Nambu mechanics, n-ary operations and their quantization *

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April 27, 1997

Abstract. We start with an overview of the "generalized Hamiltonian dynamics" introduced in 1973 by Y. Nambu, its motivations, mathematical background and subsequent developments – all of it on the classical level. This includes the notion (not present in Nambu's work) of a generalization of the Jacobi identity called Fundamental Identity. We then briefly describe the difficulties encountered in the quantization of such n-ary structures, explain their reason and present the recently obtained solution combining deformation quantization with a "second quantization" type of approach on $\mathbb{R}^n$. The solution is called "Zariski quantization" because it is based on the factorization of (real) polynomials into irreducibles. Since we want to quantize composition laws of the determinant (Jacobian) type and need a Leibniz rule, we need to take care also of derivatives and this requires going one step further (Taylor developments of polynomials over polynomials). We also discuss a (closer to the root, "first quantized") approach in various circumstances, especially in the case of covariant star products (exemplified by the case of $\mathfrak{su}(2)$). Finally we address the question of equivalence and triviality of such deformation quantizations of a new type (the deformations of algebras are more general than those considered by Gerstenhaber).

1 Introduction

In 1973, after having kept this very original idea in his files for many years since he did not arrive at what he could have considered a really satisfactory paper, Nambu decided to publish [26] the result "as is". The underlying idea of this new formalism was that in statistical mechanics the basic result is Liouville theorem, which follows from but does not require Hamiltonian dynamics. In geometrical terms, symplectic structures are unimodular but the latter are more general. The lowest dimensional example to start with is $\mathbb{R}^3$, with triplets of

*The work reported here benefited from scientific contacts supported by the EU Network Contract ERBCHRXCT940701.

dynamical variables and the Poisson bracket of two functions replaced by a "triple bracket", the Jacobian of three functions.

The very mention of the notion of triplets (even more so in those days) is reminiscent of quarks for which satisfactory statistics explaining e.g. confinement is yet to be found – and this was in the back of Nambu's mind. But in order to even think about such a connection one needs to solve the problem of quantization and Nambu was not able to get very far in solving this question. In fact quantization proved to be a very difficult question and a solution was given only last year [12], at least for polynomials on $\mathbb{R}^n$, using an elaborate construction based on arithmetic properties of polynomials and methods of second quantization.

Nambu's paper was followed by a number of papers [1] [5] [25] which essentially "chilled" the idea from the physical point of view by showing that the "classical part" of it could be cast into the framework of classical mechanics with Dirac constraints [9], for which there is a usual quantization scheme. The latter is more or less what Nambu got (and was not satisfied with). Though perfectly correct, these papers had for unwanted result that the whole formalism remained in limbo for another twenty years (and we are in part responsible for that).

On the mathematical and mathematical physics side people had recurrently been interested in Jordan algebras, which Nambu had also considered in his abovementioned attempts to quantization and (justly, it turns out) rejected: nonassociativity is not the solution. More recently (re)appeared a much more general framework, operads [21] [24] [27]. The solution we shall present here is affiliated with that framework, but we shall not develop here this point which would take us too far.

We had never quite forgotten Nambu's interesting formalism and in 1992, independently, the underlying structure of n-ary brackets was studied and the analogue of the Jacobi identity for Poisson brackets (called *Fundamental Identity*, in short FI) discovered [14] [32]. That identity, which somehow Nambu (and others) had missed in the 70's, together with an explicit expression of the Leibniz rule, was the basis of the modern developments. It is truly fundamental in many respects. In particular it selects the really interesting generalization of Poisson brackets to n-tuples from other a priori possible generalizations which give rise to essentially trivial mathematical developments (which is why we shall not quote any paper in those directions). From that identity emerged the notion of Nambu-Poisson manifolds [3] [4] [32], a manifold endowed with a n-bracket satisfying the obvious properties of skew-symmetry and Leibniz rule and the less obvious FI. Locally such manifolds are [18] foliated by manifolds on which the structure is given by a functional determinant.

In this paper we shall first present (Section 2) the earlier developments, until and including the Fundamental Identity. Then (Section 3) we shall describe in great lines the abovementioned solution to quantization of such n-ary operations based on a subtle adaptation of deformation quantization [2] [16] [33]. The basic idea is to replace in the computation of the determinant the ordinary product of (real) polynomials by a symmetrized Moyal star product of their decomposition into irreducibles (what we call their Zariski product, because of the decomposition into irreducible polynomials) – but the full solution is not so simple because of linearity requirements and the fact that the totally symmetrized Moyal product of linear functions is the usual product. In particular (which is consistent with the Barr triviality of Harrison cohomology [20] for algebras of polynomials) we get algebra deformations which are no more of the type studied by Gerstenhaber which we call here "DrG-deformations" (as a tribute to the fundamental contribution of "Dr. G." [19]): for the new type of deformations we have only $\mathbb{R}$-linearity instead of $\mathbb{R}[[\nu]]$–linearity (ν being the deformation parameter), and it is a *generalized deformation* in the sense that the limit for $\nu = 0$ of the deformed algebra is the original algebra. The algebra product which we "deform" is here, in the spirit of second quantization, that of an algebra of "fields", the algebra of the semi-group generated by irreducible polynomials (polynomials over the set of normalized irreducible polynomials). Moreover, since we need to quantize composition laws of the functional determinant (Jacobian) type and need a Leibniz rule, we must also take care of derivatives and this requires going one step further (Taylor developments of polynomials over polynomials).

When we deal with covariant star products for which the totally symmetrized star product of elementary factors is no more trivial, some variants can be developed, of the "first quantized" type; we present them here in Section 4.1, essentially in the case of $\mathfrak{su}(2)$ [11]. A similar procedure can be introduced also on $\mathbb{R}^m$ when the ordering used in the corresponding "Zariski" product (e.g. of standard type) is not the same as the ordering for the star product. A second part of the last Section is devoted to some remarks on the important question of equivalence of such generalized deformations (which is not straightforward) and to the related (a little simpler) question of triviality, already tackled in [11] where the generalized deformations introduced are not trivial relatively to the more natural notion of "strong triviality" (though they are trivial for the less demanding "weak triviality"). The physical aspect of triviality requires to address also the problem of spectrality of observables, by which we conclude that section.

2 Nambu mechanics and its foundations

2.1 Nambu's original paper (classical part)

Having in mind the abovementioned motivation, Nambu [26] started with the following "Hamilton equations" on $\mathbb{R}^3$ of the form:

$$\frac{dr}{dt} = \nabla g(r) \wedge \nabla h(r), \quad r = (x, y, z) \in \mathbb{R}^3, \tag{1}$$

where x, y, z are the dynamical variables and g, h are two functions of r. Liouville theorem follows directly from the identity $\nabla \cdot (\nabla g(r) \wedge \nabla h(r)) = 0$, which tells us that the velocity field in Eq. (1) is divergenceless. From (1) follows that the evolution of a function f on $\mathbb{R}^3$ is given by:

$$\frac{df}{dt} = \frac{\partial(f, g, h)}{\partial(x, y, z)}, \tag{2}$$

where the right-hand side is the Jacobian of the mapping $\mathbb{R}^3 \to \mathbb{R}^3$ given by $(x,y,z) \mapsto (f,g,h)$.

For example, in this "baby model for integrable systems", Euler equations for the angular momentum of a rigid body are obtained when the dynamical variables are taken to be the components of the angular momentum vector $L = (L_x, L_y, L_z)$, and g and h are, respectively, the total kinetic energy $\frac{L_x^2}{2I_x} + \frac{L_y^2}{2I_y} + \frac{L_z^2}{2I_z}$ and the square of the angular momentum $L_x^2 + L_y^2 + L_z^2$. Other examples can be given, in particular Nahm's equations for static $\mathfrak{su}(2)$ monopoles, $\dot{x}_i = x_j x_k$ $(i,j,k = 1,2,3)$ in $\mathfrak{su}(2)^* \sim \mathbb{R}^3$, with $h = x_1^2 - x_2^2$, $g = x_1^2 - x_3^2$, etc.

In this framework the analogues of canonical transformations are mappings $(x,y,z) \mapsto (x',y',z')$ with $\frac{\partial(x',y',z')}{\partial(x,y,z)} = 1$. The linear ones generate $SL(3,\mathbb{R})$. The Hamilton equations generate infinitesimal canonical transformations, and two sets of "Hamiltonians" h, g and h', g' generate the same transformation if they are related by a "gauge transformation" $\frac{\partial(h',g')}{\partial(h,g)} = 1$. Here the principle of least action, which states that the classical trajectory C_1 is an extremal of the action functional $A(C_1) = \int_{C_1}(pdq - Hdt)$, is replaced by a similar one [32] with a 2-dimensional cycle C_2 and "action functional" $A(C_2) = \int_{C_2}(xdy \wedge dz - hdg \wedge dt)$ (which bears some flavor of strings and some similitude with the cyclic cocycles of Connes [6]).

