Quasi-geostrophic Type Equations with Initial Data in Morrey Spaces

Jiahong Wu
School of Mathematics
The Institute for Advanced Study
Princeton, NJ 08540

Abstract

This paper studies the well-posedness of the initial value problem for the quasi-geostrophic type equations

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + (-\Delta)^{\gamma} \theta = 0, \quad \text{on} \quad \mathbb{R}^n \times (0, \infty),$$

$$\theta(x, 0) = \theta_0(x), \quad x \in \mathbb{R}^n$$

where $\gamma (0 \leq \gamma \leq 1)$ is a fixed parameter and $u = (u_j)$ is divergence free and determined from θ through the Riesz transform $u_j = \pm \mathcal{R}_{\pi(j)} \theta$ ($\pi(j)$ being a permutation of j, $j = 1, 2, \cdots, n$). The initial data θ_0 is taken in certain Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ (see text for the definition). The local well-posedness is proved for

$$\frac{1}{2} < \gamma \leq 1, \quad 1 < p < \infty, \quad \lambda = n - (2\gamma - 1)p \geq 0$$

and the solution is global for sufficiently small data. Furthermore, the solution is shown to be smooth.

1 Introduction

We consider the initial value problem (IVP) of the dissipative quasi-geostrophic type (QGS) equations

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + (-\Delta)^{\gamma} \theta = 0, \quad \text{on} \quad \mathbb{R}^n \times (0, \infty), \tag{1.1}$$

$$\theta(x, 0) = \theta_0(x), \quad x \in \mathbb{R}^n \tag{1.2}$$

where $\gamma (0 \leq \gamma \leq 1)$ is a fixed parameter and the velocity $u = (u_1, u_2, \cdots, u_n)$ is divergence free (i.e., $\nabla \cdot u = 0$) and determined from θ by

$$u_j = \pm \mathcal{R}_{\pi(j)} \theta, \qquad \pi(j) \text{ is a permutation of } j,\ j = 1, 2, \cdots, n \tag{1.3}$$

where u_j may take either $+$ or $-$ sign and $\mathcal{R}_j = \partial_j (-\Delta)^{-\frac{1}{2}}$ are the Riesz transforms. Here Riesz potential operator $(-\Delta)^{\alpha}$ is defined through the Fourier transform [17]:

$$\widehat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x) dx$$

$$(\widehat{(-\Delta)^{\alpha} f})(\xi) = (2\pi |\xi|)^{2\alpha} \widehat{f}(\xi)$$

A particular important special case of (1.1) ,(1.3) is the two dimensional dissipative QGS equation in which the velocity $u = (u_1, u_2)$ can also be defined through the stream function ψ:

$$u = (u_1, u_2) = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1} \right), \quad (-\Delta)^{\frac{1}{2}} \psi = -\theta \tag{1.4}$$

The 2-D QGS equations are derived from more general quasi-geostrophic approximations for flow in rapidly rotating 3-D half space, which in some important cases reduce to the evolution equation for the temperature on the 2-D boundary given in (1.1), (1.2),(1.4) ([14],[3]). The scalar θ represents the potential temperature and u is the fluid velocity. These equations have been under active investigation because of mathematical importance and potential applications in meteorology and oceanography ([14],[3], [2],[7]). As pointed out in [3], the non-dissipative 2-D QGS equations are strikingly analogous to the 3-D Euler equations and thus serve as a simple model in seeking possible singular solutions.

We are mainly interested in the well-posedness result for initial data θ_0 in some Morrey spaces, $\theta_0 \in \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ (defined in Section 2). By well-posedness we mean existence, uniqueness and persistence (i.e. the solution describes a continuous curve belonging to the same space as does the initial data) and continuous dependence on the data.

Although there is a large literature on quasi-geostrophic equations ([14], [2], [7], [3]), not many rigorous mathematical results concerning the solutions have been obtained. In [3] Constantin-Majda-Tabak prove finite time existence results for smooth data and develop mathematical criteria characterizing blowup for the 2-D non-dissipative QGS equation. In [16] Resnick obtain solutions of 2-D QGS equations with L^2 data on periodic domain by using Galerkin approximation. In a previous paper [19] the vanishing dissipation limits and Gevrey class regularity [4] for the 2-D dissipative QGS equations are obtained.

In this paper we prove that if $\frac{1}{2} < \gamma \leq 1$ and $\theta_0 \in \mathcal{M}_{p,\lambda}$ with

$$1 < p < \infty, \quad \lambda = n - (2\gamma - 1)p \geq 0 \tag{1.5}$$

Then the IVP (1.1), (1.3), (1.2) is locally well-posed and solution is global for sufficiently small initial data. Furthermore, we prove that the solution is actually smooth. The precise statements are given in Theorem 3.2 of Section 3 and Theorem 4.1 of Section 4. The proof of the well-posedness is based on the contraction mapping principle. These results reduce to those in L^p theory by taking $\lambda = 0$.

