LONG TIME STABILITY OF SOME SMALL AMPLITUDE SOLUTIONS IN NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider small perturbations of the Zakharov-Shabat nonlinear Schrödinger equation on $[0, \pi]$ with vanishing or periodic boundary conditions; we prove a Nekhoroshev type result for solutions starting in the neighbourhood (in the H^1 topology) of the majority of small amplitude finite dimensional invariant tori of the linearized system. More precisely we will prove that along the considered solutions all the actions of the linearized system are approximatively constant up to times growing exponentially with the inverse of a suitable small parameter.

1. Introduction

We consider the equation

$$iu_t + u_{xx} \pm \pi u|u|^2 = \epsilon \frac{\partial H(x, u, \bar{u})}{\partial \bar{u}} \ , \tag{1.1}$$

with vanishing or periodic boundary conditions on $[0, \pi]$. Here H is a sufficiently smooth real valued function (the coefficient π in front of the third order term has been introduced for future convenience).

When $\epsilon = 0$ eq. (1.1) reduces to the Zakharov-Shabat (ZS) equation which is well known to be integrable. So, it has an infinite sequence of constant of motion and, for any finite N, there exist families of N dimensional invariant tori (finite gap solutions) which, as $N \to \infty$, fill densely the phase space. In the case $\epsilon \neq 0$, it is known that there exists a sequence ϵ_N such that, if $|\epsilon| \leq \epsilon_N$, then the majority of the above N dimensional tori can be continued to invariant tori of (1.1)[10,14,11,6] (see also [7]). However $\epsilon_N \to 0$ as $N \to \infty$, and therefore the above results give a good insight on the dynamics of (1.1) only in compact subsets of the phase space. On the contrary, very little is known[5] on the dynamics in open subsets of the phase space (a detailed discussion of the paper [5] will be given in sect. 4).

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In the present paper we give a bound on the long time diffusion of the actions for initial data in open subsets of the phase space, in the spirit of Nekhoroshev's theorem [12,13]. Actually, our result restricts to solutions which (i) are of small amplitude, and (ii) correspond to initial data close to the majority of finite dimensional invariant tori of the linearized system, namely those fulfilling a suitable (generic) nonresonance condition.

The reasons of the limitations (i) and (ii) are as follows: due to (i) we avoid all difficulties related to the actual integration of the ZS equation. Indeed, as pointed out in [11], the ZS equation can be approximated, in a neighbourhood of the origin, with the integrable nonlinear system given by the first order Birkhoff normal form of the ZS equation itself (averaged system). The advantage is that the Hamiltonian function of the averaged system can be explicitly calculated in terms of the action variables. In particular this allows one to study the dependence of the frequency on the initial datum, and to show that the majority (in a measure sense) of finite dimensional invariant tori have some good nonresonance properties. Our result concerns precisely the dynamics in a neighbourhood of such nonresonant tori. It turns out that the condition of being close to such tori can be formulated in terms of invariant tori of the linearized system; this gives rise to condition (ii).

The paper is organized as follows. In sect. 2 we give a precise statement of our main result (theorem 2.1), and in sect. 3 we present the scheme of its proof. Sect. 4 contains a discussion on the relation of the present result with some recent works on long time behaviour in nonlinear PDE's. Sections 5, 6, 7 constitute the technical part of the paper. In sect. 5 we perform a transformation putting the system in first order Birkhoff normal form, and give the Hamiltonian a form adapted for the subsequent steps of the proof. In sect. 6 we state the abstract normal form theorem which constitutes the main technical result of the paper as well as the main step for the proof of theorem 2.1; in our opinion this could have some independent interest. In that section we also give the original elements leading to the proof of the normal form result. In sect. 7 we apply the normal form theorem to the nonlinear Schrödinger equation, and complete the proof of theorem 2.1. We also give a slightly more precise description of the dynamics.

Acknowledgements I would like to thank Antonio Giorgilli for many fruitful discussions. This work has been developed with the partial support of the grant EC contract ERBCHRXCT940460 for the project "Stability and universality in classical mechanics", and of the grant CE n. CHRX-CT96-0330/DG.

2. Statement of the main result

We come now to a precise statement of our result. For definiteness we first consider eq. (1.1) with vanishing boundary conditions on $[0,\pi]$; concerning the function H we assume that the application

$$\mathbf{C}^2 \ni (y,z) \mapsto H(.,y,z)$$

takes values in $H^1_0([0,\pi],\mathbf{C})$ (Sobolev space of L^2 functions having L^2 derivatives and satisfying $u(0)=u(\pi)=0$) and moreover is analytic in a neighbourhood of the origin as an application from $\mathbf{C}^2$ to $H^1_0([0,\pi],\mathbf{C})$. We recall that this simply means that, considering the Taylor expansion of H in y and z, namely $H(x,y,z)=\sum_{k,l\geq 0}H_{kl}(x)y^k z^l$, the series

$$\sum_{k,l\geq 0} \|H_{kl}(.)\|_{H^1}\, |y|^l |z|^k \tag{2.1}$$

is convergent in a neighbourhood of the origin.

Consider now the Fourier expansion of u, namely

$$u(x)=\sum_{k\geq 1} u_k \sqrt{\frac{2}{\pi}}\sin kx\ , \tag{2.2}$$

and define the action variables $Z_k := |u_k|^2/2$ of the linearized system. Fix an arbitrary N dimensional diophantine vector $(\omega_1,...,\omega_N)\in \mathrm{I\!R}^N$, namely a vector such that there exist $\tau\geq 0$ and $\beta>0$ with the property that

$$|\sum_{i=1}^{N}\omega_i k_i| \geq \frac{\beta}{|k|^\tau}\ ,\quad \forall k\in \mathbf{Z}^N\setminus\{0\}\ ; \tag{2.3}$$

we also assume $\omega_i>0$, $i=1,...,N$. A rescaling of the vector $\omega:=(\omega_1,...,\omega_N,0,0,0,0...)\in \mathrm{I\!R}^\infty$, will be used as reference value of the actions. As anticipated above, our main result deals with small amplitude solutions. In the statement of the forthcoming theorem the role of order of magnitude of the solution will be played by the parameter R.

Theorem 2.1. *Under the above assumptions there exist positive constants $\mu_*, C_1,...,C_4$ with the following property: fix $R>0$ and assume that*

$$\mu := R+\frac{|\epsilon|}{R^5}\leq \mu_*\ ,$$

then provided the initial value of the actions is sufficiently close to $R^2\omega$, precisely satisfies

$$\sum_{k\geq 1} k^2 \left| Z_k(0)-R^2\omega_k\right| \leq C_1 \mu^{1/(\tau+1)} R^3 \tag{2.4}$$

where $\omega_i := 0$ for $i\geq N+1$, then the value of the actions remains close to $R^2\omega$, precisely satisfies

$$\sum_{k\geq 1} k^2 \left| Z_k(t)-R^2\omega_k\right| \leq C_2 \mu^{1/(\tau+1)} R^3 \tag{2.5}$$

for exponentially long times, namely for all times t satisfying

$$|t|\leq \frac{C_3}{\mu R^2}\exp\left[\left(\frac{C_4}{\mu}\right)^{1/(\tau+1)}\right]\ . \tag{2.6}$$

□

Consider now the N-dimensional torus

$$\mathcal{M}_{R,\omega} := \left\{ u \in H_0^1([0,\pi], \mathbf{C}) \ : \ \frac{|u_k|^2}{2} = R^2\omega_k \ , k = 1, ..., N \ ; \ u_k = 0 \ , \ k \geq N+1 \right\} \ , \tag{2.7}$$

and denote by $d_{H^1}(.,.)$ the distance in the H^1 norm, then the following corollary holds

Corollary 2.2. *Provided $\mu \leq \mu_*$ there exist positive constants C_5, C_6 such that, solutions of (1.1) corresponding to initial data u_0 satisfying*

$$d_{H^1}(u_0, \mathcal{M}_{R,\omega}) \leq C_5 R^2 \mu^{1/(\tau+1)} \ ,$$

remain close to $\mathcal{M}_{R,\omega}$, namely satisfy

$$d_{H^1}(u(t), \mathcal{M}_{R,\omega}) \leq C_6 R^2 \mu^{1/(\tau+1)} \ ,$$

for exponentially long times, namely for times satisfying (2.6). □

A particularly interesting case of (1.1) is

$$iu_t + u_{xx} = ug(|u|^2) \ , \tag{2.8}$$

where $g : \mathbb{R} \to \mathbb{R}$ is analytic in a neighbourhood of the origin, and satisfies $g(0) = 0$, $g'(0) \neq 0$. For such an equation we have the following

