Long Range Intervortex Forces

J.M. Speight
Department of Mathematics
University of Texas at Austin
Austin, Texas 78712, U.S.A.

Abstract

A point particle approximation to the classical dynamics of well separated vortices of the abelian Higgs model is developed. The asymptotic static intervortex potential is calculated and used to model type II vortex scattering. A velocity dependent interaction Lagrangian for critically coupled vortices is derived, and reinterpreted geometrically to obtain a conjecture for the asymptotic form of the metric on the two-vortex moduli space, as used in the geodesic approximation. The scattering of critically coupled vortices is calculated.

1 Introduction

Solitons find numerous applications in many branches of physics. In theories of elementary particles, Lorentz invariance is a fundamental requirement, one which appears to be incompatible with integrability in spacetimes of dimension higher than $(1+1)$. Thus, it is topological solitons whose field equations are nonintegrable which are most interesting and relevant from a particle theory standpoint. A basic problem in the study of such objects is to understand classical two soliton dynamics. In the absence of integrability there is no hope of solving the initial value problem exactly. In fact, usually the only exact solutions available are static in some inertial frame, and so contain no dynamics at all. Often, even the static problem is intractable. To make progress, one must resort to numerical analysis, or devise some approximation scheme.

In this paper we study the interaction between two well separated abelian Higgs vortices in $\mathbb{R}^{2+1}$ by means of a point particle approximation. The idea is that, viewed from afar, a static vortex looks like a solution of a linear field theory in the presence of a singular point source at the vortex centre. As will be shown, the appropriate point source is a composite scalar monopole and magnetic dipole in a Klein-Gordon/Proca theory. The monopole charge and dipole moment must be fixed numerically. If physics is to be model independent, then the forces between well separated vortices should approach those between the corresponding point particles in the linear theory as the separation grows. Proceeding on this assumption, we calculate the asymptotic static two vortex potential, previously found by Bettencourt and Rivers using a field superposition ansatz [1]. As an application, the scattering of type II vortices is calculated and comparison made with numerical experiment.

The asymptotic potential reproduces the familiar trichotomy of vortex dynamics into type I, critical and type II regimes, according to the mass of the Higgs field. In particular, at critical coupling there is no net force between static point vortices, just as in the nonlinear field theory. However, *moving* point vortices *do* exert forces on one another, and we calculate these to lowest order in velocity by adapting the method of linear retarded potentials. This involves calculating the interaction of one moving point particle with the retarded field induced by another, the difficulty here being that the linear theory is massive, so conventional retarded potentials are not available to us (field disturbances do not travel uniformly at the speed of light). The technique was first used by

Manton [2] to find the asymptotic forces between two BPS monopoles, and has since been applied to the Skyrme model [3] and extreme Reissner-Nordstrom black holes [4]. In all these systems, the linear theory is massless. Having obtained an interaction Lagrangian, we go on to solve the scattering problem.

For so-called Bogomol'nyi field theories, there is another approximation scheme, also due to Manton [5], wherein low energy soliton dynamics is approximated by geodesic motion on the moduli space of static multisoliton solutions with respect to the metric induced by the field kinetic energy functional. This geodesic approximation was used by Samols [6] to study the scattering of two critically coupled vortices, and it is with these results that we compare our scattering data. Since no static vortex solutions are known exactly, Samols' work is necessarily partly numerical. However, the present work allows us to conjecture an asymptotic formula for the metric on the two vortex moduli space, reasoning as follows. Manton found that the equations of motion of two monopoles in the point particle approximation could be reinterpreted as the geodesic equation on the two monopole moduli space with a Taub NUT metric [2], precisely the asymptotic form of the Atiyah-Hitchin metric [7]. Assuming the same model independence occurs for vortices, we can make a similar reinterpretation of our point particle interaction Lagrangian, yielding new geometric information about the two vortex moduli space.

2 The abelian Higgs model

The abelian Higgs model [8] is a field theory in $(2+1)$ dimensional Minkowski space consisting of a complex scalar field ϕ coupled to a $U(1)$ gauge field A_μ. The scalar field is given a Higgs symmetry-breaking self interaction which allows topologically stable solitons called vortices to exist. The action functional is

$$S = \int d^3x \left[\frac{1}{2}D_\mu\phi\overline{D^\mu\phi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\mu^2}{8}(|\phi|^2-1)^2\right] \tag{1}$$

where

$$D_\mu\phi = (\partial_\mu + iA_\mu)\phi \tag{2}$$

is the gauge covariant derivative,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3}$$

is the field strength tensor, and Minkowski space has the standard metric, $\mathrm{diag}(1,-1,-1)$. Note that the electric charge and vacuum magnitude of the Higgs field have been normalized to unity, leaving only one parameter μ, the Higgs mass.

The Euler-Lagrange equations derived from S are

$$\begin{aligned} D_\mu D^\mu\phi - \frac{1}{2}\mu^2\phi(|\phi|^2-1) &= 0 \\ \partial_\mu F^{\mu\nu} + |\phi|^2A^\nu + \frac{i}{2}(\phi\partial^\nu\overline{\phi} - \overline{\phi}\partial^\nu\phi) &= 0, \end{aligned} \tag{4}$$

a set of coupled, nonlinear, hyperbolic partial differential equations of which no nontrivial solutions are known. To make progress, one breaks Lorentz and gauge covariance, and concentrates on the kinetic and potential energy functionals

$$T = \frac{1}{2}\int d^2\mathbf{x}\,(|\dot{\phi}|^2 + |\dot{\mathbf{A}}|^2) \tag{5}$$

$$V = \frac{1}{2}\int d^2\mathbf{x}\left[\mathbf{D}\phi\cdot\overline{\mathbf{D}\phi} + F_{12}^2 + \frac{1}{4}\mu^2(|\phi|^2-1)^2\right] \tag{6}$$

in the gauge $A_0 = 0$. The variational problem with action $\int dt\,(T-V)$ is the same as that with action S, provided one imposes the $\nu = 0$ field equation (4) as a constraint. Examining (6) one sees that for a configuration to have finite energy, the fields should satisfy the following boundary conditions as $r = |\mathbf{x}| \to \infty$,

$$|\phi| \;\to\; 1 \tag{7}$$

$$\nabla\phi \;\to\; \mathbf{0} \Rightarrow \mathbf{A} \to \frac{\nabla\phi}{i\phi}. \tag{8}$$

So from (7), $\phi_\infty(\theta) = \lim_{r\to\infty}\phi(r,\theta)$ takes values on the unit circle in $\mathbb{C}$ and hence is a map $S^1 \to S^1$. It need not be constant, provided $\mathbf{A}$ has the right form (8). Finite energy solutions fall into disjoint homotopy classes, each labelled by the degree of ϕ_∞, some integer n. This integer, usually called the winding number, cannot change under any continuous deformation of $(\phi, \mathbf{A})$ preserving finite energy (time evolution for example).

Static solutions with $n = 1$ are called vortices. By continuity there must be at least one point in the physical plane at which $\phi = 0$, and given the form of the potential energy density, there is a lump of energy located at or near this point. So vortices are topologically stable lumps of energy, solitons in the loose sense. Using Stokes' Theorem and the boundary conditions (7,8) it is easily seen that a winding n configuration has total magnetic flux

$$\Phi = -\int d^2\mathbf{x}\, F_{12} = 2n\pi. \tag{9}$$

A vortex may be visualized as a flux tube in $\mathbb{R}^{3+1}$ with translation symmetry along the x^3 axis. It has a total flux of 2π penetrating the physical plane.

