ideas can be pushed beyond the realm of "uniform non-integrability" which is not satisfied if the foliation is only Hölder continuous (see appendix III for more details). I would like to remark that theorem B is not optimal and that a look at the proof suggests that there may be several ways to improve it, whereby obtaining faster decays and results for less regular foliations. It is instead unclear if the strategy of the proof of theorem A can be modified to yield a sharper bound.

The idea of the proof essentially consists in introducing a random perturbation of the flow. This provides us with the missing "hyperbolicity" in the flow direction. Consequently, it is possible to compute the rate of decay for the random perturbation. The cornerstone of the approach is the possibility to remove the random perturbation while still keeping a control on the rate of decay of the correlations, whereby obtaining information on the rate of mixing for the deterministic flow.

It seems to me that this approach has good potentiality of yielding results also in other "non-uniform" situations. In addition, given to the possibility of avoiding Markov partitions, it is conceivable that one can apply similar ideas to non-smooth flows (e.g. billiards).

The content of the paper is as follows. Section one contains few preliminaries concerning contact flows and hyperbolicity. In section two a special random perturbation of the flow is introduced and studied assuming a key estimate (theorem 2.1), that it is proven in section three. Section four shows how to use the knowledge gained in the previous sections to study the decay of correlations for the deterministic flow. Finally, in appendix I it is proven that random perturbations with the properties required in section two exist. Appendix II contains some measure–geometrical estimates used in appendix I; while in appendix III we recall the idea and properties of the C-frames, which are the essential geometrical tool used in section three.[2] Appendix IV contains more measure-geometric estimates used in section three, while appendix V concludes the paper with some considerations on a change of coordinates used in section four.

§1 Preliminaries.

We will consider contact flows on a $2d+1$ connected compact Riemannian manifold $\mathtt{M}$.[3]

[2] These are the H-frames introduced by Chernov.

[3] For the reader convenience here is an almost verbatim quote concerning contact flows, taken from [KB]: "A contact form on $\mathtt{M}$ is a C^1 differential 1-form ω such that the $(2d+1)$-form $\omega \wedge (d\omega)^d$ is non-zero at every point. The kernel of ω is a codimension 1 distribution on $\mathtt{M}$. The restriction of the 2-form $d\omega$ to Ker ω determines a symplectic structure there. There is a unique vector field X on $\mathtt{M}$ such that $d\omega(X, Y) = 0$ for all vector fields Y and $\omega(X) = 1$. The flow ϕ_t defined by X is called the *contact flow on* $\mathtt{M}$. It preserves the contact form ω. Conversely, any flow on $\mathtt{M}$ that preserves ω is a constant reparametrization of ϕ_t. The contact flow preserves the distribution Ker ω, the symplectic structure there and the measure μ on $\mathtt{M}$ determined by the volume form $\omega \wedge (d\omega)^d$."

For simplicity we assume that the Riemannian volume μ and the contact one coincide, moreover, if X is the vector field generating the flow, we assume $\|X\| = 1$ (if this is not the case, one can always change the Riemannian structure to obtain such properties).

We assume that the flow is uniformly hyperbolic, namely at every point $p \in \mathtt{M}$, the tangent space $\mathcal{T}_p\mathtt{M}$ can be written as $E^0_p \oplus E^s_p \oplus E^u_p$, $\dim E^0_p = 1$, $\dim E^s_p = \dim E^u_p$. In addition, there exists $\lambda > 0$, $c > 1$ such that, for each $p \in \mathtt{M}$, E^0_p is the flow direction; for each $t > 0$, $d_p\phi_t E^s_p = E^s_{\phi_t p}$, $d_p\phi_t E^u_p = E^u_{\phi_t p}$ and, for each $v \in E^s_p$, $w \in E^u_p$,[4]

$$
\begin{aligned}
\|d\phi_t v\| &\leq c e^{-\lambda t}\|v\|, \\
\|d\phi_t w\| &\geq c^{-1} e^{\lambda t}\|w\|.
\end{aligned}
$$

The above hypotheses imply that the contact flow is Bernoulli (hence, mixing and ergodic) [KB]. Contact flows arise naturally in many contexts, e.g. Hamiltonian dynamics [Arn] and differential geometry. In particular, it is well known that the geodesic flow on a d dimensional Riemannian manifold $\mathcal{M}$ can be viewed as a contact flow on the unitary tangent bundle $\mathtt{M}$ [Arn]. If the manifold $\mathcal{M}$ has strictly negative curvature then the geodesic flow is uniformly hyperbolic [An], [Po2].

It is a general result of hyperbolic theory [Pe] that E^u_p, E^s_p are Hölder continuous distributions.[5] In addition, they are integrable giving rise to the stable $(W^s(p))$, the weak-stable $(W^{0s}(p))$, the unstable $(W^u(p))$ and the weak-unstable $(W^{0u}(p))$foliations. On the contrary the distribution $E^s_p \oplus E^u_p$ is not integrable (due to the contact structure, [KB]) and, in fact, this is a key ingredient in the proof that ergodic contact flows are Bernoulli.

The distribution may be more regular in special cases. The geodesic flow in strictly negative curvature for example. In this case the distributions E^s_p and E^u_p turns out to be $C^{(1)}$ for two dimensional manifolds [HK], while in higher dimensions the distributions are $C^{(1)}$ if the metric satisfies the so called $\frac{1}{4}$-pinching conditions [HP] (see [Has] for a review on regularity results on Anosov splittings). In principle it could be possible to take advantage of this to improve the results presented here but no attempt is done in this direction in the present work.[6]

In the following, by "random perturbation" of the flow we will not mean a random flows. Instead, we will consider the map T defined by the flow at some time t and construct a random process that is a small perturbation of T. Typically, moving

[4]By $d_p\phi_t$ is meant the differential of ϕ_t at the point p, some time I will write only $d\phi_t$ if no confusion arises.

[5]The result states that the distributions are Hölder continuous of some order α depending on the rate between the minimal and the maximal expansion in the unstable directions and the minimal and maximal contraction in the stable directions. In fact, there are several Hölder structures besides the stable and unstable distributions (e.g., the holonomy map and its Jacobian), here, and in the following, by "α" we will mean the smallest of all the relevant Hölder exponents (actually, the holonomy map and the distributions have the same regularity, crf. [Has], [SS], but here we are not concerned with the optimal estimate for α: more work in this direction may be needed).

[6]At the moment there exists a claim of D.Dolgopiat along such lines [D1], [D2]. He proposes a different approach to prove that, if the foliation is $C^{(1)}$, then the decay of correlations is exponential. His technique seems to be extremely sensitive to the regularity of the foliation: for Lipschitz (but not $C^{(1)}$) foliations he obtains only a decay faster than any polynomial, which it is worst than Chernov result.

the points by T and then by spreading them around according to some distribution having a small support.

Nevertheless, I think that it would be interesting to explore other possibilities. For example, in the case of geodesic flows it would seem very natural to consider the generator $\mathcal{L}_0$ of the flow (simply the vector field viewed as an operator on $L^2(\mathtt{M},\, \mu)$) and use, as a perturbation, the stochastic process defined by the generator

$$\mathcal{L}_\varepsilon = \mathcal{L}_0 + \varepsilon\Delta,$$

where Δ is the Laplace-Beltrami operator on $\mathtt{M}$. In analogy with what is done here it should be possible to study the rate of decay for the process generated by $\mathcal{L}_\varepsilon$ and to obtain informations on the rate of decay of correlations for the deterministic flow by sending ε to zero. In my opinion the development of the techniques necessary to carry out such a program would have an interest in itself.

§2 Random Perturbations.

Here we will define a suitable random perturbation of a contact flow and derive some results on the decay of correlations for such a random perturbation; eventually, this will enable us to obtain an estimate for the decay of correlations for the flow itself.

The strategy is quite general and one can probably use a large class of random perturbations to carry it out; nevertheless, its implementation turns out to be quite delicate and it appears to be convenient to choose a very special perturbation. To define such a perturbation a few preliminaries are needed.

Let d be the metric associated to the Riemannian structure, define

$$d_\varepsilon(x,\, y) = \max\left\{ \sup_{t\in[0,\, n_\varepsilon]} d(\phi_t x,\, \phi_t y);\; \varepsilon^{-1} d(\phi_{n_\varepsilon} x,\, \phi_{n_\varepsilon} y) \right\}, \tag{2.1}$$

where

$$n_\varepsilon = \inf\left\{ n\in\mathbb{N} \;\middle|\; \sup_{\substack{x\in\mathtt{M}\\ v\in E^s(x)}} \frac{\|d_x\phi_n v\|}{\|v\|} \leq \varepsilon^2 \right\}.$$

Let $B^\varepsilon_\delta(x)$ be the set of points with distance, from x, less than δ with respect to the distance d_ε.[7]

We are now in a position to define the class of averages we are interested in.

$$(\mathbb{A}^\varphi_{\varepsilon,\delta} g)(x) = \int_{B^\varepsilon_\delta(x)} d\mu(y) \frac{(1+\delta\varphi(x))(1+\delta\varphi(y))}{\mu(B^\varepsilon_\delta(x))^{\frac12}\mu(B^\varepsilon_\delta(y))^{\frac12}} g(y).$$

[7] The ball $B^1_\delta(x)$ ($\varepsilon = 1$) is just the usual dynamical ball of size δ over the trajectory from time zero to time n_1; nevertheless, we will be interested in the case where δ is small but fixed and ε is arbitrarily small. In this situation $B^\varepsilon_\delta(x)$ is more or less a tiny neighborhood of a disk of diameter 2δ, centered at x, on the strong stable manifold of x. Such a neighborhood has size ε along the flow direction and size smaller than ε^3 in the strong unstable direction.

Note that $\mathbb{A}^{\varphi}_{\varepsilon,\delta}$ is well defined both as a bounded operator on $L^2(\mathtt{M},\,\mu)$ and on $C^{(0)}(\mathtt{M})$, if $\varphi \in C^{(0)}(\mathtt{M})$. Moreover, as an operator on $L^2(\mathtt{M},\,\mu)$ it is self-adjoint. In appendix I we will see that there exist functions $\varphi^{\varepsilon}_{\delta} \in C^{(0)}(\mathtt{M})$ such that, for some fixed $c_0 > 0$, $\|\varphi^{\varepsilon}_{\delta}\|_\infty \le c_0$, for each δ and ε sufficiently small, and

$$\mathbb{A}^{\varphi^{\varepsilon}_{\delta}}_{\varepsilon,\delta} 1 = 1.$$

¿From now on we will fix some sufficiently small $\delta_0 > 0$ and set, for each $g \in L^2(\mathtt{M},\,\mu)$,

$$\mathbb{A}_\varepsilon g(x) := \mathbb{A}^{\varphi^{\varepsilon}_{\delta_0}}_{\varepsilon,\delta_0} g(x) := \int_{\mathtt{M}} a_\varepsilon(x,\,y) g(y) d\mu(y).$$

Now we can define the wanted random perturbation of the flow: for each $f \in L^2(\mathtt{M},\,\mu)$

$$T_{t,\varepsilon} f = (\mathbb{A}_\varepsilon f) \circ \phi_t.$$

This corresponds to first evolving a point for a time t, by the flow, and then spreading it on a neighborhood of radius δ_0 (in the metric d_ε) according to the distribution specified by the average $\mathbb{A}_\varepsilon$. To such a stochastic process is naturally associated a Markov semigroup (simply the adjoint):

$$\mathbb{P}_{t,\,\varepsilon} g = \mathbb{A}_\varepsilon (g \circ \phi_{-t}).$$

By construction, $T_{t,\varepsilon}1 = 1$ and $\mathbb{P}_{t,\varepsilon}1 = 1$.[8]

The key idea–introduced in a different language by Chernov–is to investigate the operator $\mathbb{C}_{t,\,\varepsilon} = (\mathbb{P}^*_{t,\,\varepsilon})^2 \mathbb{P}^2_{t,\,\varepsilon}$. Such an operator is easily seen to be

$$\mathbb{C}_{t,\,\varepsilon} g(x) = \int_{\mathtt{M}} C_{t,\,\varepsilon}(x,\,y) g(y) d\mu(y)$$

where

$$C_{t,\varepsilon}(x,\,y) = \int_{\mathtt{M}^3} d\mu(z_1) d\mu(z_2) d\mu(z_3) a_\varepsilon(\phi_t x, z_1) a_\varepsilon(\phi_t z_1, z_2) a_\varepsilon(z_3, z_2) a_\varepsilon(\phi_t y, \phi_{-t} z_3).$$

The relevance of the previous operator is due to the next estimate.

Theorem 2.1. *There exists $\gamma,\, c_1,\, \varepsilon_0 \in \mathbb{R}^+$ such that for each $\varepsilon < \varepsilon_0$ and for each $x,\, y \in \mathtt{M}$, setting $t_\varepsilon := c_1 \log \varepsilon^{-1}$,*

$$C_{t_\varepsilon,\,\varepsilon}(x,\,y) \ge \gamma \varepsilon^{\frac{1}{\alpha} - 1} > 0;$$

where α is the Hölder continuity of the foliations.

The proof of theorem 2.1 is the content of section 3.

¿From now on let us set $T_\varepsilon := T_{t_\varepsilon,\,\varepsilon}$, $\mathbb{P}_\varepsilon := \mathbb{P}_{t_\varepsilon,\,\varepsilon}$. Define the projector

$$\Pi g = \int_{\mathtt{M}} g.$$

Notice that $\Pi^* = \Pi$.

A very important, but standard, consequence of theorem 2.1 is the following.

[8]In fact, the reason of the special choice of the averaging is motivated only by the convenience of having the same invariant measure for the deterministic flow and its random perturbation.

Lemma 2.2. *The spectral radius of* $\mathbb{C}_\varepsilon - \Pi$*, as an operator on* $L^2(\mathtt{M}, \mu)$*, is less than* $1 - \gamma\varepsilon^{\frac{1}{\alpha}-1}$.