Theorem 2.3. *Consider eq. (2.8); there exist positive constants $R_*, C_7, ..., C_{10}$ such that for any positive $R < R_*$ the following holds true: if the initial value of the actions is sufficiently close to $R^2\omega$, namely satisfies*

$$\sum_{i \geq 1} i^2 \left| Z_i(0) - R^2\omega_i \right| \leq C_7 R^3 R^{1/(\tau+1)} \ , \tag{2.9}$$

then one has

$$\sum_{i \geq 1} i^2 \left| Z_i(t) - R^2\omega_i \right| \leq C_8 R^3 R^{1/(\tau+1)} \ , \tag{2.10}$$

for exponentially long times, namely for

$$|t| \leq \frac{C_9}{R^3} \exp\left[\left(\frac{C_{10}}{R} \right)^{1/(\tau+1)} \right] \ . \tag{2.11}$$

□

Remark 2.4. Theorem 2.3, exactly in the same form, holds for eq. (1.1) with $\epsilon = 1$, if the series (2.1) starts with terms of the 6^{th} order $(l + k \geq 6)$. □

Remark 2.5. Strictly speaking theorem 2.3 is not a corollary of theorem 2.1, indeed in this case one has $\epsilon := R^6$, and

$$H(u,\bar{u}) = \frac{1}{R^6}[G(|u|^2) - G'(0)\,|u|^2] \ ,$$

where G is a primitive of g with $G(0) = 0$, and therefore H depends also on ϵ. However, as it will be clear, the proof of theorem 5.1 holds also in the case of eq. (2.8). $\square$

In the case of periodic boundary conditions theorem 2.1 (and also corollary 2.2 and theorem 2.3) hold essentially unchanged, the only difference being that the Fourier expansion has to be done on the basis e^{ikx}, $k \in \mathbf{Z}$, so that the index in (2.4) and (2.5) runs over $\mathbf{Z}$, and the coefficient of $\left|Z_0 - R^2\omega_0\right|$ is 1; moreover, the vector ω has to satisfy $\omega_0 > 0$.

The proof of theorem 2.1 will be obtained using a method which we expect to work in quite general situations (see sect. 3 for more details).

For definiteness in all the rest of the paper we will consider only eq. (1.1) with vanishing boundary conditions.

3. Scheme of the proof

As anticipated above we first consider equation (1.1) as a perturbation of the completely resonant linearized system $i\dot{u} + u_{xx} = 0$, and, following [11] (see also [6]), we perform a first order averaging. Then we use the nonlinearity of the averaged system to tune suitably the frequency of the unperturbed motions around which we study the dynamics of equation (1.1). So, we consider the invariant torus of the averaged system on which the actions Z_k are equal to $R^2\omega$ (ω as in theorem 2.1). Expanding the Hamiltonian in a neighbourhood of this torus we obtain

$$H = \sum_{k=1}^{N} \tilde{\nu}_k I_k + \sum_{k\geq N+1} \tilde{\omega}_k Z_k + \text{higher order terms} \ , \tag{3.1}$$

where $\tilde{\nu}_k$ are the frequencies of the motions on the torus and $\tilde{\omega}_k$ are the frequencies the small oscillations around it (and $I_k := Z_k - R^2\omega_k$); in terms of ω such frequencies are given by

$$\tilde{\nu}_k := k^2 + R^2\sum_{i=1}^{N}\omega_i(4-\delta_{ik}) \ , \quad \tilde{\omega}_k := k^2 + R^2\sum_{i=1}^{N} 4\omega_i \ . \tag{3.2}$$

Subsequently we develop normal form theory for system (3.1). From the technical point of view this is the most difficult step, the main difficulty being related to the fact that $\tilde{\omega}_k \to \infty$ when $k \to \infty$. To illustrate this point remark that (at least formally) it is possible to introduce suitable action angle variables $(J_1, ..., J_\infty; \psi_1, ..., \psi_\infty)$ for the main part of (3.1) by defining $J_k := -I_k+4(\sum_{i=1}^N I_i+\sum_{i\geq N+1} Z_i)$, $k=1,...,N$, $J_{N+1} := \sum_{i=1}^N i^2 I_i+\sum_{i\geq N+1} i^2 Z_i$ and completing to a canonical transformation. In such a way system (3.1) takes the form

$$R^2\sum_{k=1}^N \omega_k J_k + J_{N+1} + \text{higher order terms}\ . \tag{3.3}$$

Provided $(\omega_1,...,\omega_N)$ is diophantine we are thus reduced to a perturbation of an integrable isochronous system with a finite number of nonresonant frequencies[1]. Thus one expects that it should be possible to put the system in normal form up to an exponentially small remainder in order to conclude that $J_1,...,J_{N+1}$ are approximate constant of motion. However, it is immediate to realize that, since the flow generated by J_{N+1} is essentially the flow of the linear Schrödinger equation, the angle ψ_{N+1} is *a continuous non differentiable* function of the phase point. As a consequence the perturbation, when expressed in terms of these action angle variables depends only continuously on the angle ψ_{N+1}. In particular this has the consequence that one cannot use Fourier series to solve the homological equation, and to develop normal form theory.

We solve the above difficulty by introducing a new method for the solution of the homological equation which avoids the introduction of the angle ψ_{N+1}, and even an implicit use of the Fourier expansion in such an angle (see lemma 6.4 below). Such a method is based on the fact that there is only *one* bad angle.

We point out that one could proceed also in a different way, namely exploiting the fact that the dynamics generated by J_{N+1} is much faster than the dynamics generated by $R^2\sum_{k=1}^N \omega_k J_k$. We think that theorem 6.2 giving a normal form result for Hamiltonian systems of the form (3.3) (without making use of the different time scales quoted above) could be interesting in itself.

Besides the above difficulty there is the well known problem that action variables are singular at the origin, but such a problem can be solved by the same method developed in [3]. So, we are able to prove a normal form result in the vicinity of a nonresonant torus. This allows to conclude that the functions $J_1,...,J_{N+1}$ are small for exponentially long times provided the solution does not leave the set where the normal form holds.

We point out that the above procedure can be used also to study perturbation of integrable hamiltonian PDE's which (i) are nonlinear and (ii) have an equilibrium point with the property that the spectrum of small oscillations is an infinite sequence of real numbers which are linear combination with integer coefficients of a *finite* number of fundamental frequencies.

The final step to prove theorem 2.1 is to use the functions $J_1,...,J_{N+1}$ as Lyapunof functions in order to prove that *all* the actions are approximate constants of motions (see lemma 7.4) and the solution does not leave the set where the normal form holds up to exponentially long times.

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[1] Actually there could be some resonances between $R^2\omega_i$ on the one side and 1 on the other side, however we will prove that due to the R^2 factor such resonances are of so high order that they do not matter.

4. Discussion

We compare here our result with some recent works on long time behaviour in nonlinear PDE's. In particular we will consider [5] and [4].

In [5] (see also [6]) Bourgain proved a general theorem on the long time behaviour in the nonlinear Schrödinger equation

$$i\dot{u} + u_{xx} - V(x)u = \epsilon \frac{\partial H(u,\bar{u})}{\partial \bar{u}} \ , \quad u(0) = u(\pi) = 0 \ , \tag{4.1}$$

where V is a typical smooth potential. Bourgain's theorem applies to H^s ($s \geq 1$) initial data such that "the norm stored in the high frequency modes" is very small, namely such that the relation

$$\sqrt{\sum_{i \geq N+1} i^{2s} \frac{|u_i(0)|^2}{2}} \leq C\epsilon^M \tag{4.2}$$

holds for some positive (large) M (here $u_i(0)$ is the component of the initial datum on the $i-$th eigenvector of $-\partial_{xx} + V$). The conclusion is that the solution is $O(\epsilon^M)$ close (in the H^s topology) to a quasiperiodic motion up to times of order ϵ^{-M}. Bourgain's theorem does not apply directly to eq. (1.1), but we think that it should apply to a neighbourhood of the manifold $\mathcal{M}_{R,\omega}$, at least in the case of vanishing boundary conditions.

