SU-4240-657
February 1997

Bringing Up a Quantum Baby*

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Abstract

Any two infinite-dimensional (separable) Hilbert spaces are unitarily isomorphic. The sets of all their self-adjoint operators are also therefore unitarily equivalent. Thus if all self-adjoint operators can be observed, and if there is no further major axiom in quantum physics than those formulated for example in Dirac's 'Quantum Mechanics', then a quantum physicist would not be able to tell a torus from a hole in the ground. We argue that there are indeed such axioms involving vectors in the domain of the Hamiltonian: The "probability densities" (hermitean forms) $\psi^\dagger\chi$ for ψ, χ in this domain generate an algebra from which the classical configuration space with its topology (and with further refinements of the axiom, its C^K- and $C^\infty-$ structures) can be reconstructed using Gel'fand - Naimark theory. Classical topology is an attribute of only certain quantum states for these axioms, the configuration space emergent from quantum physics getting progressively less differentiable with increasingly higher excitations of energy and eventually altogether ceasing to exist. After formulating these axioms, we apply them to show the possibility of topology change and to discuss quantized fuzzy topologies. Fundamental issues concerning the role of time in quantum physics are also addressed.

*Talk presented at the Workshop on Frontiers in Field Theory, Quantum Gravity and String Theory, Puri, December 12-21, 1996.

1 Introduction

This note will report on certain ongoing research with several colleagues concerning the nature of space and time in quantum physics. Some of our ideas have already been published elsewhere [1, 2, 3]. Our work touches both on issues of relevance to quantum gravity such as the meaning of "quantized topology" and the possibility of topology change, and on topics of significance for foundations of quantum physics. I think that we have progressively approached a measure of precision in the formulation of relatively inarticulated questions, but our responses are still tentative and lacking in physical and mathematical completeness and rigor.

2 The Problem as a Parable

The problem to be addressed here is best introduced as a little story about a quantum baby. The story will set the framework for the rest of the talk. Its proper enjoyment calls for a willing suspension of disbelief for the moment.

All babies are naturally quantum, so my adjective for the baby can be objected to as redundant and provocative, but it invites attention to a nature of infants of central interest to us, so let us leave it there.

Parable of the Quantum Baby

Entertain the conjecture of a time, long long ago, when there lived a quantum baby of cheerful semblance and sweet majesty. It was brought up by its doting parents on a nourishing diet of self-adjoint operators on a Hilbert space. All it could experience as it grew up were their mean values in quantum states. It did not have a clue when it was little that there is our classical world with its topology, dimension and metric. It could not then tell a torus from a hole in the ground.

Yet the baby learned all that as it grew up.

And the wise philosopher is struck with wonder: How did the baby manage this amazing task?

For the problem is this: Even in a quantum theory emergent from a smooth classical configuration space Q^∞, there is no need for a wave function ψ, or a probability density $\psi^*\psi$, to be continuous on Q^∞. It is enough that the integral $\int \omega\psi^*\psi$ over Q^∞ for an appropriate volume form ω is finite. Probability interpretation requires no more.

But if the baby can observe all self-adjoint operations with equal ease, and thereby prepare all sorts of discontinuous quantum states, how then does it ever learn of Q^∞, its topology and its differential attributes?

This then is our central question. All that follows is charged with its emotional content, and comes from trying to find its answer.

3 Another Formulation

We can explain the baby problem in yet another way.

In classical physics, we generally deal with dynamics on a ($C^\infty-$) manifold, the classical configuration space Q^∞. Observables are (suitably) smooth functions in the cotangent bundle T^*Q^∞ while states are similarly smooth probability distributions on that space. Time evolution preserves these smooth structures. There is no particular need to go beyond them and abandon the emphasis on smoothness. In this sense, classical physics incorporates the ideas of underlying manifolds, it knows about them all along. They are generally *a priori* and irreducible concepts for classical physics, with no model for their emergence and immune to analysis, but all the same essential in the formulation of its physical laws.

In quantum physics instead, time evolution is specified by a unitary operator $U(t)$, continuous in t, on a (separable [4]) Hilbert space $\mathcal{H}$. [We will deal only with separable Hilbert spaces.] The latter is generally infinite-dimensional.

But unlike Q^∞, all infinite-dimensional Hilbert spaces are isomorphic, in fact unitary so. If $|n>^{(i)}$ $(n \in \mathbb{N})$ gives the orthonormal basis for the Hilbert space $\mathcal{H}^{(i)}(i = 1, 2)$, we can achieve this equivalence by setting $|n>^{(2)}= V|n>^{(1)}$. That being so, any operator $A^{(1)}$ on $\mathcal{H}^{(1)}$ has a corresponding operator $A^{(2)} = VA^{(1)}V^{-1}$ on $\mathcal{H}^{2)}$.

How then does a quantum baby tell a torus from a hole in the ground?

Without further structure in quantum physics besides those to be found in standard text books, this task is in fact entirely beyond the baby.

In conventional quantum physics, we generally start from smooth functions (or smooth sections of hermitean vector bundles) on Q^∞ and complete them into a Hilbert space $\mathcal{H}$ using a suitable scalar product. In this way, we somehow incorporate knowledge about Q^∞ right at the start.

But this approach requires realizing $\mathcal{H}$ in a particular way, as square integrable functions on Q^∞. The presentation of $\mathcal{H}$ in this manner is reminiscent of the presentation of a manifold in a preferred manner, as for instance using a particular coordinate chart.

Can we give a reconstruction of Q^∞ in an intrinsic way? What new structures are needed for this purpose?

In the scheme we develop as a response to these questions, Q^∞ emerges with its $C^\infty-$ structure only from certain states, *topology and differential features being attributes of particular classes of states and not universal properties of all states.* Thus Q^∞ emerges as a manifold only if the high energy components in the observed states are strongly suppressed. When higher and higher energies are excited, it gets more rough and eventually altogether ceases to exist as a topological space modelled on a manifold. Here by becoming more rough we mean that C^∞ becomes C^K and correspondingly the $C^\infty-$ manifold Q^∞ becomes a C^K- space Q^K.