Proof. The first, completely standard, consequence of theorem 2.1 is the estimate[9]

$$\|(\mathbb{C}_\varepsilon - \Pi)^n\|_1 \leq (1 - \gamma\varepsilon^{\frac{1}{\alpha}-1})^n.$$

The operator $\mathbb{C}_\varepsilon - \Pi$ is self-adjoint, hence

$$\begin{aligned}
\|(\mathbb{C}_\varepsilon - \Pi)^n\|_2 &= \sup_{g\in\mathcal{C}^{(0)}} \frac{\langle(\mathbb{C}_\varepsilon - \Pi)^n g,\, g\rangle}{\langle g,\, g\rangle} = \sup_{g\in\mathcal{C}^{(0)}} \frac{\langle(\mathbb{C}_\varepsilon - \Pi)^{n-1} g,\, (\mathbb{C}_\varepsilon - \Pi)\, g\rangle}{\langle g,\, g\rangle} \\
&\leq \sup_{g\in\mathcal{C}^{(0)}} \frac{\|(\mathbb{C}_\varepsilon - \Pi)^{n-1} g\|_1 \|(\mathbb{C}_\varepsilon - \Pi) g\|_\infty}{\langle g,\, g\rangle} \\
&\leq \sup_{g\in\mathcal{C}^{(0)}} \frac{K_\varepsilon(1 - \gamma\varepsilon^{\frac{1}{\alpha}-1})^{n-1}\|g\|_1^2}{\langle g,\, g\rangle} \leq \frac{K_\varepsilon}{1 - \gamma\varepsilon^{\frac{1}{\alpha}-1}}(1 - \gamma\varepsilon^{\frac{1}{\alpha}-1})^n,
\end{aligned}$$

where $K_\varepsilon = \|C_{t_\varepsilon,\,\varepsilon}\|_\infty + 1$. □

Lemma 2.3. *The* $L^2(\mathtt{M}, \mu)$ *norm of* $\mathbb{P}_\varepsilon^2 - \Pi$ *is less than* $(1 - \gamma\varepsilon^{\frac{1}{\alpha}-1})^{\frac{1}{2}}$.

Proof. For each $f \in L^2(\mathtt{M}, \mu)$, we have

$$\|(\mathbb{P}_\varepsilon^2 - \Pi)f\|_2^2 = \langle f,\, (\mathbb{C}_\varepsilon - \Pi)f\rangle \leq \|\mathbb{C}_\varepsilon - \Pi\|_2 \|f\|_2^2,$$

and the Lemma is proven since, for self-adjoint operators, the norm equals the spectral radius. □

Lemma 2.3 implies a precise control on the rate of correlation decay for the random flow. Certainly the alert reader has noticed that the estimate holds for all L^2 functions, while the correlations for the flow will have a fast decay only for "smooth" observables. Here lies the strength and the weakness of the present method. Strength because it uses rough, but easy to obtain, estimates. Weakness because such estimates are likely to be non optimal.

---

[9]Let us briefly recall the argument: consider $f \in L^1(\mathtt{M}, \mu)$ with $\int_{\mathtt{M}} f = 0$. Remember that $\mathbb{C}_\varepsilon 1 = \mathbb{C}_\varepsilon^* 1 = 1$ and let $\mathtt{M}_\varepsilon^+ = \{x \in \mathtt{M} \mid \mathbb{C}_\varepsilon f \geq 0\}$; $\mathtt{M}_+ = \{x \in \mathtt{M} \mid f \geq 0\}$, $\gamma_\varepsilon := \gamma\varepsilon^{\frac{1}{\alpha}-1}$, then

$$\begin{aligned}
\|\mathbb{C}_\varepsilon f\|_1 &= 2\int_{\mathtt{M}_\varepsilon^+} d\mu(x) \int_{\mathtt{M}} d\mu(y) C_\varepsilon(x,\, y) f(y) = 2\int_{\mathtt{M}} d\mu(y) f(y) \int_{\mathtt{M}_\varepsilon^+} d\mu(x)[C_\varepsilon(x,\, y) - \gamma_\varepsilon] \\
&\leq 2\int_{\mathtt{M}_+} d\mu(y) f(y) \int_{\mathtt{M}} d\mu(x)[C_\varepsilon(x,\, y) - \gamma_\varepsilon] = (1 - \gamma_\varepsilon) 2\int_{\mathtt{M}_+} d\mu(y) f(y) \\
&= (1 - \gamma_\varepsilon)\|f\|_1.
\end{aligned}$$

Accordingly, for each $f \in L^1(\mathtt{M}, \mu)$, holds

$$\|(\mathbb{C}_\varepsilon - \Pi)^n f\|_1 = \|\mathbb{C}_\varepsilon^n (\mathbb{1} - \Pi) f\|_1 \leq (1 - \gamma_\varepsilon)^n \|f\|_1.$$

§3 Estimating the Kernel.

Here we prove Theorem 2.1. Actually, it would be easier to follow the argument by simply drawing few pictures. Unfortunately, such pictures are clearer to the one who draws them than to the one who merely looks. Hence, here I provide the algebra+analysis and I strongly invite the reader to draw her/his own pictures.

Let us consider the set

$$\begin{aligned}\Omega_{xy} = \{(z_1,\, z_2,\, z_3) \in \mathtt{M}^3 \mid d_\varepsilon(x,\, \phi_{-t} z_1) \leq \delta_0;\; d_\varepsilon(z_1,\, z_2) \leq \delta_0;\\ d_\varepsilon(z_3,\, z_2) \leq \delta_0;\; d_\varepsilon(y,\, \phi_{-t} z_3) \leq \delta_0\},\end{aligned}$$

where, through this section, $t = c_1 \log \varepsilon^{-1}$ (c_1 will be chosen later, just after lemma 3.1).

There exists $c_2 > 0$ such that[10]

$$C_{t,\,\varepsilon}(\phi_{-t}x,\, \phi_{-t}y) \geq \int_{\Omega_{xy}} d\mu_3(z_1,\, z_2,\, z_3) \frac{c_2}{\mu(B^\varepsilon_{\delta_0}(x))\mu(B^\varepsilon_{\delta_0}(y))\mu(B^\varepsilon_{\delta_0}(z_1))^2};$$

where μ_i is the product measure on $\mathtt{M}^i$.

Next, let

$$\widetilde{\Omega}_{xy} = \left\{(z_1,\, z_3) \in \mathtt{M}^2 \mid d_\varepsilon(x,\, \phi_{-t}z_1) \leq \delta_0;\; d_\varepsilon(z_1,\, z_3) \leq \frac{\delta_0}{2};\; d_\varepsilon(y,\, \phi_{-t}z_3) \leq \delta_0\right\}.$$

Clearly,

$$\begin{aligned}C_{t,\,\varepsilon}(\phi_{-t}x,\, \phi_{-t}y) &\geq c_2 \int_{\widetilde{\Omega}_{xy}} d\mu_2(z_1,\, z_3) \frac{\mu\left(\{z_2 \in \mathtt{M} \mid d_\varepsilon(z_2,\, z_1) \leq \frac{\delta_0}{2}\}\right)}{\mu(B^\varepsilon_{\delta_0}(x))\mu(B^\varepsilon_{\delta_0}(y))\mu(B^\varepsilon_{\delta_0}(z_1))^2}\\ &\geq c_3 \int_{\widetilde{\Omega}_{xy}} d\mu_2(z_1,\, z_3) \frac{1}{\mu(B^\varepsilon_{\delta_0}(x))\mu(B^\varepsilon_{\delta_0}(y))\mu(B^\varepsilon_{\delta_0}(z_1))},\end{aligned}$$

where, again, we have used lemma I.1 of appendix I.

Let U be a neighborhood containing a C-frame $(W_1,\, W_2,\, W_3)$ of size $\frac{\delta_0}{8}$.[11] Let $W^u_\varepsilon(x)$ be the piece of strong unstable manifold of x contained in $B^\varepsilon_{\delta_0}(x)$. If $(z_1,\, z_3) \in \widetilde{\Omega}_{xy}$ then z_1 must be in a small neighborhood of $\phi_t W^u_\varepsilon(x)$ and z_3 must be in a small neighborhood of $\phi_t W^u_\varepsilon(y)$.

Lemma 3.1. *There exists $L(\delta_0) > 0$ such that each piece of strong unstable manifold with diameter larger than L intersects properly*[12] *the given C-frame.*

Proof. Here we use the fact that smooth hyperbolic contact flows are mixing [KB]. Let $(W_1,\, W_2,\, W_3)$, $W_1 \cap W_3 = \{\bar{x}\}$, $W_2 \cap W_3 = \{\bar{y}\}$ be the considered C-frame. We will discuss a proper intersection near $\bar{x}$, the same arguments hold near $\bar{y}$. Consider a covering $\mathcal{P}$ of $\mathtt{M}$ made of elements approximately rectangular (that is

[10] See Lemma I.1 in appendix I.

[11] See appendix III for the definition and properties of C-frames.

[12] See appendix III for the definition of "proper" intersection.

with faces almost parallel to the weakly stable and unstable directions) such that there exists an element P_0 which is contained in a ball of radius $\delta_0^{3/\alpha}/100$ and contains a ball of radius $\delta_0^{4/\alpha}$ around $\bar{x}$. Clearly, if an unstable manifold intersects P_0 and is sufficiently large then it intersects properly the C-frame. Consider a piece of strong unstable manifold W. Because of the mixing property there exists $T > 0$ such that $\phi_T P \cap P' \neq \emptyset$ for all $P,\, P' \in \mathcal{P}$. Consider $\phi_{-T}W$ and assume it crosses completely one element P of the covering[13] (this will always be the case if $\phi_{-T}W$, and hence W, has sufficiently large diameter, provided we have chosen the covering with sufficient overlapping, see [LW] for similar constructions), then $W \cap P_0 \neq \emptyset$. □

It is now clear that the constant c_1 must be chosen such that the minimal expansion along a piece of trajectory of length $t - n_\varepsilon$ is at least $\varepsilon^{-1}\delta_0^{-1}L$, whereby $\phi_t W^u_\varepsilon(x)$ and $\phi_t W^u_\varepsilon(y)$ have diameter larger than L and therefore intersect properly the C-frame at least once, because of lemma 3.1.

We can then write $\phi_t W^u_\varepsilon(x) = \sum_{i=1}^{k_x} W_i(x)$ and $\phi_t W^u_\varepsilon(y) = \sum_{i=1}^{k_y} W_i(y)$ where $W_i(x) \cap C_1 \neq \emptyset$ and $W_i(y) \cap C_2 \neq \emptyset$, $W_i(x)$ and $W_i(y)$ intersect properly the C-frame, and the diameter of $W_i(x)$ and $W_i(y)$ is contained in the interval $[L,\, c^*L]$ (where c^* depends only on $(\mathtt{M},\, \phi_t)$ and L).[14]

Given any two $W_i(x)$, $W_j(y)$ we know (cfr. theorem III.5) that there exists W^s_{ij} such that $W^s_{ij} \cap W_i(x) \neq \emptyset$, $W^s_{ij} \cap W_j(y) \neq \emptyset$. In fact, both intersections consist of only one point; let $\{\xi_{ij}(x)\} := W^s_{ij} \cap W_i(x)$ and $\{\xi_{ij}(y)\} := W^s_{ij} \cap W_j(y)$. Moreover, the distance (on W^s_{ij}), between $\xi_{ij}(x)$ and $\xi_{ij}(y)$ is shorter, in the Riemannian metrics, than $\frac{\delta_0}{8}$ (that is, $(W_i(x),\, W_j(y),\, W^s_{ij})$ form a C-frame of size less than $\frac{\delta_0}{2}$).

In addition, we can partition $\phi_t B^\varepsilon_{\delta_0}(x)$ by sets $\{B_i(x)\}_{i=1}^{k_x}$ and $\phi_t B^\varepsilon_{\delta_0}(y)$ by sets $\{B_i(y)\}_{i=1}^{k_y}$ such that $B_i(x)$ contains $W_i(x)$, and the analogous requirement holds for $B_j(y)$.[15]

¿From appendix III (lemma III.3) we know that, for each $\xi_{ij}(x)$, there exists a set $\Sigma_{ij} \subset W^u(\xi_{ij}(x))$ such that, for each $z \in \Sigma_{ij}$, $W^s(z)$ intersects $W^u(\xi_{ij}(y))$ and, calling Σ^ε_{ij} its $c_4\delta_0^{\frac{1}{\alpha}}\varepsilon^{\frac{1}{\alpha}}$-neighborhood (for some c_4 sufficiently small) in $W^u(\xi_{ij}(x))$, Σ^ε_{ij} has measure[16] larger than $c_5\delta_0^{\frac{1}{\alpha}}\varepsilon^{\frac{1}{\alpha}}$. Thus, setting $\widetilde{\Sigma}^\varepsilon_{ij} := \bigcup_{z\in\Sigma^\varepsilon_{ij}} W^{0s}_{\frac{\varepsilon}{2}}(z)$, it follows that, if $z_1 \in \widetilde{\Sigma}^\varepsilon_{ij}$, then $\emptyset \neq W^s(z_1) \cap W^{0u}(\xi_{ij}(y)) := \{\tilde{z}\}$; $d_\varepsilon(z_1,\, \tilde{z}) \leq \frac{\delta_0}{4}$ (hence, $B^\varepsilon_{\frac{\delta_0}{4}}(\tilde{z}) \subset B^\varepsilon_{\frac{\delta_0}{2}}(z_1)$) and there exists $\tau < \frac{\varepsilon\delta_0}{8}$ such that $\phi_\tau\tilde{z} \in W^{0s}(\xi_{ij}(y))$ (due to the Hölder continuity of the stable foliation). Hence

$$\widetilde{\Omega}_{xy} \supset \left\{(z_1,\, z_3) \in \mathtt{M}^2 \mid z_1 \in \phi_t B^\varepsilon_{\delta_0}(x);\; z_3 \in \phi_t B^\varepsilon_{\delta_0}(y);\; z_1 \in \widetilde{\Sigma}^\varepsilon_{ij};\; z_3 \in B^\varepsilon_{\frac{\delta_0}{4}}(\tilde{z})\right\},$$

[13] By this we mean that $\partial P \cap \phi_{-T}W$ is contained in the faces of P almost parallel to the weak stable directions.