Vortex dynamics splits into three regimes, depending on the value of μ^2, the most interesting mathematically being the case of critical coupling, $\mu^2 = 1$. In this case, by means of an ingenious argument due to Bogoml'nyi [9], one can find an optimal topological lower bound on the energy of a configuration of given winding, and reduce the static field equations to a pair of first order partial differential equations. Let $(\phi, \mathbf{A})$ be a static configuration with winding $n \geq 0$. Since $\dot\phi = 0$, $\dot{\mathbf{A}} = \mathbf{0}$, the constraint arising from the A_0 field equation is trivially satisfied, and $T = 0$, so the field variational problem reduces to minimizing the potential energy functional within the winding n sector. Starting from the trivial inequality,

$$\begin{aligned} 0 &\leq \frac{1}{2}\int d^2\mathbf{x}\left\{|D_1\phi + iD_2\phi|^2 + \left[-F_{12} + \frac{1}{2}(|\phi|^2 - 1)\right]^2\right\} \\ &= V + \frac{1}{2}\int d^2\mathbf{x}\,\left[i\left(D_2\phi\overline{D_1\phi} - D_1\phi\overline{D_2\phi}\right) - F_{12}(|\phi|^2 - 1)\right], \end{aligned} \tag{10}$$

we rearrange to obtain,

$$V \geq \frac{i}{2}\int d^2\mathbf{x}\,\left[\partial_1(\phi\overline{D_2\phi}) - \partial_2\overline{D_1\phi}\right] - \frac{1}{2}\Phi, \tag{11}$$

where Φ is the total magnetic flux, as defined above. Using Stokes' Theorem with the boundary condition (8) and flux quantization (9), this inequality becomes

$$V \geq n\pi, \tag{12}$$

equality holding if and only if the Bogomol'nyi equations

$$\begin{aligned} (D_1 + iD_2)\phi &= 0, \\ -F_{12} + \frac{1}{2}(|\phi|^2 - 1) &= 0 \end{aligned} \tag{13}$$

are satisfied. No explicit nontrivial solutions of (13) are known, but Taubes has rigorously proved the existence of such solutions [10]. Since they are minimals of V they are static solutions of the field equations. Hence, the mass of a $\mu^2 = 1$ vortex (its rest energy) is π, a result which will be of use to us.

3 Vortex asymptotics

The first task in the point particle approximation is to find out what a static vortex looks like far from its core [11, 1]. We place the vortex at the origin, and use plane polar coordinates. Substituting the ansatz

$$\begin{aligned} \phi &= \sigma(r)e^{i\theta} \\ (A_0, A_r, A_\theta) &= (0, 0, -a(r)) \end{aligned} \tag{14}$$

(σ is real) the field equations reduce to two coupled nonlinear ordinary differential equations,

$$\begin{aligned} \frac{d^2\sigma}{dr^2} + \frac{1}{r}\frac{d\sigma}{dr} - \frac{1}{r^2}\sigma(1-a) - \frac{1}{2}\mu^2\sigma(\sigma^2 - 1) &= 0 \\ \frac{d^2 a}{dr^2} - \frac{1}{r}\frac{da}{dr} + (1-a)\sigma^2 &= 0, \end{aligned} \tag{15}$$

the equations for A_0 and A_r being trivially satisfied. Regularity demands that $\sigma(0) = a(0) = 0$ while the boundary conditions (7,8) become

$$\lim_{r\to\infty} \sigma(r) = \lim_{r\to\infty} a(r) = 1. \tag{16}$$

Note that the ansatz has unit winding by construction. No exact solutions of (15) with these boundary conditions are known, but numerical solutions suggest that both σ are a are monotonic functions of r, and that the ansatz produces an isolated lump like structure.

We are interested in the asymptotic forms of σ and a, and for these explicit expressions do exist. Define the functions α and β such that

$$\sigma(r) = 1 + \alpha(r), \quad a(r) = 1 + \beta(r). \tag{17}$$

Then (16) implies that α and β are small at large r, so we substitute (17) into (15) and linearize in α and β,

$$\begin{aligned} \mu^2\left(\frac{d^2\alpha}{d(\mu r)^2} + \frac{1}{\mu r}\frac{d\alpha}{d(\mu r)} - \alpha\right) &= 0 \\ \frac{d^2}{dr^2}\left(\frac{\beta}{r}\right) + \frac{1}{r}\frac{d}{dr}\left(\frac{\beta}{r}\right) - \left(1 + \frac{1}{r^2}\right)\frac{\beta}{r} &= 0. \end{aligned} \tag{18}$$

These are the modified Bessel's equations of zeroth order for α in μr and first order for β/r in r respectively. Hence, at large r,

$$\begin{aligned} \alpha &\sim \frac{q}{2\pi}K_0(\mu r) \\ \beta &\sim \frac{m}{2\pi}rK_1(r), \end{aligned} \tag{19}$$

where K_n is the n–th modified Bessel's function of the second kind [12]. Note that $K_1 \equiv -K_0'$.

Since we have linearized the field equations, the asymptotic solutions contain unknown scale constants q and m which can only be fixed by solving (15) numerically. Rather than solving the

μ^2	q	m	r_c
0.4	-7.54	-14.92	4.23
0.6	-8.71	-12.61	3.75
0.8	-9.70	-11.31	3.43
0.9	-10.14	-10.89	3.22
1.0	-10.58	-10.57	-
1.1	-10.98	-10.31	2.98
1.2	-11.43	-10.06	3.07
1.3	-11.80	-9.85	2.96
1.4	-12.23	-9.66	2.95
1.6	-13.04	-9.34	2.88
1.8	-13.97	-9.09	2.87
2.0	-14.50	-8.86	2.72

Table 1: Numerical values of vortex scalar charge q and magnetic dipole moment m. The other data are the critical points of the static intervortex potential.

boundary value problem $\sigma(0) = a(0) = 0$, $\sigma(\infty) = a(\infty) = 1$, we solve the initial value problem $\sigma(0) = a(0) = 0$ using $\sigma'(0)$ and $a'(0)$ as shooting parameters. In fact, this is a slight oversimplification: due to the singularities of equations (15) at the origin, we must shoot from $r = r_0$, some small positive number, rather than $r = 0$. Substituting Taylor expansions for σ and a into (15) we find that, near the origin,

$$\begin{aligned} \sigma &= a_1 r + \frac{1}{4} a_1 \left(b_2 + \frac{\mu^2}{4} \right) r^3 + O(r^5) \\ a &= b_2 r^2 - \frac{a_1^2}{8} r^4 + O(r^5). \end{aligned} \tag{20}$$

We use a_1 and b_2 as shooting parameters, adjusting them until the numerical solution has $\sigma(r_\infty) \approx 1 \approx a(r_\infty)$, where r_∞ is some large positive number, the effective infinity. Having generated such a numerical solution, we compare it at large r to the asymptotic forms (19) and deduce q and m.

The results of this procedure using a fourth order Runge-Kutta method with $r_0 = 10^{-8}$ and $r_\infty = 10$ for various values of μ^2 are presented in table 1. That r_∞ is so small is unfortunate but necessary: at large r the field equations reduce to Bessel's equations, which have two independent solutions, one exponentially decaying and the other exponentially growing. We seek to pick out the former and completely exclude the latter, an impossible task. Hence, all numerical solutions blow up at large r, and even though a_1 and b_2 were tuned to six decimal places, the Runge-Kutta algorithm could not shoot beyond $r = 10$. The results in table 1 should thus be treated with some caution, particularly where μ^2 is far from 1.

Two points about the numerical charges are noteworthy. First, at critical coupling ($\mu^2 = 1$), $q \approx m$. In fact, one can prove that $q \equiv m$ exactly in this case, because the $\mu^2 = 1$ vortex satisfies (13), a pair of *first* order field equations. Substituting the vortex ansatz (14) into (13) one can solve for a in terms of σ and σ', eliminate a and obtain a second order ordinary differential equation for σ alone which, on linearization, reduces to Bessel's equation. The asymptotic form of a can then be deduced from that of σ. Second, $|q|$ and $|m|$ are monotonic functions of μ^2, $|q|$ increasing and $|m|$ decreasing. Bettencourt and Rivers [1] also find the asymptotic forms (19), but leave their charges analogous to q and m undetermined. For purposes of calculation, they make two assumptions about the charges which, in the light of table 1, may prove ill-justified. First, they assume that $q = m$ is approximately true away from $\mu^2 = 1$, whereas in our results, q/m varies between 0.50 and 1.64. Second, they impose the condition that the magnetic flux of a vortex should vanish at $r = 0$ and deduce that $m = -2\pi$. This result may be valid for very large μ^2, but is certainly flawed close to $\mu^2 = 1$ (the $\mu^2 = 1$ vortex has *maximum* magnetic flux at the origin, as is easily seen from (13), the Bogomol'nyi equations). So, they combine assumptions which are individually true only in widely disparate physical regimes.