4 What is Our Quantum System?

The system we consider is that of a single particle, not that of quantum fields or quantized strings. The configuration space of a quantum field has a very complex topology and its relation to our three-dimensional spatial world is also nontrivial [5]. Quantum strings too share features of quantum fields. For these reasons, we will not examine these systems. For simplicity, we will also assume that the particle has no "internal" attributes like spin or flavor.

5 Time is Special

We have to assume that time evolution is given as a unitary operator $U(t)$ which is continuous in t. Our analysis needs this input. Time therefore persists as an *a priori* irreducible notion even in our quantum approach. It would be very desirable to overcome this limitation. (See [6] in this connection.)

There is more to be said on time, its role in measurement theory and as the mediation between quantum and classical physics. The appendix has brief remarks on these matters.

It is true that in so far as our main text is concerned, $U(t)$ or the Hamiltonian can be substituted by spatial translations, momenta or other favorite observables. But we think that time evolution is something special, being of universal and central interest to science. It is for this reason that we have singled out $U(t)$.

6 The Gel'fand-Naimark Theory

The principal mathematical tool of our analysis involves this remarkable theory [7] and, to some extent its developments in Noncommutative Geometry [8, 9, 10, 11, 12]. We shall now give a crude and short sketch of this theory.

A C^*-algebra $\overline{\mathcal{C}}$ with elements c has the following properties: a) It is an algebra over $\mathbb{C}$. b) It is closed under an antinvolution $*$:

$$* : c_j \in \overline{\mathcal{C}} \Rightarrow c_j^* \in \overline{\mathcal{C}},\ c_j^{**} = c_j,\ (c_1 c_2)^* = c_2^* c_1^*,\ (\xi c_j)^* = \xi^* c_j^*, \tag{6.1}$$

where ξ is a complex number and ξ^* is its complex conjugate. c) It has a norm $||.||$ with the properties $||c^*|| = ||c||$, $||c^*c|| = ||c||^2$ for $c \in \overline{\mathcal{C}}$. d) It is complete under this norm. (Hence the bar over $\mathcal{C}$).

A $*-$ representation ρ of $\overline{\mathcal{C}}$ on a Hilbert space $\mathcal{H}$ is the representation of $\overline{\mathcal{C}}$ by a C^*-algebra of bounded operators on a Hilbert space [13] with the following features: i) The $*$ and norm for $\rho(\overline{\mathcal{C}})$ are the operator adjoint $\dagger$ and operator norm (also denoted by $||\cdot||$). ii) $\rho(c^*) = \rho(c)^\dagger$.

ρ is said to be a $*$-homomorphism because of ii). We can also in a similar manner speak of $*$-isomorphisms.

We will generally encounter $\overline{\mathcal{C}}$ concretely as an algebra of operators. In any case, we will usually omit the symbol ρ.

Note that a $*$-algebra (even if it is not C^*) is by definition closed under an antinvolution $*$.

Let $\overline{\mathcal{A}}$ denote a commutative C^*-algebra. [There is no need for now to think of it as the closure of some $\mathcal{A}$.] Let $\{x\}$ denote its space of inequivalent irreducible $*$-representations (IRR's) or its spectrum. [So $a \in \overline{\mathcal{A}} \Rightarrow x(a) \in \mathbb{C}$.] The Gel'fand-Naimark theory then makes the following striking assertions: α) There is a natural topology on $\{x\}$ making it into a Hausdorff topological space [14] Q^0. [We will denote the IRR's prior to introducing topology by $\{x\}$ and after doing so by $Q^{\cdot}$ with suitable superscripts.] β) Let $\overline{\mathcal{A}}_c$ be the

C^*-algebra of $\mathbb{C}$-valued continuous functions on Q^0. Its $*$ is complex conjugation and its norm $||\cdot||$ is the supremum norm, $||\phi|| = \sup_{x\in Q^0} |\phi(x)|$. Then $\overline{\mathcal{A}}_c$ is $*$-isomorphic to $\overline{\mathcal{A}}$.

We can thus identify $\overline{\mathcal{A}}_c$ with $\overline{\mathcal{A}}$, as we will often do.

The above results can be understood as follows. By "duality", the collection of $x(a)$'s for all x defines a function a_c on $\{x\}$ by $a_c(x) := x(a)$. a_c is said to be the Gel'fand transform of a.

$\{x\}$ is as yet just a collection of points with no topology. How can we give it a natural topology? We want a_c to be C^0 in this topology. Now the set of zeros of a continuous function is closed. So let us identify the set of zeros C_a of each a_c with a closed set:

$$C_a = \{x : x(a) \equiv a_c(x) = 0\}. \tag{6.2}$$

The topology we seek is given by these closed sets. The Gel'fand-Naimark theorem then asserts α) and β) for this topology, the isomorphism $\overline{\mathcal{A}} \to \overline{\mathcal{A}}_c$ being $a \to a_c$.

A Hausdorff topological space can therefore be equally well described by a commutative C^* -algebra $\overline{\mathcal{A}}$, presented for example using generators. That would be an intrinsic coordinate-free description of the space and an alternative to using coordinate charts.

A C^K - structure can now be specified by identifying an appropriate subalgebra $\mathcal{A}^K$ of $\overline{\mathcal{A}} \equiv \mathcal{A}^0$ and declaring that the C^K - structure is the one for which $\mathcal{A}^K$ consists of K-times differentiable functions. [$\mathcal{A}^K$ is a $*-$, but not a C^*-, algebra for $K > 0$.] The corresponding C^K-space is Q^K. For $K = \infty$, we get the manifold Q^∞. We have the inclusions

$$\mathcal{A}^\infty \subset ... \subset \mathcal{A}^K ... \subset \mathcal{A}^0 \equiv \overline{\mathcal{A}} \tag{6.3}$$

where

$$\overline{\mathcal{A}}^{(\infty)} = \overline{\mathcal{A}}^{(K)} = \overline{\mathcal{A}}, \tag{6.4}$$

the bar as usual denoting closure.In contrast, Q^∞ and Q^K are all the same as sets, being $\{x\}$.