[14] Remember that $C_1 = B_{(\delta_0/8)^{3/\alpha}} \cap W^{0s}(\bar{x})$ and $C_2 = B_{(\delta_0/8)^{3/\alpha}} \cap W^{0s}(\bar{y})$.

[15] Think to, and draw, $B_i(x)$, $B_j(y)$ as long, distorted, cylinders around $W_i(x)$ and $W_j(y)$, respectively.

[16] Here we mean the measure μ restricted to $W^u(\xi_{ij}(x))$.

and we can write

$$C_{t,\varepsilon}(\phi_{-t}x,\,\phi_{-t}y)\geq c_3\sum_{ij}\int_{B_i(x)\cap\widetilde{\Sigma}^\varepsilon_{ij}}d\mu(z_1)\frac{\mu(B_j(y)\cap B^\varepsilon_{\frac{\delta_0}{4}}(\tilde z))}{\mu(B^\varepsilon_{\delta_0}(x))\mu(B^\varepsilon_{\delta_0}(y))\mu(B^\varepsilon_{\delta_0}(z_1))}.$$

To conclude other two estimates are needed.

Lemma 3.2. *For each $z_1\in\widetilde{\Sigma}^\varepsilon_{ij}\cap B_i(x)$, holds*

$$\mu(B^\varepsilon_{\frac{\delta_0}{4}}(\tilde z)\cap B_j(y))\geq c_5\delta_0^{-2d}\varepsilon^{-1}\mu(B^\varepsilon_{\delta_0}(z_1))\mu(B_j(y)).$$

In addition,

$$\mu(\widetilde{\Sigma}^\varepsilon_{ij}\cap B_i(x))\geq c_6\varepsilon^{\frac{1}{\alpha}}\mu(B_i(x)).$$

Dear reader, if you have drawn a picture now you will "see" lemma 3.2 just by looking at your picture; at any rate, the non-believer can find the proof in appendix IV.

Accordingly,

$$\begin{aligned}C_{t,\varepsilon}(\phi_{-t}x,\,\phi_{-t}y)\geq&\sum_{ij}c_{16}\int_{\widetilde{\Sigma}^\varepsilon_{ij}\cap B_i(x)}d\mu(z_1)\frac{\mu(B_j(y))}{\delta_0^{2d}\varepsilon\mu(B^\varepsilon_{\delta_0}(x))\mu(B^\varepsilon_{\delta_0}(y))}\\ \geq&c_{17}\delta_0^{-2d}\varepsilon^{\frac{1}{\alpha}-1}\sum_{ij}\frac{\mu(B_i(x))\mu(B_j(y))}{\mu(B^\varepsilon_{\delta_0}(x))\mu(B^\varepsilon_{\delta_0}(y))}\\ =&c_{17}\delta_0^{-2d}\varepsilon^{\frac{1}{\alpha}-1}:=\gamma\varepsilon^{\frac{1}{\alpha}-1}.\end{aligned}$$

§4 Decay of correlations.

In this section we will see that the results collected up to now yield a deep knowledge on the behavior of the correlations of the flow ϕ_t.

Let us define the following Hölder norms:

$$\begin{aligned}\|f\|_{u,\beta}&=\sup_{\substack{x\in\mathtt{M};\;y\in W^u(x)\\ d_\varepsilon(x,y)\leq\delta_0}}\frac{|f(x)-f(y)|}{d_\varepsilon(x,\,y)^\beta}\\ \|f\|_{s,\beta}&=\sup_{\substack{x\in\mathtt{M};\;y\in W^s(x)\\ d(x,y)\leq\delta_0}}\frac{|f(x)-f(y)|}{d(x,\,y)^\beta}\\ \|f\|_{0,\beta}&=\sup_{\substack{x\in\mathtt{M}\\ \tau\in[-\delta_0,\,\delta_0]}}\frac{|f(x)-f(\phi_\tau x)|}{\tau^\beta}\\ \|f\|_{0s,\beta}&=\|f\|_{s,\beta}+\|f\|_{0,\beta}\end{aligned}$$

The basic approximation, without which all the present approach would be useless, is contained in the following Proposition.

Proposition 4.1. *For each $f,\, g\in C^{(1)}(\mathtt{M})$ holds:*

$$\left|\int_{\mathtt{M}} fg\circ\phi_{-2kn_\varepsilon}-\int_{\mathtt{M}} f(\mathbb{P}_\varepsilon)^{2k}g\right|\leq kc_{18}\{\|g\|_1\|f\|_{s,1}\\ +\|g\|_1\|f\|_\infty+\|f\|_{01}\|g\|_\infty+\|g\|_{u,\beta}\|f\|_1\}.$$

Proof. Define $C^u_\beta(\mathtt{M})=\{f\in C^{(0)}(\mathtt{M})\mid \|f\|_\infty+\|f\|_{u,\beta}<\infty\}$ and $C^{0s}(\mathtt{M})=\{f\in C^{(0)}(\mathtt{M})\mid \|f\|_\infty+\|f\|_{0s,1}<\infty\}$. Clearly, $\|f\circ\phi_t\|_{s,\beta}\leq\lambda^{\beta t}\|f\|_{s,\beta}$; $\|f\circ\phi_t\|_{0,\beta}=\|f\|_{0,\beta}$, hence $\phi_t: C^{0s}(\mathtt{M})\to C^{0s}(\mathtt{M})$. Moreover, $\mathbb{P}_\varepsilon: C^u_\beta(\mathtt{M})\to C^u_\beta(\mathtt{M})$ due to the following lemma.

Lemma 4.2. *For each $\rho<1$, $\beta<\alpha$ there exists $c_{19}>0$ such that, for δ_0 sufficiently small, holds*

$$\|\mathbb{P}_\varepsilon f\|_{u,\,\beta}\leq c_{19}\left(\varepsilon^\rho\|f\|_\infty+\varepsilon^{2\beta}\|f\|_{u,\beta}\right).$$

Proof. ¿From appendix I (at the very end) we know that, choosing δ_0 sufficiently small,

$$\|\varphi\|_{u,\,\beta}\leq c_{20}\varepsilon^\rho.$$

Using lemma I.3 we can compute

$$\begin{aligned}\|\mathbb{P}_\varepsilon f\|_{u,\beta}&=\|\mathbb{A}_\varepsilon f\circ\phi_{-t_\varepsilon}\|_{u,\beta}\\ &\leq c_{21}\varepsilon^\rho\|f\|_\infty+(1+c_{21}\delta)\|(1+\delta\varphi)f\circ\phi_{-t_\varepsilon}\|_{u,\beta}\\ &\leq c_{22}\varepsilon^\rho\|f\|_\infty+(1+c_{22}\delta)\|f\circ\phi_{-t_\varepsilon}\|_{u,\beta}\\ &\leq c_{19}\varepsilon^\rho\|f\|_\infty+c_{19}\varepsilon^{2\beta}\|f\|_{u,\beta}\end{aligned}$$

□

Lemma 4.3. *For each $f_1\in C^{0s}(\mathtt{M})$ and $f_2\in C^u(\mathtt{M})$ hold*

$$\left|\int_{\mathtt{M}} f_1f_2-\int_{\mathtt{M}} f_1\mathbb{A}_\varepsilon f_2\right|\leq c_{24}\varepsilon^\rho\|f_1\|_\infty\|f_2\|_1\\ +c_{24}(\|f_1\|_{s,1}\|f_1\|_1+\varepsilon\|f_1\|_{0,1}\|f_2\|_1+\|f_2\|_{u,\beta}\|f_1\|_\infty).$$

Proof.

The result follows by direct computation:[17]

$$
\begin{aligned}
\int_{\mathtt{M}} f_1\mathbb{A}_\varepsilon f &= \int_{\Omega^*_{\delta_0}} d\mu_2(x,\,y)\frac{f_1(x)(1+\delta_0\varphi(x))(1+\delta_0\varphi(y))f_2(y)}{\mu(B^\varepsilon_{\delta_0}(x))^{\frac12}\mu(B^\varepsilon_{\delta_0}(y))^{\frac12}}\\
&= \int_{\Psi(\Omega^*_{\delta_0})} d\mu_2(\xi,\,\eta)\frac{f_1([\xi,\,\eta])(1+\delta_0\varphi([\xi,\,\eta]))(1+\delta_0\varphi([\eta,\,\xi]))f_2([\eta,\,\xi])}{\mu(B^\varepsilon_{\delta_0}([\xi,\,\eta]))^{\frac12}\mu(B^\varepsilon_{\delta_0}([\eta,\,\xi]))^{\frac12}}\\
&\quad + \mathcal{O}(\varepsilon^{3\alpha}\|f_1\|_\infty\|f_2\|_1)\\
&= \int_{\Omega^*_{\delta_0}} d\mu_2(\xi,\,\eta)\frac{f_1(\xi)(1+\delta_0\varphi(\eta))(1+\delta_0\varphi(\xi))f_2(\xi)}{\mu(B^\varepsilon_{\delta_0}(\xi))^{\frac12}\mu(B^\varepsilon_{\delta_0}(\eta))^{\frac12}}\\
&\quad + \mathcal{O}(\varepsilon^{\rho}\|f_1\|_\infty\|f_2\|_1) + \mathcal{O}(\|f_1\|_{s,1}\|f_2\|_1+\varepsilon\|f_1\|_{0,1}\|f_2\|_1+\|f_2\|_{u,\beta}\|f_1\|_1).
\end{aligned}
$$

□

Then,

$$
\begin{aligned}
\left|\int_{\mathtt{M}} fg\circ\phi_{-2kn_\varepsilon} - \int_{\mathtt{M}} f\mathbb{P}^{2k}_\varepsilon g\right| \le \sum_{i=0}^{k-1}\Bigg|\int_{\mathtt{M}} & f\circ\phi_{2(k-i)n_\varepsilon}\mathbb{P}^{2i}_\varepsilon g\\
&- \int_{\mathtt{M}} f\circ\phi_{2(k-i-1)n_\varepsilon}\mathbb{P}^{2(i+1)}_\varepsilon g\Bigg|.
\end{aligned}
$$

Using lemma 4.3 we get

$$
\begin{aligned}
\left|\int_{\mathtt{M}} fg\circ\phi_{-2kn_\varepsilon} - \int_{\mathtt{M}} f\mathbb{P}^{2k}_\varepsilon g\right| &\le kc_{18}\big\{\|f\|_{s,1}\|g\|_1\\
&\quad + \varepsilon^\rho\left[(\|f\|_\infty\|g\|_1+\|f\|_{0,1}\|g\|_1+\|g\|_{u,\beta}\|f\|_1)\right]\big\}.
\end{aligned}
$$

□

In conclusion, for each $f,\,g\in C^{(1)}(\mathtt{M})$, $\int_{\mathtt{M}} g = 0$, we have

$$
\begin{aligned}
\left|\int_{\mathtt{M}} fg\circ\phi_{-t}\right| &\le \left|\int_{\mathtt{M}} f\circ\phi_{n_\varepsilon} g\circ\phi_{-t+n_\varepsilon} - \int_{\mathtt{M}} f\circ\phi_{n_\varepsilon}\mathbb{P}^{[\frac{t}{n_\varepsilon}]-1}g\circ\phi_{-t+[\frac{t}{n_\varepsilon}]n_\varepsilon+n_\varepsilon}\right|\\
&\quad + \left|\int_{\mathtt{M}} f\circ\phi_{n_\varepsilon}\mathbb{P}^{[\frac{t}{n_\varepsilon}]-1}g\circ\phi_{-t+[\frac{t}{n_\varepsilon}]n_\varepsilon+n_\varepsilon}\right|\\
&\le \frac{t}{n_\varepsilon}\varepsilon^\rho c_{25}(\|g\|_{u,\beta}\|f\|_\infty+\|f\|_{0s}\|g\|_\infty)+\|f\|_\infty\|g\|_\infty e^{-\gamma\varepsilon^{\frac{1}{\alpha}-1}\frac{t}{2n_\varepsilon}}.
\end{aligned}
$$

[17] Given a couple of points $x,\,y\in\mathtt{M}$, sufficiently close, it exists only one point that belongs to $W^{0s}(x)\cap W^u(y)$, let us designate it by $[x,\,y]$. If we consider the set $\Omega^*_{\delta_0}=\{(x,\,y)\in\mathtt{M}^2\mid d_\varepsilon(x,\,y)\le\delta_0\}$, then, for δ_0 sufficiently small, we can define the function $\Psi:\Omega^*_{\delta_0}\to\mathtt{M}^2$ by

$$\Psi(x,\,y) = ([x,\,y],\,[y,\,x]) := (\xi,\,\eta).$$

In appendix v it is shown that the Jacobian of Ψ differs from 1 by $\mathcal{O}(\varepsilon^{3\alpha})$; while

$$\mu_2(\Psi(\Omega^*_{\delta_0})\Delta\Omega^*_{\delta_0}) \le \text{const}\varepsilon\mu_2(\Omega^*_{\delta_0}).$$

If $\alpha = 1$, then $\gamma_\varepsilon = \gamma$ does not depend on ε and it suffices to choose $\varepsilon = e^{-\sqrt{\gamma t}}$ to conclude the argument obtaining

$$\left|\int_{\mathtt{M}} fg\circ\phi_{-t}\right| \leq c_{20}\sqrt{t}e^{-\sqrt{\gamma t}}\left(\|g\|_{u,\beta}\|f\|_\infty + \|f\|_{0s,1}\|g\|_\infty + \|f\|_\infty\|g\|_\infty\right),$$

which proves theorem A (the result is easily extended to Hölder observables via an approximation argument).