4 The point vortex

The next task is to replicate the vortex asymptotics found above, using point sources in the linear theory. To linearize the abelian Higgs model, we choose gauge so that ϕ is real. Defining the field $\psi = 1 - \phi$, the vacuum is then $\psi = 0$, and the linear Lagrangian density is obtained by expanding (1) up to quadratic order in ψ and A_μ,

$$\mathcal{L}_{\text{free}} = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}\mu^2\psi^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}A_\mu A^\mu. \tag{21}$$

Including the external source Lagrangian density,

$$\mathcal{L}_{\text{source}} = \rho\psi - j_\mu A^\mu, \tag{22}$$

with scalar density ρ and vector current j_μ, we obtain the following massive, inhomogeneous wave equations for ψ and A_μ,

$$\begin{aligned}(\Box + \mu^2)\psi &= \rho &(23)\\ (\Box + 1)A_\mu &= j_\mu + \partial_\mu(\partial_\nu A^\nu). &(24)\end{aligned}$$

All gauge freedom has been exhausted, and there is no global $U(1)$ symmetry of $\mathcal{L}_{\text{free}}$ with whose Noether current we can identify j_μ because ψ is real. Hence there is no reason to assume that j_μ is a conserved current, and we cannot set the extra "fictitious current" term in the Proca equation (24) to zero.

To make comparison with the asymptotic vortex fields, these must first be converted to the real ϕ gauge. Since ϕ has non-zero winding, there is no gauge transformation regular on all $\mathbb{R}^2$ which will accomplish this. However, we only require comparison at large r, so for our purposes it is sufficient that the transformation be regular on $\mathbb{R}^2\backslash\{\mathbf{0}\}$. Since a singular point source will be introduced into the linear theory, this is from the outset regular only on $\mathbb{R}^2\backslash\{\mathbf{0}\}$. So, we unwind the static vortex (14) with gauge transformation $\phi \mapsto e^{-i\theta}\phi$, $A^\mu \mapsto A^\mu + \partial^\mu\theta$ to obtain

$$\begin{aligned}\phi &= \sigma(r) \sim 1 + \frac{q}{2\pi}K_0(\mu r) &(25)\\ A_\theta &= -a(r) + 1 \sim \frac{m}{2\pi}rK_1(r), &(26)\end{aligned}$$

while $A_r = A_0 = 0$. It is convenient to introduce a unit vector $\widehat{\mathbf{k}}$ in a fictitious third direction perpendicular to the physical plane, so that the $\mathbb{R}^3$ vector product can be defined. In terms of the 2-vector field $\mathbf{A}$, the unwound asymptotic behaviour is

$$\mathbf{A} \sim -\frac{m}{2\pi}K_0'(r)\widehat{\boldsymbol{\theta}} = -\frac{m}{2\pi}\widehat{\mathbf{k}} \times \nabla K_0(r). \tag{27}$$

We thus seek sources ρ and j_μ such that the solutions of (23,24) are

$$\begin{aligned}\psi &= \frac{q}{2\pi}K_0(\mu r) &(28)\\ (A^0, \mathbf{A}) &= \left(0, -\frac{m}{2\pi}\widehat{\mathbf{k}} \times \nabla K_0(r)\right). &(29)\end{aligned}$$

The static Klein-Gordon equation in $(2+1)$ dimensions has Green's function K_0,

$$(-\Delta + \mu^2)K_0(\mu r) = 2\pi\delta(\mathbf{x}). \tag{30}$$

Substituting (28) into (23) and using (30) one finds that

$$\rho = q\delta(\mathbf{x}). \tag{31}$$

Similarly, substitution of (29) into (24) yields

$$\mathbf{j} - \nabla(\nabla\cdot\mathbf{j}) = -m\widehat{\mathbf{k}} \times \nabla\delta(\mathbf{x}). \tag{32}$$

Taking the divergence of (32) one sees that $\nabla\cdot\mathbf{j}$ is a solution of the *homogeneous* static Klein-Gordon equation, so if $\mathbf{j}$ is a point source (meaning $\mathbf{j} = \mathbf{0}$ except at $\mathbf{x} = \mathbf{0}$) then $\nabla\cdot\mathbf{j} = 0$ everywhere. Thus the unique point source satisfying (32) is

$$\mathbf{j} = -m\widehat{\mathbf{k}} \times \nabla\delta(\mathbf{x}). \tag{33}$$

Since $A^0 = 0$ we take $j^0 = 0$.

The physical interpretation of these expressions for ρ and $\mathbf{j}$ is that the point source consists of a scalar monopole of charge q and a magnetic dipole of moment m perpendicular to the physical plane. Both q and m are negative (see table 1). We refer to this composite point source as the point vortex. One should note an important difference between the present endeavour and the analogous analysis for BPS monopoles [2]. Both systems (abelian Higgs and Yang-Mills-Higgs) have topologically quantized magnetic flux Φ. In the monopole calculation, the Dirac monopole charge is assumed to have precisely the topological value, while we must fix the dipole moment of the point vortex by numerical solution of the static field equations. The difference lies in the fact that monopoles exist in $\mathbb{R}^3$ where Φ is defined as flux through a large spherical shell centred on the monopole, an asymptotic quantity, whereas vortices exist in $\mathbb{R}^2$ where Φ is flux through the whole plane, including the vortex core. Such a quantity is not asymptotic, and so is not reflected in the properties of the point vortex. In fact, although a vortex is a flux tube in the $\widehat{\mathbf{k}}$ direction, to reproduce it asymptotically, one needs a dipole of moment $|m|$ in the $-\widehat{\mathbf{k}}$ direction in the linear theory.

5 The static intervortex potential

Having found the scalar charge and magnetic dipole moment carried by a point vortex, it is straightforward to calculate the force between two such vortices held at rest, in the framework of the linear theory. The interaction Lagrangian for two arbitrary (possibly time dependent) sources $(\rho_1, j_{(1)})$ and $(\rho_2, j_{(2)})$ is

$$L_{\text{int}} = L_\psi + L_A = \int d^2\mathbf{x}\, \rho_1\psi_2 - \int d^2\mathbf{x}\, j^\mu_{(1)} A^{(2)}_\mu \tag{34}$$

where $(\psi_i, A_{(i)})$ are the fields induced by source $(\rho_i, j_{(i)})$ according to the wave equations (23,24). This is found by extracting the cross terms in $\int d^2\mathbf{x}\,(\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{source}})$ where (ρ, j) is the superposition of the two sources, and (ψ, A) is a superposition of the induced fields (using linearity). The expression (34) looks asymmetric under interchange of sources $1 \leftrightarrow 2$, but in fact L_{int} is symmetric as may be shown using the wave equations (23,24) and integration by parts.