The technique of the present paper allows to improve Bourgain's result in the case of H^1 initial data; the improvement being essentially that condition (4.2) on the initial datum is removed, and that the time scales on which the dynamics is controlled are much longer than those considered by Bourgain. Moreover the present method allows to deal with the case of periodic boundary conditions, which seems difficult to be treated with Bourgain's technique. However, in the general case of perturbations of linear systems (like eq. (4.1)) Bourgain's method is much more general (in a measure sense) than that of the present paper.

We come now to the work [4], where the perturbation of a completely resonant linear system was studied. In that paper a method was given for constructing a family of closed phase curves which are approximate solutions of the considered system, and are stable over exponentially long times. Such a method was also applied to the nonlinear wave equation. We expect that the method of [4] applies also to eq. (1.1), allowing to prove the existence of a family of closed curves with the property that solutions starting $O(R^b)$ close (in the H^1 topology) to one of these curves remain $O(R^{b/2})$ close to it (in the same topology) up to times of order $R^b \exp(c/R^2)$.

The result of the present paper is much stronger since it applies to initial data close to the majority of finite dimensional tori, instead of initial data close to the particular curves selected by the method of [4]. On the other hand we point out that the method of [4] applies to generic perturbations of the linear Schrödinger equation, while the method of the present paper strongly rely on the integrability and on the nonlinearity of the ZS equation.

We also recall the works [1],[2],[3]; the technique of the present paper is essentially a generalization of the techniques developed in those papers; it turns out that the main abstract theorems of [1] and [3] are contained in theorem 6.2 of the present paper.

5. First Normalization

It is well known that eq. (1.1) with vanishing boundary conditions is Hamiltonian, e.g., on the phase space $\mathcal{P}=H_0^1([0,\pi],\mathbf{C})$ endowed with the scalar product $\langle .,.\rangle$, and with the symplectic form $\Omega(.,.)$ defined respectively by

$$\langle u,v\rangle := \mathrm{Re}\int_0^\pi \bar u_x v_x dx\ ,\quad \Omega(u,v):=\mathrm{Re}\int_0^\pi ui\bar v dx\ . \tag{5.1}$$

We recall that the Hamiltonian vector field $\nabla^\Omega f$ corresponding to a differentiable function $f:\mathcal{P}\to \mathbb{R}$ is defined by

$$\Omega(\nabla^\Omega f(u),X)=df(u)X\ ,\quad \forall X\in\mathcal{P}\ , \tag{5.2}$$

so that the Hamiltonian function corresponding to (1.1) is given by

$$\frac{1}{2}\int_0^\pi |u_x|^2\,dx+\frac{\pi}{4}\int_0^\pi |u|^4 dx+\epsilon\int_0^\pi H(x,u(x),\bar u(x))dx\ . \tag{5.3}$$

We will also denote

$$h_0(u):=\frac{1}{2}\int_0^\pi |u_x|^2\,dx\ .$$

It is useful (and possible) to introduce canonical coordinates. To this end consider the Fourier expansion of u (cf. eq. (2.2)) and define

$$q_k:=\mathrm{Im}\,u_k\ ,\quad p_k:=\mathrm{Re}u_k\ ,$$

then (p_k,q_k) are the canonical conjugated coordinates (which obviously are not orthonormal). Remark that in terms of these coordinates one has

$$h_0(u)=\frac{1}{2}\sum_{k\ge1}k^2|u_k|^2=\sum_{k\ge1}k^2\frac{p_k^2+q_k^2}{2}\ . \tag{5.4}$$

Following [11] we now put (5.3) in normal form up to quantities of order $|u|^6+|\epsilon|$.

So, consider the complexification $\mathcal{P}^{\mathbf{C}}$ of $\mathcal{P}$, i.e. allow p_k and q_k to assume complex values, and denote by $B_R\subset\mathcal{P}^{\mathbf{C}}$ the (complex) ball of radius R and center at the origin. *We will also use the notation $a\preceq b$ to mean "there exists a positive constant C independent of R and ϵ such that $a\le Cb$".*

Lemma 5.1. *Consider the Hamiltonian system (5.3) in the domain B_R. There exists a positive $\mu_\sharp$ such that, if $|\epsilon|+R^6\le\mu_\sharp$, then the following holds true: there exists an analytic canonical transformation $\mathcal{T}_1$ such that*

1. $\mathcal{T}_1$ is defined on $B_{3R/4}$, one has $B_{R/2}\subset\mathcal{T}_1(B_{3R/4})\subset B_R$ and

$$\sup_{B_{3R/4}}\|u-\mathcal{T}_1(u)\|_{H^1}\preceq R^3, \tag{5.5}$$

2.

$$H \circ \mathcal{T}_1 = h_0 + \langle f \rangle + \tilde{f} \tag{5.6}$$

where

$$\langle f \rangle(u) := \frac{1}{2}\Big(\sum_{k\geq 1} |u_k|^2 \Big)^2 - \frac{1}{8} \sum_{k\geq 1} |u_k|^4 \ , \tag{5.7}$$

$\tilde{f}$ is analytic on $B_{R/2}$ and satisfies the estimates

$$\begin{aligned} \frac{1}{R} \sup_{u\in B_{R/2}} \left\| \nabla^{\Omega} \tilde{f}(u) \right\|_{H^1} &\preceq R^4 + \frac{|\epsilon|}{R^2} \ , \\ \sup_{u\in B_{R/2}} \left| \tilde{f}(u) \right| &\preceq R^6 + |\epsilon| \end{aligned} \tag{5.8}$$

3. For any analytic function $g : B_R \to \mathcal{B}$, with $\mathcal{B}$ a Banach space, one has

$$\sup_{u\in B_{R/2}} \| g(u) - g(\mathcal{T}_1(u)) \|_{\mathcal{B}} \preceq R^2 \sup_{u\in B_R} \| g(u) \|_{\mathcal{B}} \ , \tag{5.9}$$

and the same estimate holds for $\mathcal{T}_1^{-1}$. □

Such a lemma is just a reformulation of lemma 4 of [11]. We gave it a form suitable for our purpose.

Proof. The proof can be obtained by standard perturbation theory. For example one can define the transformation $\mathcal{T}_1$ as the time one flow of a Hamiltonian system with Hamiltonian function χ given by

$$\chi(u) := \frac{1}{2\pi} \int_0^{2\pi} t \left(f(e^{At}u) - \langle f \rangle (e^{At}u) \right) dt \ , \tag{5.10}$$

where e^{At} is the flow generated by the linearized system ($A = -\partial_{xx}$), and

$$f(u) := \frac{\pi}{4} \int_0^{\pi} |u|^4 dx \quad , \qquad \langle f \rangle(u) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{At}u) dt \ , \tag{5.11}$$

are respectively the Hamiltonian of the nonlinear part of the ZS equation and its average with respect to the unperturbed flow. The estimate (5.8) can be obtained exploiting the relations

$$\nabla^{\Omega} \chi = \frac{1}{2\pi} \int_0^{2\pi} t e^{-At} \left[\nabla^{\Omega} f(e^{At}u) - \nabla^{\Omega} \langle f \rangle (e^{At}u) \right] dt \ , \tag{5.12}$$

(which follows from the fact that e^{At} is a canonical transformation) and

$$\sup_{B_{R(1-d)}} \left\| \nabla^{\Omega} \{ g, g_1 \} \right\| \leq \frac{1}{Rd} \left(\sup_{B_r} \left\| \nabla^{\Omega} g \right\| \right) \left(\sup_{B_r} \left\| \nabla^{\Omega} g_1 \right\| \right) \ , \tag{5.13}$$

which holds for any two functions whose Hamiltonian vector field is analytic. Eq. (5.13) follows from $\nabla^\Omega\{g,g_1\}=[\nabla^\Omega g,\nabla^\Omega g_1]$ (for the complete proof of (5.13) see [1], lemma 5.2). For more details on the estimate (5.8) see [11] or [1].