A dense $*$-subalgebra of a C^*-algebra $\overline{\mathcal{C}}$ will be denoted by $\mathcal{C}$ or $\mathcal{C}^\cdot$, the superscript highlighting some additional property. The algebras $\mathcal{A}^K$ are examples of such $\mathcal{C}^K$ for $\overline{\mathcal{C}} = \overline{\mathcal{A}}$. We will also reserve the symbols $\mathcal{A}$ and $\mathcal{A}^\cdot$ for the algebras supposedly or otherwise leading to the classical configuration space, the latter if it is C^K being Q^K.

Example 1: Consider the algebra $\mathcal{A}$ generated by the identity, an element u and its inverse u^{-1}. Its elements are $a = \sum_{N\in\mathbb{Z}} \alpha_N u^N$ where α_N's are complex numbers vanishing rapidly in N at ∞. The $*$ is defined by $u^* = u^{-1}, a^* = \sum a_N^* u^{-N}$. As $\mathcal{A}$ has identity $\mathbb{1}$, there is a natural way to define inverse a^{-1} too : a^{-1} is that element of $\mathcal{A}$ such that

$a^{-1}a = aa^{-1} = \mathbf{1}$. There is also a canonical norm $||.||$ compatible with properties c) [8, 10]: $||a||$ = Maximum of $|\lambda|$ such that $a^*a - |\lambda|^2$ has no inverse.

The space Q^∞ for this $\mathcal{A}$ is just the circle S^1, u_c being the function with value $e^{i\theta}$ at $e^{i\theta} \in S^1$.

If similarly we consider the algebra associated with N commuting unitary elements, we get the N-torus T^N. If for $N = 2$, the generating unitary elements do not commute, but fulfill $u_1 u_2 = \omega u_2 u_1, \omega$ being any phase, we get the noncommutative torus [15, 8]. It is the "rational" or "fuzzy" torus if $\omega^K = 1$ for some $K \in \mathbb{Z}$, otherwise it is "irrational" [12, 16].

7 What Time Evolution Tells Us

Let $\mathcal{H}^0 \equiv \overline{\mathcal{H}}^0$ denote the Hilbert space of state vectors. [The bar has been introduced as usual to emphasize norm-completeness.] The unitary operator $U(t)$ is defined on all of $\mathcal{H}^0$ and is continuous there. Let us assume that $U(t)$ has a discrete spectrum with orthonormal eigenvectors ϕ_n spanning $\mathcal{H}^0$:

$$U(t)\phi_n = e^{-iE_n t}\phi_n,\ \ (\phi_n, \phi_m) = \delta_{nm}, n \in \mathbb{N}. \tag{7.1}$$

The completeness of $\{\phi_n\}$ means that every $\psi^0 \in \mathcal{H}^0$ has the expansion

$$\psi^0 = \sum_{n\in\mathbb{N}} a_n^0 \phi_n \ \text{ with } \sum_{n\in\mathbb{N}} |a_n^0|^2 = (\psi^0, \psi^0) < \infty. \tag{7.2}$$

The time evolution of ψ^0 is

$$U(t)\psi^0 = \sum a_n^0 e^{-iE_n t}\psi^0. \tag{7.3}$$

But $U(t)$ is not always differentiable on all of $\mathcal{H}^0$. That requires that a_n^0 fulfill $\sum |a_n^0|^2 E_n^2 < \infty$. [Otherwise the norm of $\sum a_n^0 \frac{d}{dt}(e^{-iE_n t}\psi^0)$ diverges.] Let a_n^1 denote these a_n^0's, ψ^1's the corresponding vectors and $\mathcal{H}^1$ the subspace of these vectors. Then

$$\mathcal{H}^1 = \{\psi^1 \in \mathcal{H}^0 : \psi^1 = \sum a_n^1 \phi_n \text{ with } \sum |a_n^1|^2 |E_n^2| < \infty\ \}, \tag{7.4}$$

$$\frac{dU(t)}{dt}\psi^1 = -i \sum a_n^1 e^{-iE_n t}\phi_n. \tag{7.5}$$

$\mathcal{H}^1$ is a dense subspace of $\mathcal{H}^0$:

$$\mathcal{H}^1 \subseteq \mathcal{H}^0, \quad \overline{\mathcal{H}}^1 = \mathcal{H}^0 \equiv \overline{\mathcal{H}}^0. \tag{7.6}$$

It is the domain [13] of the Hamiltonian H, the latter being defined by $H\psi^1 = -i\frac{dU(t)}{dt}\psi^1|_{t=0}$.

It could of course happen that $\mathcal{H}^1 = \mathcal{H}^0$. That would be the case if $\mathcal{H}^0$ is finite dimensional, or if $|E_n|$ does not diverge as $n \to \infty$ (that is if H is bounded [13]). In either case, there is no classical underlying manifold in our approach, *unbounded operators being of essential significance for the recovery of classical attributes.* So we consider only systems not covered by the above possibilities. Then we have the strict inclusion $\mathcal{H}^1 \subset \mathcal{H}^0$.

The subspace $\mathcal{H}^K$ where $U(t)$ is K-times differentiable is found in a similar way,

$$\mathcal{H}^K =< \psi^K \in \mathcal{H}^0 : \psi^K = \sum a_n^K \phi_n, \sum |a_n^K|^2 |E_n|^M < \infty \ \text{ for all } \ M \in \{0, 1, \ldots, K\} >, \tag{7.7}$$

while

$$\mathcal{H}^\infty =< \psi^\infty \in \mathcal{H}^0 : \psi^\infty = \sum a_n^\infty \phi_n, \sum |a_n^\infty|^2 |E_n|^M < \infty \ \text{ for all } \ M \in \mathbb{N} > . \tag{7.8}$$

We have

$$\mathcal{H}^\infty \subset \ldots \subset \mathcal{H}^K \subset \ldots \subset \mathcal{H}^0 \equiv \overline{\mathcal{H}}^0. \tag{7.9}$$

Note how higher and higher energies are suppressed in $\mathcal{H}^K$ as we go up in K.