If $\alpha < 1$ then the choice $\varepsilon = bt^{-\frac{\alpha}{1-\alpha}}(\log t)^{\frac{2\alpha}{1-\alpha}}$, for some b sufficiently large, yields

$$\left|\int_{\mathtt{M}} fg\circ\phi_{-t}\right| \leq c_{20}(1+|t|)^{-\frac{(\rho+1)\alpha-1}{1-\alpha}}\left(\|g\|_{u,\beta}\|f\|_\infty + \|f\|_{0s,1}\|g\|_\infty + \|f\|_\infty\|g\|_\infty\right),$$

which proves theorem B.

Appendix I (Averages).

In this appendix we prove that there exists a special average satisfying the requirement stated in section 2 (that is $\mathbb{A}_\varepsilon^\varphi 1 = 1$).

In the following we will use c_i to designate any constant that depends only on $(\mathtt{M}, \phi_1, \mu)$.[18]

First we need some informations on the measure of the balls in the d_ε metrics.

Lemma I.1. *There exists $\delta_0 > 0$, $c_0 > 1$ and $\theta \in (0, 1)$, such that for each $\delta \leq \delta_0$; ε sufficiently small; $x, y \in \mathtt{M}$*

$$\mathrm{Exp}[-c_0 d_\varepsilon(x, y)] \leq \frac{\mu(B_\delta^\varepsilon(x))}{\mu(B_\delta^\varepsilon(y))} \leq \mathrm{Exp}[c_0 d_\varepsilon(x, y)]$$

and, if $y \in W^u(x)$,

$$\mathrm{Exp}\big[-c_0\varepsilon\delta^{-1} d_\varepsilon(x, y)\big] \leq \frac{\mu(B_\delta^\varepsilon(x))}{\mu(B_\delta^\varepsilon(y))} \leq \mathrm{Exp}\big[c_0\varepsilon\delta^{-1} d_\varepsilon(x, y)\big];$$

in addition,[19]

$$\frac{\mu(B_\delta^\varepsilon(x)\Delta B_\delta^\varepsilon(y))}{\mu(B_\delta^\varepsilon(x))} \leq c_0\delta^{-1}d_\varepsilon(x, y),$$

and

$$\frac{\mu(B_{c_0^{-1}\delta}^\varepsilon(x))}{\mu(B_\delta^\varepsilon(x))} \geq \theta.$$

The proof of the above Lemma is the content of appendix II.

---

[18] The c_i in this appendix have no relation with the constants in the main text or in other appendices bearing the same name.

[19] Given two sets A and B we will use $A\Delta B$ to designate the symmetric difference, i.e. $A\Delta B = (A\cup B)\backslash(A\cup B)$.

Let us get to the point: a little algebra shows that the equation $\mathbb{A}_\varepsilon^\varphi 1 = 1$ is equivalent to

$$\varphi = \frac{1}{2}(\mathbb{1} - \mathbb{B}_\delta^\varepsilon)\varphi - \frac{1}{2}h_\delta^\varepsilon - \frac{\delta}{2}H_\delta^\varepsilon(\varphi) := \mathbb{L}_\delta^\varepsilon\varphi \tag{I.1}$$

where

$$\begin{aligned}
\mathbb{B}_\delta^\varepsilon f(x) &= \int_{B_\delta^\varepsilon(x)} \frac{f(y)}{\mu(B_\delta^\varepsilon(x))^{\frac12}\mu(B_\delta^\varepsilon(y))^{\frac12}} d\mu(y);\\
h_\delta^\varepsilon(x) &= \delta^{-1}[\mathbb{B}_\delta^\varepsilon 1 - 1];\\
H_\delta^\varepsilon(f) &= f\mathbb{B}_\delta^\varepsilon f + f h_\delta^\varepsilon.
\end{aligned}$$

Note that[20]

$$\begin{aligned}
&\|h_\delta^\varepsilon\|_\infty \leq c_0;\\
&\|H_\delta^\varepsilon(f)\|_\infty \leq c_0(\|f\|_\infty^2 + \|f\|_\infty).
\end{aligned}$$

Let us define, for each $f \in C^{(0)}(\mathtt{M})$, the norm

$$\|f\|_* = \sup_{\substack{x,\, y\in\mathtt{M}\\ d_\varepsilon(x,\,y)\leq\delta_0}} \frac{|f(x)-f(y)|}{d_\varepsilon(x,\,y)}.$$

¿From now on I will suppress all the subscripts and superscripts involving ε and δ provided it can be done without creating confusion.

The most relevant fact about the operator $\mathbb{B}$ is contained in the following.

Lemma I.2. *There exists $\theta_* < 1$, independent on ε and $\delta \leq \delta_0$, such that, viewing $\mathbb{B}$ as an operator on $\mathcal{C}^{(0)}(\mathtt{M})$,*

$$\|(\mathbb{1} - \mathbb{B})^2\|_\infty \leq 4\theta_*.$$

Proof. For each $f \in C^{(0)}(\mathtt{M})$ and $x \in \mathtt{M}$ holds[21]

$$\begin{aligned}
|(\mathbb{1} - \mathbb{B})\mathbb{B}f(x)| &\leq \frac{1}{\mu(B_\delta^\varepsilon(x))}\int_{B_\delta^\varepsilon(x)} d\mu(y)|\mathbb{B}f(x) - \mathbb{B}f(y)| + c_1\delta\|f\|_\infty\\
&\leq \frac{1}{\mu(B_\delta^\varepsilon(x))}\int_{B_\delta^\varepsilon(x)} d\mu(y)\min\{2\|\mathbb{B}f\|_\infty;\, \|\mathbb{B}f\|_* d_\varepsilon(x,\,y)\} + c_1\delta\|f\|_\infty\\
&\leq \frac{1+c_1\delta}{\mu(B_\delta^\varepsilon(x))}\int_{B_\delta^\varepsilon(x)} d\mu(y)\min\{2\|f\|_\infty;\, c_0\delta^{-1}\|f\|_\infty d_\varepsilon(x,\,y)\} + c_1\delta\|f\|_\infty\\
&\leq \frac{(1+c_1\delta)\|f\|_\infty}{\mu(B_\delta^\varepsilon(x))}\left\{\mu(B_{c_0^{-1}\delta}^\varepsilon(x)) + 2\mu(B_\delta^\varepsilon(x)\backslash B_{c_0^{-1}\delta}^\varepsilon(x))\right\} + c_1\delta\|f\|_\infty\\
&\leq (2-\theta)(1+c_1\delta)\|f\|_\infty < 2\|f\|_\infty.
\end{aligned}$$

Then, by choosing δ_0 sufficiently small,

$$\|(\mathbb{1} - \mathbb{B})^2\|_\infty \leq \|\mathbb{1} - \mathbb{B}\|_\infty + \|(\mathbb{1} - \mathbb{B})\mathbb{B}\|_\infty \leq 2 + (2-\theta)(1+c_1\delta) \leq 4\theta_*.$$

[20] The following estimates rest on Lemma I.1.

[21] Notice that lemma I.1 implies $\|\mathbb{B}f\|_* \leq c_0\delta^{-1}\|f\|_\infty$.

□

By using the above estimates follows immediately that the operator $\mathbb{L}^2$ is a contraction on $K_a = \{f \in C^{(0)}(\mathtt{M}) \mid \|f\|_\infty \leq a\}$ for δ_0 sufficiently small and a appropriately chosen. Therefore, equation (I.1) has a unique solution in K_a. Since (I.1) has a solution, we can rewrite it in the equivalent form

$$\varphi = -h_\delta^\varepsilon - \mathbb{B}_\delta^\varepsilon \varphi - \delta H_\delta^\varepsilon(\varphi) \tag{I.2}$$

For it holds[22]

$$\|h_\delta^\varepsilon\|_* \leq c_0\delta^{-1};$$
$$\|H_\delta^\varepsilon(f)\|_* \leq c_0\delta^{-1}(\|f\|_\infty^2 + \|f\|_\infty) + c_0\|f\|_*(\|f\|_\infty + c_0);$$

equation (I.2) immediately implies that $\varphi \in \mathcal{C}^{(1)}(\mathtt{M})$ and

$$\|\varphi\|_* \leq b\delta^{-1}$$

for some b independent on δ and ε.

The only property of the function φ that is used in the paper and it is not a consequence of what we have done so far is the estimate on the unstable derivative used in section four. To gain this further inside more work is needed.

Given a function f let us define a Hölder norm in the unstable direction by

$$\|f\|_{u,\beta} := \sup_{x\in\mathtt{M}} \sup_{\substack{y\in W^u(x)\\ d_\varepsilon(x,y)\leq\delta_0}} \frac{|f(x)-f(y)|}{d_\varepsilon(x,y)^\beta}.$$

Lemma I.3. *For each $f \in \mathcal{C}^{(1)}(\mathtt{M})$, $\beta \leq \alpha$,*

$$\|\mathbb{B}f\|_{u,\beta} \leq (1+c_3\delta)\|f\|_{u,\beta} + c_3\varepsilon\delta^{-\beta}\|f\|_\infty.$$

---

[22]The proof is simple and relies on Lemma I.1. Here is the idea:

$$\begin{aligned}
|\mathbb{B}_\delta^\varepsilon 1(x) - \mathbb{B}_\delta^\varepsilon 1(y)| &= \frac{\left|\int_{B_\delta^\varepsilon(x)} \frac{d\mu(z)}{\mu(B_\delta^\varepsilon(z))^{\frac12}} - \int_{B_\delta^\varepsilon(y)} \frac{d\mu(z)}{\mu(B_\delta^\varepsilon(z))^{\frac12}}\right|}{\mu(B_\delta^\varepsilon(x))^{\frac12}} + \left|\frac{\mu(B_\delta^\varepsilon(y))^{\frac12}}{\mu(B_\delta^\varepsilon(x))^{\frac12}} - 1\right| \mathbb{B}1(y)\\
&\leq \frac{\left|\int_{B_\delta^\varepsilon(x)} d\mu(z)\left[1 - \frac{\mu(B_\delta^\varepsilon(x))^{\frac12}}{\mu(B_\delta^\varepsilon(z))^{\frac12}}\right] - \int_{B_\delta^\varepsilon(y)} d\mu(z)\left[1 - \frac{\mu(B_\delta^\varepsilon(x))^{\frac12}}{\mu(B_\delta^\varepsilon(z))^{\frac12}}\right]\right|}{\mu(B_\delta^\varepsilon(x))^{\frac12}}\\
&\quad + \frac{|\mu(B_\delta^\varepsilon(x)) - \mu(B_\delta^\varepsilon(y))|}{\mu(B_\delta^\varepsilon(x))} + e^{c_0\delta}c_0\delta d_\varepsilon(x,y)\\
&\leq \frac{c_0\delta\mu(B_\delta^\varepsilon(x)\Delta B_\delta^\varepsilon(y))}{\mu(B_\delta^\varepsilon(x))} + 2e^{c_0\delta}c_0\delta d_\varepsilon(x,y)\\
&\leq (c_0^2 + 2e^{c_0\delta}c_0\delta)d_\varepsilon(x,y) \leq c_2 d_\varepsilon(x,y).
\end{aligned}$$

The proof of Lemma I.3 is contained in appendix II. As an immediate consequence

$$\|h\|_{u,\,\alpha} \le c_4 \varepsilon \delta^{-\alpha}$$
$$\|\mathbb{L} f\|_{u,\,\alpha} \le (1 + c_5(\|f\|_\infty + 1)\delta)\|f\|_{u,\alpha} + c_5(1 + \|f\|_\infty)\delta\varepsilon\delta^{-\alpha}.$$

Iterating the second of the above inequalities yields, remembering that $\|f\|_\infty \le a$ implies $\|\mathbb{L}^i f\|_\infty \le a$,

$$\|\mathbb{L}^n f\|_{u,\,\alpha} \le e^{nc_6(1+a)\delta} \left(\|f\|_{u,\,\alpha} + c_7 \varepsilon \delta^{-1-\alpha}\right).$$

We have already seen that $\mathbb{L}^2$ is a contraction, hence by defining $\tilde\varphi_n = \mathbb{L}^n(0)$ we have a sequence converging exponentially fast, in the sup norm, to φ. Whence, for each $x \in \mathtt{M}$ and $y \in W^u(x)$, setting $n_{xy} = c_8 \ln[\varepsilon d_\varepsilon(x,\,y)^{\alpha-\epsilon}]^{-1}$ (where ϵ will be chosen later),

$$\frac{|\varphi(x) - \varphi(y)|}{d_\varepsilon(x,\,y)^{\alpha-\epsilon}} \le \frac{|\tilde\varphi_{n_{xy}}(x) - \tilde\varphi_{n_{xy}}(y)|}{d_\varepsilon(x,\,y)^{\alpha-\epsilon}} + c_9\varepsilon$$

where c_8 has been chosen as to obtain the last term in the above expression.

Accordingly, for each $x,\,y \in \mathtt{M}$, $d_\varepsilon(x,\,y) \le \delta_0$,

$$\begin{aligned}
\frac{|\varphi(x) - \varphi(y)|}{d_\varepsilon(x,\,y)^{\alpha-\epsilon}} &\le d_\varepsilon(x,\,y)^\epsilon \|\mathbb{L}^{n_{xy}}(0)\|_{u,\,\alpha} + c_9\varepsilon \\
&\le d_\varepsilon(x,\,y)^\epsilon e^{c_{10}\delta \ln \varepsilon^{-1} d_\varepsilon(x,y)^{-1}} c_7 \varepsilon \delta^{-1-\alpha} + c_9\varepsilon \\
&\le d_\varepsilon(x,\,y)^{\epsilon - c_{10}\delta} \varepsilon^{1-c_{10}\delta} \delta^{-1-\alpha} + c_9\varepsilon \le c_{11}\varepsilon^{1-2\epsilon},
\end{aligned}$$

where we have chosen $\epsilon = c_{10}\delta$ and ε sufficiently small($\varepsilon < \delta^{\frac{1+\alpha}{c_{10}\delta}}$). That is

$$\|\varphi\|_{u,\,\alpha - c_{10}\delta} \le c\varepsilon^{1-c_{12}\delta}.$$

Appendix II (Balls).