As we shall find out in Section 3, the requirement $1 < p < \infty$ in (1.5) comes from estimating the singular integrals of Calderon-Zygmund type when we bound u in terms of θ in Morrey spaces. $p = 1$ or $p = \infty$ is excluded since the Calderon-Zygmund type singular operator is not bounded either in L^1 (but of weak type $(1,1)$) or in L^∞ (but taking to BMO) [17]. For the Navier-Stokes equations, the singularity of the Biot-Savart kernel is milder and the borderline case $p = 1$ or $p = \infty$ is often included in the well-posedness results([6],[8]).

The index $\gamma = \frac{1}{2}$ seems special as we can see from this well-posedness and other related results concerning the QGS equation ([16],[19],[20]). For $\gamma > \frac{1}{2}$, the existence, uniqueness and regularity are obtained, but the same questions remain open for $\gamma \leq \frac{1}{2}$. Now the important question is if the index $\gamma = \frac{1}{2}$ is actually sharp and how this index is related to the singularity of the integral operator determining the velocity field. The answer to this question may provide a clue to the solution of the outstanding problem concerning the existence of smooth solutions for the Navier-Stokes equations.

This paper is organized as follows. In Section 2 the definition and some properties of Morrey spaces are given as well as the solution operator of the linear equation and its properties. The well-posedness results are stated and proved in Section 3 while the smoothness of the solution is established in Section 4.

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2 Morrey spaces and the linear equation

In this section we shall list some basic properties of Morrey spaces and then study the the solution operator for the linear equation over Morrey spaces.

Definition 2.1 *For $1 \leq p < \infty$, $0 \leq \lambda < n$, the Morrey space $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ is defined as*

$$\mathcal{M}_{p,\lambda} \equiv \{f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{p,\lambda} < \infty\}$$

where the norm is given by

$$\|f\|_{p,\lambda} = sup_{\{x\in\mathbb{R}^n, R>0\}} R^{-\frac{\lambda}{p}} \left(\int_{|y-x|\leq R} |f|^p(y)dy \right)^{\frac{1}{p}}$$

$\ddot{\mathcal{M}}_{p,\lambda}$ is defined to be a subspace of $\mathcal{M}_{p,\lambda}$:

$$\ddot{\mathcal{M}}_{p,\lambda} = \{f \in \mathcal{M}_{p,\lambda} : \quad \|f(\cdot - y) - f(\cdot)\|_{p,\lambda} \to 0, \quad as\ y \to 0\}$$

$\mathcal{M}_{p,\lambda}$ is a Banach space and $\ddot{\mathcal{M}}_{p,\lambda}$ is a closed subspace of $\mathcal{M}_{p,\lambda}$. For $p > 1$, $\mathcal{M}_{p,0} = L^p$ and $\mathcal{M}_{1,0} = \mathcal{M}$, where $\mathcal{M}$ is the space of finite measures. We shall also use the following properties of Morrey spaces (see e.g., [1],[15],[8]). When not specified, the indices in the notation $\mathcal{M}_{p,\lambda}$ will be restricted in $1 \leq p < \infty$, $0 \leq \lambda < n$. Occasionally we consider $p = \infty$ and then $\mathcal{M}_{\infty,\lambda}$ simply means L^∞.

Lemma 2.2 *Let $1 \leq p, q, r \leq \infty$.*

(i) *Inclusion relations:*

$$\mathcal{M}_{p,\lambda} \subset \mathcal{M}_{q,\mu}, \quad if \quad \frac{n-\lambda}{p} = \frac{n-\mu}{q}, \quad q \leq p$$

(ii) *The Hölder inequality:*

$$\|fg\|_{p,\lambda} \leq \|f\|_{q,\mu}\|g\|_{r,\nu}, \quad if \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r}, \quad \frac{n-\lambda}{p} = \frac{n-\mu}{q} + \frac{n-\nu}{r}$$

(iii) *Continuous embedding in weighted Lebesgue space:*

$$\mathcal{M}_{p,\lambda} \subset L_{-\frac{\mu}{p},p}, \quad for\ p > 1, \quad \mu > \lambda$$

where $L_{s,p}$ is the weighted Lebesgue space consisting of functions f such that $(-\Delta)^{s/2} f \in L^p$.

We now consider the solution operator for the linear equation

$$\partial_t\theta + \Lambda^{2\gamma}\theta = 0, \qquad \Lambda = (-\Delta)^{\frac{1}{2}}$$

on the whole space $\mathbb{R}^n$. For a given initial data θ_0, the solution of this equation is given by $\theta = K(t)\theta_0 = e^{-\Lambda^{2\gamma}t}\theta_0$, where $K(t)$ is a convolution operator with the kernel $k_t(x) = k_t^{(\gamma)}(x)$ being defined through its Fourier transform

$$\widehat{k}_t^{(\gamma)}(\xi) = e^{-|\xi|^{2\gamma}t}, \tag{2.1}$$

In particular, k_t is the heat kernel for $\gamma = 1$ and the Poisson kernel for $\gamma = \frac{1}{2}$.