Example 2: For a nonrelativistic particle on $\mathbf{R}^3$ with Hamiltonian $H = -\frac{1}{2m}\vec{\nabla}^2 + a$ a smooth bounded potential, $\mathcal{H}^\infty = C^\infty(\mathbf{R}^3) \cap L^2(\mathbf{R}^3)$.

8 Axioms on Probability Densities

Let us first note a few heuristic results.

Let W be a commutative $*$-algebra of bounded operators on $\mathcal{H}^0$. Let $\{x\}$ be its spectrum, $|x>$ the corresponding states (assumed nondegenerate) in $\mathcal{H}^0$ and $I = \int \omega |x><x|$ the resolution of identity.

For $\psi^0,\ \chi^0 \in \mathcal{H}^0$, we can now define the analogue of a probability density, a hermitean form $\psi^{0\ \dagger}\chi^0$ which is a function on $\{x\}$, by

$$\psi^{0\ \dagger}\chi^0(x) =< \psi^0|x >< x|\chi^0 > . \tag{8.1}$$

It is L^1 since

$$\int \omega\psi^{0\ \dagger}\chi^0(x) = (\psi^0, \chi^0). \tag{8.2}$$

We now come to our central axioms for the recovery of classical topology from quantum physics. *They are*, in so far as we can tell, *new*, *to be added on to existing quantum principles whenever we desire an emergent classical topology.*

Let $\mathcal{B}$ be a *-algebra with properties a) to c) of Section 6, spectrum $Q^{\mathcal{B}}$ with points x, corresponding state vectors $|x>$ (assumed non-degenerate) and resolution of unity $I = \int \omega|x >< x|$.

We concentrate now on $\mathcal{H}^\infty$. For $\psi^\infty,\ \chi^\infty$ in $\mathcal{H}^\infty$, we have then an integrable function $\psi^{\infty\dagger}\chi^\infty$ on $\{x\}$ as in the above construction. It will not be in the Gel'fand transform $\mathcal{B}_c$ of $\mathcal{B}$ for an arbitrary choice of $\mathcal{B}$.

The Axioms

- A1. There exists at least one choice $\mathcal{A}^\infty$ for $\mathcal{B}$ such that $\psi^{\infty\dagger}\chi^\infty$ as constructed above is in $\mathcal{A}^\infty$ and generates it. [We are here identifying $\mathcal{A}^\infty$ with $\mathcal{A}_c^\infty$.]
- A2. (Locality): $\mathcal{A}^\infty$ is *local*, that is, there exists an $M \in \mathbb{N}^+$ such that

$$[a_K, [a_{K-1}, \ldots [a_0, H] \ldots] = 0 \ \text{ for } \ \forall a_i \in \mathcal{A}^{(\infty)} \ \text{ and } \ K \geq M. \tag{8.3}$$

- A3. If the choice of $\mathcal{A}^\infty$ with properties A1,2 is not unique for a particular system, it does become unique when a sufficiently large number of systems are considered, $\mathcal{A}^\infty$ being the common algebra to be found among all of them.
- A4. The classical configuration space as a topological space is the spectrum Q^0 of $\overline{\mathcal{A}}^\infty = \overline{\mathcal{A}}$ whereas it is given as a manifold Q^∞ by treating $\mathcal{A}^\infty$ as C^∞-functions.

Explanations

- A1: For example 1, $\mathcal{A}^\infty$ is just $C^\infty(\mathbf{R}^3) \cap L^1(\mathbf{R}^3)$.

- A2: Equation (8.3) with $K = M$ of course implies its validity for all $K > M$.

  For a particle on S^1 say, with A1 alone, $\mathcal{A}^\infty$ can be smooth functions on *either* S^1 *or* $\mathbb{Z}$ (the quantum momentum space). A2 attempts to sort out this sort of ambiguity since interactions generally are local only on configuration space. When the latter is a standard manifold, the meaning of A2 is that H must be a differential operator of finite order. The *order* of H then is just the *least* value of K. We shall adopt this always as our definition of order of H: *It is the least value of K for which* (8.3) *is true.*

- A3: The algebra $\mathcal{A}^\infty$ may not be fixed even with A2. For a free particle on S^1, we still have the possibility of considering functions on S^1 or $\mathbb{Z}$ as $\mathcal{A}^\infty$. The intention of A3 is to resolve this ambiguity by considering several interactions, the assumption being that they will always be local on configuration space, but not so on other spaces.

  The Hamiltonian $[\underline{p}^2 + m^2]^{\frac{1}{2}}$ for a relativistic particle of mass m $[\vec{p} =$ momentum$]$ does not fit in our scheme, being nonlocal on configuration space. It is also generally rejected in the presence of interactions for this nonlocality, so perhaps we need not become anxious thinking of this operator.

  In any case, a better principle than our locality to fix $\mathcal{A}^\infty$ uniquely would be desirable.

- A4: I fixes only $\omega|x><x|$, and not ω and $|x><x|$ separately. This fact creates uncertainties in the definition of $\psi^{\infty\,\dagger}\chi^\infty$. One way to resolve this uncertainty adequately is to choose ω as some particular smooth volume form ω_0 on Q^∞, thereby defining $|x><x|$. Equivalently, we can fix a smooth volume form ω_0 on Q^∞ and define $\psi^{\infty\,\dagger}\chi^\infty$ by the equality $\omega<\psi^\infty|x><x|\chi^\infty>=\omega_0\psi^{\infty\,\dagger}\chi^\infty(x)$. Alternatively, we can modify A1 to the requirement that there exists a choice $\mathcal{A}^\infty$ for $\mathcal{B}$ such that the set of $\omega\psi^{\infty\,\dagger}\chi^\infty$ forms an $\mathcal{A}^\infty$-module.