Now consider the case of two static point vortices, vortex 1 at $\mathbf{y}$ and vortex 2 at $\mathbf{z}$. Then $\rho_1 = q\delta(\mathbf{x}-\mathbf{y})$, while the scalar field due to ρ_2 is $\psi_2 = qK_0(\mu|\mathbf{x}-\mathbf{z}|)/2\pi$. Hence,

$$L_\psi = \int d^2\mathbf{x}\, \frac{q^2}{2\pi}\delta(\mathbf{x}-\mathbf{y})K_0(\mu|\mathbf{x}-\mathbf{z}|) = \frac{q^2}{2\pi}K_0(\mu|\mathbf{y}-\mathbf{z}|). \tag{35}$$

The magnetic interaction is similar: $j^0_{(1)} = 0$, $\mathbf{j}_{(1)} = -m\widehat{\mathbf{k}} \times \nabla\delta(\mathbf{x}-\mathbf{y})$ while $A^0_{(2)} = 0$, $\mathbf{A}_{(2)} = -m\widehat{\mathbf{k}} \times \nabla K_0(|\mathbf{x}-\mathbf{z}|)$, so

$$\begin{aligned} L_A &= \int d^2\mathbf{x}\, \frac{m^2}{2\pi}[\widehat{\mathbf{k}} \times \nabla\delta(\mathbf{x}-\mathbf{y})] \cdot [\widehat{\mathbf{k}} \times \nabla K_0(|\mathbf{x}-\mathbf{z}|)] \\ &= -\frac{m^2}{2\pi}\Delta_y K_0(|\mathbf{y}-\mathbf{z}|) \\ &= -\frac{m^2}{2\pi}K_0(|\mathbf{y}-\mathbf{z}|) \end{aligned} \tag{36}$$

using (30) with $\mathbf{y} \neq \mathbf{z}$. The total interaction Lagrangian is a function of $|\mathbf{y}-\mathbf{z}|$ only, so we interpret $-L_{\text{int}}$ as the potential energy of the interaction,

$$U = \frac{1}{2\pi}[m^2 K_0(r) - q^2 K_0(\mu r)] \tag{37}$$

where r is the vortex separation, that is $\mathbf{r} = r(\cos\vartheta, \sin\vartheta) := \mathbf{y} - \mathbf{z}$. This is the same potential as found in [1], but we arrived at it via a different route.

This potential is consistent with the partition into type I, critical and type II regimes. The central force due to U is

$$-U'(r) = \frac{1}{2\pi}[m^2 K_1(r) - \mu q^2 K_1(\mu r)]. \tag{38}$$

If $\mu < 1$, then $K_1 \to 0$ at large r faster than $K_1(\mu r)$, so scalar attraction dominates over magnetic repulsion and the force is negative, consistent with type I behaviour. If $\mu > 1$, the reverse is true and the force is positive at large r, consistent with type II behaviour. At $\mu = 1$, $m \equiv q$, as explained in section 3 so $U \equiv 0$ and there is no net force at all. This consistency at large r emerges regardless of the specific values of m and q away from $\mu = 1$, and may be attributed to the inverse relationship between a field's mass and its range. At moderate r, the μ dependance of q/m becomes important. Given that K_1 is a strictly decreasing function, it is clear from (38) that there exists a unique critical point of U for each $\mu \neq 1$ if and only if $m/q > \sqrt{\mu}$ when $\mu < 1$ and $m/q < \sqrt{\mu}$ when $\mu > 1$. Our numerical work suggests that m/q easily passes these criteria. The rightmost column of table 1 presents the approximate critical vortex separation r_c for each value of μ^2. Potentials for $\mu = 0.4$ (type I) and $\mu = 2.0$ (type II) are plotted in figure 1. Rebbi and Jacobs [14] have found approximate static intervortex potentials by numerically minimizing the potential energy functional (6) subject to the constraint that ϕ has two simple zeros separated by a given distance d, for a range of values of d. In the type I case, they find that the dominance of scalar attraction over magnetic repulsion subsides as the vortex separation gets small, but not to the extent of producing a stable equilibrium at non-zero d. Similarly, they find that magnetic repulsion of type II vortices is increasingly counteracted by scalar attraction (though they do not use this terminology) as d becomes small. It would appear that $\mathcal{U}$ is in broad agreement with their results when $r > r_c$, but that the asymptotic approximation breaks down for $r < r_c$. Of course, this is to be expected – vortices are not point particles, as in our picture, and when they approach one another closely enough their overlap produces significant effects.

6 Type II vortex scattering

The interaction potential $\mathcal{U}$ provides a very simple model of two-vortex dynamics: the dynamics of two point particles each of mass M (the energy of a single vortex at rest, a μ^2 dependent quantity) interacting via the potential (37). Ignoring the (trivial) centre of mass motion, the Lagrangian of such a mechanical system is

$$L = \frac{1}{4}M(\dot{r}^2 + r^2\dot{\vartheta}^2) - U(r), \tag{39}$$

since the reduced mass of the system is $M/2$. This is a manifestly bad model if $\mu = 1$, because it would predict that there is no scattering at all, in conflict with the results of numerical simulations [13] and the geodesic approximation [6]. Away from critical coupling, one might expect the potential U to dominate over velocity dependent corrections, at least at moderately low speeds, so the above model, although simple, may give a good quantitative account of vortex interactions. We choose to study type II vortices because these provide a simple, clear-cut dynamic problem: vortex scattering. Type I dynamics is slightly more complicated in that vortices can scatter or form bound states depending on the initial conditions. The coupling chosen for the type II numerical simulations of [13] is $\mu^2 = 2$, a choice which we follow for purposes of comparison. In the Lagrangian (39), the constants q and m are already known for $\mu^2 = 2$, but the vortex mass M is not. Rather than attempt to calculate M from our numerical solution, we use the careful numerical analysis of Rebbi and Jacobs [14]. Unfortunately, they found M for each of a regular sequence of μ values, rather than μ^2 values, so the $\mu = \sqrt{2}$ value is not quoted. However, a graph of μ against M is very nearly linear, so we use linear interpolation to estimate the $\mu = 1.41421\ldots$ mass from the $\mu = 1.4$ and $\mu = 1.5$ masses given. The result is $M = 1.51230\pi$. The potential for this coupling is plotted in figure 1.

¿From the plot of $U(r)$ we see that all trajectories which do not encroach on the interior region $r < r_c$ are scattering trajectories, that is as $t \to \pm\infty$, $r \to \infty$. At some point on the trajectory, the particle achieves its closest approach r_0 to the origin. By time-translation and rotational symmetries, we can, without loss of generality, take this point to lie on the $\vartheta = 0$ ray and occur at time $t = 0$. It is then straightforward to show that $\lim_{t\to\infty}\vartheta$ is

$$\vartheta_\infty = J^2 \int_{r_0}^{\infty} \frac{dr}{r^2}\left[\frac{4}{M}(U(r_0) - U(r)) + J^2\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right)\right]^{-\frac{1}{2}} \tag{40}$$

where $J = r^2\dot{\vartheta}$ is the conserved angular momentum conjugate to ϑ. The deflection angle Θ is $\pi - 2\vartheta_\infty$ (see figure 2).

To make comparison with the numerical simulations described in [13] we calculate Θ as a function of impact parameter b for scattering at four different impact speeds ($v_\infty = 0.1, 0.2, 0.3$ and 0.4). One might worry that at high impact speed and low angular momentum the vortices will penetrate the $r < r_c$ zone and become unrealistically captured. In fact even in a head on collision, the speed required for this is greater than 0.4, so the problem is never encountered. Given a particular pair (v_∞, b), the angular momentum is $J = 2v_\infty b$ while r_0 is found by solving the equation obtained by equating the total energy of the system at $t = 0$ and as $t \to -\infty$. The parameter pair (J, r_0) is then substituted into (40) and the integral evaluated numerically. Note that there is a $(r - r_0)^{-\frac{1}{2}}$ singularity in the integrand at $r = r_0$. This presents no problem in principle, because of the integration with respect to r, but it must be treated carefully in any numerical algorithm. Schematically, we handle the integral as follows,

$$\begin{aligned} \vartheta_\infty &= \int_{r_0}^{r_0+\delta} + \int_{r_0+\delta}^{\Delta} + \int_{\Delta}^{\infty} \\ &\approx \vartheta_\delta + \vartheta_{\text{num}} + \vartheta_\Delta, \end{aligned} \tag{41}$$

where δ is small ($\delta = 0.1$) and Δ is large ($\Delta = 15$). The contribution ϑ_δ is calculated by Taylor expansion of the integrand about $r = r_0$, while ϑ_{num} is evaluated using the Newton-Cotes rule. At large r the potential falls off exponentially, so for $r > \Delta$ we set $U \equiv 0$ and calculate ϑ_Δ in the free vortex approximation. The results of this algorithm are shown in figure 3. The fit to the numerical simulations of [13] could be improved by adjusting the values of q and m. Given the warning attached to these charges in section 3, this may well be justified. However, such adjustment corrupts the deductive nature of the model, so we prefer not to make it.