Expression (2) was easily generalized to n functions $f_i, i = 1, \ldots, n$. As was already noticed by Nambu (and in apparent contradistinction with the definition of Poisson brackets), the correct way to proceed (see [26], [1]) is not by a direct sum of m triples, where the linear canonical transformations are $SL(3,\mathbb{R})^m$ and the pairs of "Hamiltonians" (h_i, g_i), $i = 1, \ldots, m$, are no longer constants of motion. One should instead introduce a n-tuple of functions on $\mathbb{R}^n$ with composition law given by their Jacobian, with linear canonical transformations $SL(n,\mathbb{R})$ and a corresponding $(n-1)$-form which is the analogue of the Poincaré-Cartan integral invariant. The Jacobian has to be interpreted as a generalized Poisson bracket: it is skew-symmetric with respect to the f_i's and a derivation of the algebra of smooth functions on $\mathbb{R}^n$, i.e., the Leibniz rule is verified in each argument (e.g. $\{f_1 f_2, f_3, \ldots, f_{n+1}\} = f_1\{f_2, \ldots, f_{n+1}\} + \{f_1, f_3, \ldots, f_{n+1}\} f_2$, etc.). Hence there is a complete analogy with the Poisson bracket formulation of Hamilton equations (including the fact that the components of the $(n-1)$-tuple of "Hamiltonians" $(f_2, \ldots f_n)$ are constants of motion) except, at first sight, for the equivalent of the Jacobi identity which seems to be lacking.

2.2 Nambu's mechanics as constrained mechanics

As we said in the Introduction, it turns out [1] that Nambu mechanics on $\mathbb{R}^n$ can be cast into the framework of classical mechanics with Dirac constraints [9] on $\mathbb{R}^{2n}$. Take $n = 3$ to fix ideas. Motivated by (1), which can be obtained using the intuitive Hamiltonian $\mathcal{H} = p \cdot (\nabla g(q) \times \nabla h(q))$, $q = r = (x,y,z) \in \mathbb{R}^3$, we consider the "Nambu Lagrangian" $L_N = h.\sum_{i=1}^3 \dot{q}_i \frac{\partial g}{\partial q_i}$ for which the Euler-Lagrange equations can be written as $\frac{dg}{dt}\nabla h = \frac{dh}{dt}\nabla g$. Then (in an open set $\Omega \in \mathbb{R}^3$ where $\nabla h \times \nabla g$ is nonzero, by vector multiplication by ∇h and ∇g) we see that h and g are constants of motion, as required.

But now in addition to the equation for $\dot{q}$ there is the "other half" for $\dot{p}$ and thus we need [1] to introduce three Dirac constraints $\phi_i(q,p) = p_i - h(q)\partial_i g(q)$, $i = 1,2,3$, where $\partial_i = \frac{\partial}{\partial q_i}$ (these are primary constraints which give a pair of second class constraints and one first class constraint; there are no secondary constraints). We get (with 0 for "Hamiltonian") $\dot{q}_i = u_i$ and $\dot{p}_i = \sum_{j=1}^{3} u_i \partial_i(h(\partial_j g))$ for some functions u_i to be determined. Expressing $\dot{\phi}_i = 0$ we get $\sum_{j=1}^{3} u_i(\partial_i h \partial_j g - \partial_j h \partial_i g) = 0$ hence

$$\dot{q}_i = u_i = v(t,q)[\nabla h \times \nabla g]_i \tag{3}$$

with an arbitrary function v and finally, with a position-dependent time rescaling $\tau = \int_0^t v(s, q(s,q_0))ds$ we obtain $\frac{dq}{d\tau} = \nabla h \times \nabla g$. Thus for a fixed v (supposed to be nonzero in Ω) we have a one-to-one correspondence between solutions of (1) and (3) which becomes an identity by a time rescaling. Since the choice of the time axis is arbitrary, the two formulations contain the same dynamical information. $\mathcal{H}_T = v(t,q)\ p \cdot (\nabla h \times \nabla g)$ is now the total Hamiltonian and it vanishes weakly since it is proportional to the first class constraint $\phi = \mathcal{H} = [\nabla h \times \nabla g] \cdot [p - h\nabla g]$ (the effect of the two second class constraints is to reduce the problem to some $\mathbb{R}^4 \subset \mathbb{R}^6$). We can now try and quantize along the lines of constrained mechanics as proposed by Dirac and recover the usual Heisenberg quantization already proposed by Nambu. But such an approach is not unambiguous and it is not a quantization of the triple bracket. As we shall see in Section 3, there is a much more fruitful and original approach which applies directly to all n-tuple brackets ($n \geq 2$).

2.3 Fundamental Identity and Nambu-Poisson manifolds

2.3.1 Nambu-Poisson brackets and the Fundamental Identity

In the usual Poisson formulation, the Jacobi identity is the infinitesimal form of the Poisson theorem which states that the bracket of two integrals of motion is also an integral of motion. If we want a similar theorem for Nambu mechanics there must be an infinitesimal form of it which will provide a generalization of the Jacobi identity. Denote by $\{f,g,h\}$ the Jacobian appearing in (2). Let $\phi_t : r \mapsto \phi_t(r)$ be the flow for (1). Then a generalization of the Poisson theorem would imply that ϕ_t is a "canonical transformation" for the generalized bracket:

$$\{f_1 \circ \phi_t, f_2 \circ \phi_t, f_3 \circ \phi_t\} = \{f_1, f_2, f_3\} \circ \phi_t.$$

Differentiation of this equality with respect to t yields the desired generalization of the Jacobi identity:

$$\begin{aligned}\{\{g,h,f_1\},f_2,f_3\} + \{f_1,\{g,h,f_2\},f_3\} + \{f_1,f_2,\{g,h,f_3\}\} \\ = \{g,h,\{f_1,f_2,f_3\}\}, \quad \forall g,h,f_1,f_2,f_3 \in C^\infty(\mathbb{R}^3).\end{aligned}$$

This identity and its generalization to $\mathbb{R}^n$, called Fundamental Identity (FI), was introduced by Flato, Frønsdal [14] and Takhtajan [32] as a consistency condition for Nambu Mechanics (this consistency condition was also formulated in [30]) and allows a generalized Poisson theorem: the generalized bracket of n

integrals of motion is an integral of motion. It turns out that the Jacobian on $\mathbb{R}^n$ satisfies the FI. We are thus lead to the following generalization, denoting by S_n the group of permutations of the set $\{1,\ldots,n\}$ and by $\epsilon(\sigma)$ the sign of the permutation $\sigma \in S_n$:

Definition 1 *A Nambu bracket of order n on a m-dimensional manifold M ($2 \leq n \leq m$) is defined by a n-linear map on $A = C^\infty(M)$ taking values in A:*

$$\{\cdot,\ldots,\cdot\}\colon A^n \to A,$$

such that the following relations are verified $\forall f_0,\ldots,f_{2n-1} \in A$:

a) Skew-symmetry

$$\{f_1,\ldots,f_n\} = \epsilon(\sigma)\{f_{\sigma_1},\ldots,f_{\sigma_n}\}, \quad \forall \sigma \in S_n; \tag{4}$$

b) Leibniz rule

$$\{f_0 f_1, f_2,\ldots,f_n\} = f_0\{f_1,f_2,\ldots,f_n\} + \{f_0,f_2,\ldots,f_n\}f_1; \tag{5}$$

c) Fundamental Identity

$$\begin{aligned}
&\{f_1,\ldots,f_{n-1},\{f_n,\ldots,f_{2n-1}\}\} \\
&\quad = \{\{f_1,\ldots,f_{n-1},f_n\},f_{n+1},\ldots,f_{2n-1}\} \\
&\qquad + \{f_n,\{f_1,\ldots,f_{n-1},f_{n+1}\},f_{n+2},\ldots,f_{2n-1}\} \\
&\qquad + \cdots + \{f_n,f_{n+1},\ldots,f_{2n-2},\{f_1,\ldots,f_{n-1},f_{2n-1}\}\}.
\end{aligned} \tag{6}$$

Properties a) and b) imply that there exists a n-vector field η on M such that:

$$\{f_1,\ldots,f_n\} = \eta(df_1,\ldots,df_n), \quad \forall f_1,\ldots,f_n \in A. \tag{7}$$