Further Remarks

Once $\mathcal{A}^\infty$ has been fixed, we can construct the functions $\psi^{K^\dagger}\chi^K$ from $\psi^K,\ \chi^K \in \mathcal{H}^K$. With decreasing K, they would give vectors which are fewer and fewer times differentiable. Eventually for $K = 0$, they would all be defined only in the L^1-sense. Thus when higher and higher energies are increasingly excited, things fall apart and the topology of configuration space increasingly gets rougher. It disappears altogether as a topological space modeled on a manifold when $K = 0$ is reached. This interesting point was first discussed in ref. 2.

This comment can be rephrased in terms of modules of forms as in the comments above on A4.

The tie-up between $\mathcal{H}^\infty$ and $\mathcal{A}^\infty$ in our approach is not prompted by random fancy. It is this connection that links time evolution to classical spatial topology.

Incidentally it can checked in standard examples, as for instance the commutative tori of example 1, that $\mathcal{H}^\infty$ are $\mathcal{A}^\infty$ - modules.

9 Dimension and Metric

The topological space $Q^{\cdot}$ constructed with the level of generality maintained hitherto need not even resemble a manifold for our limited axioms. Further conditions like those discussed by Connes [10] are necessary to avoid what a classical physicist may consider as pathologies.

Suppose that such conditions are also met and that Q^∞ is a manifold. We can then find its dimension in the usual way.

There is also a novel manner to find its dimension d from the spectrum $\{\lambda_n\}$ of H: If H is of order N, $|\lambda_n|$ grows like $n^{N/d}$ as $n \to \infty$ [8, 9, 10, 17].

We can find a metric as well for Q^∞ [8, 10, 17]: It is specified by the distance

$$d(x, y) = \{\sup_a |a_c(x) - a_c(y)| : \frac{1}{N!} || \underbrace{[a, [a, \ldots [a, H] \ldots]}_{N\ a's} || \leq 1\}. \tag{9.1}$$

This remarkable formula gives the usual metric for the Dirac operator $[N = 1]$ [8, 9] and the Laplacian $[N = 2]$ [17].

10 What is Quantum Topology?

A question of the following sort often suggests itself when encountering discussions of topology in quantum gravity: If Q is a topological space, possibly with additional differential and geometric structures ["classical" data], what is meant by *quantizing* Q?

It is perhaps best understood as: *finding an algebra of operators on a Hilbert space from which Q and its attributes can be reconstructed* [much as in the Gel'fand-Naimark theorem].

11 Topology Change

We now use the preceding ideas to discuss topology change, following ref. 2. [See ref. [18] for related work.]

There are indications from theoretical considerations that spatial topology in quantum gravity cannot be a time-invariant attribute, and that its transmutations must be permitted in any eventual theory.

The best evidence for the necessity of topology change comes from the examination of the spin-statistics connection for the so-called geons [19, 20, 21]. Geons are solitonic excitations caused by twists in spatial topology. In the absence of topology change, a geon can neither annihilate nor be pair produced with a partner geon, so that no geon has an associated antigeon.

Now spin-statistics theorems generally emerge in theories admitting creation-annihilation processes [5, 20, 22]. It can therefore be expected to fail for geons in gravity theories with no topology change. Calculations on geon quantization in fact confirm this expectation [20, 23].

The absence of a universal spin-statistics connection in these gravity theories is much like its absence for a conventional nonrelativistic quantum particle which too cannot be pair produced or annihilated. Such a particle can obey any sort of statistics including parastatistics regardless of its intrinsic spin. But the standard spin-statistics connection can be enforced in nonrelativistic dynamics also by introducing suitable creation-annihilation processes [24].

There is now a general opinion that the spin-statistics theorem should extend to gravity as well. Just as this theorem emerges from even nonrelativistic physics once it admits pair production and annihilation [5], quantum gravity too can be expected to become compatible with this theorem after it allows suitable topology change [22]. In this manner, the desire for the usual spin-statistics connection leads us to look for a quantum gravity with transmuting topology.

Canonical quantum gravity in its elementary form is predicated on the hypothesis that spacetime topology is of the form $\Sigma \times \mathbf{R}$ (with $\mathbf{R}$ accounting for time) and has an eternal spatial topology. This fact has led to numerous suggestions that conventional canonical gravity is inadequate if not wrong, and must be circumvented by radical revisions of spacetime concepts [25], or by improved approaches based either on functional integrals and cobordism [22] or on alternative quantization methods.

Ideas on topology change were first articulated in quantum gravity, and more specifically in attempts at semiclassical quantization of classical gravity. Also it is an attribute intimately linked to gravity in the physicist's mind. These connections and the apparently revolutionary nature of topology change as an idea have led to extravagant speculations about twinkling topology in quantum gravity and their impact on fundamental concepts in physics.

Here we show that models of quantum particles exist which admit topology change or contain states with no well-defined classical topology. *This is so even though gravity does not have a central role in our ideas and is significant only to the extent that metric is important for a matter Hamiltonian.* These models use only known physical principles and have no revolutionary content, and at least suggest that topology change in quantum gravity too may be achieved with a modest physical input and no drastic alteration of basic laws.

We consider particle dynamics as usual. The configuration space of a particle being ordinary space, we are thus imagining a physicist probing spatial topology using a particle.

Let us consider a particle with no internal degrees of freedom living on the union Q' of two intervals which are numbered as 1 and 2:

$$Q' = [0, 2\pi] \bigcup [0, 2\pi] \equiv Q'_1 \bigcup Q'_2 \ . \tag{11.1}$$

It is convenient to write its wave function ψ as (ψ_1, ψ_2), where each ψ_i is a function on $[0, 2\pi]$ and $\psi_i^* \psi_i$ is the probability density on Q'_i. The scalar product between ψ and another wave function $\chi = (\chi_1, \chi_2)$ is

$$(\psi, \chi) = \int_0^{2\pi} dx \sum_i (\psi_i^* \chi_i)(x) \ . \tag{11.2}$$

It is interesting that we can also think of this particle as moving on $[0, 2\pi]$ and having an internal degree of freedom associated with the index i.