7 Velocity dependent forces

So far all the results obtained could be found without using the point source formalism. Introducing the point vortex provides a compact and convenient way of organizing essentially the same calculations as arise in the superposition ansatz approach. (The connexion between these two viewpoints is explored in the context of the Skyrme model in [3].) The utility of the point source formalism becomes more apparent when we go on to calculate velocity dependent intervortex forces. It is not obvious how to approach such a calculation using a superposition ansatz, but there is a large collection of analogous calculations for point sources in linear field theory, most notably, classical electrodynamics. Although one could calculate a velocity dependent interaction Lagrangian at arbitrary μ^2, there are three reasons why we choose to consider only the case of critical coupling. First, when $\mu^2 = 1$ there are no static intervortex forces, so the velocity dependent forces are leading, whereas when μ^2 is far from 1 they will likely be swamped by the static potential. Second, the resulting mechanical system has the simplifying property of Galilei symmetry if $\mu^2 = 1$, but not otherwise. Third, in the absence of a static potential we can justify our expansion *a posteriori*, as will be shown. The argument does not work if $U \neq 0$, so our approximation may not be self consistent away from critical coupling.

The first task is to find expressions for the sources ρ and j for a point vortex moving on some arbitrary trajectory $\mathbf{y}(t)$. We employ a quasi-adiabatic approximation: we assume that at each point $\mathbf{y}(t)$, the source is a static point vortex Lorentz boosted with velocity $\dot{\mathbf{y}}$. (An adiabatic approximation would be to assume that the point vortex always has the static form, and simply translates along the trajectory $\mathbf{y}(t)$.) The strategy is then to expand this moving source in powers of $|\dot{\mathbf{y}}|$, truncating at order $|\dot{\mathbf{y}}|^2$, calculate the time dependent fields it induces, truncating similarly, then use these to find an interaction Lagrangian.

7.1 Moving sources

We seek expressions for the scalar charge density ρ and vector current j of a point vortex moving along some curve $\mathbf{y}(t)$ in $\mathbb{R}^2$. At time $t = 0$, let the point vortex be at $\mathbf{x} = \mathbf{0}$, moving with velocity $\mathbf{u}$. Introduce rest frame coordinates η^μ, related to laboratory coordinates x^μ by a Lorentz boost (on x) with velocity $\mathbf{u}$. Explicitly,

$$\begin{aligned} \eta^\parallel &= \gamma(u)(x^\parallel - ux^0) = \gamma x^\parallel \\ \eta^\perp &= x^\perp \end{aligned} \tag{42}$$

at time $x^0 = 0$, where $\gamma(u) = (1-u^2)^{-\frac{1}{2}}$. The components $\eta^\parallel$ and $\eta^\perp$ can be combined into a single 2-vector equation for $\boldsymbol{\eta}(\mathbf{x})$ by decomposing $\boldsymbol{\eta}$ into parallel and perpendicular components:

$$\begin{aligned} \boldsymbol{\eta} &= \frac{(\boldsymbol{\eta}\cdot\mathbf{u})\mathbf{u}}{u^2} + \left[\boldsymbol{\eta} - \frac{(\boldsymbol{\eta}\cdot\mathbf{u})\mathbf{u}}{u^2}\right] \\ &= \frac{\gamma(\mathbf{x}\cdot\mathbf{u})\mathbf{u}}{u^2} + \left[\mathbf{x} - \frac{(\mathbf{x}\cdot\mathbf{u})\mathbf{u}}{u^2}\right] \\ &= \mathbf{x} + \frac{1}{2}(\mathbf{x}\cdot\mathbf{u})\mathbf{u} + \ldots \end{aligned} \tag{43}$$

where the ellipsis denotes discarded terms of order u^4 or greater (we shall not persist in so noting these discarded terms; henceforth an ellipsis will only be included where we have discarded further negligible terms in deriving the given expression). Now generalize to the situation of a point vortex located at $\mathbf{y}(t)$ moving with velocity $\dot{\mathbf{y}}(t)$. The rest frame coordinates of a general point $\mathbf{x}$ on the surface $\mathbf{x}^0 = t$ are

$$\boldsymbol{\eta}(t,\mathbf{x}) = \mathbf{x} - \mathbf{y} + \frac{1}{2}[(\mathbf{x}-\mathbf{y})\cdot\dot{\mathbf{y}}]\dot{\mathbf{y}} \tag{44}$$

by mapping $\mathbf{x} \mapsto \mathbf{x} - \mathbf{y}(t)$, $\mathbf{u} \mapsto \dot{\mathbf{y}}(t)$ in equation (43).

We must now transform the sources, applying Lorentz boosts with velocity $-\dot{\mathbf{y}}$ to the rest frame distributions $\rho_{(0)}$ and $j_{(0)}$. Consider first the scalar distribution, which, from previous work, in the rest frame has the form

$$\rho_{(0)}(\boldsymbol{\eta}) = q\delta(\boldsymbol{\eta}). \tag{45}$$

Since ρ transforms as a Lorentz scalar, $\rho(x) = \rho_{(0)}(\boldsymbol{\eta}(\mathbf{x}))$ in the laboratory frame. To find $\delta(\boldsymbol{\eta})$ as a function of $\mathbf{x}$ one notes that

$$\int d^2\mathbf{x}\, f(\mathbf{x})\delta(\boldsymbol{\eta}) = \int d^2\boldsymbol{\eta} \left|\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}\right| f(\mathbf{x})\delta(\boldsymbol{\eta}) \tag{46}$$

for any function f, where $|\partial\mathbf{x}/\partial\boldsymbol{\eta}|$ is the determinant of the Jacobian of the transformation $\mathbf{x} \mapsto \boldsymbol{\eta}$. From (44),

$$\begin{aligned} \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{x}} &= \mathbb{I} + \frac{1}{2}\dot{\mathbf{y}} \otimes \dot{\mathbf{y}} \\ \Rightarrow \frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}} &= \mathbb{I} - \frac{1}{2}\dot{\mathbf{y}} \otimes \dot{\mathbf{y}} + \ldots \\ \Rightarrow \left|\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}\right| &= 1 - \frac{1}{2}|\dot{\mathbf{y}}|^2 + \ldots \end{aligned} \tag{47}$$

Substituting into (46),

$$\begin{aligned} \int d^2\mathbf{x}\, f(\mathbf{x})\delta(\boldsymbol{\eta}) &= \left(1 - \frac{1}{2}|\dot{\mathbf{y}}|^2\right) \int d^2\boldsymbol{\eta} f(\mathbf{x})\delta(\boldsymbol{\eta}) \\ &= \left(1 - \frac{1}{2}|\dot{\mathbf{y}}|^2\right) f(\mathbf{x})\Big|_{\boldsymbol{\eta}=\mathbf{0}} \\ &= \left(1 - \frac{1}{2}|\dot{\mathbf{y}}|^2\right) f(\mathbf{y}) \end{aligned} \tag{48}$$

since $\boldsymbol{\eta} = \mathbf{0} \Leftrightarrow \mathbf{x} = \mathbf{y}$. Thus,

$$\delta(\boldsymbol{\eta}) = \left(1 - \frac{1}{2}|\dot{\mathbf{y}}|^2\right) \delta(\mathbf{x} - \mathbf{y}) \tag{49}$$

$$\Rightarrow \rho(\mathbf{x}) = \left(1 - \frac{1}{2}|\dot{\mathbf{y}}|^2\right) q\delta(\mathbf{x} - \mathbf{y}). \tag{50}$$

The term scalar charge for q is something of a misnomer since, as shown by (50) it is not a scalar. A plate of area C in its rest frame, carrying uniform scalar charge density ρ has total scalar charge $q = C\rho$. Looked at from a boosted frame, the plate is squashed along the boost direction by a factor $1/\gamma$ due to Lorentz-Fitzgerald contraction, so the area of the plate in this frame is $C/\gamma \approx (1-u^2/2)C$. The charge density is invariant, so the total scalar charge is $q' \approx (1 - u^2/2)q$ in agreement with the calculation above.