2.3.2 Nambu-Poisson manifolds and associated Nambu mechanics

Of course the FI imposes constraints on η, analyzed in [32]. A n-vector field on M is called a Nambu tensor if its associated Nambu bracket defined by (7) satisfies the FI. This brings us to:

Definition 2 *A Nambu-Poisson manifold (M,η) is a manifold M on which is defined a Nambu tensor η. Then M is said to be endowed with a Nambu-Poisson structure.*

The dynamics associated with a Nambu bracket on M is specified by $n-1$ Hamiltonians $H_1,\ldots,H_{n-1} \in A$ and the time evolution of $f \in A$ is given by:

$$\frac{df}{dt} = \{H_1,\ldots,H_{n-1},f\}. \tag{8}$$

Suppose that the flow ϕ_t associated with (8) exists and let U_t be the one-parameter group acting on A by $f \mapsto U_t(f) = f \circ \phi_t$. It follows from the FI that:

Proposition 1 *The one-parameter group U_t is an automorphism of the Nambu bracket structure on A.*

Definition 3 *$f \in A$ is called an integral of motion for the system defined by (8) if it satisfies $\{H_1, \ldots, H_{n-1}, f\} = 0$.*

It follows from the FI that a Poisson-like theorem exists for Nambu-Poisson manifolds:

Proposition 2 *The Nambu bracket of n integrals of motion is also an integral of motion.*

For the case $n = 2$, the FI is the Jacobi identity and one recovers the usual definition of a Poisson manifold. On $\mathbb{R}^2$, the canonical Poisson bracket of two functions $\mathcal{P}(f, g)$ is simply their Jacobian, and Nambu defined his bracket on $\mathbb{R}^n$ as a Jacobian of n functions $f_1, \ldots, f_n \in C^\infty(\mathbb{R}^n)$ of n variables $x_1, \ldots, x_n$:

$$\{f_1, \ldots, f_n\} = \sum_{\sigma \in S_n} \epsilon(\sigma) \frac{\partial f_1}{\partial x_{\sigma_1}} \cdots \frac{\partial f_n}{\partial x_{\sigma_n}}, \tag{9}$$

which gives the *canonical Nambu bracket* of order n on $\mathbb{R}^n$. Other examples of Nambu-Poisson structures have been found [4]. One of them is a generalization of linear Poisson structures and is given by the following Nambu bracket of order n on $\mathbb{R}^{n+1}$:

$$\{f_1, \ldots, f_n\} = \sum_{\sigma \in S_{n+1}} \epsilon(\sigma) \frac{\partial f_1}{\partial x_{\sigma_1}} \cdots \frac{\partial f_n}{\partial x_{\sigma_n}} x_{\sigma_{n+1}}. \tag{10}$$

In general any manifold endowed with a Nambu-Poisson structure of order n is *locally foliated* [18] by Nambu-Poisson manifolds of dimension n endowed with the canonical Nambu-Poisson structure (9). In particular it is shown in [18] that any Nambu tensor is decomposable (this fact was conjectured by Takhtajan, eventually proved in a very elegant way by Gautheron and then discovered by chance by Takhtajan to follow from a result going back to 1923, reproduced in a textbook by Schouten!).

We now have a complete parallel between Poisson classical mechanics and Nambu classical mechanics: we have a bracket (the Nambu n-bracket, with $n = 2$ in the former case) satisfying the required properties (skew symmetry, Leibniz rule and the FI), equations of evolution of a similar form (8) with solutions which can be written as $f_t = \exp(t\{H_1, \ldots, H_{n-1}, .\}) f_0$, and an *action functional* for which the classical trajectory is an extremal [32]. That action is the integral $A(C_{n-1}) = \int_{C_{n-1}} \omega^{(n-1)}$ over $(n-1)$-chains $C_{(n-1)}$ in the extended phase space $M \times \mathbb{R}$ of the generalized Poincaré-Cartan integral invariant action form $\omega^{(n-1)} = x_1 dx_2 \wedge \ldots \wedge dx_n - H_1 dH_2 \wedge \ldots \wedge dH_{n-1} \wedge dt$. We refer to [32] for more details (especially for the case $n = 3$).

3 Quantization of Nambu mechanics

3.1 First attempts and problems raised

As realized already by Nambu ([26]), the first problem in the intriguing question of quantization (of such a system of equations) is how to define a quantization. A simplistic answer is to say that, since the classical system can be treated as a (Dirac) constrained Hamiltonian system, it is enough to quantize the latter. Even within this framework a rigorous solution (in particular due to the presence of a first class constraint which turns the physical manifold into a Poisson, not symplectic, manifold) is not so easy to obtain. But there is more: how can we be sure that the classical equivalence can be carried over to the quantum case? Quantum anomalies pop up like mushrooms nowadays, so one must attack directly the problem of quantization of the original classical system.

Nambu [26], assuming only (4) and (5), could not find a satisfactory solution. Assuming the "usual framework" (operator algebras) to quantize (8) he arrives at the "disappointing" (albeit not so surprising in view of the classical equivalence with a constrained system) result that the quantum analogue of (8) is equivalent to a Heisenberg equation. Indeed, skew-symmetry (4) gives what is called in [32] ($n = 3$ case) the Nambu–Heisenberg commutation relation:

$$[A_1, A_2, A_3] \equiv \sum_{\sigma \in S_3} \epsilon(\sigma) A_{\sigma_1} A_{\sigma_2} A_{\sigma_3} = cI, \tag{11}$$

where c is a constant and I is the unit operator. Representations could be found (a more complete treatment is given in [32]), but already the Leibniz rule (5) brings back to the Heisenberg case. Using associative algebras to find realizations of the Poisson bracket for the canonical triplet leads again nowhere. He thus naturally turns to the use of nonassociative algebra. The Cayley–Dickson algebra (an algebra with alternating associator and 8 generators) is similarly rejected. So Nambu turns to Jordan algebras with the same conclusion, which is no surprise since all but one of them (the exceptional algebra, 3×3 matrices with Cayley coefficients) arise from associative algebras and thus bring back to Heisenberg formalism. Takhtajan [32], with the additional constraint of the FI, could certainly do no better. His main positive contribution there (for the quantization problem) was to find an explicit realization of the Nambu–Heisenberg relation (the case of general n was also discussed).

3.2 Deformation Quantization

Deformation quantization was introduced [16] a few years after Nambu's paper. It is well adapted to constrained systems but (for the same reasons as in the previous subsection) using it in this way for Nambu mechanics will not give anything new. A more radical approach is needed if one wants to get a solution expressing the specificity of Nambu mechanics.

As we have mentioned in Section 2.3.2, the Jacobian is the typical case of Nambu bracket. It is thus natural to try and replace in (9) usual products of observables by star products. For the sake of self-completeness we shall present here a very brief overview of deformation quantization (see also [17] [33]).

Let M be a Poisson manifold, a differentiable manifold endowed with a (possibly degenerate) 2-tensor τ satisfying $[\tau, \tau] = 0$ in the sense of Schouten-Nijenhuis brackets. The latter ensures that the corresponding Poisson bracket $\mathcal{P}(f, g) = \tau(df \wedge dg)$ of two functions $f, g \in A = C^\infty(M)$ is a Lie algebra law. We denote by $A[[\nu]]$ the space of formal power series in the parameter ν with coefficients in A. A *star product* $*_\nu$ on M is an associative (non-Abelian) deformation of the usual product of the algebra A defined on A by:

$$f *_\nu g = \sum_{r \geq 0} \nu^r C_r(f, g), \quad \forall f, g \in A, \tag{12}$$

where $C_0(f, g) = fg$, $f, g \in A$, and $C_r \colon A \times A \to A$ $(r \geq 1)$ are bidifferential operators (bipseudodifferential operators can sometimes be considered) on A satisfying:

a) $*_\nu$ extends by linearity in ν to $A[[\nu]]$ and the associativity condition $(f *_\nu g) *_\nu h = f *_\nu (g *_\nu h)$ is satisfied for all $f, g, h \in A[[\nu]]$.

b) $C_1(f, g) - C_1(g, f) = 2\mathcal{P}(f, g)$, $f, g \in A$.

(12) and condition a) express that we have an associative algebra deformation in the sense of Gerstenhaber [19] (in short, a *DrG-deformation*) while condition b) ensures that the corresponding commutator $[f, g]_{*_\nu} \equiv (f *_\nu g - g *_\nu f)/2\nu$ is a DrG-deformation of the Lie algebra $(A, \mathcal{P})$.