This appendix is dedicated to the task of proving Lemma I.1 and Lemma I.3. The reader be advised that throughout this appendix "c" will stand for a generic constant (not always the same) depending on the manifold $\mathtt{M}$, on ϕ_1 but <u>not</u> on ε or δ.

To address the problem it is convenient to start by introducing appropriate foliations.[23]

Given a point $x \in \mathtt{M}$ and a ball $B_{\delta_*}(x)$ (the ball, centered at x, of radius δ_* in the metric d) we will call a smooth foliation $\mathcal{F} = \{\mathcal{F}\}$ of a neighborhood of x "unstable-like" if it consists of $(d+1)$-dimensional manifolds uniformly transversal to $W^s(x)$ and "stable-like" if it consists of d-dimensional manifolds uniformly transversal to $W^{0u}(x)$. In addition, we will call an unstable-like foliation $\mathcal{F}$ "adapted to $B_{\delta_*}(x)$"

[23]The following computations may look a bit cumbersome; yet, on the one hand, I do not know of a simpler approach; on the other hand, the idea to compute volumes by foliating them is a very efficient and very old one [Arc].

if the neighborhood foliated by the leaves of $\mathcal{F}$ contains the neighborhood $B^{\varepsilon}_{4\delta_*}(x)$ and if, for each $\gamma \in \mathcal{F}$, $\gamma \cap \partial B_{\delta_*}(x) \neq \emptyset$ implies $\gamma \subset \partial B_{\delta_*}(x)$;[24] moreover, if $\gamma \cap W^s(x) \neq \emptyset$, then $\partial(B^{\varepsilon}_{4\delta_*}(x) \cap \gamma) \subset \partial B^{\varepsilon}_{4\delta_*}(x)$ and $\gamma \subset B^{\varepsilon}_{8\delta_*}(x)$. Analogously, we will call a stable-like foliation $\mathcal{F}$ "adapted to $B_{\delta_*}(x)$" if the neighborhood foliated by the leaves of $\mathcal{F}$ contains the neighborhood $\phi_{n_\varepsilon} B^{\varepsilon}_{4\delta_*}(\phi_{-n_\varepsilon} x)$ and if for each $\gamma \in \mathcal{F}$ $\gamma \cap \partial\phi_{n_\varepsilon} B_\delta(\phi_{-n_\varepsilon} x) \neq \emptyset$ implies $\gamma \subset \partial\phi_{n_\varepsilon} B_\delta(\phi_{-n_\varepsilon} x)$; moreover, if $\gamma \cap W^{0u}(x) \neq \emptyset$, then $\partial(\phi_{n_\varepsilon} B^{\varepsilon}_{4\delta_*}(\phi_{-n_\varepsilon} x) \cap \gamma) \subset \partial\phi_{n_\varepsilon} B^{\varepsilon}_{4\delta_*}(\phi_{-n_\varepsilon} x)$ and $\gamma \subset \phi_{n_\varepsilon} B^{\varepsilon}_{8\delta_*}(\phi_{-n_\varepsilon} x)$.

Proof of Lemma I.1.

Clearly it suffices to discuss the situation $d_\varepsilon(x, y) \ll \delta$. We will consider three cases: i) $y \in W^u(x)$ ii) $y = \phi_\tau x$ iii) $y \in W^{0s}(x)$. The first inequality of Lemma I.1 follows from a trivial combination of similar inequalities for the cases (i), (ii) and (iii). Here we will discuss explicitly only (i), since it yields the second inequality of Lemma I.1 (stronger than what it is needed to prove the first inequality) and the proofs of (ii), (iii) follow along the same lines with only minor, and obvious, changes.

The idea is to use a stable-like foliation $\mathcal{F}$ adapted to $B_{\delta\varepsilon}(\phi_{n_\varepsilon} x)$. Given such a foliation we can obtain an associated stable-like foliation adapted to $B_{\delta\varepsilon}(\phi_{n_\varepsilon} y)$ by introducing a diffeomorphism $\psi : \mathtt{M} \to \mathtt{M}$ such that $\psi(B_{\delta\varepsilon}(\phi_{n_\varepsilon} x)) = B_{\delta\varepsilon}(\phi_{n_\varepsilon} y)$. A moment reflection shows that it can be arranged so that ψ, restricted to the set $\phi_{n_\varepsilon} B^{\varepsilon}_{16\delta}(x)$, is measure preserving, and

$$\|D\psi - \mathbb{1}\|_\infty \leq c d(\phi_{n_\varepsilon} x, \phi_{n_\varepsilon} y) \leq c\varepsilon d_\varepsilon(x, y).$$

The foliation $\mathcal{F}' = \psi\mathcal{F}$ is the wanted foliation of $B_{\delta\varepsilon}(\phi_{n_\varepsilon} y)$.

The next step is to induce foliations in a neighborhood of x. Let us define $\mathcal{F}_* = \phi_{-n_\varepsilon}\mathcal{F}$; $\mathcal{F}'_* = \phi_{-n_\varepsilon}\mathcal{F}'$; $\widetilde{\mathcal{F}}_* = \{\gamma_* \cap B_\delta(x)\}_{\gamma_* \in \mathcal{F}_*}$; $\widetilde{\mathcal{F}}'_* = \{\gamma'_* \cap B_\delta(y)\}_{\gamma'_* \in \mathcal{F}_*}$ and $\widetilde{\mathcal{F}} = \phi_{n_\varepsilon}\widetilde{\mathcal{F}}_*$, $\widetilde{\mathcal{F}}' = \phi_{n_\varepsilon}\widetilde{\mathcal{F}}'_*$.[25] By construction $\widetilde{\mathcal{F}}_*$ is a foliation of $B^{\varepsilon}_{\delta}(x)$ and $\widetilde{\mathcal{F}}'_*$ is a foliation of $B^{\varepsilon}_{\delta}(y)$. Moreover $\widetilde{\psi} = \phi_{-n_\varepsilon} \circ \psi \circ \phi_{n_\varepsilon}$ establishes a correspondence among the leaves of $\mathcal{F}_*$ and $\mathcal{F}'_*$. Unfortunately $\widetilde{\psi}$ is not suitable to establish a correspondence point by point since, due to the possibility of different expansion rates (in higher dimensions), it is not possible to bound $d(z, \widetilde{\psi}(z))$ effectively in terms of $d_\varepsilon(x, y)$. It is therefore more convenient to establish, between corresponding leaves, a pointwise correspondence by using an unstable-like foliation.[26] To be concrete one can introduce a coordinate system in a neighborhood of x (e.g. the one induced by the exponential map) and consider a foliation $\mathcal{F}^u$ made of planes (with respect to the Euclidean structure of the chart) parallel to the weak-unstable direction at x. One can then define $\theta_{\gamma_*} : \gamma_* \to \gamma'_* := \widetilde{\psi}(\gamma_*)$ by $\theta_{\gamma_*}(z) := \gamma^u(z) \cap \gamma'_*$ (where $\gamma^u(z) \in \mathcal{F}^u$ is the leaf containing z).

[24] Note that this implies that the curvature of the manifolds in $\mathcal{F}$ cannot be less than δ_*^{-1}, yet it can be arranged so that it is less than $c\delta_*^{-1}$, also it can be arranged for the conditional measure on the fibers to have smooth densities (both along the same fiber and from fiber to fiber), with respect to the restriction of μ to the fibers.

[25] It is essential to remark that $\gamma_* \cap B^{\varepsilon}_{\delta}(x)$, where $\gamma_* \in \mathcal{F}_*$, is the connected component of $\gamma_* \cap B_\delta(x)$ intersecting $\phi_{-n_\varepsilon} W^{0u}_{\varepsilon\delta}(\phi_{n_\varepsilon} x)$ and the same holds for $\gamma'_* \in \mathcal{F}'_*$.

[26] To use the weak-unstable foliation itself it is not advisable since such a foliation, in more than three dimensions, may be only Hölder continuous. In fact, we will be forced to use it in the proof of Lemma I.3, due to that we will be able to obtain only Hölder estimates.

The first important property of the above construction is that for each $z \in \tilde{\gamma}_* \in \widetilde{\mathcal{F}}_*$

$$d(z,\, \theta_{\gamma_*}(z)) \leq c\varepsilon d_\varepsilon(x,\, y). \tag{II.1}$$

To see this define $z_1 = \phi_{n_\varepsilon}(z)$ and $w_1 = W^{0u}(z_1) \cap \gamma'$ ($\gamma' = \phi_{n_\varepsilon}(\gamma'_*)$). Since the foliations $\mathcal{F}$ and $\mathcal{F}'$ are smooth and uniformly transversal to the unstable direction, it follows, by construction, that $d(z_1,\, w_1) \leq c\varepsilon d_\varepsilon(x,\, y)$. Accordingly, setting $w = \phi_{-n_\varepsilon}(w_1)$, $d(z,\, w) \leq c\varepsilon d_\varepsilon(x,\, y)$. Since the stable and unstable foliation are, at worst, Hölder we have $d(w,\, \theta_{\gamma_*}(z)) \leq cd(z,\, w)$ as well, which implies (II.1).

Let us define $\hat{B}(x) = \{z \in \mathtt{M} \mid \gamma_*(z) \cap B^\varepsilon_\delta(x) \neq \emptyset\}$ and $\hat{B}(y) = \{z \in \mathtt{M} \mid \gamma'_*(z) \cap B^\varepsilon_\delta(y) \neq \emptyset\}$.[27] By construction $\hat{B}(x)$ is $\mathcal{F}_*$ measurable and $\hat{B}(y)$ is $\mathcal{F}'_*$ measurable (i.e., they are measurable with respect to the σ-algebra associated to the partition of $\mathtt{M}$ induced by $\mathcal{F}_*$ and $\mathcal{F}'_*$, respectively); moreover, $B^\varepsilon_\delta(x) = \hat{B}(x) \cap B_\delta(x)$, $B^\varepsilon_\delta(y) = \hat{B}(y) \cap B_\delta(y)$ and $\widetilde{\psi}\hat{B}(x) = \hat{B}(y)$.

The second key fact is[28]

$$\sup_{\xi \in \hat{B}(y)} |1 - \frac{d(\theta_{\gamma_*})_*\nu_{\gamma_*}}{d\nu_{\gamma'_*}}(\xi)| \leq c\varepsilon d_\varepsilon(x,\, y), \tag{II.2}$$

where ν_γ is the measure on γ obtained by restricting the Riemannian metric to γ and, in general, for a map θ and a measure ν we use the notation $\theta_*\nu(f) := \nu(f \circ \theta)$.

We are now in position to compute.[29]

$$\begin{aligned} \mu(B^\varepsilon_\delta(y)) =& \mu(\mu(\chi_{B^\varepsilon_\delta(y)} \mid \mathcal{F}'_*)) = \mu(\chi_{\hat{B}(y)}\mu(\chi_{B_\delta(y)} \mid \mathcal{F}'_*)) \\ =& \mu(\chi_{\hat{B}(x)}\mu(\chi_{B_\delta(y)} \mid \mathcal{F}'_*) \circ \widetilde{\psi}). \end{aligned} \tag{II.3}$$

If we adopt the convention of calling $\gamma(z)$ the fiber of a generic smooth foliation $\mathcal{F}$ containing the point z ($\gamma(z) = \emptyset$ if z is not covered by $\mathcal{F}$), then it is well known that for each integrable function f

$$\mu(f \mid \mathcal{F})(z) = \int_{\gamma(z)} f(\xi) J_{\gamma(z)}(\xi) d\nu_{\gamma(z)}(\xi). \tag{II.4}$$

[27] Note that, by construction, both $\hat{B}(x)$ and $\hat{B}(y)$ consist of a "pile" of fibers of diameter larger than 4δ.

[28] The point here is that the tangent spaces $\mathcal{T}_z\gamma_*$ and $\mathcal{T}_{z'}\gamma'_*$ ($z' = \theta_{\gamma_*}(z)$) form an "angle" (e.g. in the above mentioned chart) smaller than $c\varepsilon^2 d_\varepsilon(x,\, y)$. The proof of this fact is left to the reader but can be obtained straightforwardly by the same estimates that allow to prove the Hölder continuity of the stable distribution (see [KH]). Once such an estimate on the tangent spaces is obtained the result follows by direct computation.

[29] Notice that we use indifferently the same symbol for a partition and for the associated σ-algebra; hence $\mu(\cdot \mid \mathcal{F})$ means the conditional measure with respect to the σ-algebra associated to the partition of $\mathtt{M}$ induced by the foliation $\mathcal{F}$ (that is $\mathcal{F} \cup \{\cup_{\gamma \in \mathcal{F}}\gamma\}^c$). Also, here and in the following we will use χ_A to indicate the characteristic function of the set A.