One has also to prove that $\langle f\rangle$, as defined by (5.11) coincides with the function (5.7). This can be obtained by explicit calculation exactly as in [11] (see in particular lemma 5); the details are omitted. □

Remark *5.2.* The proof has been obtained by performing a resonant normal form, i.e. by averaging the first order part of the perturbation with respect to the linear flow. The resonant normal form thus obtained turns out to coincide with the nonresonant normal form obtained in [11]. This is not surprising since along a period of the flow of the linearized system all the angles conjugated to the linear actions vary by an integer multiple of 2π. □

Remark *5.3.* Particular useful cases of (5.9) are

$$\sup_{B_{R/2}}|Z_k(u)-Z_k(\mathcal{T}_1(u))|\preceq R^4\ , \tag{5.14}$$

$$\sup_{B_{R/2}}\left|\sum_{k\geq N+1}k^2Z_k(u)-\sum_{k\geq N+1}k^2Z_k(\mathcal{T}_1(u))\right|\preceq R^4\ , \tag{5.15}$$

□

We concentrate now on system (5.6). Remark that it is a perturbation of the integrable system $h_0+\langle f\rangle$, for which action variables are $Z_k=(p_k^2+q_k^2)/2$. We will use perturbation theory to construct a local normal form around a nonresonant finite dimensional invariant torus of such an integrable system.

We give now a suitable form to our Hamiltonian. Fix a positive integer N and introduce the first N action angle variables for our system, namely define (Z_i,ϕ_i) by

$$p_i=\sqrt{2Z_i}\cos\phi_i\ ,\quad q_i=\sqrt{2Z_i}\sin\phi_i\ ,\quad i=1,...,N\ .$$

Remark *5.4.* Outside the set $Z_i=0$ for some i, this is an analytic change of coordinates, and therefore $\tilde f$ is analytic in terms of these coordinates. □

As in the introduction we fix an N-dimensional vector $(\omega_1,...,\omega_N)$ with positive entries ($\omega_i>0$, $\forall i=1,...,N$), which we assume to be diophantine, precisely to satisfy (2.3); we also assume $\sum_{i=1}^N i^2\omega_i\leq 1/8$, the general case in which such a quantity is arbitrary can be dealt with minor modifications.

In what follows we will denote

$$I_i:=Z_i-R^2\omega_i\ .$$

Remark *5.5.* By our assumption on ω, the torus

$$\mathcal{M}:=\{(I,\phi,p,q)\in\mathcal{P}\ :\ I_k=p_l=q_l=0,\ k=1,...,N,\ l\geq N+1\} \tag{5.16}$$

is in $B_{R/4}$, therefore it is in the domain of definition of $\mathcal{T}_1$; by (5.5) it is $O(R^3)$ close in the H^1 norm to the torus $\mathcal{M}_{R,\omega}$ defined by (2.7). $\square$

We also introduce the notations

$$r := R^{1.5} \ , \qquad \zeta = (\{I_i, \phi_i\}_{i=1}^N, \{p_k, q_k\}_{k\geq N+1}) \ . \tag{5.17}$$

In what follows the role of perturbative parameter will be played by r.

We introduce now a new $r-$dependent norm in $\mathcal{P}$, by

$$\|\zeta\| := \sum_{i=1}^N \frac{i^2|I_i|}{r} + \sup_{i=1,\dots,N} r|\phi_i| + \sqrt{\sum_{i\geq N+1} i^2 \frac{|p_i|^2+|q_i|^2}{2}} \ , \tag{5.18}$$

(recall that $r = R^{1.5}$) which is clearly equivalent to the H^1 norm used before. *From now on we will use only this new norm, except when explicitly specified.* In what follows we will consider domains of the form

$$\mathcal{G}_\rho := \bigcup_{\phi\in\mathbf{T}^N} \mathcal{B}_\rho(I,\phi,p,q) \ ,$$

where

$$\mathcal{B}_\rho(I,\phi,p,q) = \mathcal{B}_\rho(\zeta) := \{\zeta_1 \in \mathcal{P} \ : \ \|\zeta-\zeta_1\| \leq \rho\} \ . \tag{5.19}$$

The norm (5.18) is useful since in the domain $\mathcal{G}_r$ the actions I and the coordinates (p,q) are small, but the angles ϕ can change by quantities of order one also in the complex directions.

Remark *5.6.* For R (and therefore r) small enough we have

$$\mathcal{G}_{4r} \subset B_{R/2} \ , \tag{5.20}$$

and therefore $\mathcal{G}_{4r}$ (which will be of particular interest) is contained in the domain of definition of $\mathcal{T}_1$. $\square$

Remark *5.7.* Let $u \in H_0^1$ be such that its coordinates ζ are real and belong to $\mathcal{G}_{4r}$, then one has (for R small enough)

$$d_{H^1}(u,\mathcal{M}) \preceq d_{\mathcal{P}}(\zeta,\mathcal{M})$$
$$d_{\mathcal{P}}(\zeta,\mathcal{M}) \preceq \frac{R}{r} d_{H^1}(u,\mathcal{M}) \ ,$$

where $d_{\mathcal{P}}(.,.)$ is the distance in the norm (5.18). $\square$

Proposition **5.8.** *In terms of the variables (I,ϕ,p,q), the Hamiltonian (5.6) can be written in the form*

$$H(\zeta) = \sum_{i=1}^{N+1} \nu_i J_i(\zeta) + \bar{f}(\zeta) \tag{5.21}$$

where

$$J_i(\zeta) := -I_i + 4\left(\sum_{k=1}^{N} I_k + \sum_{k\geq N+1} \frac{p_k^2+q_k^2}{2}\right) \ , \qquad i=1,...,N \ ,$$

$$J_{N+1}(\zeta) := \sum_{i=1}^{N} i^2 I_i + \sum_{i\geq N+1} i^2 \frac{p_i^2+q_i^2}{2} \ , \tag{5.22}$$

$$\nu_i := R^2\omega_i, \quad i=1,...,N \ ; \qquad \nu_{N+1} := 1 \ .$$

Moreover there exists r_* such that, for $r \leq r_*$ the estimate

$$\frac{1}{r}\sup_{\mathcal{G}_{2r}} \left\|\nabla^\Omega \bar{f}\right\| \preceq r^2 + \frac{|\epsilon|}{r^2} \tag{5.23}$$

holds, and $\bar{f}$ is analytic in $\mathcal{G}_{2r}$. □

Remark 5.9. The flow of $\nu_i J_i$, is periodic with period $T_i := 2\pi/\nu_i$. □

Proof. A simple calculation shows that (apart from a constant) $\langle f\rangle = \sum_{i=1}^{N}\nu_i J_i + \hat{h}$, where $\langle f\rangle$ is the function (5.7) expressed in terms of the new variables, and

$$\hat{h} := 2(\sum_{k=1}^{N} I_k + \sum_{k\geq N+1} Z_k)^2 - \frac{1}{2}\left(\sum_{k=1}^{N} I_k^2 + \sum_{k\geq N+1} Z_k^2\right) \ . \tag{5.24}$$

Define $\bar{f} := \hat{h} + \tilde{f}$. We estimate now $\nabla^\Omega \tilde{f}$ in the new norm. One has

$$\sqrt{\sum_{k\geq N+1}\frac{k^2}{2}\left(\left|\frac{\partial \tilde{f}}{\partial p_k}\right|^2 + \left|\frac{\partial \tilde{f}}{\partial q_k}\right|^2\right)} \leq \sqrt{\sum_{k\geq 1}\frac{k^2}{2}\left(\left|\frac{\partial \tilde{f}}{\partial p_k}\right|^2 + \left|\frac{\partial \tilde{f}}{\partial q_k}\right|^2\right)} \preceq R^5 + \frac{|\epsilon|}{R} < r^3 + \frac{|\epsilon|}{r} \ .$$

Consider now

$$\sum_{i=1}^{N}\frac{i^2}{r}\left|\frac{\partial \tilde{f}}{\partial \phi_i}\right| + \sup_i \left|\frac{\partial \tilde{f}}{\partial I_i}\right| r \ .$$

Notice that, due to (5.20) and (5.18), $\tilde{f}$, as a function of ϕ_i is analytic in the complex strip $|\mathrm{Im}\,\phi_i| \preceq 1$. It follows, by Cauchy inequality, that

$$\sup_{\mathcal{G}_{2r}}\left|\frac{\partial \tilde{f}}{\partial \phi_i}\right| \preceq \sup_{\mathcal{G}_{3r}} |\tilde{f}| \preceq R^6 + |\epsilon| = r^4 + |\epsilon| \ .$$

Analogously

$$\sup_{\mathcal{G}_{2r}}\left|\frac{\partial \tilde{f}}{\partial I_i}\right| \preceq \frac{1}{r^2}\sup_{\mathcal{G}_{3r}} |\tilde{f}| \preceq r^2 + \frac{|\epsilon|}{r^2} \ ,$$

and therefore $\left\|\nabla^\Omega \tilde{f}\right\| \preceq r^3 + |\epsilon|/r$. The estimate of $\left\|\nabla^\Omega \hat{h}\right\|$ is a straightforward calculation that we omit. □

6. Perturbation theory for Hamiltonians of the form (5.21)

System (5.21) has the form of a small perturbation of an integrable isochronous system whose dynamics is continuous but not differentiable and quasiperiodic with a finite number of frequencies; we will find a canonical transformation putting it in resonant normal form up to an exponentially small remainder. We will prove a result for abstract systems of the form (5.21) without taking into account the fact that ν is of order R^2; in our opinion such a result could have some independent interest.