The moving magnetic dipole is rather more complicated, because j^μ transforms as a vector itself. The rest frame source is

$$(j^0_{(0)}, \quad \mathbf{j}_{(0)}) = (0, -m\widehat{\mathbf{k}} \times \nabla_\eta \delta(\boldsymbol{\eta})). \tag{51}$$

Again, we perform a Lorentz boost on this with velocity $-\dot{\mathbf{y}}$, that is

$$j^\mu(x) = \Lambda^\mu{}_\nu j^\nu_{(0)}(\boldsymbol{\eta}(x)), \tag{52}$$

where

$$\begin{aligned}\Lambda^0{}_0 = \Lambda^{\parallel}{}_{\parallel} &= \gamma \\ \Lambda^0{}_{\parallel} = \Lambda^{\parallel}{}_0 &= \gamma|\dot{\mathbf{y}}| \\ \Lambda^{\perp}{}_{\perp} &= 1,\end{aligned} \tag{53}$$

all other $\Lambda^\mu{}_\nu = 0$. Explicitly,

$$j^0(x) = |\dot{\mathbf{y}}|\gamma j^{\parallel}_{(0)}(\eta(x)) = \dot{\mathbf{y}} \cdot \mathbf{j}_{(0)}(\eta(x)) + \dots \tag{54}$$

$$\begin{aligned} j^{\parallel}(x) &= \gamma j^{\parallel}_{(0)}(\eta(x)) = \left(1 + \frac{1}{2}|\dot{\mathbf{y}}|^2\right) j_{(0)}(\eta(x)) + \dots \\ j^{\perp}(x) &= j^{\perp}_{(0)}(\eta(x)). \end{aligned} \tag{55}$$

Repeating the algebraic trick of (43),

$$\mathbf{j}(x) = \mathbf{j}_{(0)}(\boldsymbol{\eta}(x)) + \frac{1}{2}(\mathbf{j}_{(0)}(\boldsymbol{\eta}(x)) \cdot \dot{\mathbf{y}})\dot{\mathbf{y}} + \dots \tag{56}$$

Now,

$$\begin{aligned}\nabla_\eta = \left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}\right)\nabla &= \left(\mathbb{I} - \frac{1}{2}\dot{\mathbf{y}} \otimes \dot{\mathbf{y}}\right)\nabla \\ &= \nabla - \frac{1}{2}\dot{\mathbf{y}}(\dot{\mathbf{y}} \cdot \nabla),\end{aligned} \tag{57}$$

so, using (49), we find that

$$\begin{aligned}\mathbf{j}_{(0)}(\boldsymbol{\eta}(x)) &= -m\widehat{\mathbf{k}} \times \nabla_\eta \delta(\boldsymbol{\eta}) \\ &= -m\left(1 - \frac{1}{2}|\dot{\mathbf{y}}|^2\right)\widehat{\mathbf{k}} \times \nabla\delta(\mathbf{x}-\mathbf{y}) + \frac{1}{2}m(\widehat{\mathbf{k}} \times \dot{\mathbf{y}})\dot{\mathbf{y}} \cdot \nabla\delta(\mathbf{x}-\mathbf{y}) + \dots\end{aligned} \tag{58}$$

Substituting (58) into (54) and (56),

$$\begin{aligned} j^0(x) &= m(\widehat{\mathbf{k}} \times \dot{\mathbf{y}}) \cdot \nabla\delta(\mathbf{x}-\mathbf{y}), \\ \mathbf{j}(x) &= -m\left(1 - \frac{1}{2}|\dot{\mathbf{y}}|^2\right)\widehat{\mathbf{k}} \times \nabla\delta(\mathbf{x}-\mathbf{y}) \\ &\quad + \frac{1}{2}m\left[\dot{\mathbf{y}}(\widehat{\mathbf{k}} \times \dot{\mathbf{y}}) \cdot \nabla\delta(\mathbf{x}-\mathbf{y}) + (\widehat{\mathbf{k}} \times \dot{\mathbf{y}})\dot{\mathbf{y}} \cdot \nabla\delta(\mathbf{x}-\mathbf{y})\right],\end{aligned} \tag{59}$$

the final result.

One should note that

$$\partial_\mu j^\mu = \frac{\partial j^0}{\partial t} + \nabla \cdot \mathbf{j} = m(\widehat{\mathbf{k}} \times \ddot{\mathbf{y}}) \cdot \nabla\delta(\mathbf{x}-\mathbf{y}), \tag{60}$$

so that $\partial_\mu j^\mu \neq 0$ unless $\ddot{\mathbf{y}} = \mathbf{0}$ (in which case the rest frame is inertial and $\partial_\mu j^\mu = 0$ follows from the vanishing of $\nabla \cdot \mathbf{j}$ for a static point vortex). If $\ddot{\mathbf{y}} = \mathbf{0}$ the current is conserved, so we can visualize the current density of a vortex moving with constant velocity as an ordinary electric current. In analogy with standard electrodynamics, we identify $j^0 = \varrho$ as the electric charge density of the distribution. From (59) we see that $\varrho \neq 0$ for a moving magnetic dipole, but that ϱ corresponds to an electric dipole of moment $-m\widehat{\mathbf{k}} \times \dot{\mathbf{y}} = |m|\widehat{\mathbf{k}} \times \dot{\mathbf{y}}$. It is helpful to think of the magnetic dipole at rest as a small clockwise current loop consisting of interpervading gases of oppositely charged current carriers confined to a circle in $\mathbb{R}^2$, travelling in opposite senses, as in figure 4, so that the whole has $\varrho = 0$

everywhere. Consider such an object viewed in a frame in which it is moving with constant velocity $\dot{\mathbf{y}}$. In the figure, the positive gas in the upper half experiences greater Lorentz-Fitzgerald contraction than the negative due to its velocity of circulation, and *vice-versa* in the lower half. Thus the upper half acquires a net positive charge density, while the lower half acquires an equal net negative charge density (it is *charge* not charge *density* which is scalar here), this charge splitting being the origin of our electric dipole ϱ. Note the agreement of orientation. Turning now to the current density $\mathbf{j}$, the first term of (59) represents a current loop, while the second and third represent anisotropies due to the aforementioned charge splitting. The transport of the net positive charge (upper half) and net negative charge (lower half) with velocity $\dot{\mathbf{y}}$ produces the current represented by the second term in (59). Also, as the loop moves along, the charge must split in front of the loop and recombine behind it in order to create the electric dipole. The current due to this process is represented by the final term.

7.2 Interaction Lagrangians

In order to compute the interaction Lagrangians L_ψ and L_A in the case of arbitrarily moving vortices, one must find the fields ψ and A induced by time varying sources ρ and j. Were the linear theory massless, this would involve the use of retarded potentials, since disturbances of the fields due to time varying sources would propagate uniformly at the speed of light. For example, the potential induced by a moving point charge in classical electrodynamics has been well studied, and explicit formulae can be found in the literature [15]. Not surprisingly, the analogous problem in *massive* electrodynamics (or scalar field theory) has not received such attention: the only fundamental physical force transmitted by massive quanta is the weak force, which has no rôle to play in the classical dynamics of point particles. We handle the problem by introducing formal temporal Fourier transforms, as follows. Let $\psi(t,\mathbf{x})$ be the field induced by time-varying source $\rho(t,\mathbf{x})$, according to the inhomogeneous Klein-Gordon equation. Define Fourier transforms $\widetilde{\psi}$ and $\widetilde{\rho}$ with variable ω dual to t. That is,

$$\psi(t,\mathbf{x}) =: \int_{-\infty}^{\infty} d\omega e^{i\omega t}\widetilde{\psi}(\omega,\mathbf{x}), \qquad \rho(t,\mathbf{x}) =: \int_{-\infty}^{\infty} d\omega e^{i\omega t}\widetilde{\rho}(\omega,\mathbf{x}). \tag{61}$$