A direct computation yields[30]

$$J_{\gamma(z)}(\xi)=J_{\gamma_*(\phi_{-n_\varepsilon}z)}(\phi_{-n_\varepsilon}\xi)\left|\det\left(d_{(\phi_{-n_\varepsilon}\xi)}\phi_{n_\varepsilon}\big|_{\gamma_*(\phi_{-n_\varepsilon}z)}\right)\right|^{-1}. \tag{II.5}$$

In addition, by construction of $\mathcal{F}$, there exists a smooth function $j:\mathtt{M}\to\mathbb{R}$ such that

$$J_\gamma(\xi)=\frac{j(\xi)}{Z_\gamma};\quad Z_\gamma=\int_\gamma j(\xi)d\nu_\gamma(\xi).$$

Then, for the foliation $\mathcal{F}'$, holds

$$J_{\gamma'}(\xi)=\frac{j'(\xi)}{Z_\gamma};\quad j'(\xi)=j(\psi^{-1}(\xi))|\det(D_{\psi^{-1}\xi}\psi|_\gamma)|^{-1}.$$

Let $\widetilde{\gamma}_*\in\widetilde{\mathcal{F}}_*$ and $\widetilde{\gamma}'_*=\widetilde{\psi}\widetilde{\gamma}_*\in\widetilde{\mathcal{F}}'_*$, then

$$\left|1-\frac{\nu_{\gamma_*}(\widetilde{\gamma}_*)}{\nu_{\gamma'_*}(\widetilde{\gamma}'_*)}\right|\leq c\varepsilon d_\varepsilon(x,y).$$

To see this, on the one hand, calling $\bar{\gamma}=\theta_{\gamma_*}(\tilde{\gamma}_*)\subset\gamma'_*$, (II.2) implies

$$\left|1-\frac{\nu_{\gamma_*}(\widetilde{\gamma}_*)}{\nu_{\gamma'_*}(\bar{\gamma})}\right|\leq c\varepsilon d_\varepsilon(x,y). \tag{II.6}$$

On the other hand, geometric considerations yield immediately

$$\frac{\nu_{\gamma'_*}(\widetilde{\gamma}'_*\Delta\bar{\gamma})}{\nu_{\gamma'_*}(\widetilde{\gamma}'_*)}\leq c\varepsilon d_\varepsilon(x,y). \tag{II.7}$$

The usual distortion estimates [Ma] imply

$$\left|1-\frac{\nu_{\gamma}(\widetilde{\gamma})}{\nu_{\gamma'}(\widetilde{\gamma}')}\right|\leq c\varepsilon d_\varepsilon(x,y). \tag{II.8}$$

In addition, for each $\xi\in\widetilde{\gamma}$ and $\xi'\in\widetilde{\gamma}'$, $d(\xi,\xi')\leq\varepsilon d_\varepsilon(x,y)$,

$$\left|1-\frac{J_{\gamma}(\xi)}{J_{\gamma'}(\xi')}\right|\leq c\varepsilon d_\varepsilon(x,y). \tag{II.9}$$

---

[30] Just remember that for each foliation $\mathcal{F}$, $\gamma\in\mathcal{F}$, each function f and diffeomorphism $G:\mathtt{M}\to\mathtt{M}$, letting $\mathcal{F}'=G\mathcal{F}$, $\gamma'=G\gamma$, (see [KS])

$$\int_\gamma fd\nu_\gamma=\int_{\gamma'}f\circ G^{-1}|\det(DG^{-1}|_{\gamma'})|d\nu_{\gamma'};$$
$$\mu(f|\mathcal{F})\circ G^{-1}=\frac{\mu(f\circ G^{-1}|\det DG^{-1}||\mathcal{F}')}{\mu(|\det DG^{-1}||\mathcal{F}')},$$

and use the obvious change of variable in the representation (II.4).

Using again distortion estimates, the definitions of J, J' and (II.5), we have

$$\left|1-\frac{J_{\gamma_*}(\xi_*)}{J_{\gamma'_*}(\xi'_*)}\right| \leq c\varepsilon d_\varepsilon(x,\, y),$$

for each $\xi_* \in \widetilde{\gamma}_*$ and $\xi'_* \in \widetilde{\gamma}'_*$, $d(\phi_t\xi_*,\, \phi_t\xi'_*) \leq c\varepsilon d_\varepsilon(x,\, y)$ for each $t \in [0,\, n_\varepsilon]$. Further, it is easy to show that both J_{γ_*} and $J_{\gamma'_*}$ are uniformly bounded by $c\nu_{\gamma_*(x)}(\widetilde{\gamma}_*(x))^{-1}$.

All the above facts together allow to continue the computation started in (II.3)

$$\begin{aligned}
\mu(\chi_{B_\delta(y)}|\mathcal{F}'_*)(\tilde{\psi}z) &= \int_{\gamma'_*(\tilde{\psi}z)} \chi_{B_\delta(y)}(\xi) J_{\gamma'_*(\tilde{\psi}z)}(\xi)d\nu_{\gamma'_*(\tilde{\psi}z)}(\xi)\\
&= \int_{\tilde{\gamma}'_*(\tilde{\psi}z)} J_{\gamma'_*(\tilde{\psi}z)}(\xi)d\nu_{\gamma'_*(\tilde{\psi}z)}(\xi)\\
&= (1+\mathcal{O}(\varepsilon d_\varepsilon(x,\, y)))\int_{\tilde{\gamma}(\tilde{\psi}z)} J_{\gamma'_*(\tilde{\psi}z)}(\xi)d\nu_{\gamma'_*(\tilde{\psi}z)}(\xi)\\
&= (1+\mathcal{O}(\varepsilon d_\varepsilon(x,\, y)))\int_{\tilde{\gamma}_*(z)} J_{\gamma'_*(\tilde{\psi}z)}(\theta\xi)d\nu_{\gamma_*(z)}(\xi)\\
&= (1+\mathcal{O}(\varepsilon d_\varepsilon(x,\, y)))\int_{\tilde{\gamma}_*(z)} J_{\gamma_*(z)}(\xi)d\nu_{\gamma_*(z)}(\xi)\\
&= (1+\mathcal{O}(\varepsilon d_\varepsilon(x,\, y)))\mu(\chi_{B_\delta(x)}|\mathcal{F}_*)(z).
\end{aligned}$$

Whence,

$$\mu(B^\varepsilon_\delta(y)) = (1+\mathcal{O}(\varepsilon d_\varepsilon(x,\, y)))\mu(B^\varepsilon_\delta(x)). \tag{II.10}$$

This concludes the proof of case (i). Case (ii) and (iii) can be dealt in the same way, only in (iii) one must start with an unstable-like foliation $\mathcal{F}$ adapted to $B_\delta(x)$ and perform the computation in $B_\delta(\phi_{n\varepsilon}x)$.

The first and second inequality of the Lemma are then proven by interchanging the role of x and y and thanks to the compactness of $\mathtt{M}$.

The third inequality is much simpler; just notice that[31]

$$B^\varepsilon_\delta(x)\Delta B^\varepsilon_\delta(y) \subset \{z \in \mathtt{M} \mid \delta - d_\varepsilon(x,\, y) \leq d_\varepsilon(x,\, z) \leq \delta + d_\varepsilon(x,\, y)\}. \tag{II.11}$$

That is to say that the set we are interested in is contained in a small neighborhood of $\partial B^\varepsilon_\delta(x)$ which measure is easily estimated, using the same arguments employed in deriving (II.6)–(II.9), by a constant times $\delta^{-1}\mu(B^\varepsilon_\delta(x))d_\varepsilon(x,\, y)$. The fourth inequality is more of the same. $\square$

Proof of Lemma I.3.

The already obtained estimates on the measure of balls allow us to write for each $x \in M$, $y \in W^u(x)$ and $d_\varepsilon(x,\, y) \leq \delta_0$,

$$\|\mathbb{B}f(x) - \mathbb{B}f(y)\| \leq c\varepsilon\delta^{-1}d_\varepsilon(x,\, y)\|f\|_\infty + \frac{\left|\int_{B^\varepsilon_\delta(x)} \frac{f(z)}{\mu(B^\varepsilon_\delta(z))^{\frac12}} - \int_{B^\varepsilon_\delta(y)} \frac{f(z)}{\mu(B^\varepsilon_\delta(z))^{\frac12}}\right|}{\mu(B^\varepsilon_\delta(x))^{\frac12}}.$$

[31]In fact, if $z \in B^\varepsilon_\delta(x)\Delta B^\varepsilon_\delta(y)$, then there are two possibilities: either $d_\varepsilon(x,\, z) \leq \delta$ and $d_\varepsilon(y,\, z) \geq \delta$ (which implies $\delta \leq d_\varepsilon(y,\, z)+d_\varepsilon(x,\, y) \leq d_\varepsilon(x,\, y)+\delta$) or $d_\varepsilon(x,\, z) \geq \delta$ and $d_\varepsilon(y,\, z) \leq \delta$ (which implies $d_\varepsilon(x,\, z) \leq d_\varepsilon(x,\, y) + d_\varepsilon(y,\, z) \leq d_\varepsilon(x,\, y) + \delta$).

To proceed we will use again the previously introduced foliations, only now we will establish the pointwise correspondence among fibers by using the weak-unstable foliation. Namely, let $\theta^u_{\gamma_*} : \gamma_* \to \gamma'_*$ be defined by $\theta^u_{\gamma_*}(z) := W^u(z) \cap \gamma'_*$. The considerations previously carried out in the proof of Lemma I.1 can be applied again only remembering that the function $\theta^u_{\gamma_*}$ is now only Hölder continuous, together with its Jacobian.

Thus[32]

$$\begin{aligned}\|\mathbb{B}f(x) - \mathbb{B}f(y)\| &\leq c\varepsilon\delta^{-\alpha} d_\varepsilon(x, y)^\alpha \|f\|_\infty + \frac{1}{\mu(B^\varepsilon_\delta(x))}\bigg| \int_{B^\varepsilon_\delta(x)} \frac{\mu(B^\varepsilon_\delta(x))^{\frac{1}{2}} f(z)}{\mu(B^\varepsilon_\delta(z))^{\frac{1}{2}}} \\ &\quad - \int_{B^\varepsilon_\delta(x)} \frac{\mu(B^\varepsilon_\delta(x))^{\frac{1}{2}} f(\theta^u z)}{\mu(B^\varepsilon_\delta(\theta^u z))^{\frac{1}{2}}} \bigg| \\ &\leq c\varepsilon\delta^{-\alpha} d_\varepsilon(x, y)^\alpha \|f\|_\infty + (1 + c\delta)\|f\|_{u,\alpha} d_\varepsilon(x, y)^\alpha,\end{aligned}$$

since $d_\varepsilon(z, \theta^u(z)) = \varepsilon^{-1} d(\phi_{n_\varepsilon}(z), \phi_{n_\varepsilon}(\theta^u(z)) \leq (1 + c\varepsilon) d_\varepsilon(x, y)$ (and, at the same time, $d_\varepsilon(z, \theta^u(z)) \geq (1 - c\varepsilon) d_\varepsilon(x, y)$). We finally obtain the wanted inequality:

$$\|\mathbb{B}^\varepsilon_\delta f\|_{u,\beta} \leq (1 + c\delta)\|f\|_{u,\beta} + c\varepsilon\delta^{-\beta}\|f\|_\infty.$$

□

Appendix III (C-frames).

This appendix is dedicated to the definition and the study of the properties of C-frames.

The notion of C-frames has been introduced by N. Chernov [Ch] (he called them H-frames) in the two dimensional case; here we generalize the construction to arbitrary dimensions. Also, we do not use the notion of "uniform non-integrability." Chernov expresses such a condition in terms of the function τ defined in the proof of Lemma III.3. His condition reads

$$c^{-1}\|z\|\|v\| \leq \tau(z, v) \leq c\|z\|\|v\|.$$

---

[32] As already remarked the computation carried out in (II.10), and all the previous relevant inequalities, can be extended to the present context. The only substantial difference is a consequence of the lack of regularity of the unstable foliation that allows only the weaker estimate [Ma]

$$|1 - \frac{d\theta^u_* \nu_{\gamma_*}}{d\nu_{\gamma'_*}}| \leq c d_\varepsilon(x, y)^\alpha \varepsilon^{2\alpha}.$$

This means that, for each $g \in L^\infty(\mathtt{M})$,

$$\begin{aligned}\mu(\chi_{B^\varepsilon_\delta(y)} g) &= \mu(\chi_{\hat{B}(y)} \mu(\chi_{B_\delta(y)} g | \mathcal{F}'_*)) = \mu(\chi_{\hat{B}(x)} \circ \tilde{\psi}^{-1} \mu(\chi_{B_\delta(y)} g | \mathcal{F}'_*)) \\ &= \mu(\chi_{\hat{B}(x)} \mu(\chi_{B_\delta(y)} g | \mathcal{F}_*) \circ \tilde{\psi}) = (1 + \mathcal{O}(\varepsilon^{2\alpha} d_\varepsilon(x, y)^\alpha)) \mu(\chi_{\hat{B}(x)} \mu(\chi_{B_\delta(x)} g \circ \theta^u | \mathcal{F}_*)) \\ &= (1 + \mathcal{O}(\varepsilon d_\varepsilon(x, y)^\alpha)) \mu(\chi_{B^\varepsilon_\delta(x)} g \circ \theta^u),\end{aligned}$$

where, in the last equality, we have assumed $\alpha \geq \frac{1}{2}$, which is the only case in which theorem B yields an interesting result.

For contact flows on three dimensional manifolds (or, more generally, for contact flows with Lipschitz foliations) it is possible to see that such a condition is always satisfied. In the higher dimensional situation the story is different. In fact it is known that the regularity of τ is related to the regularity of the foliation [Ham], hence Chernov condition cannot hold if the foliation is only Hölder.

Definition III.1. *By C-frame we mean three d-dimensional manifolds W_i such that*

$$W_1 \cap W_3 = \{x\};\ W_1 \cap W_3 = \{y\};$$

where x and y are two different points in $\mathtt{M}$. In addition, $W_1 \subset W^u(x)$ and $W_2 \subset W^u(y)$ while $W_3 \subset W^s(x) \cap W^s(y)$.

We say that a C-frame is of size δ if the diameter of W_3 is between $\frac{\delta}{2}$ and δ, $\frac{\delta}{6} \leq d(x, y) \leq \frac{\delta}{3}$, and $W_1 \subset B_{2\delta(x)}$, $\partial(W_1 \cap B_\delta(x)) = W_1 \cap \partial B_\delta(x)$; $W_2 \subset B_{2\delta(x)}$, $\partial(W_2 \cap B_\delta(y)) = W_2 \cap \partial B_\delta(y)$.

Let us call $C_1 := B_{\delta^{3/\alpha}}(x) \cap W^{0s}(x)$ and $C_2 := B_{\delta^{3/\alpha}}(y) \cap W^{0s}(y)$.[33]

Definition III.2. *We say that a connected piece of unstable manifold W_* intersects properly a C-frame if*

$$(W_* \cap C_1) \cup (W_* \cap C_2) \neq \emptyset,$$

and $\partial(W_* \cap B_{2\delta}(x)) = W_* \cap \partial B_{2\delta}(x)$.