Equivalence of two DrG-deformations is defined in the associative case (two star products $*_\nu$ and $*'_\nu$) by the existence of a formal series of (differential) operators $T = \sum_{r \geq 0} \nu^r T_r$ with $T_0 = Id$ such that $Tf * Tg = T(f *' g)$, $\quad f, g \in A[[\nu]]$. By equivalence one may consider only star products "vanishing on constants" (i.e. $C_r(f, c) = C_r(c, f) = 0$, $r \geq 1$, $c \in \mathbb{R}$, $f \in A$, since by [20] a deformation of a unital algebra is unital and equivalent to a deformation with the same unit) and assume $C_1 = \mathcal{P}$. A deformation is said trivial if it is equivalent to the original product (resp. bracket, for Lie algebra deformations); equivalence classes of DrG-deformations are classified, at each order in ν, by the second Hochschild (resp. Chevalley) cohomology spaces $H^2(A)$ (resp. $\tilde{H}^2(A)$) of A valued in itself with the natural action (product, resp. adjoint).

When τ is everywhere nondegenerate, M is *symplectic* (we denote by ω the symplectic 2-form on M, inverse of τ). In this case (and similarly for regular Poisson manifolds; cf. e.g. [13] [7] [33]) there always exist a star product, $\mathcal{P}$ is a nontrivial Hochschild 2-cocycle and $\dim \tilde{H}^2(A) = 1 + b_2$ (where $b_2 = \dim H^2(M)$, the second Betti number of M), which classifies at each order in ν inequivalent 1-differentiable deformations [15] of $(A, \mathcal{P})$ vanishing on constants (they are given by deformations of the symplectic structure). Then [13] [8] equivalence classes of star products are in one-to-one correspondence with series $[\omega] + \nu[H^2(M)][[\nu]]$. In the "flat case" $b_2 = 0$, star products are (up to equivalence) unique and given by the Moyal product $*_M$ [2] which can be expressed on $\mathbb{R}^{2n}$ by the well-known formula $f *_M g = \exp(\nu \mathcal{P})(f, g)$. It corresponds to the Weyl (totally symmetric) ordering of operators in quantum mechanics, the deformation parameter being here $\nu = i\hbar/2$.

A given Hamiltonian $H \in A$ determines the time evolution of an observable $f \in A$ by the Heisenberg equation $\frac{df_t}{dt} = [H, f_t]_{*_\nu}$, the corresponding one-parameter group of time evolution being given by the star exponential defined by:

$$\exp_*\left(\frac{tH}{2\nu}\right) \equiv \sum_{r\geq 0} \frac{1}{r!}\left(\frac{t}{2\nu}\right)^r (*H)^r, \tag{13}$$

where $(*H)^r = H * \cdots * H$ (r factors). Then the solution to the Heisenberg equation above can be expressed as $f_t = \exp_*(tH/2\nu) * f * \exp_*(-tH/2\nu)$. In many examples (cf. e.g. [2]), the star exponential is convergent as a series in the variable t in some interval ($|t| < \pi$ for the harmonic oscillator in the Moyal case) and converges as a distribution on M for fixed t. Then it makes sense to consider a Fourier-Dirichlet expansion of the star exponential:

$$\exp_*\left(\frac{tH}{2\nu}\right)(x) = \int \exp(\lambda t/2\nu)d\mu(x;\lambda), \quad x \in M, \tag{14}$$

the "measure" μ being interpreted as the Fourier transform (in the distribution sense) of the star exponential in the variable t. Equation (14) permits to define [2] the spectrum of the Hamiltonian H as the support Λ of the measure μ. In the discrete case, $\exp_*(tH/2\nu)(x) = \sum_{\lambda\in\Lambda} \exp(\lambda t/2\nu)\pi_\lambda(x)$, $x \in M$. The functions π_λ on M are interpreted as eigenstates of H associated with the eigenvalues λ, and satisfy $H * \pi_\lambda = \pi_\lambda * H = \lambda\pi_\lambda$, with $\pi_\lambda * \pi_{\lambda'} = \delta_{\lambda\lambda'}$ and $\sum_{\lambda\in\Lambda} \pi_\lambda = 1$.

In the Moyal case, the Feynman path integral can be expressed [28] as the Fourier transform over momentum space of the star exponential. In field theory, where the normal star product (which is the exponential of "half of the Poisson bracket" in the variables $p \pm iq$) is relevant, the Feynman path integral is given (up to a multiplicative factor) [10] by the star exponential.

We have limited ourselves here to some highlights (most of which will be needed later) of deformation quantization to give the flavor of how it provides a completely autonomous quantization scheme of a classical Hamiltonian system. See [31] for a more extensive list of relevant references on the subject. We shall now use it as a tool for the quantization of Nambu-Poisson structures.

3.3 Zariski product: a generalized Abelian deformation

Our starting point here is a simple remark: the Jacobian of n functions on $\mathbb{R}^n$ is a Nambu bracket because the usual product of functions is Abelian, associative, distributive and respects the Leibniz rule. This is what permits us to work out the functional determinant and the required properties of a Nambu bracket (including the FI) will be satisfied. Therefore, if we replace the usual product in the Jacobian by a new product having the preceding properties, we get a "modified Jacobian" which is still a Nambu bracket in the sense of Definition 1 – but which has "built in" quantization if the new product has it. In [12] the interested reader will find more in detail how we arrived at the solution which we shall now describe.

3.3.1 Deformation of the usual product law

To fix ideas, take the space N of polynomials on $\mathbb{R}^3$ and denote by $*$ the Moyal star product in (e.g.) the first two variables. To get an Abelian product the natural idea is to symmetrize the Moyal product. But if one does it brutally (i.e. $\frac{1}{2}(a*b+b*a)$, $a,b \in N$) one loses associativity (this is the star analogue of the Jordan algebra formalism already studied by Nambu).

One can then try, using linearity, to bring down the problem to monomials and then to perform total symmetrization in star monomials (star products of the coordinates functions): symmetrization will give back the usual product. (This however would not necessarily be true with different orderings or nonlinear elementary factors; we shall come back to this point in Section 4).

But algebra tells us there is another way to decompose polynomials: into irreducible factors, which can be of any degree. The latter is true for polynomials in several complex variables and a fortiori in the real domain, which is of interest for us since we want e.g. the harmonic oscillator Hamiltonian p^2+q^2 to be "elementary". Symmetrization adapted to the number of irreducible factors will restore associativity of the product, but at the same time there will be "built in" quantization because the factors can be of any degree. That decomposition is the core of Zariski topology on the manifold ($\mathbb{R}^3$ here), hence the name of Zariski product.

More precisely, we start with $N = \mathbb{R}[x_1,x_2,x_3]$ and consider $\mathcal{S}(N) = \bigoplus_{n=1}^{\infty} N^{\overset{n}{\otimes}}$, its *symmetric* tensor algebra without scalars (a kind of Fock space over N). We denote by N_1 the semi-group of normalized polynomials, those for which the "maximal" monomial (defined in a natural manner: highest total degree, then maximal with respect to the lexicographic ordering in the variables (x_1,x_2,x_3)) has coefficient 1, with the convention that $0 \in N_1$. In $N[\nu]$ we consider similarly the semi-group (under usual product) N_1^ν of normalized polynomials, those for which the non-vanishing coefficient of lowest degree in ν belongs to N_1. Now we define a map $\alpha : N_1^\nu \to \mathcal{S}(N)$ by $\alpha(P) = P_1 \otimes \cdots \otimes P_n$ where $P_1 \cdots P_n$ is the decomposition into irreducible factors of the coefficient of degree 0 in ν of $P \in N_1^\nu$ and $\otimes$ denotes the *symmetric* tensor product. In other words, α takes into account only the classical part (which can be 0) of the polynomial in N_1^ν, and then replaces the usual product by the symmetric tensor product.

In order to get our "built in" quantization we use now an *"evaluation map"* $T : \mathcal{S}(N) \to N[\nu]$ defined by replacing the symmetric tensor product by a symmetrized Moyal product. In the above notations the result of all this is

$$T(\alpha(P)) = \frac{1}{n!} \sum_{\sigma \in S_n} P_{\sigma_1} * \cdots * P_{\sigma_n} \tag{15}$$

and we can define an Abelian product by

$$P \times_\alpha Q = T(\alpha(P) \otimes \alpha(Q)), \quad \forall P,Q \in N_1^\nu. \tag{16}$$

Obviously the product (16) maps $N_1^\nu \times N_1^\nu$ into N_1^ν and if $P,Q \in N_1$, then $P \times_\alpha Q|_{\nu=0} = PQ$. In this sense, $\times_\alpha$ is a generalized deformation of the usual product on the semi-group N_1, but this is far from being a DrG-deformation of the algebra N. In [12] simple examples are given to make this more explicit. We only have a deformed Abelian semi-group law, and if we plug this law into the Jacobian then the Leibniz rule and the FI will fail. What is lacking (and obviously so since addition and decomposition into irreducible factors of polynomials are highly noncommutative operations) is distributivity of the product $\times_\alpha$ with respect to addition.