So, consider a weakly symplectic [8] Banach space $\mathcal{P}$ endowed by the symplectic structure Ω. Let $\mathcal{P}^{\mathbf{C}}$ be the complexification of $\mathcal{P}$, and let $\mathcal{G} \subset \mathcal{P}$ be a domain. For each $\rho > 0$ define the complex extension of $\mathcal{G}$ by

$$\mathcal{G}_\rho := \bigcup_{\zeta \in \mathcal{G}} B_\rho(\zeta) ,$$

where $B_\rho(\zeta)$ is the (complex) ball of radius ρ and center ζ. We fix once for all a real number $r > 0$ and consider, in $\mathcal{G}_{2r}$ an analytic Hamiltonian function of the form

$$\sum_{i=1}^{N+1} h^i + f . \tag{6.1}$$

Essentially we will assume that the h^i's commute and that each of them generates a periodic flow Φ^i, $i = 1, ..., N+1$; finally we assume that the flows Φ^i $i = 1, ..., N$ are analytic in time, while Φ^{N+1} *is only continuous.*

The precise assumptions we need are

1) For $i = 1, ..., N$ the symplectic gradient $\nabla^\Omega h^i$ (defined by (5.2)) exists in $\mathcal{G}_{2r}$ and is analytic, moreover there exists a domain D dense in $\mathcal{G}_{2r}$ where $\nabla^\Omega h^{N+1}$ exists
2) there exist positive constants T_i, $i = 1, ..., N+1$ such that the flow Φ^i generated by $\nabla^\Omega h^i$ is periodic with period T_i, namely

$$\Phi^i_{t+T_i} = \Phi^i_t , \quad i = 1, ..., N+1 ,$$

   moreover $\Phi^i_t(\mathcal{G}_\rho) \subset \mathcal{G}_\rho$ $\forall \rho < 2r$.
3) The functions h^i commute each other, namely one has

$$\left\{h^i, h^j\right\} \equiv 0 , \quad i,j = 1, ..., N+1 , \quad i < j . \tag{6.2}$$

Remark 6.1. Using the flows Φ^i, $i = 1, ..., N$ one can define an action Ψ of $\mathbf{T}^N$ on $\mathcal{G}_\rho$, namely

$$\Psi_{\psi_1,...,\psi_N}(\zeta) := \Phi^1_{\psi_1/\nu_1} \circ \Phi^2_{\psi_2/\nu_2} \circ ... \circ \Phi^N_{\psi_N/\nu_N}(\zeta) ,$$

where $\nu_i = 2\pi/T_i$, $i = 1, ..., N$. □

We assume

4) denote by $\mathbf{T}^N + i\sigma$ the set of the ψ's belonging to the complexified N-dimensional torus such that $|\mathrm{Im}\,\psi_i| \leq \sigma$, then we assume that there exists a $\sigma > 0$ such that $\forall \zeta \in \mathcal{G}_r$, the

function $\mathbf{T}^N \ni \psi \mapsto \Psi_\psi(\zeta) \in \mathcal{G}_r$ can be extended to a complex analytic function from $\mathbf{T}^n + i\sigma$ to $\mathcal{G}_{2r}$.

5) $\Phi_t^{N+1}(D) \subset D$, the map Φ_t^{N+1} leaves invariant $\mathcal{G}_\rho$ for each $\rho \le 2r$ and is analytic (for each fixed t); moreover it is a canonical transformation, i.e. its differential $(\Phi_t^{N+1})'(\zeta)$ satisfy

$$\Omega\left[\left(\Phi_t^{N+1}\right)'(\zeta)X, \left(\Phi_t^{N+1}\right)'(\zeta)Y\right] = \Omega(X,Y) \ ,$$

and satisfies also the estimate

$$\sup_{\zeta\in\mathcal{G}_r}\left\|\left(\Phi_t^{N+1}\right)'(\zeta)\right\|_{\mathcal{P},\mathcal{P}} \le 1 \ .$$

6) The different periodic motions have frequencies which form a vector close to a diophantine one, precisely there exist $\gamma > 0$, $\tau \ge 0$ and $M > 0$ such that

$$\left|\sum_{i=1}^N \nu_i k_i + \omega n\right| \ge \frac{\gamma}{|k|^\tau} \ , \quad \forall (k,n) \in \mathbf{Z}^{N+1}\setminus\{0\} \ , \qquad |k| := \sum_{i=1}^N |k_i| \le M \ , \tag{6.3}$$

here $\omega := 2\pi/T_{N+1}$, $\nu_i = 2\pi/T_i$, $i = 1, ..., N$.

Theorem 6.2. *Consider the Hamiltonian system (6.1), assume that it is analytic over $\mathcal{G}_{2r}$, that it satisfies (1–6) above; assume also that $\nabla^\Omega f$ is analytic on $\mathcal{G}_{2r}$, and that there exist constants ω_f, c_T, such that*

$$\frac{1}{r}\sup_{\zeta\in\mathcal{G}_{2r}}\left\|\nabla^\Omega f\right\| \le \omega_f \ ,$$
$$\sup_{\zeta\in\mathcal{G}_r}\left\|\Psi'_\psi(\zeta)\right\|_{\mathcal{P},\mathcal{P}} \le c_T \ , \forall\psi \in \mathbf{T}^N + i\sigma \ .$$

Define the dimensionless parameter

$$\mu := 24(K_1)^\tau e^{2\tau}\frac{\pi\tilde{\omega}_f}{\gamma} \ ,$$

where

$$\tilde{\omega}_f := c_T\left(\frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}}\right)^N \omega_f \ , \quad K_1 := \max\left\{1, \frac{\tau+1+\log 2}{\sigma/2}\right\} \ .$$

Then the following holds true: if $\mu < 1$ and

$$M \ge \frac{K_1}{\mu^{1/(\tau+1)}} \ , \tag{6.4}$$

then there exists an analytical canonical transformation $\mathcal{T} : \mathcal{G}_{r/2} \to \mathcal{G}_{3r/4}$, with $\mathcal{T}(\mathcal{G}_{r/2}) \supset \mathcal{G}_{r/4}$, such that $H \circ \mathcal{T}$ turns out to be of the form

$$H(\mathcal{T}(\zeta)) = \sum_{i=1}^{N+1} h^i(\zeta) + Z(\zeta) + \mathcal{R}(\zeta) \ , \tag{6.5}$$

where

R1) Z is analytic together with $\nabla^\Omega Z$; it is in normal form, namely satisfies $\{Z, h^i\} \equiv 0$, $i = 1, ..., N+1$, and moreover satisfies the estimate

$$\frac{1}{r} \sup_{\zeta \in \mathcal{G}_{r/2}} \left\| \nabla^\Omega Z(\zeta) \right\| \leq \tilde{\omega}_f(e+1) \; ; \tag{6.6}$$

R2) $\mathcal{R}$ is an exponentially small remainder, namely one has

$$\frac{1}{r} \operatorname*{Sup}_{\zeta \in \mathcal{G}_{r/2}} \left\| \nabla^\Omega \mathcal{R}(\zeta) \right\|_{\mathcal{P}} \leq e^{\tau+2} \tilde{\omega}_f \mu^{1/(\tau+1)} \exp\left[-(\tau + 1 - \log 2) \left(\frac{1}{\mu} \right)^{1/(\tau+1)} \right] ;$$

R3) for any analytic function $g : \mathcal{G}_r \to \mathcal{B}$, where $\mathcal{B}$ is a Banach space, one has

$$\operatorname*{Sup}_{\zeta \in \mathcal{G}_{r/2}} \| g(\zeta) - g(\mathcal{T}(\zeta)) \|_{\mathcal{B}} \leq \mu^{1/(\tau+1)} \operatorname*{Sup}_{\zeta \in \mathcal{G}_r} \| g(\zeta) \|_{\mathcal{B}} \; . \tag{6.7}$$

□

The remainder of this section is devoted to the scheme of the proof of this theorem. It is almost identical to the proof of theorem 4.1 of [3], the only variant being the technique used to solve the homological equation.