Let us investigate a bit more the structure of the C-frames. Consider a sufficiently small neighborhood U. For δ_0 small enough, construct a C-frame of size δ_0 in U.

Lemma III.3. *Let Σ be the set of points $z \in W_1$ such that $\{W_1, W^s(z), W_2\}$ form a C-frame. Then there exists ε_0, $c' > 0$ such that for each $\varepsilon \leq \varepsilon_0$ the ε-neighborhood of Σ in W_1 has measure larger than $\varepsilon c'$.*[34]

Proof. In the following we introduce a Riemannian metric in which $E^s(x)$ and $E^u(x)$ are orthogonal and $\mathrm{Exp}[\cdot]$ sends a neighborhood of 0, in $E^u(x)$, in a neighborhood of x, in $W^u(x)$ and a neighborhood of 0, in $E^s(x)$, in a neighborhood of x, in $W^s(x)$. From now on $\mathrm{Exp}[\cdot]$ will always be referred to such a metric.

Consider $z \in E^s(x)$ such that $\mathrm{Exp}[z] = y$. Let $v \in E^u(x)$ and $\tau(z, v)$ be the distance, along the flow direction, between $W^u(y)$ and $W^s(\mathrm{Exp}[v])$. By construction $\tau(z, 0) = 0$. Moreover, by using the contact structure one can obtain the formula [KB]

$$\tau(z, v) = d\omega(z, v) - f(z, v)$$

where f is a continuous function such that $|f(z,v)| = o(\|z\|^2 + \|v\|^2)$. Since in the following z is fixed we will use $f(v)$ to designate $f(z,v)$. We want to study $\Sigma_* = \{v \in E^u(x) \mid \tau(z, v) = 0\}$. The set Σ_* is then uniquely determined by the equation

$$d\omega(z, v) = f(v). \tag{III.1}$$

[33] Here "α" is the Hölder regularity of the distributions $E^u(x)$, $E^s(x)$ (see section one for more details).

[34] Of course, we mean the measure μ restricted to W_1. In the case $d = 1$, Chernov's case, Σ would consist of only one point and the Lemma would be trivial.

It is well known that there exists a $d-1$ dimensional linear space $\mathbb{V} \subset E^u(x)$ such that $d\omega(z, w) = 0$ for all $w \in \mathbb{V}$. Let $\bar{w} \in E^u(x)$ be perpendicular to $\mathbb{V}$ and such that $d\omega(z, \bar{w}) = \delta_0^2$. Setting $c_1 = \frac{d\omega(z, \bar{w})}{\|z\|\|\bar{w}\|}$, we can choose δ_0 so small that $|f(v)| \leq \frac{c_1}{100}(\|z\|^2 + \|v\|^2)$.

For each $\xi \in E^u(x)$ we will use the decomposition $\xi = a\bar{w} + \eta$ with $\eta \in \mathbb{V}$. Then (III.1) reads

$$\delta_0^2 a = f(a\bar{w} + \eta). \tag{III.2}$$

We will show that (III.2) has at least one solution for each $\eta \in B_{\frac{\delta_0}{6}}(0)$. To see this define $g_\eta(a) = \delta_0^{-2} f(a\bar{w} + \eta) - a$.

Sub-Lemma III.4. *For each $\eta \in \mathbb{V}$ with $\|\eta\| \leq \|z\|$, there exists $\bar{a} \in \mathbb{R}$, $|\bar{a}| \leq \frac{c_1}{36}$, such that*

$$g_\eta(\bar{a}) = 0.$$

Proof. There are three possibilities to consider: $g_\eta(0) = 0$, $g_\eta(0) > 0$, $g_\eta(0) < 0$. If the first possibility occurs, then the Lemma is trivially true. The other two cases can be treated in complete analogy, we will consider explicitly $g_\eta(0) > 0$. By construction $\|z\| \in [\frac{\delta_0}{6}, \frac{\delta_0}{3}]$, so for each $|a| \leq \frac{c_1}{36}$

$$|a|\|\bar{w}\| \leq \frac{\delta_0}{6} \leq \|z\|.$$

Hence $\mathrm{Exp}[a\bar{w} + \eta] \in W_1$ and

$$\begin{aligned}
|f(a\bar{w} + \eta)| &\leq \frac{c_1}{100}\left[(|a|\|\bar{w}\| + \|\eta\| - \|z\|)^2 + 2\|z\|(|a|\|\bar{w}\| + \|\eta\|)\right] \\
&\leq \frac{c_1}{100}(3\|z\|^2 + \frac{2|a|\delta_0^2}{c_1}) \leq (\frac{c_1}{300} + \frac{|a|}{50})\delta_0^2.
\end{aligned}$$

Therefore, $g_\eta(\frac{c_1}{36}) < 0$. Since g_η is a continuous function the lemma is proven. □

Next, let $\theta(\eta)$ be the smallest solution provided by the above theorem. Clearly the set

$$\Sigma = \{\mathrm{Exp}[\theta(\eta)\bar{w} + \eta] \mid \eta \in \mathbb{V},\ \|\eta\| \leq \frac{\delta_0}{6}\}$$

is made up of points whose stable manifold intersect the unstable manifold of y; moreover

$$d(x, \mathrm{Exp}[\theta(\eta)\bar{w} + \eta]) \leq \|\theta(\eta)\bar{w} + \eta\| \leq \delta_0.$$

We are left with the task of estimating the measure of and ε-neighborhood of Σ. By the exponential map we can reduce it to an estimate in $\mathcal{T}_x W^u(x)$. Namely there exists c_0, c_1 such that (by $m(\cdot)$ we mean the Lebesgue measure in $\mathbb{R}^d$)

$$\begin{aligned}
\mu_u(\Sigma_\varepsilon) &\geq c_0 m(\{v \in E^u(x) \mid \exists \eta \in \mathbb{V} : \|\theta(\eta)\bar{w} + \eta - v\| \leq c_1\varepsilon\}) \\
&\geq c_0 m(\{a\bar{w} + \eta \in E^u(x) \mid v \in \mathbb{V},\ \|\eta\| \leq \delta_0,\ |\theta(\eta) - a| \leq c_1\varepsilon\}) \\
&= c_2 \delta^{d-1}\varepsilon.
\end{aligned}$$

□

There exits another way to construct a multitude of C-frames near a given one. The next theorem is due to Chernov in the two dimensional case and very little changes in higher dimensions.

Theorem III.5. *Consider a C-frame of size δ and two pieces of unstable manifold $\widetilde{W}_1$ and $\widetilde{W}_2$ that intersects C_1 and C_2, respectively. Then there exists a piece of stable manifold $\widetilde{W}_3$ such that $(\widetilde{W}_1, \widetilde{W}_2, \widetilde{W}_3)$ form a C-frame of size larger than $\delta/2$ and smaller than 2δ.*

Proof. Consider $\hat{x} = W^{0s}(x) \cap \widetilde{W}_1$ and $\hat{y} = W^{0s}(x) \cap \widetilde{W}_2$. Call Ψ the projection, along the stable manifold, of $\widetilde{W}_1$ into $W^{0u}(\hat{y})$. By construction, the distance between $\Psi(\hat{x})$ and $\hat{y}$ it is $\mathcal{O}(\delta^3)$ (it follows from the Hölder continuity of the stable foliation). Due to the possible lack of regularity of the stable foliation $\Psi(\widetilde{W}_1)$ may not be a regular manifold, yet (in analogy with lemma III.3) we can construct an Hölder continuous chart (namely $\Psi \circ \operatorname{Exp}[] : E^u(\hat{x}) \to \Psi(\widetilde{W}_1)$), also $\Psi(\widetilde{W}_1)$ is clearly a codimension one set in $W^{0u}(\hat{y})$ which separate it into two connected components. Lemma III.3 implies that $\Psi(\widetilde{W}_1)$ moves in the flow direction by at least $\mathcal{O}(\delta^2)$ so it will necessarily intersect $W^u(\hat{y})$ at some point z. Whereby, we have the C-frame $(\widetilde{W}_1, \widetilde{W}_2, \widetilde{W}_3)$, and its size is easily checked. $\square$

Appendix IV (Product Sets).

The proof of lemma 3.2 is the content of this appendix.

First remember that, by construction, $d(\tilde{z}, W^u(\xi_{ij}(y)) \leq \frac{\varepsilon}{8}\delta_0$. If we define $z_* = \{\phi_t \tilde{z}\}_{t \in [-\varepsilon\delta_0, \varepsilon\delta_0]} \cap W^u(\xi_{ij}(y))$, then $B^{\varepsilon}_{\frac{\delta_0}{8}}(z_*) \subset B^{\varepsilon}_{\frac{\delta_0}{4}}(\tilde{z})$, and $z_* \in W_j(y) \subset B_j(y)$, $d_\varepsilon(z_*, z_1) \leq \frac{3}{8}\delta_0$.

We will obtain the first inequality by using the product structure of $B_i(y)$. To be more precise

$$\mu(B^{\varepsilon}_{\frac{\delta_0}{4}}(\tilde{z}) \cap B_j(y)) \geq \mu(B^{\varepsilon}_{\frac{\delta_0}{8}}(z_*) \cap B_j(y)) = \mu(\phi_{n_\varepsilon}(B_j(y) \cap B_{\frac{\delta_0}{8}}(z_*)) \cap B_{\frac{\delta_0\varepsilon}{8}}(\phi_{n_\varepsilon} z_*)).$$

In analogy with the approach used in appendix II we can consider an unstable like foliation $\mathcal{F}$ adapted to $B_{\delta_0}(y)$ (see appendix II for definitions). Let $\hat{\mathcal{F}} = \phi_t \mathcal{F}$ and $\hat{\mathcal{F}}_* = \{p \cap B_{\delta_0/8}(z_*)\}_{p \in \hat{\mathcal{F}}}$. We can then extend $\hat{\mathcal{F}}_*$ to a smooth foliation $\mathcal{F}_0$ such that $B_{\frac{\delta_0}{8}}(z_*)$ is measurable in the associated σ-algebra; finally, we can consider the foliation $\mathcal{F}_1 = \phi_{n_\varepsilon} \mathcal{F}_0$.

Since $\phi_{n_\varepsilon} B_j(y)$ is foliated by $\mathcal{F}_1$, which is almost the weak-unstable foliation, it follows that there exists c_7 such that for each $p \in \mathcal{F}_1$,

$$c_7^{-1} \geq \frac{\mu(B_{\delta_0\varepsilon/8}(\phi_{n_\varepsilon} z_*) \mid \mathcal{F}_1)(p)}{\mu(B_{\delta_0\varepsilon/8}(\phi_{n_\varepsilon} z_*) \mid \mathcal{F}_1)(p_*)} \geq c_7$$

where $p_* \in \mathcal{F}_1$ and $\phi_{n_\varepsilon} z_* \in p_*$.

In addition, by the usual distortion estimates, there exists c_8 such that, for all $\hat{p} \in \hat{\mathcal{F}}_*$ and $\hat{p}_* \in \phi_{-n_\varepsilon} p_*$,

$$c_8 \leq \frac{\mu(B^{\varepsilon}_{\frac{\delta_0}{8}}(z_1) \mid \mathcal{F}_0)(\hat{p})}{\mu(B^{\varepsilon}_{\frac{\delta_0}{8}}(z_1) \mid \mathcal{F}_0)(\hat{p}_*)} \leq c_8^{-1}.$$

Thus,

$$\mu(B^{\varepsilon}_{\frac{\delta_0}{8}}(z_*)) = \mu(\chi_{B_{\frac{\delta_0}{8}}(z_*)}\mu(B^{\varepsilon}_{\frac{\delta_0}{8}}(z_*)|\mathcal{F}_0)) \leq c_8\mu(B^{\varepsilon}_{\frac{\delta_0}{8}}(z_*)|\mathcal{F}_0)(p_*)\mu(B_{\frac{\delta_0}{8}}(z_*)).$$

Next, let $\hat{B} = \bigcup_{p\in\hat{\mathcal{F}}_*} p \supset B_j(y) \cap B_{\frac{\delta_0}{8}}(z_*)$. By construction $\hat{B}$ is $\mathcal{F}_0$ measurable. In the same way as before

$$\mu(B_j(y) \cap B_{\frac{\delta_0}{8}}(z_*)) = \mu(\chi_{\hat{B}}\mu(B_j(y)|\mathcal{F}_0)) \leq c_9\mu(\hat{B})\mu(B_j(y)|\mathcal{F}_0)(p_*) \leq c_{10}\varepsilon\mu(\hat{B}),$$

since $B_j(y) \cap p_*$ consists of a strip of width $\varepsilon\delta_0$ (in the flow direction) in p_* (which is approximately a $d+1$ dimensional disk of size δ_0) and the conditional measure $\mu(\cdot|\mathcal{F}_0)$ is uniformly equivalent to the restriction, on the fibers, of the Riemannian measure (the proof is standard, see appendix II for more details).

The last fact to take into account is that, for each $p \in \mathcal{F}_0$, $p \subset \hat{B}$, holds $p\cap B_j(y) \supset p \cap B^{\varepsilon}_{\frac{\delta_0}{8}}(z_*)$.