As observed in [16], $k_t^{(\gamma)}$ can be expressed as an average of the heat kernel,

$$k_1^{(\gamma)}(x) = \int_0^\infty k_s^{(1)}(x)dP_\gamma(s), \qquad k_t^{(\gamma)} = t^{-\frac{n}{2\gamma}}k_1^{(\gamma)}\left(\frac{x}{t^{\frac{1}{2\gamma}}}\right) \tag{2.2}$$

where P_γ is some probability distribution. Furthermore,

Lemma 2.3 *For* $0 < \gamma \leq 1$,

(a) $k_t^{(\gamma)} \geq 0$, *and is a non-increasing radial function on* $\mathbb{R}^n$ *with*

$$|x|^l k_t^{(\gamma)}(|x|) \longmapsto 0, \quad \text{as} \quad |x| \to \infty$$

for any real power l.

(b) *For* $t > 0$, $\|k_t^{(\gamma)}\|_{L^1} = 1$ *and thus for* $f_t = k_t^{(\gamma)} * f$,

$$|f_t|^p \leq k_t^{(\gamma)} * |f|^p, \qquad 1 \leq p < \infty$$

(c) *For* $s \in \mathbb{R}$ *and* $q \in [1, \infty)$,

$$k_t^{(\gamma)} * f \longmapsto f, \quad \text{in } L_{s,q} \quad \text{as } t \to 0$$

for any $f \in Ls, q$.

Proof. The proof of (a) is an easy consequence of equation (2.2) and the corresponding properties of the heat kernel. (b) can be seen from the Fourier transform (2.1) of $k_t^{(\gamma)}$. The proof of (c) involves merely the definition of the norm in $L_{s,q}$ and the dominated convergence theorem.

We now establish estimates for the operator $K(t)$ between Morrey spaces. The following proposition is a straightforward generalization of Lemma 2.1 of Kato [8].

Proposition 2.4 *Let $1 \leq q_1 \leq q_2 \leq \infty$ and $0 \leq \lambda_1 = \lambda_2 < n$. For any $t > 0$, the operators $K(t)$, $W(t) = \nabla K(t)$ and $\partial_t K(t)$ are bounded operators from $\mathcal{M}_{q_1,\lambda_1}$ to $\ddot{\mathcal{M}}_{q_2,\lambda_2}$ and depend on t continuously , where ∇ denotes the space derivarive. Furthermore, we have for $f \in \mathcal{M}_{q_1,\lambda_1}$,*

$$t^{\frac{1}{2\gamma}(\alpha_1-\alpha_2)}\|K(t)f\|_{q_2,\lambda_2} \leq C\|f\|_{q_1,\lambda_1}, \tag{2.3}$$

$$t^{\frac{1}{2\gamma}+\frac{1}{2\gamma}(\alpha_1-\alpha_2)}\|W(t)f\|_{q_2,\lambda_2} \leq C\|f\|_{q_1,\lambda_1}, \tag{2.4}$$

$$t^{1+\frac{1}{2\gamma}(\alpha_1-\alpha_2)}\|\partial_t K(t)f\|_{q_2,\lambda_2} \leq C\|f\|_{q_1,\lambda_1}, \tag{2.5}$$

where $\alpha_i = \frac{n-\lambda_i}{p_i}(i = 1, 2)$ and constants C depend on $\gamma, q_1, q_2, \lambda_1, \lambda_2$.

The proof of this proposition is a modification of that for Lemma 2.1 in [8], but for reader's convenience, it will be given in the Appendix.

3 Well-posedness in Morrey spaces

In this section we deal with the IVP (1.1),(1.3), (1.2) for the QGS equations with initial data in Morrey spaces. The solutions are found to be in the spaces of weighted continuous functions in time, which we now introduce. Kato and collaborators first define such spaces in solving the IVP for the Navier-Stokes equations ([8],[10],[12],[13]).

Definition 3.1 *Let $0 < T < \infty$. For a given Banach space X and a real number $\alpha \geq 0$, we denote by $C_\alpha((0,T);X)$ the space of $X-$valued continuous functions f on $(0,T)$ with the norm*

$$\|f\|_{\alpha,X} = \sup_{0<t<T} t^\alpha \|f(\cdot,t)\|_X < \infty$$

In particular, $C_0((0,T);X) = BC((0,T);X)$ is the space of bounded continuous functions (Note here that $C_0((0,T);X) \neq C((0,T);X)$, the space of continuous functions.).

$\dot{C}_\alpha = \dot{C}_\alpha((0,T);X)$ denotes a subspace of $C_\alpha((0,T);X)$ consisting of all functions f with

$$\limsup_{t\to 0} t^\alpha \|f(\cdot,t)\|_X = 0$$

In the following Morrey spaces will play the role of X and the norm in $C_\alpha((0,T);\mathcal{M}_{p,\lambda})$ will be abbreviated as $\| \quad \|_{\alpha,p,\lambda}$.

Now we state the main theorem of this section and then prove it.