First we introduce a suitable Fourier expansion for functions: related to the action Ψ we define the "$k-$th Fourier coefficient" $\hat{g}_k(\zeta)$ of a function $g : \mathcal{P}^{\mathbf{C}} \to \mathcal{B}$, where $\mathcal{B}$ is a Banach space, by

$$\hat{g}_k(\zeta) := \frac{1}{(2\pi)^N} \int_{\mathbf{T}^N} g\left(\Psi_\psi(\zeta)\right) e^{ik\cdot\psi} d^N\psi \; . \tag{6.8}$$

It is easy to see that, for smooth functions one has

$$g(\Psi_\psi(\zeta)) = \sum_{k \in \mathbf{Z}^N} e^{-ik\cdot\psi} \hat{g}_k(\zeta) \; ,$$

so that, in particular

$$g(\zeta) = \sum_{k \in \mathbf{Z}^N} \hat{g}_k(\zeta) \; .$$

Remark *6.3.* Using standard arguments and hypothesis 4) of the theorem it is possible to prove exponential decay of the Fourier coefficients of an analytic function; moreover, if a function g has symplectic gradient which is analytic then the following estimate holds

$$\sup_{\mathcal{G}_r} \left\| \nabla^\Omega \hat{g}_k \right\| \leq c_T e^{-\sigma|k|} \sup_{\mathcal{G}_{2r}} \left\| \nabla^\Omega g \right\| \tag{6.9}$$

(for the proof see [3] Lemma 7.4) □

Then we make the so called ultraviolet cutoff. So we fix $K \in \mathbf{N}^+$, and correspondingly we give the following

Definition *A function $g : \mathcal{G}_r \to \mathbf{C}$ is said to be of class Π_l if one has*

$$g(\zeta) = \sum_{|k|<Kl} \hat{g}_k(\zeta) \ .$$

Next, decompose the perturbation in terms of class Π_s:

$$f = \sum_{s\geq 1} f_s \ ,$$

where

$$f_s := \sum_{(s-1)K\leq|k|<sK} \hat{f}_k \ .$$

From (6.9) it follows

$$\sup_{\mathcal{G}_r} \left\| \nabla^{\Omega} f_s \right\| \leq \tilde{\omega}_f e^{-\sigma(s-1)K/2}$$

(for the proof see [9] lemma 8), which allows (provided K is sufficiently large) to consider f_s of order s.

At this point, provided one is able to solve the homological equation he can use any formal algorithm to develop perturbation theory for the Hamiltonian

$$H = \sum_{i=1}^{N+1} h^i + \sum_{s\geq 1} f_s \ ,$$

where $f_s \in \Pi_s$ is of order s.

We recall that the homological equation has the form

$$\left\{ h_\omega, \chi \right\}(\zeta) + Z(\zeta) = g(\zeta) \ , \tag{6.10}$$

where $h_\omega := \sum_{i=1}^{N+1} h^i$, and g are given, while the unknowns are Z which has to satisfy $\left\{Z, h^i\right\} \equiv 0$, and χ.

Concerning the homological equation we have the following

Lemma 6.4. *Let $g : \mathcal{G}_r \to \mathbf{C}$ be an analytic function of class Π_s with analytic symplectic gradient, then the homological equation (6.10) has a solution*

$$Z(\zeta) := \frac{1}{T_{N+1}} \int_0^{T_{N+1}} \hat{g}_0(\Phi_s^{N+1}(\zeta)) ds \tag{6.11}$$

where $\hat{g}_0$ is the 0-th Fourier coefficients of g, and $\chi \in \Pi_s$, which is defined by

$$\hat{\chi}_0(\zeta) := \frac{1}{T_{N+1}} \int_0^{T_{N+1}} s \hat{\mathcal{Z}}_0(\Phi_s^{N+1}(\zeta)) ds \ , \tag{6.12}$$

where $\mathcal{Z} := g - Z$, *and*

$$\hat{\chi}_k(\zeta) := \frac{e^{-i\frac{\nu\cdot k}{\omega}2\pi}}{1-e^{-i\frac{\nu\cdot k}{\omega}2\pi}}\int_0^{T_{N+1}} e^{i\nu\cdot ks}\hat{g}_k(\Phi_s^{N+1}(\zeta))ds \ , \quad 0\neq |k|\leq Ks \ , \tag{6.13}$$

The function χ *has the same analyticity properties of* g*, moreover for any* $\rho\leq r$ *the following inequalities hold*

$$\begin{aligned} \sup_{\zeta\in\mathcal{G}_\rho}\left\|\nabla^\Omega\hat{\chi}_k(\zeta)\right\| &\leq \frac{\pi(Ks)^\tau}{\gamma}\sup_{\zeta\in\mathcal{G}_\rho}\left\|\nabla^\Omega\hat{g}_k(\zeta)\right\| \ , \quad \forall k\in\mathbf{Z}\setminus\{0\} \ , \\ \sup_{\zeta\in\mathcal{G}_\rho}\left\|\nabla^\Omega Z(\zeta)\right\| &\leq \sup_{\zeta\in\mathcal{G}_\rho}\left\|\nabla^\Omega\hat{g}_0(\zeta)\right\| \ . \end{aligned} \tag{6.14}$$

□

Proof. First we define Z by (6.11). Then we will solve the equation

$$\{h_\omega,\chi\}(\zeta) = \mathcal{Z}(\zeta) \ . \tag{6.15}$$

Consider $\{h_\omega,\chi\}$: one has

$$\{h_\omega,\chi\} = \sum_{j=1}^N \nu_j \left.\frac{\partial}{\partial\psi_j}\right|_{\psi=0}(\chi\circ\Psi_\psi) + \left.\frac{d}{dt}\right|_{t=0}(\chi\circ\Phi_t^{N+1}) \ , \tag{6.16}$$

passing to Fourier series we get that the r.h.s. of (6.16) is equal to

$$\sum_{j=1}^N\sum_{k\in\mathbf{Z}^N} i\nu_j k_j\hat{\chi}_k + \left.\frac{d}{dt}\right|_{t=0}\sum_{k\in\mathbf{Z}^N}\left(\chi\circ\Phi_t^{N+1}\right)_k^\wedge = \sum_{k\in\mathbf{Z}^N} i\nu\cdot k\hat{\chi}_k + \left.\frac{d}{dt}\right|_{t=0}\sum_{k\in\mathbf{Z}^N}\hat{\chi}_k\circ\Phi_t^{N+1} \ , \tag{6.17}$$

where the last equality follows from (6.2). So, writing equation (6.15) in Fourier components we get

$$i\nu\cdot k\hat{\chi}_k + \left.\frac{d}{dt}\right|_{t=0}\hat{\chi}_k\circ\Phi_t^{N+1} = \hat{\mathcal{Z}}_k \ . \tag{6.18}$$

Actually we will solve the equation

$$i\nu\cdot k\left(\hat{\chi}_k\circ\Phi_t^{N+1}\right) + \frac{d}{dt}\left(\hat{\chi}_k\circ\Phi_t^{N+1}\right) = \hat{\mathcal{Z}}_k\circ\Phi_t^{N+1} \ . \tag{6.19}$$