Putting together all the above estimates yields

$$\begin{aligned}
\mu(B_j(y) \cap B^{\varepsilon}_{\frac{\delta_0}{8}}(z_*)) &\geq \mu(\chi_{\hat{B}}\mu(B^{\varepsilon}_{\frac{\delta_0}{8}}(z_*)|\mathcal{F}_0)) \geq c_8\mu(\hat{B})\mu(B^{\varepsilon}_{\frac{\delta_0}{8}}(z_*)|\mathcal{F}_0)(p_*) \\
&\geq c_8^2\mu(\hat{B})\mu(B_{\frac{\delta_0}{8}}(z_*))^{-1}\mu(B^{\varepsilon}_{\frac{\delta_0}{8}}(z_*)) \\
&\geq c_{11}\delta_0^{-2d-1}\varepsilon^{-1}\mu(B_j(y) \cap B_{\frac{\delta_0}{8}}(z_*))\mu(B^{\varepsilon}_{\frac{\delta_0}{8}}(z_*)).
\end{aligned}$$

The first inequality of the Lemma follows by using the foliation $\hat{\mathcal{F}}$ and distortion estimates.[35]

The second inequality is obtained in a similar way. Construct the foliations $\hat{\mathcal{F}}$, $\hat{\mathcal{F}}_*$, $\mathcal{F}_0$ as before, but with respect to x instead than to y. Then notice that for each $p \in \hat{\mathcal{F}}_*$

$$\mu(\widetilde{\Sigma}^{\varepsilon}_{ij} \mid \hat{\mathcal{F}}_*)(p) \geq c_{13}\varepsilon^{\frac{1}{\alpha}+1}$$

due to the results of appendix III. In addition, $\hat{B}(x)$ is defined as $\hat{B}(y)$,

$$\mu(\widetilde{\Sigma}^{\varepsilon}_{ij} \cap B_i(x)) \geq \mu(\chi_{\hat{B}(x)}\mu(\widetilde{\Sigma}^{\varepsilon}_{ij} \mid \mathcal{F}_0)) \geq c_{14}\varepsilon^{\frac{1}{\alpha}+1}\mu(\hat{B}(x)) \geq c_{15}\varepsilon^{\frac{1}{\alpha}}\mu(B_i(x)).$$

[35] Just choose $m \in \mathbb{N}$ such that $\phi_{-m}B_j(y)$ is contained in a ball of radius δ_0 (this can be achieved by a fixed m, independent of y and ε). It is then easy to compare the measure of $\phi_{-m}B_j(y)$ with $\phi_{-m}(B_j(y)\cap B_{\frac{\delta_0}{8}}(z_*))$ by using the foliation $\phi_{-m}\hat{\mathcal{F}}$. It follows that $\mu(B_j(y)\cap B_{\frac{\delta_0}{8}}(z_*))$ is proportional to $\mu(B_j(y))$. Moreover, by Lemma I.1 $\mu(B^{\varepsilon}_{\frac{\delta_0}{8}}(z_*)) \geq e^{-c_0\frac{\delta_0}{4}}\mu(B^{\varepsilon}_{\frac{\delta_0}{8}}(z_1)) \geq c_{12}\mu(B^{\varepsilon}_{\delta_0}(z_1))$.

Appendix V (A change of coordinates).

In this appendix we study the change of coordinates used in section four to prove lemma 4.3.

Given a couple of points $x,\, y \in \mathtt{M}$, sufficiently close, it exists only one point that belongs to $W^{0s}(x)\cap W^u(y)$, let us designate it by $[x,\, y]$. If we consider the set $\Omega^*_\delta = \{(x,\, y)\in \mathtt{M}^2 \mid d_\varepsilon(x,\, y)\leq \delta\}$, then, for δ sufficiently small, we can define the function $\Psi : \Omega^*_\delta \to \mathtt{M}^2$ by

$$\Psi(x,\, y) = ([x,\, y],\, [y,\, x]).$$

The above function is used as a change of coordinates in section four. The properties of Ψ needed in the paper are summarized by the following.

Lemma V.1. *There exists a constant c such that*[36]

$$\sup_{\Omega^*_\delta} |1 - \frac{d\Psi_*(\mu_2)}{d\mu_2}(x,\, y)| \leq c\varepsilon^{3\alpha};$$

and, for each $x,\, y\in \mathtt{M}$, $d_\varepsilon(x,\, y)\in [\frac{\delta}{2},\, 2\delta]$,

$$e^{-c\varepsilon d_\varepsilon(x,\, y)} \leq \frac{d_\varepsilon(x,\, y)}{d_\varepsilon([x,\, y],\, [y,\, x])} \leq e^{c\varepsilon d_\varepsilon(x,\, y)}.$$

Proof. To study the first inequality it is convenient to introduce a new system of coordinates. Namely, given $\xi\in\mathtt{M}$ we can define Θ_ξ from $\mathbb{R}^{2d+1}$ to a neighborhood of ξ, by introducing local coordinates ζ on $W^{0s}(\xi)$ and η on $W^u(\xi)$ and then defining $\Theta_\xi(\zeta,\, \eta) = W^u(\zeta)\cap W^{0s}(\eta)$. By the general theory of Anosov systems [Ma], [HK] follows that Θ_ξ it is Hölder continuous and that

$$\frac{d(\Theta_\xi)_*\nu}{d\mu} = \rho_\xi,$$

where ν is Lebesgue measure $\mathbb{R}^{2d+1}$, and ρ_ξ is Hölder continuous as well.

For each $x\in\mathtt{M}$ we can use such coordinates. Let $\Theta_2(\zeta_1,\, \zeta_2) := (\Theta_x(\zeta_1),\, \Theta_x(\zeta_2))$; $\rho_2(\zeta_1,\, \zeta_2) := \rho_x(\zeta_1)\rho_x(\zeta_2)$; then, for each $f\in C^{(0)}(\mathtt{M}^2)$, supp $f\subset\Omega^*_\delta$

$$\Psi_*\mu_2(f) = \mu_2(f\circ\Psi) = \Theta_{2*}\nu_2(\rho_2^{-1} f\circ\Psi).$$

Since, for each $\zeta_1,\, \zeta_2\in\Omega^*_\delta$,

$$\begin{aligned}\frac{\rho_2(\Psi(\zeta_1,\, \zeta_2))}{\rho_2(\zeta_1,\, \zeta_2)} &= \frac{\rho_x([\zeta_1,\, \zeta_2])\rho_x([\zeta_2,\, \zeta_1])}{\rho_x(\zeta_1)\rho_x(\zeta_2)}\\ &= 1 + \mathcal{O}(d(\zeta_1,\, [\zeta_2,\, \zeta_1])^\alpha) + d(\zeta_2,\, [\zeta_1,\, \zeta_2])^\alpha)) = 1 + \mathcal{O}(\varepsilon^{3\alpha}\delta^\alpha),\end{aligned}$$

where we have used the contraction in the unstable direction to get 3α.

[36] By $\Psi_*(\mu)$ we mean the measure defined by $\Psi_*(\mu)(f) := \mu(f\circ\Psi)$ for each $f\in\mathcal{C}^{(0)}(\Omega^*_\delta)$. Here, again, $\mu_2 = \mu\times\mu$.

Therefore,[37]

$$\Psi_*\mu_2(f) = \nu_2((\rho_2^{-1}f)\circ\Theta_2\circ(\Theta_2^{-1}\circ\Psi\circ\Theta_2)) + \mathcal{O}(\varepsilon^{3\alpha}\|f\|_\infty)$$
$$= \nu_2((\rho_2^{-1}f)\circ\Theta_2) + \mathcal{O}(\varepsilon^{3\alpha}\|f\|_\infty) = \mu_2(f) + \mathcal{O}(\varepsilon^{3\alpha}\|f\|_\infty).$$

Let us consider now the second inequality: here it is convenient to divide the argument in different cases.

If $d(x, y) \geq \varepsilon^{-1}d(\phi_{n_\varepsilon}x, \phi_{n_\varepsilon}y)$, then

$$d(x, y) \leq d(x, [y, x]) + d([y, x], [x, y]) + d([x, y], y)$$
$$\leq c\varepsilon^2\delta + d([y, x], [x, y]).$$

In an analogous way we get

$$|d(x, y) - d([y, x], [x, y])| \leq c\varepsilon^2\delta.$$

If $d(x, y) \leq \varepsilon^{-1}d(\phi_{n_\varepsilon}x, \phi_{n_\varepsilon}y)$, then

$$|d(\phi_{n_\varepsilon}x, \phi_{n_\varepsilon}y) - d(\phi_{n_\varepsilon}[y, x], \phi_{n_\varepsilon}[x, y])| \leq c\varepsilon^2\delta.$$

A fast look at the different cases yields the wanted result. □

Finally we have.

Lemma V.2.

$$\mu_2(\Psi(\Omega^*_{\delta_0})\Delta\Omega^*_{\delta_0}) \leq c\varepsilon\mu_2(\Omega^*_{\delta_0}).$$

Proof. Due to Lemma V.1

$$\Psi(\Omega^*_{\delta_0})\Delta\Omega^*_{\delta_0} \subset \{(\xi, \eta) \in \mathtt{M}^2 \mid d_\varepsilon(\xi, \eta) \in [\delta_0(1 - c\varepsilon), \delta_0(1 + c\varepsilon)]\}.$$

In addition,

$$\mu_2(\Omega^*_{\delta_0}) = \int_{\mathtt{M}} d\mu(x)\mu(B^\varepsilon_{\delta_0}(x))$$

and

$$\mu_2(\Psi(\Omega^*_{\delta_0})\Delta\Omega^*_{\delta_0}) \leq \int_{\mathtt{M}} d\mu(x)\mu(B^\varepsilon_{\delta_0}(x)\backslash B^\varepsilon_{\delta_0(1-c\varepsilon)}(x)).$$

Accordingly the result follows in the same way discussed in Lemma I.1. □

References

[An] Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Steklov Inst. Math. **90** (1967).

[Arc] Archimede, *Le Méthode*, Archimède Tome III, Texte étabili et traduit par C.Mugler, Colletion des Université de France, Société D'Édition "Les Belles Lettres", Paris, 1971.

[Arn] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, Heidelberg, Berlin, 1978.

[37] Note that $\Theta_2^{-1}\circ\Psi\circ\Theta_2$ is linear and its Jacobian is 1.

[AS] D.V. Anosov, Ya.G. Sinai, *Some smooth ergodic systems*, Russian Math. Surveys **22:5** (1967), 103-167.

[Bo] R.Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics **470**, Springer-Verlag, New York, Heidelberg, Berlin, 1978.

[BY] M.Benedicks, L.-S. Young, *Decay of Correlations for Certain Henon Maps*, preprint.

[Ch] N. Chernov, *Markov approximations and decay of correlations for Anosov flows*, preprint.

[CEG] P.Collet, H. Epstein, G. Gallavotti, *Perturbations of geodesic flows on surfaces of constant negative curvature*, Communications in Mathematical Physics **95** (1984), 61–112.

[Do1] D. Dolgopiat, *Decay of correlations for Anosov flows*, preprint.

[Do2] D. Dolgopiat, *Prevalence of rapid mixing in hyperbolic flows*, preprint.

[KB] A.Katok in collaboration with K. Burnes, *Infinitesimal Lyapunov functions, invariant cone families and stochastic properties of smooth dynamical systems*, Ergodic Theory and Dynamical Systems **14** (1994), 757–758.

[KH] A.Katok, B.Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications **54**, edited by G.-C.Rota, Cambridge University Press, New York, 1995.

[KS] A.Katok, J.M. Strelcyn with collaboration of F. Ledrappier and F. Przytycki, *Invariant Manifolds, Entropy and Billiards, Smooth Maps with Singularities*, Lectures Notes in Mathematics **1222** (1986).

[Ham] U.Hamenstädt, *Regularity at infinity of compact negatively curved manifolds*, Ergodic Theory & Dynamical Systems **14** (1994), 493–514.

[Has] B. Hasselblatt, *Regularity of Anosov Splitting and Horospheric Foliations*, Ergodic Theory & Dynamical Systems **14** (1994), 645–666.

[HP] M.Hirsh, C.Pugh, *Smoothness of Horocycle Foliations*, Journal of Differential Geometry **10** (1975), 225–238.

[Li1] C. Liverani, *Decay of correlations*, Annals of Mathematics **142** (1995), 239-301.

[Li2] C. Liverani, *Central Limit Theorem for Deterministic Systems*, International Congress on Dynamical Systems, Montevideo 95, Proceedings, Research Notes in Mathematics series, Pitman, 1996.

[LW] C. Liverani, M. P. Wojtkowski, *Ergodicity of Hamiltonian Systems*, Dynamics Reported, vol. 4, 1995, pp. 130-202.

[Ma] R. Mañe, *Ergodic Theory and Differentiable Dynamics*, A Series of Modern Surveys in Mathematics, Springer–Verlag, Berlin Heidelberg New York, 1978.

[Mo] C. Moore, *Exponential decay of correlation coefficients for geodesic flows*, Group Representation Ergodic Theory, Operator Algebra and Mathematical Physics, Springer, Berlin, 1987.

[Pe] Y.Pesin, *Characteristic exponents and smooth ergodic theory*, Russian Mathematical Surveys **32 (n.4)** (1977), 55–114.

[Po1] M.Pollicott, *Exponential mixing for geodesic flow on hyperbolic three manifold*, Journal of Statistical Physics **67** (1992), 667-673.

[Po2] M.Pollicott, *Lectures on ergodic theory and Pesin theory on compact manifolds*, London Mathematical Society, Lecture Note Series **180**, Cambridge University Press, 1993.

[Ra] M. Ratner, *The rate of mixing for geodesic and horocycle flows*, Ergodic Theory and Dynamical Systems **7** (1987), 267–288.

[SS] J.Schmeling, R.Siegmund–Schultze, *Hölder Continuity of the Holonomy Map for Hyperbolic Basic Sets I.*, Ergodic Theory and Related Topics III, Proceedings International Conference (Güstrow, Germany, 1990), Springer Lecture notes in Mathematics, eds. U.Krengel, K.Richter and V.Warstat, vol. 1514, Springer, Berlin, 1992, pp. 174–191.

[Yo] Lai-Sang Young, *Statistical Properties of Dynamical Systems with Some Hyperbolicity*, preprint.

LIVERANI CARLANGELO, MATHEMATICS DEPARTMENT, UNIVERSITY OF ROME II, TOR VERGATA, 00133 ROME, ITALY.

E-mail address: liverani@mat.utovrm.it