In fact it follows from general results of Harrison cohomology (cf. e.g. [20] and references therein; compare also with Gelfand's theorem on the realization of Abelian involutive Banach algebras as algebras of functions over the spectrum) that Abelian DrG-deformations of algebras of polynomials (or even functions [29]) are trivial. To overcome this difficulty and restore distributivity we use a trick inspired by the second quantization procedure. Formally, let us look at "functions" on N_1 (e.g. formal series). Intuitively we get a deformed coproduct and the dual of this space of "functions" (polynomials on polynomials) will then have a product and a deformed product, both of which will be distributive with respect to the vector space addition. Now the product of polynomials is again a polynomial. So in the end we are getting some deformed product on an algebra generated by the polynomials.

3.3.2 The Zariski algebra and its deformations

The "Zariski" algebra generated by the polynomials is nothing but the algebra $\mathcal{Z}_0$ of the semi-group N_1, i.e. the free Abelian algebra generated by the set of normalized real irreducible polynomials $N_1^{irr} \subset N_1$ as building blocks. If we denote by Z_u the element of $\mathcal{Z}_0$ defined by $u \in N_1$, the *classical Zariski product* $\bullet^z$ in $\mathcal{Z}_0$ is given by $Z_u \bullet^z Z_v = Z_{uv}$, $u, v \in N_1$. Defining $Z_{cu} = cZ_u$ for $c \in \mathbb{R}$, we can extend the above product to all $u \in N$ and get a multiplicative injection of N into $\mathcal{Z}_0$ which is non-additive, i.e. $Z_{u+v} \neq Z_u + Z_v$ (the addition in $\mathcal{Z}_0$ is *not* related to the addition in N). It is the algebra $\mathcal{Z}_0$ that we shall now quantize.

We consider the space $\mathcal{Z}_\nu = \mathcal{Z}_0[\nu]$ of polynomials in ν with coefficients in $\mathcal{Z}_0$ and inject N_1^ν into $\mathcal{Z}_\nu$ by $\zeta : (\sum_{r\geq 0} \nu^r u_r) \mapsto \sum_{r \geq 0} \nu^r Z_{u_r}$ with $u_0 \in N_1$ and $u_i \in N, i \geq 1$. Using this injection we can define a *deformed Zariski product* $\bullet_\nu^z$ based on $\times_\alpha$, first on basis elements Z_u, $u = u_1 \cdots u_m$, $u_j \in N_1^{irr}$, $j = 1, \dots, m$, and Z_v, by $Z_u \bullet_\nu^z Z_v = \zeta(u \times_\alpha v)$; then we extend it by linearity to all of $\mathcal{Z}_\nu$, taking into account the requirement that the product $\bullet_\nu^z$ annihilates the non-zero powers of ν:

$$(\sum_{r\geq 0} \nu^r A_r) \bullet_\nu^z (\sum_{s \geq 0} \nu^s B_s) = A_0 \bullet_\nu^z B_0, \quad \forall A_r, B_s \in \mathcal{Z}_0, r, s \geq 0. \tag{17}$$

The $\bullet_\nu^z$ product is obviously associative (because $\times_\alpha$ is), distributive with respect to addition in $\mathcal{Z}_\nu$, and Abelian. Its limit for $\nu = 0$ is $\bullet^z$. Thus we have achieved a first step:

Theorem 1 *The vector space $\mathcal{Z}_\nu$ endowed with the product $\bullet_\nu^z$ is an Abelian algebra which is a (generalized) deformation of the Abelian algebra $(\mathcal{Z}_0, \bullet^z)$.*

The algebra $\mathcal{Z}_\nu$ with product $\bullet_\nu^z$ provides thus an Abelian deformation of the algebra $\mathcal{Z}_0$, a fact interesting *per se* because it gives a first example of a non-trivial Abelian deformation, however generalized and therefore not necessarily classified by the Harrison cohomology (defined on the sub-complex of the Hochschild complex consisting of symmetric cochains [20]).

3.4 Arithmetic "second" quantization of Jacobian in $\mathbb{R}^3$

3.4.1 Problems with the definition of derivatives

The next step towards our goal of deforming the Nambu bracket on $\mathbb{R}^3$ is to define the *derivatives* δ_i, $1 \leq i \leq 3$, on $\mathcal{Z}_0$ and extend them to $\mathcal{Z}_\nu$. This would allow to define first the classical Nambu bracket on $\mathcal{Z}_0$, and the quantum one on $\mathcal{Z}_\nu$. But this is not so simple to achieve. The straightforward definition $\delta_i Z_u = Z_{\partial_i u}$, $\forall u \in N$, where ∂_i is the usual derivative with respect to x^i, does not satisfy the Leibniz rule because of the different nature of the addition in N and in $\mathcal{Z}_0$ (except on the diagonal, a remark relevant for the analogue of the star exponential (13), which we shall consider in Section 4.3.1).

The next natural choice is then to "build in" the Leibniz rule by using the above definition of the derivatives δ_i only for irreducible polynomials $u \in N_1^{irr}$ and postulating the Leibniz rule on the product $v = v_1 v_2 \cdots v_m$ of irreducible polynomials :

$$\delta_i \; Z_{v_1 v_2 \cdots v_m} = Z_{(\partial_i v_1) v_2 \cdots v_m} + Z_{v_1 (\partial_i v_2) \cdots v_m} + \cdots + Z_{v_1 v_2 \cdots (\partial_i v_m)}.$$

Obviously, the maps δ_i are derivations on the algebra $\mathcal{Z}_0$, but they are not commuting maps, i.e. the Frobenius property (trivially verified with the straightforward definition above) $\delta_i \delta_j = \delta_j \delta_i$, $i \neq j$, is not verified. This comes from the fact that when one takes the derivatives of an irreducible polynomial u, the polynomials $\partial_i u$, $1 \leq i \leq 3$, do not necessarily factorize out into the same number of factors (a simple example is given in [12]).

A consequence of this fact is the following: If one defines the classical Nambu bracket on $\mathcal{Z}_0$ by replacing, in the Jacobian, the usual product by $\bullet^z$ and the usual partial derivatives by the maps δ_i, this new bracket will not satisfy the FI. There will be anomalies in the FI (even at this classical, or "prequantized" level) due to terms which cannot cancel out each other because the Frobenius property is not satisfied on $\mathcal{Z}_0$.

3.4.2 Taylor series algebra: a solution (case $n = 3$)

Denote by $\mathcal{E} = \mathcal{Z}_0[y^1, y^2, y^3]$, the algebra of polynomials in some real variables (y^1, y^2, y^3) with coefficients in $\mathcal{Z}_0$. The Taylor series development of translated polynomials $x \mapsto u(x+y)$ can now be written $u(x+y) = u(x) + \sum_i y^i \partial_i u(x) + \frac{1}{2}\sum_{i,j} y^i y^j \partial_{ij} u(x) + \cdots$. These translated polynomials are multiplied by $(uv)(x+y) = u(x+y)v(x+y)$. In accordance with our setup we shall look instead at "Taylor series" in $\mathcal{E}$, for $u \in N_1$:

$$J(Z_u) = Z_u + \sum_i y^i Z_{\partial_i u} + \frac{1}{2}\sum_{i,j} y^i y^j Z_{\partial_{ij} u} + \cdots = \sum_n \frac{1}{n!}(\sum_i y^i \partial_i)^n (Z_u), \quad (18)$$

where $\partial_i u$, $\partial_{ij} u$, etc. are the usual derivatives of $u \in N_1 \subset N$ with respect to the variables x^i, x^i and x^j, etc., $\partial_i Z_u \equiv Z_{\partial_i u}$ and, since in general the derivatives of $u \in N_1$ are in N, one has to factor out the appropriate constants in $Z_{\partial_i u}$, $Z_{\partial_{ij} u}$, etc. (i.e. $Z_{\lambda u} \equiv \lambda Z_u$, $u \in N_1$, $\lambda \in \mathbb{R}$). J defines an additive map from $\mathcal{Z}_0$ to $\mathcal{E}$ (to say that J is multiplicative is tantamount to the Leibniz property).