Applying both sides of (6.19) to a fixed ζ, we obtain a linear nonhomogenous ordinary differential equation which can be solved explicitly. We need a solution periodic with period T_{N+1}. We will simply denote $\chi_k(t)$ for $\hat{\chi}_k\circ\Phi_t^{N+1}(\zeta)$. It is convenient to study separately the case $k=0$. Indeed in such a case (6.19) reduces to $\dot{\chi}_0 = \hat{\mathcal{Z}}_0$, whose solution can be written in the form

$$\chi_0(t) := \frac{1}{T_{N+1}}\int_0^{T_{N+1}} s\hat{\mathcal{Z}}_0(s+t)ds \ ,$$

from which we get (6.12). Such a quantity is analytic and actually solves our equation. Moreover, from hypothesis 5 it follows that $\nabla^{\Omega}\hat{\chi}_0$ satisfies a formula similar to (5.12) (with $\left(\Phi_t^{N+1}\right)'(\zeta)$ in place of e^{At}), and therefore one has

$$\sup_{\mathcal{G}_\rho}\left\|\nabla^{\Omega}\hat{\chi}_0\right\| \leq \frac{T_{N+1}}{2}\sup_{\mathcal{G}_\rho}\left\|\nabla^{\Omega}\hat{\mathcal{Z}}_0\right\| \ .$$

In the case $k\neq 0$ the general solution of (6.19) is

$$e^{-i\nu\cdot kt}a_k+\int_0^t e^{-i\nu\cdot k(t-s)}\hat{\mathcal{Z}}_k(s)ds \ ,$$

where a_k is a constant of integration. Imposing the above function to be periodic of period T_{N+1} we get an equation for a_k whose solution is

$$a_k=\frac{e^{-i\frac{\nu\cdot k}{\omega}2\pi}}{1-e^{-i\frac{\nu\cdot k}{\omega}2\pi}}\int_0^{T_{N+1}} e^{i\nu\cdot ks}\hat{\mathcal{Z}}_k(s)ds \ ,$$

which is well defined by hypothesis 5. So, we define $\hat{\chi}_k$ by (6.13), and it actually solves the original equation (6.18). We can use (6.13) to estimate the norm of the symplectic gradient of $\hat{\chi}_k$. Exploiting hypothesis 5) we get

$$\left|e^{-i\frac{\nu\cdot k}{\omega}2\pi}-1\right|\geq\frac{2\gamma}{\omega|k|^{\tau}} \ ,$$

from which,

$$\sup_{\mathcal{G}_\rho}\left\|\nabla^{\Omega}\hat{\chi}_k(\zeta)\right\|\leq\pi\frac{|k|^{\tau}}{\gamma}\sup_{\mathcal{G}_\rho}\left\|\nabla^{\Omega}f_k\right\| \ .$$

So the thesis holds. □

Now, to make rigorous the formal normalization procedure one has to introduce a norm for analytic functions (to be precise for their vector field), obtain estimates of the norm of the Poisson brackets of two analytic functions, and finally to use one of the well established recursive algorithms which allow to show that the normalizing transformation is well defined in a suitable domain, and to obtain the exponential estimate of the remainder. All this program can be achieved *exactly* as in [3]. So we omit the details of the proof.

7. On stability over exponential times in nonlinear Schrödinger equations

We apply theorem 6.2 to the nonlinear Schrödinger equation. We get

Lemma 7.1. *Consider the Hamiltonian system (5.21). There exists μ_* and C_4, such that, if*

$$\mu := R + \frac{|\epsilon|}{R^5} \leq \mu_*$$

then there exists an analytic canonical transformation $\zeta = \mathcal{T}(\zeta')$, with $\mathcal{T} : \mathcal{G}_{r/2} \to \mathcal{G}_{3r/4}$ (recall that $r := R^{1.5}$) such that

$$H \circ \mathcal{T} = \sum_{i=1}^{N+1} \nu_i J_i + Z + \mathcal{R} \ , \tag{7.1}$$

where Z is in normal form, i.e. $\{Z, J_i\} \equiv 0$ ($i = 1, ..., N+1$), $\mathcal{R}$ is exponentially small with R, namely satisfies

$$\frac{1}{r} \sup_{\zeta' \in \mathcal{G}_{r/2}} \left\| \nabla^{\Omega} \mathcal{R}(\zeta') \right\| \preceq \mu R^2 \mu^{1/(\tau+1)} \exp \left[- \left(\frac{C_4}{\mu} \right)^{1/(\tau+1)} \right] \ ; \tag{7.2}$$

moreover, for any analytic function $g : \mathcal{G}_r \to \mathcal{B}$, where $\mathcal{B}$ is a Banach space, one has

$$\operatorname*{Sup}_{\zeta' \in \mathcal{G}_{r/2}} \|g(\zeta') - g(\mathcal{T}(\zeta'))\|_{\mathcal{B}} \preceq \mu^{1/(\tau+1)} \operatorname*{Sup}_{\zeta \in \mathcal{G}_r} \|g(\zeta)\|_{\mathcal{B}} \ . \tag{7.3}$$

□

Proof. We have

$$\Phi_t^l(\zeta) = (I_k, \phi_k + (4 - \delta_k^l)\nu_l t, u_k e^{i4\nu_l t}) \ , \quad l = 1, ..., N \ ,$$
$$\Phi_t^{N+1}(\zeta) = (I_k, \phi_k + k^2 t, u_k e^{ik^2 t})$$

where $u_k = p_k + iq_k$. It follows

$$\Psi_\psi(\zeta) = (I_k, \phi_k - \psi_k + 4 \sum_{l=1}^{N} \psi_l, u_k e^{i4 \sum_{l=1}^{N} \psi_l}) \ .$$

Then, choosing σ small enough, hypothesis 4) holds, and $c_{\mathcal{T}} \leq 2$. Hypothesis 5) is also satisfied. By (5.23) one has $\omega_f \preceq r^2 + |\epsilon|/r^2$. We come to (6.3); we show now that it holds with $\gamma = R^2 \beta$ and any M satisfying

$$M \leq \frac{1}{R^2 \sup_i \omega_i} - \beta \ . \tag{7.4}$$

The statement is obvious when $n = 0$. Consider the case $|n| \geq 1$ assuming $|k| \leq M$, we have

$$|n + R^2 \sum_{i=1}^{N} \omega_i k_i| \geq 1 - R^2 M \sup_i \omega_i \ ,$$

and the r.h.s. is larger than βR^2 provided (7.4) holds. So we have $\mu = R + |\epsilon|/R^5$. It is now clear that it is possible to choose M in such a way that both (6.4) and (7.4) hold. So we can apply theorem 6.2, and therefore the thesis holds. $\square$

Remark 7.2. Particular cases of (7.3) are

$$|J_i \circ \mathcal{T} - J_i| \preceq \mu^{1/(\tau+1)} R^3 \ . \tag{7.5}$$

$\square$

Remark 7.3. One has (by Cauchy inequality)

$$\begin{aligned}\sup_{\zeta' \in \mathcal{G}_{r/2}} \left|\dot{J}_i(\zeta')\right| = \sup_{\zeta' \in \mathcal{G}_{r/2}} \left|\left\{J^i, \mathcal{R}\right\}(\zeta')\right| \le \frac{2}{r} \sup_{\zeta' \in \mathcal{G}_r} |J_i(\zeta')| \sup_{\zeta' \in \mathcal{G}_{r/2}} \left\|\nabla^\Omega \mathcal{R}(\zeta')\right\| \\ \preceq R^5 \mu \mu^{1/(\tau+1)} \exp\left[-\left(\frac{C_4}{\mu}\right)^{1/(\tau+1)}\right] \ .\end{aligned} \tag{7.6}$$

$\square$

From lemma 7.1 we obtain that the functions $J_i(\zeta')$ move exponentially slowly. We aim to use such functions as Lyapunov functions in order to show that the distance of the solution from the torus $I' = p' = q' = 0$ increases exponentially slowly. To this end we will need the following

Lemma 7.4. *For any $\rho > 0$ consider real (p, q, I, ϕ) and assume*

$$|J_i(\zeta)| \le \rho \ , \quad i = 1, ..., N+1 \ ; \tag{7.7}$$

then there exists a constant C_{11} such that

$$\sum_{i \ge N+1} i^2 \frac{p_k^2 + q_k^2}{2} \le C_{11}\rho \ , \quad |I_k| \le C_{11}\rho \ , \quad k = 1, ..., N. \tag{7.8}$$

$\square$

Proof. Define

$$A := 4\sum_{l=1}^{N} I_l \ , \quad J := 4 \sum_{k \ge N+1} \frac{p_k^2 + q_k^2}{2} \ ,$$

so that, one has $J_i = -I_l + A + J$, $i = 1, ..., N$. The first N of (7.7) are therefore equivalent to

$$A + J - \rho \le I_l \le A + J + \rho \ ,$$

inserting these inequalities in the definition of A solving with respect to A and substituting again in the above inequalities we get

$$-\frac{1}{4N-1} J - \frac{8N-1}{4N-1}\rho \le I_l \le -\frac{1}{4N-1} J + \frac{8N-1}{4N-1}\rho \ .$$