Then,

$$\begin{aligned} (\Box+\mu^2)\psi &= \rho \qquad (62)\\ \Rightarrow (-\omega^2-\Delta+\mu^2)\widetilde{\psi} &= \widetilde{\rho} \\ \Rightarrow [-\Delta+(\mu^2-\omega^2)]\widetilde{\psi} &= \widetilde{\rho}, \qquad (63) \end{aligned}$$

so $\widetilde{\psi}(\omega,\mathbf{x})$ satisfies the static inhomogeneous Klein-Gordon equation with squared mass $\mu^2-\omega^2$ and source $\widetilde{\rho}(\omega,\mathbf{x})$. Equation (63) is solved (at least formally) by convolution of $\widetilde{\rho}$ with the Green's function $K_0(\sqrt{\mu^2-\omega^2}|\mathbf{x}-\mathbf{x}'|)/2\pi$,

$$\widetilde{\psi}(\omega,\mathbf{x}) = \frac{1}{2\pi}\int d^2\mathbf{x}'\, K_0(\sqrt{\mu^2-\omega^2}|\mathbf{x}-\mathbf{x}'|)\widetilde{\rho}(\omega,\mathbf{x}'). \tag{64}$$

Now expand the Green's function in ω/μ, truncating at order ω^2/μ^2,

$$\begin{aligned} K_0(\sqrt{\mu^2-\omega^2}|\mathbf{x}-\mathbf{x}'|) &= K_0(\mu(1-\frac{1}{2}\frac{\omega^2}{\mu^2}+\cdots)|\mathbf{x}-\mathbf{x}'|) \\ &= K_0(\mu|\mathbf{x}-\mathbf{x}'|)-\frac{\omega^2}{2\mu^2}|\mathbf{x}-\mathbf{x}'|K_0'(\mu|\mathbf{x}-\mathbf{x}'|)+\cdots \qquad (65) \end{aligned}$$

so that substitution into (64) yields

$$\widetilde{\psi}(\omega,\mathbf{x}) = \frac{1}{2\pi}\int d^2\mathbf{x}'\, K_0(\mu|\mathbf{x}-\mathbf{x}'|)\widetilde{\rho}(\omega,\mathbf{x}')$$

$$-\frac{\omega^2}{4\pi\mu}\int d^2\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|K_0'(\mu|\mathbf{x}-\mathbf{x}'|)\widetilde{\rho}(\omega,\mathbf{x}'), \tag{66}$$

whence we obtain $\psi(t,\mathbf{x})$ by (61),

$$\begin{aligned}
\psi(t,\mathbf{x}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\int d^2\mathbf{x}'\,K_0(\mu|\mathbf{x}-\mathbf{x}'|)\widetilde{\rho}(\omega,\mathbf{x}') \\
&\quad -\frac{1}{4\pi\mu}\int_{-\infty}^{\infty} d\omega\,\omega^2 e^{i\omega t}\int d^2\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|K_0'(\mu|\mathbf{x}-\mathbf{x}'|)\widetilde{\rho}(\omega,\mathbf{x}') \\
&= \frac{1}{2\pi}\int d^2\mathbf{x}'\,K_0(\mu|\mathbf{x}-\mathbf{x}'|)\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\widetilde{\rho}(\omega,\mathbf{x}') \\
&\quad +\frac{1}{4\pi\mu}\int d^2\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|K_0'(\mu|\mathbf{x}-\mathbf{x}'|)\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\widetilde{\rho}(\omega,\mathbf{x}') \\
&= \frac{1}{2\pi}\int d^2\mathbf{x}'\,K_0(\mu|\mathbf{x}-\mathbf{x}'|)\rho(t,\mathbf{x}') \\
&\quad +\frac{1}{4\pi\mu}\int d^2\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|K_0'(\mu|\mathbf{x}-\mathbf{x}'|)\ddot{\rho}(t,\mathbf{x}').
\end{aligned} \tag{67}$$

Note that truncating the expansion in ω is, in effect, the same as neglecting higher time derivatives of ρ in general, eventually acting on $\mathbf{z}(t)$ in our application. No claim of rigour is attached to the above Fourier transform manoeuvre. One should regard it as a convenient algebraic short-hand for obtaining a perturbative ansatz for (62). Substitution of (67) into (62) explicitly verifies that ψ is indeed a solution, up to higher derivative terms ($d^3\rho/dt^3$ etc.).

Using this procedure, we find ψ_2, the field induced by time varying source ρ_2. The interaction of such a field with another time varying source ρ_1 is given by (34),

$$\begin{aligned}
L_\psi &= \int d^2\mathbf{x}\,\rho_1\psi_2 \\
&= \frac{1}{2\pi}\int d^2\mathbf{x}\,d^2\mathbf{x}'\,K_0(\mu|\mathbf{x}-\mathbf{x}'|)\rho_1(t,\mathbf{x})\rho_2(t,\mathbf{x}') \\
&\quad -\frac{1}{4\pi\mu}\int d^2\mathbf{x}\,d^2\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|K_0'(\mu|\mathbf{x}-\mathbf{x}'|)\dot{\rho}_1(t,\mathbf{x})\dot{\rho}_2(t,\mathbf{x}'),
\end{aligned} \tag{68}$$

where a total time derivative has been discarded. The vector field calculation is essentially identical, yielding interaction Lagrangian

$$\begin{aligned}
L_A &= -\frac{1}{2\pi}\int d^2\mathbf{x}\,d^2\mathbf{x}'\,K_0(|\mathbf{x}-\mathbf{x}'|)j^\mu_{(1)}(t,\mathbf{x})J^{(2)}_\mu(t,\mathbf{x}') \\
&\quad +\frac{1}{4\pi}\int d^2\mathbf{x}\,d^2\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|K_0'(|\mathbf{x}-\mathbf{x}'|)\frac{\partial j^\mu_{(1)}}{\partial t}(t,\mathbf{x})\frac{\partial J^{(2)}_\mu}{\partial t}(t,\mathbf{x}'),
\end{aligned} \tag{69}$$

the only new feature being that one of the sources includes the "fictitious current" due to non-conservation of $j^{(2)}_\mu$. That is, the field equation for A_μ with source j_μ (24) looks like the Proca equation (a triplet of Klein-Gordon equations) with source $j_\mu+\partial_\mu(\partial_\nu j^\nu)$, so we define the pseudocurrent

$$J^{(2)}_\mu := j^{(2)}_\mu + \partial_\mu(\partial_\nu j^\nu_{(2)}), \tag{70}$$

and it is this source which appears in the algebra analogously to ρ_2 in the scalar calculation above. There is an apparent asymmetry in (69)– it looks asymmetric under the interchange of sources $1\leftrightarrow 2$ – but this is easily removed, given the form of the pseudocurrent (70), by integration by parts; taking the first integral in (69), the extra term due to the fictitious current is

$$\begin{aligned}
&-\frac{1}{2\pi}\int d^2\mathbf{x}\,d^2\mathbf{x}'\,K_0(|\mathbf{x}-\mathbf{x}'|)\,j^\mu_{(1)}(x)\,\partial'_\mu(\partial'_\nu j^\nu_{(2)}(x'))\Big|_{t'=t} \\
&\qquad = \frac{1}{2\pi}\int d^2\mathbf{x}\,d^2\mathbf{x}'\,j^\mu_{(1)}\,\partial'_\mu K_0\,\partial'_\nu j^\nu_{(2)}\Big|_{t'=t} \\
&\qquad = \frac{1}{2\pi}\int d^2\mathbf{x}\,d^2\mathbf{x}'\,j^\mu_{(1)}\,\partial_\mu K_0\,\partial'_\nu j^\nu_{(2)}\Big|_{t'=t} \\
&\qquad = -\frac{1}{2\pi}\int d^2\mathbf{x}\,d^2\mathbf{x}'\,K_0(|\mathbf{x}-\mathbf{x}'|)\,\partial_\mu j^\mu_{(1)}(x)\,\partial'_\nu j^\nu_{(2)}(x')\Big|_{t'=t}
\end{aligned} \tag{71}$$

where, as usual, all boundary integrals have vanished. A similar calculation for the second integral yields the extra term

$$\frac{1}{4\pi}\int d^2\mathbf{x}\,d^2\mathbf{x}'\,|x-\mathbf{x}'|K_0'(|\mathbf{x}-\mathbf{x}'|)\frac{\partial}{\partial t}\partial_\mu j^\mu_{(1)}(x)\frac{\partial}{\partial t'}\partial'_\nu j^\nu_{(2)}(x')\Big|_{t'=t} \tag{72}$$