That algebra is however too large and for our purpose we need to restrict to the smaller algebra $\mathcal{A}_0$, the subalgebra of $\mathcal{E}$ generated by elements of the form (18). We shall denote by $\bullet$ the product in $\mathcal{A}_0$ which is naturally induced by the product in $\mathcal{E}$. In order to define the (classical) Nambu-Poisson structure on $\mathcal{A}_0$, we need a correct definition of the derivative of an element of $\mathcal{A}_0$. Notice that the derivative $\partial_i u(x+y)$ is again a Taylor series of the form $\partial_i u(x) + \sum_j y^j \partial_{ij} u(x) + \cdots$. We shall thus define the derivative Δ_a, $1 \leq a \leq 3$, of an element of the form (18) by the natural extension to $\mathcal{A}_0$ of the previous straightforward definition, i.e.,

$$\Delta_a(J(Z_u)) = J(Z_{\partial_a u}) = Z_{\partial_a u} + \sum_i y^i Z_{\partial_{ai} u} + \frac{1}{2} \sum_{i,j} y^i y^j Z_{\partial_{aij} u} + \cdots, \tag{19}$$

for $u \in N_1$, $1 \leq a \leq 3$. One can look at definition (19) of Δ_a as the restriction, to the subset of elements of the form $J(Z_u)$, of the formal derivative with respect to y^a in the ring $\mathcal{E} = \mathcal{Z}_0[y^1, y^2, y^3]$. Since $\Delta_a(J(Z_u)) = J(Z_{\partial_a u})$, we have $\Delta_a(\mathcal{A}_0) = \mathcal{A}_0$ and we get a family of maps $\Delta_a : \mathcal{A}_0 \to \mathcal{A}_0$, $1 \leq a \leq 3$, restriction to $\mathcal{A}_0$ of the derivations with respect to y^a, $1 \leq a \leq 3$, in $\mathcal{E}$. We summarize the properties of Δ_a in the following:

Lemma 1 *The maps $\Delta_a : \mathcal{A}_0 \to \mathcal{A}_0$, $1 \leq a \leq 3$, defined by Eq. (19) constitute a family of commuting derivations (satisfying the Leibniz rule) of the algebra $\mathcal{A}_0$.*

The definition of derivatives on $\mathcal{A}_0$ leads to the following natural definition of the classical Nambu bracket on the Abelian algebra $\mathcal{A}_0$ (it is obvious that this defines a Nambu-Poisson structure on $\mathcal{A}_0$).

Definition 4 *The classical Nambu bracket on $\mathcal{A}_0$ is the trilinear map taking values in $\mathcal{A}_0$ given, $\forall A, B, C \in \mathcal{A}_0$, by:*

$$(A, B, C) \mapsto [A, B, C]_\bullet \equiv \sum_{\sigma \in S_3} \epsilon(\sigma) \Delta_{\sigma_1} A \bullet \Delta_{\sigma_2} B \bullet \Delta_{\sigma_3} C. \tag{20}$$

Now that we have a classical Nambu-Poisson structure on $\mathcal{A}_0$, we shall construct a quantum Nambu-Poisson structure by defining a generalized Abelian deformation $(\mathcal{A}_\nu, \bullet_\nu)$ of $(\mathcal{A}_0, \bullet)$. The construction is based on the map α introduced in Section 3.3.1 and we shall extend the definition of the product $\bullet_\nu^z$ defined in Section 3.3.2 to the present setting for the Nambu-Poisson structure on $\mathcal{A}_0$.

Let $\mathcal{E}[\nu]$ be the algebra of polynomials in ν with coefficients in $\mathcal{E}$. We consider the subspace $\mathcal{A}_\nu$ of $\mathcal{E}[\nu]$ consisting of series $\sum_{r \geq 0} \nu^r A_r$ for which the coefficient A_0 is in $\mathcal{A}_0$. Then we define a map $\bullet_\nu : \mathcal{A}_\nu \times \mathcal{A}_\nu \to \mathcal{E}[\nu]$ by extending the product $\bullet_\nu^z$ defined previously, with $u, v \in N_1$ (it is sufficient to define it on $\mathcal{A}_0$ since $\bullet_\nu^z$ annihilates the non-zero powers of ν):

$$J(Z_u) \bullet_\nu J(Z_v) = Z_u \bullet_\nu^z Z_v + \sum_i y^i (Z_{\partial_i u} \bullet_\nu^z Z_v + Z_u \bullet_\nu^z Z_{\partial_i v}) + \cdots. \tag{21}$$

Actually $\bullet_\nu$ defines a product on $\mathcal{A}_\nu$. Moreover, for $A = \sum_{r\geq 0} \nu^r A_r$ and $B = \sum_{s\geq 0} \nu^s B_s$ in $\mathcal{A}_\nu$, we have $A \bullet_\nu B = A_0 \bullet_\nu B_0$ and the coefficient of ν^0 of the latter is $A_0 \bullet B_0$ which is in $\mathcal{A}_0$ since $A_0, B_0 \in \mathcal{A}_0$. This shows that $\bullet_\nu$ is actually a product on $\mathcal{A}_\nu$ which is Abelian by definition. Finally, for $\nu = 0$, we have $A \bullet_\nu B|_{\nu=0} = A_0 \bullet B_0$. Thus:

Theorem 2 *The vector space $\mathcal{A}_\nu$ endowed with the product $\bullet_\nu$ is a (generalized) Abelian algebra deformation of the Abelian algebra $(\mathcal{A}_0, \bullet)$.*

The derivatives Δ_a, $0 \leq a \leq 3$, are naturally extended to $\mathcal{A}_\nu$. Every element $A \in \mathcal{A}_\nu$ can be written as $A = \sum_I y^I A_I$, where $I = (i_1, \ldots, i_n)$ is a multi-index and $A_I \in \mathcal{Z}_\nu$. Then we have, for $A, B \in \mathcal{A}_\nu$, $A \bullet_\nu B = \sum_{I,J} y^I y^J A_I \bullet_\nu^z B_J$. Since $(\mathcal{Z}_\nu, \bullet_\nu^z)$ is an Abelian algebra and the derivative Δ_a acts as a formal derivative with respect to y^a on the product $A \bullet_\nu B$, the usual properties (linearity, Leibniz, Frobenius) of a derivative are still satisfied on $\mathcal{A}_\nu$ and we can define the quantum Nambu bracket on $\mathcal{A}_\nu$.

Definition 5 *The quantum Nambu bracket on $\mathcal{A}_\nu$ is the trilinear map taking values in $\mathcal{A}_\nu$ defined, $\forall A, B, C \in \mathcal{A}_\nu$, by:*

$$(A, B, C) \mapsto [A, B, C]_{\bullet_\nu} \equiv \sum_{\sigma \in S_3} \epsilon(\sigma) \Delta_{\sigma_1} A \bullet_\nu \Delta_{\sigma_2} B \bullet_\nu \Delta_{\sigma_3} C. \tag{22}$$

It is now straightforward to show:

Theorem 3 *The quantum Nambu bracket (22) endows $\mathcal{A}_\nu$ with a Nambu-Poisson structure, a (generalized) deformation of the classical Nambu structure on $\mathcal{A}_0$.*

3.4.3 General case

The above procedure can be easily generalized to $\mathbb{R}^n$, $n \geq 2$. The only non-straightforward modification to be done appears in the evaluation map. One has to distinguish the cases n even and n odd. If $n = 2p$, $p \geq 1$, then one replaces the partial Moyal product in (15) by the usual Moyal product on $\mathbb{R}^{2p}$. If $n = 2p + 1$, $p \geq 1$, one uses the partial Moyal product $*_{1\cdots 2p}$ on (e.g.) the hyperplane defined by $x_{2p+1} = 0$. The other definitions and properties are directly generalized to $\mathbb{R}^n$. More generally, on a manifold M endowed with a star product (e.g. a regular Poisson manifold), the same procedure can be developed, using the a priori given star product in Formula (15).

The question of spectrum of observables can be treated for Zariski products [12] as we have done for deformation quantization. We shall not develop this point here but shall briefly discuss the question in Section 4.3.1 together with the "first quantized" approach to which we devote Section 4.

Note that the canonical Nambu-Poisson structure of order 2 on $\mathbb{R}^2$ is the usual Poisson structure; there our procedure gives a quantization of the Poisson bracket $\mathcal{P}$ different from Moyal, however not on $N[\nu]$ but on $\mathcal{A}_\nu$; this quantization will be somewhat like in field theory. The same applies to $\mathbb{R}^{2n}$ by starting with a sum of Poisson brackets on the various $\mathbb{R}^2$.