Theorem 3.2 *Let $\frac{1}{2} < \gamma \leq 1$ and $\theta_0 \in \mathcal{M}_{p,\lambda}$ with*

$$1 < p < \infty \quad \text{and} \quad \lambda = n - (2\gamma - 1)p \geq 0.$$

Then there is a $\delta > 0$ such that if $\|\theta_0\|_{p,\lambda} < \delta$, the IVP (1.1), (1.3), (1.2) has a solution θ on $(0,T)$ for some $T > 0$ satisfying

$$\theta \in BC((0,T);\ddot{\mathcal{M}}_{p,\lambda}) \cap (\cap_{1<q\leq p} \cap_{k>n-(2\gamma-1)q} BC([0,T);L_{-k/q,q})) \tag{3.1}$$

Furthermore, for any $p < q \leq \infty$

$$\theta \in \mathbb{Y} \equiv C_{m'}((0,T);\ddot{\mathcal{M}}_{q,\lambda}), \quad m' = \left(1 - \frac{1}{2\gamma}\right)\left(1 - \frac{p}{q}\right) \tag{3.2}$$

and θ is the unique solution in the class of functions satisfying (3.2) with small norm $\|u\|_{q,\lambda}$ for $p < q \leq \infty$. Moreover, the mapping

$$\mathcal{Q} : \theta_0 \longmapsto \theta, \qquad V \longmapsto \mathbb{Y}$$

is Lipschitz for some neighborhood V of θ_0.

Remark 3.3 *It is easy to see from the proof of this theorem that the solution is actually global (i.e., $T = \infty$) if the norm $\|\theta_0\|_{p,\lambda}$ is sufficiently small.*

Remark 3.4 *Note that (3.1), in particular, implies that for $1 < q \leq p$*

$$\theta \to \theta_0 \qquad \text{in } L_{-\frac{k}{q},q} \text{ as } t \to 0$$

In general we don't anticipate $\theta \to \theta_0$ in $\mathcal{M}_{p,\lambda}$ for any $\theta_0 \in \mathcal{M}_{p,\lambda}$ since $K(t)$ is not a C_0 semigroup on $\mathcal{M}_{p,\lambda}$. Furthermore, we have

Proposition 3.5 *Let $f \in \mathcal{M}_{q,\lambda}$. Then $f \in \ddot{\mathcal{M}}_{q,\lambda}$ if and only if*

$$K(t)f - f \to 0, \quad \text{as } t \to 0 \quad \text{in} \quad \mathcal{M}_{q,\lambda}$$

This proposition shows that $\ddot{\mathcal{M}}_{p,\lambda}$ is the maximal closed subspace of $\mathcal{M}_{p,\lambda}$ on which $K(t)$ is a C_0 semigroup. As a consequence of this proposition, (3.1) can be improved to

$$\theta \in BC([0,T); \ddot{\mathcal{M}}_{p,\lambda})$$

if the initial data $\theta_0 \in \ddot{M}_{p,\lambda}$.

Remark 3.6 *For $\|\theta_0\|_{p,\lambda} < \delta$, (3.2) also gives us the decay rate of θ for large t, especially,*

$$\|\theta\|_{L^\infty} = O(t^{\frac{1}{2\gamma}-1})$$

We prove this theorem by the method of integral equations and contraction mapping arguments. Following standard practice ([5], [6], [8], [9], [11], [12]), we write the QGS equation (1.1) into the integral form:

$$\theta(t) = K(t)\theta_0 - G(u,\theta)(t) \equiv K(t)\theta_0 - \int_0^t K(t-\tau)(u \cdot \nabla\theta)(\tau)d\tau \qquad (3.3)$$

where $K(t)$ is the solution operator of the linear equation

$$\partial_t \theta + \Lambda^{2\gamma}\theta = 0,$$

We observe that $u \cdot \nabla\theta = \sum_j u_j \partial_j \theta = \nabla \cdot (u\theta)$ provided that $\nabla \cdot u = 0$. This provides an alternative expression for G:

$$G(u,\theta)(t) = G(u\theta)(t) = \int_0^t \nabla \cdot K(t-\tau)(u\theta)(\tau)d\tau$$

We shall solve (3.3) in the spaces of weighted continuous functions in time introduced at the beginning of this section. To this end we need estimates for the operator G acting between these spaces.

Proposition 3.7 *Let $f \in C_h((0,T);\mathcal{M}_{q_1,\lambda_1})$ and $g \in C_l((0,T);\mathcal{M}_{q_2,\lambda_2})$ with*

$$1 \le q_1 \le \infty, \quad 1 \le q_2 \le \infty, \quad \frac{1}{q_1} + \frac{1}{q_2} \le 1, \quad h + l < 1$$

Assume that q and λ satisfy: $\quad 1 \le q \le \infty$

$$\frac{1}{q} \le \frac{1}{q_1} + \frac{1}{q_2}, \quad 0 \le \epsilon \equiv \frac{n-\lambda_1}{q_1} + \frac{n-\lambda_2}{q_2} - \frac{n-\lambda}{q} < 2\gamma - 1$$

Then

$$G(f,g) \in C_m((0,T);\mathcal{M}_{q,\lambda})$$

and

$$\|G(f,g)\|_{C_m((0,T);\mathcal{M}_{q,\lambda})} \le C\|f\|_{C_h((0,T);\mathcal{M}_{q_1,\lambda_1})}\|g\|_{C_l((0,T);\mathcal{M}_{q_2,\lambda_2})} \tag{3.4}$$

where $m = h + l + \frac{1}{2\gamma}(1+\epsilon) - 1$ and C is a constant.