After a convenient choice of units, we define the Hamiltonian formally by

$$(H\psi)_i(x) = -\frac{d^2\psi_i}{dx^2}(x) \tag{11.3}$$

[where ψ_i is assumed to be suitably differentiable in the interval $[0, 2\pi]$]. This definition is only formal as we must also specify its domain $\mathcal{H}^1$ [13]. The latter involves the statement of the boundary conditions (BC's) at $x = 0$ and $x = 2\pi$.

Arbitrary BC's are not suitable to specify a domain: A symmetric operator $\mathcal{O}$ with domain $D(\mathcal{O})$ will not be self-adjoint unless the following criterion is also fulfilled:

$$\mathcal{B}_{\mathcal{O}}(\psi, \chi) \equiv (\psi, \mathcal{O}\chi) - (\mathcal{O}^\dagger \psi, \chi) = 0 \ \ \textit{for all} \ \chi \in D(\mathcal{O}) \Leftrightarrow \psi \in D(\mathcal{O}) \ . \tag{11.4}$$

For the differential operator H, the form $\mathcal{B}_H(\cdot,\cdot)$ is given by

$$\mathcal{B}_H(\psi,\chi) = \sum_{i=1}^{2}\left[-\psi_i^*(x)\frac{d\chi_i(x)}{dx} + \frac{d\psi_i^*(x)}{dx}\chi_i(x)\right]_0^{2\pi} . \tag{11.5}$$

It is not difficult to show that there is a $U(4)$ worth of $D(H) \equiv \mathcal{H}^1$ here compatible with (11.4).

We would like to restrict this enormous choice for $D(H)$, our intention not being to study all possible domains for $D(H)$. So let us restrict ourselves to the domains

$$\mathcal{H}_u^{(1)} = \{\psi \in C^2(Q') : \psi_i(2\pi) = u_{ij}\psi_j(0),\ \ \frac{d\psi_i}{dx}(2\pi) = u_{ij}\frac{d\psi_j}{dx}(0),\ u \in U(2)\} . \tag{11.6}$$

These domains have the virtue of being compatible with the definition of momentum in the sense discussed in ref. 2.

There are two choices of u which are of particular interest:

$$a) \qquad u_a = \begin{pmatrix} 0 & e^{i\theta_{12}} \\ e^{i\theta_{21}} & 0 \end{pmatrix} , \tag{11.7}$$

$$b) \qquad u_b = \begin{pmatrix} e^{i\theta_{11}} & 0 \\ 0 & e^{i\theta_{22}} \end{pmatrix} . \tag{11.8}$$

In case a, the density functions $\psi_i^*\chi_i$ fulfill

$$(\psi_1^*\chi_1)(2\pi) = (\psi_2^*\chi_2)(0) , \tag{11.9}$$

$$(\psi_2^*\chi_2)(2\pi) = (\psi_1^*\chi_1)(0) . \tag{11.10}$$

Figure 1 displays (11.10), these densities being the same at the points connected by broken lines.

In case b, they fulfill, instead,

$$(\psi_1^*\chi_1)(2\pi) = (\psi_1^*\chi_1)(0) , \tag{11.11}$$

$$(\psi_2^*\chi_2)(2\pi) = (\psi_2^*\chi_2)(0) \tag{11.12}$$

which fact is shown in a similar way in Figure 2.

Continuity properties of $\psi_i^*\chi_i$ imply that we can identify the points joined by dots to get the classical configuration space Q. It is <u>not</u> Q', but rather a circle S^1 in case a and the union $S^1 \bigcup S^1$ of two circles in case b.

Figure 1: In case a, the density functions are the same at the points joined by broken lines in this Figure.

Figure 2: In case b, the density functions are the same at the points joined by broken lines in this Figure.

The requirement $H^M\mathcal{H}^\infty \subset \mathcal{H}^\infty$ for all $M \in \mathbb{N}$ means just that arbitrary derivatives of $\psi_i^*\chi_i$ are continuous at the points joined by broken lines, that is on S^1 and $S^1 \bigcup S^1$ for the two cases. We can prove this easily using (11.6). In this way we also recover S^1 and $S^1 \bigcup S^1$ as manifolds.

When u has neither of the values (11.7) and (11.8), then Q becomes the union of two intervals. The latter happens for example for

$$u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad . \tag{11.13}$$

In all such cases, Q can be regarded as a manifold with boundaries as shown by the argument above.

Summarizing, we see that the character of the underlying classical manifold depends on the domain of the quantum Hamiltonian and can change when u is changed.

It is possible to reduce the u in the BC to $\mathbf{1}$ by introducing a connection. Thus since $U(2)$ is connected, we can find a $V(x) \in U(2)$ such that

$$V(0) = \mathbf{1}, \qquad V(2\pi) = u^{-1}. \tag{11.14}$$

Using this V, we can unitarily transform H to the new Hamiltonian

$$H' = VHV^{-1}, \tag{11.15}$$

$$(H'\psi)_i(x) = -\left[\frac{d}{dx} + A(x)\right]^2_{ij} \psi_j(x) \ , \tag{11.16}$$

$$A(x) = V(x)\frac{d}{dx}V^{-1}(x) \ . \tag{11.17}$$

With suitable physical interpretation, the system defined by H' and the domain

$$D_{\mathbf{1}}(H') = VD_u(H) \equiv \{\phi : \ \phi = V\psi, \quad \psi \in \mathcal{H}_u^1\} \tag{11.18}$$

is evidently equivalent to the system with Hamiltonian H and domain $\mathcal{H}_u^1$. Note in this connection that density functions on Q_i' are $\psi_i^*\chi_i$ and not $(V\psi)_i^*(V\chi)_i$.

Dynamics for Boundary Conditions

We saw in the previous section that topology change can be achieved in quantum physics by treating the parameters in the BC's as suitable external parameters which can be varied.