4 Generalized deformations of the "first quantized" type; equivalence, triviality and spectrality for generalized deformations

At the beginning of Section 3.3.1, we mentioned another possibility for getting a generalized deformation, already at the level of the algebra N of real polynomials over $\mathbb{R}^n$. In [11] this was developed in the covariant case, which we shall present below. Let us first present in an even simpler context how this can be done.

Take $\mathbb{R}^2$, with coordinates p and q. If p (resp. q) corresponds to an operator P (resp. Q), then antistandard ordering associates to a monomial $q^j p^k$ the operator $Q^j P^k$ and we define in this way a star product $*_S$ which [17] is nothing but the product of symbols of pseudodifferential operators. It is equivalent to $*_M$ (Moyal) as a star product. The same can be done with standard (first P) and normal (with $P \pm iQ$) orderings, etc. Now change the product $\times_\alpha$ in (15) into a "sun" product $\odot_{MS}$ defined by $(q^{j_1}p^{k_1}) \odot_{MS} (q^{j_2}p^{k_2}) = q^{j_1+j_2} *_M p^{k_1+k_2}$ on monomials. This is obviously an Abelian product but now it has some "built in" quantization. We extend it by linearity to all of $N[[\nu]]$ with the convention that it annihilates nonzero powers of ν. We get in this way a generalized deformation, an associative, distributive and Abelian product with which we can proceed as before, but this time at the level of the algebra of polynomials on $\mathbb{R}^2$. If now we consider some $\mathbb{R}^2 \subset \mathbb{R}^3$, we can quantize Nambu brackets at this "quantum mechanical" level.

More generally (and in rough terms), if $*$ is some star product on some manifold M and $N \subset C^\infty(M)$ is some algebra, $\mathcal{S}$ the symmetric algebra over it, given some map $\alpha : N[[\nu]] \to \mathcal{S}$ defined in a way similar to the previous case (e.g. by performing some ordering operation on basis elements of N and extending to formal series with the convention that it annihilates nonzero powers of ν) and an evaluation map T defined by replacing $\otimes$ by $*$, we can introduce a "sun" product $\odot_\nu = T\alpha$. The question which has to be addressed is how trivial is this type of quantization, but before we shall present the more sophisticated form studied in [11].

4.1 Covariant star products and Zariski quantization

Let N be the algebra of real polynomials over $\mathbb{R}^n$ ($n = 3$ in the following) and define a sun product $\odot_\nu = T\alpha$ where α: $N[\nu] \to \mathcal{S}$ maps the monomial $x_1^{k_1} \cdots x_n^{k_n} \in N$ into a similar expression belonging to the symmetric tensor algebra $\mathcal{S}$ over N obtained by replacing the product of the coordinates x_i ($i = 1, \ldots, n$) by the symmetrized tensor product $\otimes$; we extend by linearity to N

and to $N[\nu]$ by the convention that α annihilates nonzero powers of ν (in other words, we extend by linearity to $N[\nu]$ and compose with the canonical projection of $N[\nu]$ on N). The evaluation map T: $\mathcal{S} \to N[\nu]$ is defined, for $F_i \in N$, $1 \leq i \leq k$, $\forall k \geq 1$, by

$$T(F_1 \otimes \cdots \otimes F_k) = \frac{1}{k!} \sum_{\sigma \in S_k} F_{\sigma_1} * \cdots * F_{\sigma_k} \tag{23}$$

where $*$ is some star product on N. We call $\odot_\nu$ the *sun product associated with the star product* $*$. It is obviously Abelian and associative.

In the following we consider $\mathbb{R}^3$ as the dual of the $\mathfrak{su}(2)$-Lie algebra and take $*_M$ to be the Moyal product on $\mathbb{R}^6$. The Lie algebra $\mathfrak{su}(2)$ can be realized with the functions $L_i(p,q) = \sum_{1\leq j,k \leq 3} \varepsilon_{ijk} p_j q_k$, $1 \leq i,j,k \leq 3$, on $\mathbb{R}^6$, where ε_{ijk} is the totally skew-symmetric tensor with $\varepsilon_{123} = 1$:

$$[L_i, L_j]_M \equiv \frac{1}{2\nu}(L_i *_M L_j - L_j *_M L_i) = \sum_{1\leq k \leq 3} \varepsilon_{ijk} L_k. \tag{24}$$

One can easily show that for any real polynomial $F \in N(\mathbb{R}^3)$, the polynomial on $\mathbb{R}^6$ defined by $F(L_1, L_2, L_3)$ satisfies $(1 \leq i,j,k \leq 3)$

$$L_i *_M F = L_i F + \nu \sum_{1 \leq j,k \leq 3} \varepsilon_{ijk} L_k \frac{\partial F}{\partial L_j} + \nu^2 \Big(2 \frac{\partial F}{\partial L_i} + \sum_{1 \leq j \leq 3} L_j \frac{\partial^2 F}{\partial L_i \partial L_j} \Big). \tag{25}$$

Therefore $L_{i_1} *_M \cdots *_M L_{i_k}$ is a polynomial in (L_1, L_2, L_3) and any polynomial in (L_1, L_2, L_3) can be expressed as a $*_M$-polynomial of the L_i's, so that the product $F *_M G$ of two polynomials F, G in (L_1, L_2, L_3) is again a polynomial in (L_1, L_2, L_3). Hence from the Moyal product $*_M$ on $\mathbb{R}^6$, we get a star product on $\mathbb{R}^3$ satisfying Eq. (25) for any polynomial F. It is actually an invariant (and covariant) star product on $\mathfrak{su}(2)^* \sim \mathbb{R}^3$. Herebelow we shall denote by $*$ this star product and by $\odot_\nu$ the associated $\odot_\nu$-product. Since $L_i * L_j = L_i L_j + \nu \sum_{1\leq k \leq 3} \varepsilon_{ijk} L_k + 2\nu^2 \delta_{ij}$, $1 \leq i,j \leq 3$, and $L_i \odot_\nu L_j = L_i L_j + 2\nu^2 \delta_{ij}$, this $*$-product does provide quantum terms.

Let us write $F * G = \sum_{r \geq 0} \nu^r C_r(F,G)$, where $C_1(F,G) = \mathcal{P}(F,G) \equiv \sum_{1 \leq i,j,k \leq 3} \varepsilon_{ijk} L_k \frac{\partial F}{\partial L_i} \frac{\partial G}{\partial L_j}$ is the standard Poisson bracket on $\mathfrak{su}(2)^* \sim \mathbb{R}^3$. We denote by Δ the Laplacian operator: $\Delta = \sum_{1\leq k \leq 3} \frac{\partial^2}{\partial L_k^2}$. From (25) we get $C_2(L_i, F) = (2 + \mathcal{D})(\frac{\partial F}{\partial L_i})$, in terms of the differential operator (constant on the subspace $H_n \subset N$ of homogeneous polynomials of degree n) $\mathcal{D} = \sum_{1 \leq k \leq 3} L_k \frac{\partial}{\partial L_k}$. Complicated calculations give $\forall F, G \in H_n$, $F \odot_\nu G = \sum_{r \geq 0} \nu^{2r} a(2n, r) \Delta^r (FG)$, with $\forall n \geq 1$, $a(n,r) = \frac{1}{n}((n-2r)a(n-1,r) + (n-2r+2)a(n-1,r-1))$ and $a(n,0) = 1 = a(0,r)$. After some more calculations [11] we obtain

$$a(n,r) = \sum_{\substack{j_1 + \cdots + j_{n-2r+2} = r \\ j_1, \ldots, j_{n-2r+2} \geq 0}} \gamma_{j_1} \gamma_{j_2} \tau_{j_3} \cdots \tau_{j_{n-2r+2}}, \quad n \geq 2r, \tag{26}$$

where $\gamma_n = E_{2n}/(2n)!$ and $\tau_n = 2^{2n+2}(2^{2n+2}-1) B_{2n+2}/(2n+2)!$, the constants E_k (resp. B_k) being the Euler (resp. Bernoulli) numbers. Let $B(k,p)$ be the set of partitions of k of length p, i.e. the number of ways to write k as a sum of p strictly positive integers: $k = n_1 + \cdots + n_p$, with $n_1 \geq \cdots \geq n_p$, and $A_k = \sum_{\substack{i+j=k \\ i,j\geq 0}} \gamma_i\gamma_j$, $k \geq 0$. In the above notations we then get:

Theorem 4 *The sun product $\odot_\nu$ associated with the invariant star product on $\mathfrak{su}(2)^*$ defined by Eq. (25) admits the following form:*

$$F\odot_\nu G = FG + \sum_{r\geq 1} \nu^{2r}\eta_r(FG), \quad F, G \in N \quad (\textit{polynomials on } \mathbb{R}^3), \tag{27}$$

where $\eta_r: N \to N$, $r \geq 1$, are differential operators given by

$$\eta_r(F) = \left(A_r + \sum_{p=1}^{r} z_{p,r}\mathcal{D}(\mathcal{D}-1)\cdots(\mathcal{D}-p+1) \right) \Delta^r(F), \quad F \in N, \tag{28}$$

with $z_{p,r} = \sum_{k=p}^{r} A_{r-k} \sum_{(n_1,\ldots,n_p)\in B(k,p)} (m_1!\cdots m_r!)^{-1}\tau_{n_1}\cdots\tau_{n_p}$.
This $\odot_\nu$-product has a unique extension to $F, G \in C^\infty(\mathbb{R}^3)$.