Proof. To prove this proposition, we use Proposition 2.4 and the Hölder inequality in Lemma 2.2

$$\|G(f,g)(t)\|_{q,\lambda} \le C\int_0^t (t-\tau)^{-\frac{1}{2\gamma}(1+\epsilon)}\|fg\|_{\frac{q_1q_2}{q_1+q_2}, \frac{\lambda_1 q_2}{q_1+q_2}+\frac{\lambda_2 q_1}{q_1+q_2}}(\tau)d\tau$$

$$\le C\int_0^t (t-\tau)^{-\frac{1}{2\gamma}(1+\epsilon)}\|f(\cdot,\tau)\|_{\lambda_1,q_1}\|g(\cdot,\tau)\|_{\lambda_2,q_2}d\tau$$

Therefore for $m = h + l + \frac{1}{2\gamma}(1+\epsilon) - 1$,

$$\|G(f,g)(t)\|_{q,\lambda} \le C\|f\|_{h,q_1,\lambda_1}\|g\|_{l,q_2,\lambda_2} t^{-m} \int_0^1 (1-\sigma)^{-\frac{1}{2\gamma}(1+\epsilon)} \sigma^{-(h+l)} d\sigma$$

The integral in the last inequality can be written as the Beta function

$$B\left(1 - \frac{1}{2\gamma}\left[1+\epsilon\right], 1-h-l\right)$$

Using the fact that the Beta function $B(a,b)$ is finite if $a > 0$ and $b > 0$, we obtain that

$$\|G(f,g)\|_{m,q,\lambda} \le C\|f\|_{h,q_1,\lambda_1}\|g\|_{l,q_2,\lambda_2}$$

under the assumptions of this proposition. The estimate (3.4) is thus proved. The continuity of $G(f,g)$ in t comes from applying dominated convergence theorem to the quantity

$$t^m G(f,g)(t) = t^m \int_0^t W(t-\tau)(fg)(\tau)d\tau$$

We will need a result concerning the Calderon-Zygmund type singular integral operators on Morrey spaces. The Riesz transform is a particular example of these types of singular integral operators.

Lemma 3.8 *Let $\mathcal{Z}$ be a Calderon-Zygmund type singular integral operator, i.e., $\mathcal{Z} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is a homogeneous continuous function of degree $-n$ and integral on unit sphere vanishes. Let $\mathcal{M}_{q,\mu}$ with $1 < q < \infty, 0 \le \mu < n$ be a Morrey space. Then $\mathcal{Z}$ is bounded on $\mathcal{M}_{q,\mu}$ to itself.*

This result can be found in [18]. See [8] for an elementary proof.

Proof of Theorem 3.2. Let X denote the Banach space

$$X = C_{\left(\frac{1}{2}-\frac{1}{4\gamma}\right)}((0,T); \mathcal{M}_{2p,\lambda}) \tag{3.5}$$

and X_R denote the complete metric space of the closed ball in X centered at 0 and of radius R, where T and R are yet to be determined. Consider the nonlinear map $\mathcal{A}$ on X_R given by

$$\mathcal{A}(\theta)(t) = K(t)\theta_0 - G(u\theta)(t), \quad t \in (0,T)$$

We now start proving that $\mathcal{A}$ maps X_R to itself and is a contraction. Applying Proposition 2.4 with

$$q_1 = p, \quad q_2 = 2p, \quad \lambda_1 = \lambda_2 = \lambda = n - (2\gamma - 1)p$$

we obtain

$$\|K\theta_0\|_X = \|K\theta_0\|_{\frac{1}{2}-\frac{1}{4\gamma},2p,\lambda} \le C_0\|\theta_0\|_{p,\lambda}$$

where constant C_0 may depend on n, p, γ.

The application of Proposition 3.7 with

$$q_1 = q_2 = q = 2p, \quad \lambda_1 = \lambda_2 = \lambda = n - (2\gamma - 1)p, \quad m = h = l = \frac{1}{2} - \frac{1}{4\gamma}$$

shows that

$$\|G(u\theta)\|_{\frac{1}{2}-\frac{1}{4\gamma},2p,\lambda} \le C\|u\|_{\frac{1}{2}-\frac{1}{4\gamma},2p,\lambda}\|\theta\|_{\frac{1}{2}-\frac{1}{4\gamma},2p,\lambda}$$

Using Lemma 3.8, we obtain for $\theta \in X_R$

$$\|G(u\theta)\|_X = \|G(u\theta)\|_{\frac{1}{2}-\frac{1}{4\gamma},2p,\lambda} \le C_1 R^2$$

where C and C_1 are constants.