Inserting in the last of (7.7), and remembering the definition of J we get

$$\sum_{i\geq N+1} \left[i^2 - \alpha(N)\right] Z_i \leq \left(1 + \frac{8N-1}{4}\alpha(N)\right) \rho \ ,$$

where

$$\alpha(N) := 4\frac{\sum_{l=1}^{N} l^2}{4N-1} \ .$$

Recalling that

$$\sum_{l=1}^{N} l^2 = N\left(\frac{N^2}{3} + \frac{N}{2} + \frac{1}{6}\right) \ ,$$

it is easy to see that $(N+1)^2 - \alpha(N)$ is positive, so that there exists a positive constant such that

$$\inf_{i\geq N+1} \frac{i^2 - \alpha(N)}{i^2} \geq c > 0 \ .$$

From this the thesis follows. □

Remark *7.5.* The above lemma has an interesting geometrical interpretation. Indeed it can be shown to be equivalent to the statement that J_{N+1} restricted to the surface $J_i = 0$, $i = 1, ..., N$ has a minimum at zero. □

Remark *7.6.* By lemma 7.4 one has that there exists C_{12}, such that, provided ρ is small enough

$$|J_i(\zeta)| \leq \rho \implies d(\zeta, \mathcal{M}) \leq C_{12}\left(\frac{\rho}{r} + \sqrt{\rho}\right) \ , \tag{7.9}$$

where $\mathcal{M}$ was defined by (5.16). □

We have now the following

Theorem **7.7.** *Consider the Hamiltonian system (5.21) with real initial data. There exists μ_*, C_4, and C_{13} such that, if $\mu \leq \mu_*$ then corresponding to initial data such that*

$$\sum_{i=1}^{N} i^2 |I_i(\zeta(0))| + \sum_{i\geq N+1} i^2 Z_i(\zeta(0)) \leq C_{13} R^3 \mu^{1/(\tau+1)} \tag{7.10}$$

one has

$$\sum_{i=1}^{N} i^2 |I_i(\zeta(t))| + \sum_{i\geq N+1} i^2 Z_i(\zeta(t)) \preceq R^3 \mu^{1/(\tau+1)}$$

for all times satisfying

$$|t| \preceq \frac{1}{R^2 \mu} \exp\left[\left(\frac{C_4}{\mu}\right)^{1/(\tau+1)}\right] \ . \tag{7.11}$$

□

Proof. Eq. (7.10) implies $|J_i(\zeta(0))| \preceq R^3\mu^{1/(\tau+1)}$ $\forall i = 1, ..., N+1$, and $\zeta(0) \in \mathcal{G}_{r/2}$, so we can apply lemma 7.1 implying (7.5), from which

$$|J_i(\zeta'(0))| \preceq R^3\mu^{1/(\tau+1)} , \tag{7.12}$$

and moreover (7.6) holds. It implies

$$|J_i(\zeta'(t)) - J_i(\zeta'(0))| \preceq R^3\mu^{1/(\tau+1)} \tag{7.13}$$

provided t satisfies (7.11) and $\zeta'(t) \in \mathcal{G}_{r/2}$. We prove such an inclusion. By (7.12) and (7.13) it follows

$$|J_i(\zeta'(t))| \preceq R^3\mu^{1/(\tau+1)} ,$$

which by (7.9) implies $\zeta'(t) \in \mathcal{G}_{CR\mu^{1/2(\tau+1)}}$, which provided μ is small enough implies $\zeta'(t) \in \mathcal{G}_{r/2}$. Using again (7.5) we get the thesis. □

Theorem 2.1 is a simple corollary of theorem 7.7. We have

Proof of theorem 2.1. First we prove that (2.4) implies (7.10). Indeed, denoting by u' the function whose coordinates are ζ, and by $u := \mathcal{T}_1(u')$ we have

$$\begin{aligned}
\sum_{i=1}^{N} i^2|I_i| + \sum_{i\geq N+1} i^2\frac{p_i^2+q_i^2}{2} &= \sum_{i=1}^{N} i^2 \left|Z_i(u') - \nu_i\right| + \sum_{i\geq N+1} i^2 Z_i(u') \\
&\leq \sum_{i=1}^{N} i^2 \left|Z_i(u) - Z_i(u')\right| + \sum_{i=1}^{N} i^2 \left|Z_i(u) - \nu_i\right| \\
&+ \left|\sum_{i\geq N+1} i^2 \left(Z_i(u') - Z_i(u)\right)\right| + \sum_{i\geq N+1} i^2 Z_i(u) .
\end{aligned}$$

Using (5.14) and (5.15) we obtain

$$\sum_{i=1}^{N} i^2|I_i| + \sum_{i\geq N+1} i^2\frac{p_i^2+q_i^2}{2} \preceq R^4 + \sum_{i\geq 1}|Z_i(u) - \nu_i| ,$$

which shows that indeed (2.4) implies (7.10). Using the analogous of (5.14) and (5.15) for $\mathcal{T}_1^{-1}$ it is clear that also the converse estimate holds, and therefore that theorem 7.7 implies theorem 2.1. □

A slightly different description of the dynamics is given by the following

Theorem 7.8. *There exist μ_* and C_{14} such that, if $\mu \leq \mu_*$ then there exists a manifold $\mathcal{N} \subset \mathcal{P}$ isomorphic to an N dimensional torus, with the following properties*

1) It is stable over exponentially long times, namely

$$\begin{aligned}
&d(\zeta(0), \mathcal{N}) \leq C_{14} r \qquad \text{implies} \\
&d(\zeta(t), \mathcal{N}) \preceq \sqrt{r d(\zeta(0), \mathcal{N}) + |t|\mu R^5 \mu^{1/(\tau+1)} \exp\left[-\left(\frac{C_4}{\mu}\right)^{1/(\tau+1)}\right]} ,
\end{aligned} \tag{7.14}$$

for the times satisfying (7.11).

2) $\mathcal{N}$ is close to the manifold $\mathcal{M}$ defined by (5.16) (and therefore also to $\mathcal{M}_{R,\omega}$), namely one has

$$\sup_{\zeta\in\mathcal{M}} d(\zeta,\mathcal{N}) \preceq r\mu^{1/2(\tau+1)} \ . \tag{7.15}$$

□

Proof. $\mathcal{N}$ is defined as the set such that $I'_k = p'_l = q'_l = 0$, $\forall\, k = 1,...,N$, $\forall\, l \geq N+1$. Denote for simplicity

$$\rho := d(\zeta(0),\mathcal{N}) \ ,$$

then, since the Lipschitz constant of $\mathcal{T}$ is bounded we have

$$d(\zeta'_0,\mathcal{N}) \preceq \rho \ ,$$

with $\zeta'_0 := \mathcal{T}^{-1}(\zeta(0))$; it follows, by the very definition of the norm and of $\mathcal{N}$

$$|J_i(\zeta'_0)| \preceq (r+\rho)\rho \preceq r\rho \ ,$$

and, by (7.6),

$$|J_i(\zeta'(t))| \preceq r\rho + |t|\mu R^5 \mu^{1/(\tau+1)} \exp\left[-\left(\frac{C_4}{\mu}\right)^{1/(\tau+1)}\right] \ ;$$

using (7.9) we get

$$d(\zeta'(t),\mathcal{N}) \preceq \sqrt{r\rho + |t| R^5 \mu^{1/(\tau+1)} \exp\left[-\left(\frac{C_4}{\mu}\right)^{1/(\tau+1)}\right]} \ ,$$

and, by the boundedness of the Lipschitz constant of $\mathcal{T}_1^{-1}$ we get (7.14).

To prove (7.15) use (7.3) which gives

$$I_k = p_l = q_l = 0 \Longrightarrow |I'_k| \preceq \mu^{1/(\tau+1)} R^3 \ , \qquad \sum_{i\geq N+1} i^2 \frac{p_i'^2 + q_i'^2}{2} \preceq \mu^{1/(\tau+1)} R^3 \ ,$$

from which the thesis immediately follows. □

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