4.2 Two notions of equivalence and triviality

4.2.1 Sun products

Definition 6 *Two products $\odot_\nu$ and $\odot'_\nu$ are said to be A-equivalent, if there exists a $\mathbb{R}[[\nu]]$-linear (formally invertible) map $S_\nu: N[[\nu]] \mapsto N[[\nu]]$ of the form $S_\nu = \sum_{r\geq 0} \nu^r S_r$, where the $S_r: N \mapsto N$, $r \geq 1$, are differential operators and $S_0 = Id$, such that*

$$S_\nu(F\odot_\nu G) = S_\mu(F)\odot'_\nu S_\mu(G)\big|_{\mu=\nu}, \quad \forall F, G \in N. \tag{29}$$

We say that they are B-equivalent if the map S_ν satisfies

$$S_\nu(F\odot_\nu G) = S_\nu(F)\odot'_\nu S_\nu(G), \quad \forall F, G \in N. \tag{30}$$

It is straightforward to check that both notions are indeed equivalence relations (between $\odot_\nu$-products). Note that there is no order relation between both; in particular two $\odot_\nu$-products which are B-equivalent are not necessarily A-equivalent. Relation (30) can be written as $S_\nu(F\odot_\nu G) = F\odot'_\nu G$ and relation (29) amounts to

$$\sum_{r,s\geq 0} \nu^{r+s} S_s(\rho_r(FG)) = \sum_{r,s,s'\geq 0} \nu^{r+s+s'} \rho'_r(S_s(F)S_{s'}(G)), \quad F, G \in N, \tag{31}$$

where ρ_r (resp. ρ'_r) are the cochains of the product $\odot_\nu$ (resp. $\odot'_\nu$).

Triviality is equivalence with the original product of the algebra, the usual product. That is:

Definition 7 *A sun product $\odot_\nu$ is said strongly (resp., weakly) trivial if it is A-equivalent (resp., B-equivalent) with the usual product.*

In other words, if we take for $\odot'_\nu$ the usual product, $\odot_\nu$ is strongly trivial if there exists a (formally invertible) $\mathbb{R}[[\nu]]$-linear map $S_\nu \colon N[[\nu]] \mapsto N[[\nu]]$ of the form $S_\nu = \sum_{r\geq 0} \nu^r S_r$, where the $S_r \colon N \mapsto N$, $r \geq 1$, are differential operators and $S_0 = Id$, such that (denoting by $\cdot$ the usual product):

$$S_\nu(F \odot_\nu G) = S_\nu(F) \cdot S_\nu(G), \quad \forall F, G \in N. \tag{32}$$

$\odot_\nu$ is weakly trivial if (with the same notations)

$$S_\nu(F \cdot G) = F \odot_\nu G, \quad \forall F, G \in N, \tag{33}$$

or equivalently $F \cdot G = S_\nu^{-1}(F \odot_\nu G)$, which corresponds to taking for $\odot'_\nu$ (instead of $\odot_\nu$) in (30) the usual product. We have here manifest symmetry in the substitution of the usual product. If in (29) we take for $\odot_\nu$ the usual product we get $S_\nu(F \cdot G) = S_\mu(F) \odot_\nu S_\mu(G)|_{\mu=\nu}$, which (using (31)) is equivalent to (32) with S^{-1} instead of S. From Definition 7 we see that a strongly trivial sun product is weakly trivial (hence the terminology). As a matter of fact one proves easily:

Proposition 3 *A $\odot_\nu$-product is strongly trivial if and only if it coincides with the usual product. A $\odot_\nu$-product is weakly trivial whenever its cochains are given by differential operators.*

The proof of the first part is by elementary calculation. For the second part one writes (for $F, G \in N$) $F \odot_\nu G = \rho(F \cdot G)$ where $\rho = Id + \sum_{r\geq 1} \nu^r \rho_r$ is formally invertible and the ρ_r's are differential operators acting on N. From Theorem 4 we get:

Corollary 1 *The sun product constructed in Section 4.1 is weakly trivial but not strongly trivial.*

4.2.2 Zariski products

The previous treatment (with the obvious modifications) applies to the case of Zariski quantization. But in this case we see easily that *the Zariski product $\bullet_\nu$ is never trivial in either sense* (except when the defining product $*_\nu$ is not a star product but the usual product).

It is true that the map $T\alpha$ intertwines the Zariski product $\bullet_\nu$ with the usual product $\bullet$ on the classical algebra $\mathcal{A}_0$. But it is neither invertible as a formal series nor acting by differential operators. Even if one defines S_ν by taking the $\mathbb{R}[[\nu]]$-linear extension of the restriction of $T \circ \alpha$ to $\mathcal{A}_0$, one gets an invertible map but not an intertwining map (in the sense of B-equivalence) as S_ν is not given by differential operators on $\mathcal{A}_0$. We restrict the intertwining operators to be defined by *differentiable* cochains (in spite of the fact that in general the deformations considered are not defined by differentiable cochains) because eventually the non-triviality for the Abelian generalized deformations should be linked with the usual (differentiable) Hochschild or Harrison cohomologies.

4.3 Spectrality and concluding remarks

4.3.1 Exponentials and spectrum

As we recalled in Section 3.2, the spectrum of an observable f is obtained, in deformation quantization, via the $*$-exponential (13). If we take a DrG-deformation of the form (12) which is trivial with intertwining operator T (note that a star product is never trivial because the Poisson bracket is a nontrivial Hochschild 2-cocycle), the corresponding exponential is the inverse image of the usual exponential of Tf, hence the spectrum is continuous.

For a $\odot_\nu$ product (e.g. the case of $SU(2)$) we can define $\exp_\odot(f)$ like in the star case (13) with $\odot_\nu$ instead of $*$. This exponential coincides with the star exponential for linear elements like L_3 in the case of $SU(2)$ but not for expressions like $H_1 = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2)$. In particular the Fourier decomposition of $\exp_\odot(L_3)$ is discrete but for H_1 the spectra do not coincide. Now if we take a weakly trivial $\odot_\nu$ product, the intertwiner S is not enough to trivialize the spectrum, in contradistinction with the previous case.

For the Zariski product $\bullet_\nu^z$ we define similarly the corresponding exponential. Here, in addition to linear elements, the star and $\bullet_\nu^z$ exponentials coincide for irreducible polynomials, giving e.g. the usual discrete spectrum for the harmonic oscillator Hamiltonian.

4.3.2 Quantized Nambu bracket

A quantization of the classical Nambu bracket is achieved by replacing the usual product by the $\odot_\nu$-product of Section 4. Due to the properties of the $\odot_\nu$-product, it is easy to see that actually the quantized Nambu bracket is given by:

$$[F,G,H]_{\odot_\nu} = T(\alpha(\{F,G,H\})), \quad F,G,H \in C^\infty(\mathbb{R}^3),$$

where $\{F,G,H\}$ denotes the classical Nambu bracket on $\mathbb{R}^3$, i.e., the Jacobian. Though the Leibniz rule is not satisfied for the $\odot_\nu$-product, this quantized Nambu bracket does satisfy the Fundamental Identity. Indeed only the weaker form of the Leibniz rule

$$\begin{aligned} &F\odot_\nu(\frac{\partial}{\partial L_i}(G\odot_\nu H) - G\odot_\nu\frac{\partial H}{\partial L_i} - \frac{\partial G}{\partial L_i}\odot_\nu H) \\ &= T(\alpha(F(\frac{\partial}{\partial L_i}(GH) - G\frac{\partial H}{\partial L_i} - \frac{\partial G}{\partial L_i}H))) = 0, \end{aligned}$$

is required to ensure that the Fundamental Identity is verified by the quantized Nambu bracket.

Finally let us mention that, as for the $\odot_\nu$-product case, the quantized Nambu bracket is weakly trivial, but not strongly trivial.

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