However it is not quite satisfactory to have to regard u as an external parameter and not subject it to quantum rules. We now therefore promote it to an operator, introduce its conjugate variables and modify the Hamiltonian as well to account for its dynamics. The result is a closed quantum system. It has no state with a sharply defined u. We cannot therefore associate one or two circles with the quantum particle and quantum spatial topology has to be regarded as a superposition of classical spatial topologies. Depending on our choice of the Hamiltonian, it is possible to prepare states where topology is peaked at one or two S^1's for a long time, or arrange matters so that there is transmutation from one of these states to another.

Dynamics for u which determines BC's is best introduced in the connection picture where the domain of H' is associated with $u = 1$. We assume this representation hereafter.

Quantization of u is achieved as follows. Let $T(\alpha)$ be the antihermitean generators of the Lie algebra of $U(2)$ [the latter being regarded as the group of 2×2 unitary matrices] and normalized according to $Tr\ T(\alpha)T(\beta) = -N\delta_{\alpha\beta}$, N being a constant. Let $\hat{u}$ be the matrix of quantum operators representing the classical u. It fulfills

$$\hat{u}_{ij}\hat{u}^{\dagger}_{ik} = \mathbf{1}\delta_{jk}, \quad [\hat{u}_{ij}, \hat{u}_{kh}] = 0 \ , \tag{11.19}$$

$\hat{u}^{\dagger}_{ik}$ being the adjoint of $\hat{u}_{ik}$. The operators conjugate to $\hat{u}$ will be denoted by L_α. If

$$[T_\alpha, T_\beta] = c^{\gamma}_{\alpha\beta} T_\gamma, \tag{11.20}$$

$$c^{\gamma}_{\alpha\beta} = \text{structure constants of } U(2), \tag{11.21}$$

L_α has the commutators

$$[L_\alpha, \hat{u}] = -T(\alpha)\hat{u} \ , \tag{11.22}$$

$$[L_\alpha, L_\beta] = c^{\gamma}_{\alpha\beta} L_\gamma, \tag{11.23}$$

$$[T(\alpha)\hat{u}]_{ij} \equiv T(\alpha)_{ik}\hat{u}_{kj}. \tag{11.24}$$

If $\hat{V}$ is the quantum operator for a function V of u, $[L_\alpha, \hat{V}]$ is determined by (11.17) and (11.23).

The Hamiltonian for the combined particle-u system can be taken to be, for example,

$$\hat{H} = \hat{H}' + \frac{1}{2I}\sum_\alpha L^2_\alpha, \tag{11.25}$$

$$\hat{H}' = -\left(\frac{d}{dx} + \hat{A}(x)\right)^2, \quad \hat{A}(x) \equiv \hat{V}(x)\frac{d}{dx}\hat{V}^{-1}(x), \tag{11.26}$$

I being the moment of inertia.

Quantized BC's with a particular dynamics are described by (11.19), (11.22),(11.23) and (11.26).

The general state vector in the domain of $\hat{H}$ is a superposition of state vectors $\phi \otimes_{\mathbf{C}} |u\rangle$ where $\phi \in D_{\mathbf{1}}(H')$ and $|u\rangle$ is a generalized eigenstate of $\hat{u}$:

$$\hat{u}_{ij}|u\rangle = u_{ij}|u\rangle, \quad \langle u'|u\rangle = \delta(u'^{-1}u) \ . \tag{11.27}$$

The δ-function here is defined by

$$\int du f(u)\delta(u'^{-1}u) = f(u'), \tag{11.28}$$

du being the (conveniently normalized) Haar measure on $U(2)$. Also

$$\hat{A}(x)|u\rangle = A(x)|u\rangle \ . \tag{11.29}$$

It follows that the classical topology of one and two circles is recovered on the states $\sum_\lambda C_\lambda \phi^{(\lambda)} \otimes_{\mathbf{C}} |u_a\rangle$ and $\sum_\lambda D_\lambda \phi^{(\lambda)} \otimes_{\mathbf{C}} |u_b\rangle$, $[C_\lambda, \ D_\lambda \in \mathbf{C}, \ \phi^{(\lambda)} \in D_{\mathbf{1}}(H')]$ with the two fixed values u_a, and u_b of (11.7) and (11.8) for u.

But these are clearly idealized unphysical vectors with infinite norm. The best we can do with normalizable vectors to localize topology around one or two circles is to work with the wave packets

$$\int du f(u)\phi \otimes_{\mathbf{C}} |u\rangle \ , \tag{11.30}$$

$$\int du|f(u)|^2 < \infty \tag{11.31}$$

where f is sharply peaked at the u for the desired topology. The classical topology recovered from these states will only approximately be one or two circles, the quantum topology also containing admixtures from neighboring topologies of two intervals.

A localized state vector of the form (11.31) is not as a rule an eigenstate of a Hamiltonian like $\hat{H}$. Rather it will spread in course of time so that classical topology is likely to disintegrate mostly into that of two intervals. We can of course localize it around one or two S^1's for a very long time by choosing I to be large, the classical limit for topology being achieved by letting $I \to \infty$. By adding suitable potential terms, we can also no doubt arrange matters so that a wave packet concentrated around $u = u_a$ moves in time to one concentrated around $u = u_b$. This process would be thought of as topology change by a classical physicist.

The preceding considerations on topology change admit generalizations to higher dimensions as explained in ref..2.

12 Final Remarks

In this essay, we have touched upon several issues concerning quantum topology and showed their utility for research of current interest such as topology change and fuzzy topology. Our significant contribution, if any, here has been in formulating new fundamental problems with reasonable clarity. We have also sketched a few answers, but they are tentative and incomplete.

13 Acknowledgments

The work reported in this article is part of an ongoing program with several colleagues. I have especially benefited from discussions with Jan Ambjorn,Peppe Bimonte, T.R. Govindarajan, Gianni Landi, Fedele Lizzi, Beppe Marmo, Shasanka Mohan Roy, Alberto Simoni and Paulo Teotonio-Sobrinho in its preparation.I am also deeply grateful to Arshad Momen for his extensive and generous help in the preparation of this paper. This work was supported by the U.S. Department of Energy under Contract Number DE-FG-02-85ER40231.