Furthermore, for any θ and $\tilde{\theta} \in X_R$,

$$\|\mathcal{A}(\theta) - \mathcal{A}(\tilde{\theta})\|_X = \|G(u\theta) - G(\tilde{u}\tilde{\theta})\|_X$$

where $\tilde{u} = (\tilde{u}_j)$ with $\tilde{u}_j = \pm\mathcal{R}_{\pi(j)}\tilde{\theta}(j = 1, 2, \cdots, n)$. Using Proposition 3.7 to estimate G again,

$$\|\mathcal{A}(\theta) - \mathcal{A}(\tilde{\theta})\|_X \le \|G((\tilde{u} - u)\tilde{\theta})\|_X + \|G(u(\theta - \tilde{\theta}))\|_X$$

$$\le C\left(\|\tilde{u} - u\|_X\|\tilde{\theta}\|_X + \|\theta - \tilde{\theta}\|_X\|u\|_X\right)$$

Since $(\tilde{u} - u)_j = \pm\mathcal{R}_{\pi(j)}(\tilde{\theta} - \theta)$, Lemma 3.8 implies that for some constant $C_{p,\lambda}$

$$\|u\|_X \le C_{p,\lambda}\|\theta\|_X, \quad \|\tilde{u} - u\|_X \le C_{p,\lambda}\|\tilde{\theta} - \theta\|_X$$

Therefore for constant $C_2 = 2CC_{p,\lambda}$,

$$\|\mathcal{A}(\theta, \theta_0) - \mathcal{A}(\tilde{\theta}, \theta_0)\|_X \le \frac{C_2}{2}(\|\tilde{\theta}\|_X + \|\theta\|_X)\|\tilde{\theta} - \theta\|_X \le C_2 R\|\tilde{\theta} - \theta\|_X$$

It then follows that the conditions that $\mathcal{A}$ map X_R into itself and be a contraction are

$$C_0\|\theta_0\|_{p,\lambda} + C_1 R^2 \leq R, \qquad C_2 R < 1$$

These conditions are met if

$$\|\theta_0\|_{p,\lambda} < \delta = \frac{1}{4C_0C_1}, \qquad R < \min\left\{\frac{1}{2C_1}, \frac{1}{C_2}\right\}$$

Hence for the above chosen δ and R, $\mathcal{A}$ has a unique fixed point $\theta \in X_R$ satisfying $\theta(t) = \mathcal{A}(\theta)(t)$ for $t \in (0,T)$ ($T > 0$ and $T = \infty$ if $\|\theta_0\|_{p,\lambda}$ is small enough).

To show that θ satisfies (3.1), we first note that

$$\theta = \mathcal{A}(\theta) \equiv K\theta_0 - G(u\theta)$$

For $\theta_0 \in \mathcal{M}_{p,\lambda}$ with $\lambda = n - (2\gamma - 1)p$,

$$K\theta_0 \in BC((0,T); \ddot{\mathcal{M}}_{p,\lambda})$$

as implied by Proposition 2.4. Furthermore, because of the continuous embedding for $1 < q \leq p$ and $k > n - (2\gamma - 1)q$

$$\mathcal{M}_{p,\lambda} \subset \mathcal{M}_{q,n-(2\gamma-1)q} \subset L_{-\frac{k}{q},q}$$

and the fact that $K(t)$ forms a C_0 semigroup on $L_{-k/q,q}$ for $1 < q$, we have $K\theta_0 \in BC([0,T); L_{-k/q,q})$ for $q > 1$ and $k > n - (2\gamma - 1)q$.

For the term $G(u\theta)$, we first apply Proposition 3.7 with

$$\lambda_1 = \lambda_2 = \lambda, \quad q_1 = q_2 = 2p, \quad q = p, \quad h = l = \frac{1}{2} - \frac{1}{4\gamma}, \quad m = 0$$

to show that

$$G(u\theta) \in BC((0,T); \ddot{\mathcal{M}}_{p,\lambda}) \tag{3.6}$$

Now we use (3.6) and apply Proposition 3.7 with

$$\lambda_1 = \lambda_2 = \lambda, \qquad q_1 = q_2 = p, \quad q = \frac{p}{1+\eta},$$

$$h = l = 0, \quad m = -\eta\left(1 - \frac{1}{2\gamma}\right) \quad \eta > 0 \text{ is small}$$

to obtain that

$$G(u\theta) \in C_{-\eta\left(1-\frac{1}{2\gamma}\right)}((0,T); \mathcal{M}_{q,\lambda}), \quad \text{for } q = \tfrac{p}{1+\eta} < p$$

which implies that

$$G(u\theta) \to 0, \quad \text{as } t \to 0 \quad \text{in } \mathcal{M}_{q,\lambda}$$

because of the embedding $\mathcal{M}_{q,\lambda} \subset L_{-k/q,q}$ for $k > n - (2\gamma - 1)q \geq \lambda$,

$$G(u\theta) \to 0, \quad \text{in } L_{-k/q,q}, \quad \text{as} \quad t \to 0,$$

Therefore $G(u\theta) \in BC([0,T); L_{-\frac{k}{q},q})$ for $1 < q \leq p$ and $k > n - (2\gamma - 1)q$. Summing up, we've shown that $\theta = K\theta_0 - G(u\theta)$ is exactly in the class defined by (3.1).