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A Appendix

Standard quantum measurement theory deals with instantaneous measurements which are then shown to be capable of observing only commuting sets of operators. We assume these observables to be bounded, closed under $\dagger$ and complete also. They generate a commutative normed $*$-algebra $\mathcal{B}$, the norm $||\cdot||$ being the operator norm and $*$ being $\dagger$. We can recover a Hausdorff topological space from $\mathcal{B}$ (or from its C^*-completion $\overline{\mathcal{B}}$). In this manner, we can recover the classical configuration space from the instantaneous measurement of a correct commuting set.

In the text, we were not overtly concerned with instantaneous or any other sort of measurements. Rather we were concerned with the construction of a correct algebra for the recovery of the classical configuration space. The reason why this algebra should be adapted to $\mathcal{H}^\infty$ was explained there. In that discussion, we required that the hermitean form $\psi^{\infty\ \dagger}\chi^\infty(\psi^\infty, \chi^\infty \in \mathcal{H}^\infty)$ should generate a *commutative* algebra and then tried to devise rules to pick out the right form and its algebra $\mathcal{A}^\infty$. Our insistence on this commutative nature came partly from the desire to reconstruct a Hausdorff space, as is appropriate for classical physics. Once we have this $\mathcal{A}^\infty$, it can serve as our choice of $\mathcal{B}$.

We thus see that there is a link between quantum instantaneous measurements and classical topology and that this link is mediated by $\mathcal{H}^\infty$ and $\mathcal{A}^\infty$.

There is a simple argument based on temporal continuity of experimental outputs which confirms the standard analysis that instantaneous measurements can be made only on *commuting* sets.

Suppose that we first measure an observable $a(= a^\dagger)$ and then an observable $b(= b^\dagger)$ an infinitesimal time δ later. Let us for simplicity assume that a and b have discrete spectra. We consider two cases in turn.

Case 1. $ab \neq ba$.

In this case, if $|\alpha>, |\beta>$ are eigenvectors of norm 1 for eigenvalues α, β of a and b (assumed nondegenerate for simplicity), the probability of finding α for a in the state vector $|\psi>$ $(<\psi|\psi>= 1)$ is $|<\alpha|\psi>|^2$. The vector after finding α is $|\alpha>$. The probability of finding β for b an instant later is thus $|<\beta|\alpha>|^2$. The probability $P(\beta, \alpha)$ of finding α for a and then β for b in the limit $\delta \downarrow 0$ is the product of these probabilities:

$$P(\beta, \alpha) = |<\beta|\alpha>|^2\ |<\alpha|\psi>|^2. \tag{1.1}$$

As this is not symmetric in a and b, the order of the measurements of a and b will give different answers, even if their time separation is negligible. So insistence on time-continuity excludes the possibility of their simultaneous measurement.

Case 2. $ab = ba$

Let us suppose that a and b form a complete commuting set and let $|\alpha', \beta'>$ denote the simultaneous eigenvectors of a and b for eigenvalues α' and β', with $<\alpha'', \beta''|\alpha', \beta'>= \delta_{\alpha''\alpha'}\delta_{\beta''\beta'}$. Then the probability of finding α for a in $|\psi>$ is $P(\alpha) = \sum_{\beta'} |<\alpha, \beta'|\psi>|^2$. The state vector after finding α is $\sum_{\beta'} \frac{1}{\sqrt{P(\alpha)}}|\alpha, \beta'><\alpha, \beta'|\psi>$. The probability of finding β in this state as $\delta \downarrow 0$ is $\frac{|<\alpha,\beta|\psi>|^2}{P(\alpha)}$. Hence now

$$P(\beta,\alpha) = | < \alpha, \beta|\psi > |^2. \tag{1.2}$$

This is symmetric in a and b, so time-continuity does not exclude their measurements within a vanishingly small temporal separation.

Instantaneous measurements are linked not just to classical topology, they are linked to classical physics in yet another way: We have emphasised that they observe only commutative algebras $\mathcal{B}$ [we continue to assume that $\mathcal{B}$ is of the sort indicated earlier],but the state $|\psi >< \psi|$ restricted to $\mathcal{B}$ is just a classical probability distribution. [Cf.[26]].

Thus let $|x>$ carry the IRR's of $\mathcal{B}$ on $\mathcal{H}$ with $b|x >= b_c(x)|x >, b \in \mathcal{B}, b_c(x) \in \mathbb{C}$ and let $I = \int \omega |x >< x|$ be the resolution of identity. Then the classical probability density in question is given by $\omega| < x|\psi > |^2$ for a volume form ω while the mean value of b in $|\psi >$ is $\int \omega| < x|\psi > |^2 b_c(x)$.

As $|\psi >$ thus is equivalent to a classical probability measure for an instantaneous measurement (which any way is the only sort of measurement discussed in usual quantum physics), there is no need to invoke "collapse of wave packets" or similar hypotheses for its interpretation. The uniqueness of quantum measurement theory then consists in the special relations it predicts between outcomes of measurements of different commutative algebras $\mathcal{B}_1$ and $\mathcal{B}_2$. These relations are often universal, being independent of the state vector $|\psi >$.

Such a point of view of quantum physics, or at least a view close to it, has been advocated especially by Sorkin [27].

Thus we see that instantaneous measurements are linked both to classical topology and to classical measurement theory.

But surely the notion of *instantaneous measurements* can only be an idealization. Measurements must be extended in time too, just as they are extended in space. But we know of no fully articulated theory of measurements extended in time, and maintaining quantum coherence during its duration, although interesting research about these matters exists [28].

A quantum theory of measurements extended in time, with testable predictions, could be of fundamental importance. We can anticipate that it will involve noncommutative algebras $\mathcal{N}$ instead of commutative algebras, the hermitean form $\psi^\dagger\chi$ for the appropriate vectors ψ, χ in the Hilbert space being valued in $\mathcal{N}$. Such quantum theories were encountered in [1]. Mathematical tools for their further development are probably available in Noncommutative Geometry [8, 9, 10, 11, 12].