We now prove that θ satisfies (3.2). $K\theta_0$ satisfying (3.2) is an easy consequence of Proposition 2.4. We apply Proposition 3.7 to $G(u\theta)$ with

$$h = l = \left(\frac{1}{2} - \frac{1}{4\gamma}\right), \quad m = m'$$

$$q = q, \quad q_1 = q_2 = 2p, \quad \lambda = \lambda_1 = \lambda_2$$

and use the fact that $\theta \in X$ (3.5) to show that $G(u\theta)$ is in the class defined by (3.2). This argument, combined with the uniqueness of θ in (3.5), indicates the uniqueness of θ in (3.2).

The proof of the Lipschitz property is routine (see i.e., [20]) and is therefore omitted.

4 Further regularity

In this section we prove that the solution θ obtained in Theorem 3.2 is actually smooth. More precisely, we have

Theorem 4.1 *Let θ be the solution obtained in theorem 3.2, Then for any $p \le q < \infty$ and $k, j = 0, 1, 2, \cdots$*

$$\partial_t^k \nabla^j \theta \in C((0,T); \ddot{\mathcal{M}}_{q,\lambda}), \tag{4.1}$$

where $\lambda = n - (2\gamma - 1)p$, ∇ denotes the space derivative and $C((0,T); X)$ is the space of X-valued continuous functions on $(0,T)$.

Proof The smoothness of θ is proved by standard schemes. First we consider the case when $k = 0$. For $j = 0$, (4.1) can be seen from (3.1), (3.2) in Theorem 3.2 . We now prove that (4.1) is true for $j = 1$. We take any $t_1 > 0$ and prove the results for $t > t_1$.

We take ∇ of G and apply the Leibnitz rule to obtain

$$\nabla G(u, \theta) = G_1(\nabla u, \theta) + G_2(u, \nabla \theta)$$

where G_1 and G_2 are integral operators on (t_1, T) with the same properties as G. First let $q > p$ and X be the space consisting of functions θ such that

$$\theta \in C([t_1, T); \ddot{\mathcal{M}}_{q,\lambda}), \qquad \nabla \theta \in C_{1-\frac{1}{2\gamma}}((t_1, T); \ddot{\mathcal{M}}_{q,\lambda}) \tag{4.2}$$

and X_R be the closed ball of radius R in X. The idea is to apply contraction mapping arguments to $\mathcal{A}$ on X_R with T and R to be determined. First we choose R appropriately such that $K(t)\theta_1 \in X_R$, where $\theta_1 = \theta(t_1)$ is the value of θ at t_1. As in the proof of Theorem 3.2, we apply Proposition 3.7 to G, G_1 and G_2 to show that for $\theta \in X_R$

$$G(u, \theta) \in C_{\left(-\left(1-\frac{1}{2\gamma}\right)\left(1-\frac{p}{q}\right)\right)}((0,T); \ddot{\mathcal{M}}_{q,\lambda}) \tag{4.3}$$

$$\nabla G(u, \theta) \in C_{\left(1-\frac{1}{2\gamma}\right)\left(1-\frac{p}{q}\right)}((0,T); \ddot{\mathcal{M}}_{q,\lambda}) \tag{4.4}$$

and Lemma 3.8 is used in estimating u and ∇u as usual. (4.3) and (4.4) imply not only $G \in X_R$, but also that the norm of $G(u, \theta)$ in X_R has a small factor $(T - t_1)^{\varrho}$ if $T - t_1$ is small, where

$$\varrho = \min\{\left(1 - \frac{1}{2\gamma}\right)\left(1 - \frac{p}{q}\right), \left(1 - \frac{1}{2\gamma}\right)\frac{p}{q}\}$$

If $T - t_1$ is chosen small and R taken as above, then $\mathcal{A}$ maps X_R to itself. Furthermore , it can be shown in the same spirit as in the proof of Theorem 3.2 that $\mathcal{A}$ is a contraction map on X_R. Therefore $\mathcal{A}$ has a fixed point θ in X_R, which solves 3.3. The uniqueness result in Theorem 3.2 indicates that this θ is just the original θ obtained in Theorem 3.2. Thus we've shown that $\theta \in C((t_1, T); \ddot{\mathcal{M}}_{q,\lambda})$, which implies that $\theta \in C((0, T); \ddot{\mathcal{M}}_{q,\lambda})$ because of the arbitrariness of t_1.

For the case $q = p$, the result $\nabla\theta \in C((t_1, T); \ddot{\mathcal{M}}_{p,\lambda})$ can be established by applying Proposition 3.7 to ∇G again and using the relation in (4.2).

Repeating the same argument for higher spatial derivatives of θ, we obtain the result $\nabla^j\theta \in C((0, T); \ddot{\mathcal{M}}_{q,\lambda})$. This finishes the proof for the case $k = 0$.

We now prove (4.1) for $k = 1$. It is easy to see from the regularity result we've just obtained that

$$\nabla^j\theta, \quad \nabla^j u, \quad (-\Delta)^\lambda \nabla^j\theta \in C((0, T); \ddot{\mathcal{M}}_{q,\lambda}), \quad j = 0, 1, 2, \cdots$$

for any $p \leq q < \infty$. Using Equation 1.1

$$\partial_t\theta = -u \cdot \nabla\theta - (-\Delta)^\gamma\theta$$

and the Hölder inequatlity for the Morrey spaces (i.e., (ii) of Lemma 2.2), we obtain for $j = 0, 1, 2, \cdots$

$$\partial_t\nabla^j\theta \in C((0, T); \ddot{\mathcal{M}}_{q,\lambda})$$

The result for general k can be established by induction. This concludes the proof of Theorem 4.1.

5 Appendix

We prove Proposition 2.4 in this appendix. For simplicity of notation, we'll drop γ when writing the kernel $k_t^{(\gamma)}$ and use f_t for $k_t * f$.

We first prove estimate (2.3). For $q_1 = q_2 = \infty$, (2.3) is obvious. For $q_1 = q_2 < \infty$, we use (b) of Lemma 2.3, i.e.,

$$\|k_t\|_{L^1} = 1, \qquad |f_t|^p \le k_t * |f|^p, \tag{5.1}$$

to obtain for $q_2 = q_1, \lambda_2 = \lambda_1$

$$\|f_t\|_{q_2,\lambda_2} \le \|f\|_{q_1,\lambda_1}. \tag{5.2}$$

To prove (2.3) in the general case, we first estimate $\|f_t\|_{L^\infty}$. By (5.1),

$$|f_t(0)|^{q_1} \le \int_{\mathbb{R}^n} k_t(|x|)|f(x)|^{q_1} dx = \int_0^\infty k_t(\rho) dr(\rho)$$

$$\le \int_0^\infty |k_t'(\rho)| r(\rho) d\rho \le \|f\|_{q_1,\lambda_1}^{q_1} \int_0^\infty |k_t'(\rho)| \rho^{\lambda_1} d\rho$$

where $r(\rho) = \int_{|y|<\rho} |f(y)|^{q_1} dy$ and we've used the fact that $r(\rho) \le \|f\|_{q_1,\lambda_1}^{q_1} \rho^{\lambda_1}$. The decay properties of k_t (see Lemma 2.3) are also used here. Thus,

$$\|f_t\|_{L^\infty}^{q_1} \le c\|f\|_{q_1,\lambda_1}^{q_1} t^{-\frac{n+1}{2\gamma}} \int_0^\infty |k_1'(\rho t^{-\frac{1}{2\gamma}})| \rho^{\lambda_1} d\rho \le c\|f\|_{q_1,\lambda_1}^{q_1} t^{\frac{\lambda_1 - n}{2\gamma}} \tag{5.3}$$

which also proves (2.3) for $q_1 < q_2 = \infty$.

We can now estimate $\|f_t\|_{q_2,\lambda_2}$ in terms of $\|f\|_{q_1,\lambda_1}$ for $q_1 \le q_2 < \infty$. For any real number $R > 0$ and $x \in \mathbb{R}^n$,

$$R^{-\lambda_2} \int_{|x-y|<R} |f_t(y)|^{q_2} dy \le R^{-\lambda_2} \left(\|f_t\|_{L^\infty}\right)^{(q_2-q_1)} \int_{|x-y|<R} |f_t(y)|^{q_1} dy$$

Using Estimate (5.3) and noting that $\lambda_1 = \lambda_2$,

$$R^{-\lambda_2} \int_{|x-y|<R} |f_t(y)|^{q_2} dy \le c \left(\|f\|_{q_1,\lambda_1}\right)^{(q_2-q_1)} t^{\frac{1}{2\gamma}\left(\frac{(\lambda_1-n)}{q_1}\right)(q_2-q_1)} \|f_t\|_{q_1,\lambda_1}^{q_1}$$

Because of $\|f_t\|_{q_1,\lambda_1} \le \|f\|_{q_1,\lambda_1}$ as in (5.2),

$$R^{-\lambda_2}\int_{|x-y|<R} |f_t(y)|^{q_2}dy \le c\|f\|_{q_1,\lambda_1}^{q_2} t^{\frac{q_2}{2\gamma}(\alpha_2-\alpha_1)}$$

where $\alpha_i = \frac{n-\lambda_i}{q_i}(i=1,2)$. By definition,

$$\|f_t\|_{q_2,\lambda_2} \le ct^{\frac{1}{2\gamma}(\alpha_2-\alpha_1)}\|f\|_{q_1,\lambda_1}$$

where constants c depend on the indices $\gamma, q_1, q_2, \lambda_1, \lambda_2$.

Estimates (2.4), (2.5) can be proved similarly by using the identities

$$(\partial_x k_t)(x) = ct^{-\frac{1}{2\gamma}}g_t(x), \quad (\partial_t k_t)(x) = ct^{-1}g_t(x)$$

where g_t is another radial function enjoying the same properties as $k_t(x)$ does (The properties of k_t are listed in Lemma 2.3).

$f_t \in \ddot{\mathcal{M}}_{q_2,\lambda_2}$ follows directly from (2.4). and the continuity of $K(t)$ in t is a consequence of (2.5). The continuity of W follows from a similar estimate for $\partial_t W$ as in (2.5) and that of $\partial_t K(t)$ from the identity $\partial_t K = -\Lambda^{2\gamma}K$.

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