Recall from (60) that $\partial_\mu j^\mu$ is of order $|\ddot{\mathbf{y}}|$, so neither term (71) nor (72) makes any contribution to L_A at the order to which we are calculating. We may thus discard them and work with the formula

$$\begin{aligned}
L_A &= -\frac{1}{2\pi}\int d^2\mathbf{x}\,d^2\mathbf{x}'\,K_0(|\mathbf{x}-\mathbf{x}'|)j^\mu_{(1)}(t,\mathbf{x})j^{(2)}_\mu(t,\mathbf{x}') \\
&\quad +\frac{1}{4\pi}\int d^2\mathbf{x}\,d^2\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|K_0'(|\mathbf{x}-\mathbf{x}'|)\frac{\partial j^\mu_{(1)}}{\partial t}(t,\mathbf{x})\frac{\partial j^{(2)}_\mu}{\partial t}(t,\mathbf{x}').
\end{aligned} \tag{73}$$

It remains to substitute the point vortex sources (50) and (59) into these expressions for L_ψ and L_A, noting that at critical coupling, $\mu = 1$ and $m \equiv q$. A straightforward though lengthy calculation, presented in the appendix, yields

$$\begin{aligned}
L_{\rm int} &= L_A + L_\psi \\
&= \frac{q^2}{4\pi}\left\{\left[\frac{(\mathbf{y}-\mathbf{z})\cdot(\dot{\mathbf{y}}-\dot{\mathbf{z}})}{|\mathbf{y}-\mathbf{z}|}\right]^2 - \left[\frac{(\mathbf{y}-\mathbf{z})\cdot(\widehat{\mathbf{k}}\times(\dot{\mathbf{y}}-\dot{\mathbf{z}}))}{|\mathbf{y}-\mathbf{z}|}\right]^2\right\}\Lambda(|\mathbf{y}-\mathbf{z}|),
\end{aligned} \tag{74}$$

where

$$\Lambda(\lambda) = K_0(\lambda) + \frac{2K_1(\lambda)}{\lambda}. \tag{75}$$

The Lagrangian is completed by adding to this the standard non-relativistic kinetic Lagrangian for two point particles each of mass π, the rest energy of the $\mu = 1$ vortex, as found using the Bogomol'nyi argument. Introducing the centre of mass position $\mathbf{R} := \frac{1}{2}(\mathbf{y}+\mathbf{z})$ and relative position $\mathbf{r} := \mathbf{y}-\mathbf{z}$, the two vortex Lagrangian is

$$L = \pi|\dot{\mathbf{R}}|^2 + \frac{\pi}{4}|\dot{\mathbf{r}}|^2 + \frac{q^2}{4\pi}[(\widehat{\mathbf{r}}\cdot\dot{\mathbf{r}})^2 - (\widehat{\mathbf{r}}\cdot(\widehat{\mathbf{k}}\times\dot{\mathbf{r}}))^2]\Lambda(|\mathbf{r}|). \tag{76}$$

Note that L is Galilei invariant (a feature which does not generalize to arbitrary coupling since in the presence of static intervortex forces the inertia associated with $\mathbf{R}$ depends on the vortex separation $\mathbf{r}$). So, we can work in the centre of mass frame with the reduced Lagrangian, which looks like the Lagrangian of a free particle moving geodesically on a manifold (naively $\mathbb{R}^2$) with respect to the metric

$$g = \left(1+\frac{q^2}{\pi^2}\Lambda(r)\right)dr^2 + r^2\left(1-\frac{q^2}{\pi^2}\Lambda(r)\right)d\vartheta^2, \tag{77}$$

where $r(\cos\vartheta, \sin\vartheta) := \mathbf{r}$. This idea will be pursued in section 8. For the moment, we note that the equations of motion are

$$\ddot{r}^i + \Gamma^i_{jk}\dot{r}^j\dot{r}^k = 0 \tag{78}$$

where r^i are some coordinates on the manifold, and Γ is the Levi-Civita connexion derived from g, so if $r^i(t)$ is a solution of (78), then $|\ddot{\mathbf{r}}|$ is of order $|\dot{\mathbf{r}}|^2$ for all t. Differentiating (78) with respect to time, one sees that for each integer $n \geq 2$ there is a set of position dependent coefficients $\Omega^{(n)\,i}_{j_1 j_2 \cdots j_n}$ such that

$$\frac{d^n r^i}{dt^n} = \Omega^{(n)\,i}_{j_1 j_2 \ldots j_n}\dot{r}^{j_1}\dot{r}^{j_2}\cdots\dot{r}^{j_n}, \tag{79}$$

so that $|d^n\mathbf{r}/dt^n|$ is of order $|\dot{\mathbf{r}}|^n$. This provides *a posteriori* justification for truncating the expansion in time derivatives. That is, although the assumption that higher time derivatives are negligible may turn out to be bad for real vortex dynamics, it is at least self consistent. Away from critical coupling the situation is different. There is a $\partial U/\partial r_i$ term in (78), U being the static intervortex potential (37), and consequently the above argument does not work.

8 Critical vortex scattering

As remarked previously, the approximate equations of motion of two critically coupled point vortices can be interpreted as the geodesic equation on a two-dimensional manifold (call it $\mathcal{M}$) with metric $g = G(r)dr^2 + r^2H(r)d\vartheta^2$ where

$$G(r) = 1 + \frac{q^2}{\pi^2}\Lambda(r), \qquad H(r) = 1 - \frac{q^2}{\pi^2}\Lambda(r). \tag{80}$$

Vortices are not classically distinguishable particles, so $\mathbf{r}$ and $-\mathbf{r}$ correspond to the same configuration, and should be identified. Accordingly $\vartheta \in [0,\pi]$, $\vartheta = 0$ and $\vartheta = \pi$ being identified, and g is a metric on the cone $\mathbb{R}^2/\sim$ where $\mathbf{r} \sim \mathbf{r}' \Leftrightarrow \mathbf{r} = \pm\mathbf{r}'$. The function Λ is a strictly positive, strictly decreasing function on $(0,\infty)$ and $\Lambda \to \infty$ as $r \to 0$. Hence there is one and only one value of r for which the coefficient H is zero. At this radius, $r_s \approx 2.73$, there is a metric singularity where the signature of g flips from Euclidean ($r > r_s$) to Lorentzian ($r < r_s$). This singularity is completely unphysical, merely reminding us once again that the point vortex approximation breaks down when the vortices approach one another too closely. (A similar singularity occurs in the BPS monopole calculation [2].)

Since we are really only interested in geodesics on the portion $r > r_s$, it is helpful to define a new manifold $\mathcal{M}_* \cong (\mathbb{R}^2\backslash D)/\sim$ where D is the disc of radius $r_d > r_s$ centred on $\mathbf{r} = \mathbf{0}$ excluding boundary, so that the restriction of g to $\mathcal{M}_*$, call it g_* is strictly positive. The curvature of $(\mathcal{M}_*, g_*)$ is

$$K = \frac{-1}{r\sqrt{GH}}\frac{d}{dr}\left(\frac{1}{\sqrt{G}}\frac{d}{dr}(r\sqrt{H})\right). \tag{81}$$

It is a matter of straightforward calculation, which we will not reproduce here, to prove that $K > 0$, that K is a strictly decreasing function of r and that $K \to 0$ as $r \to \infty$. These results are independent of the specific value of q. One may therefore visualize $(\mathcal{M}_*, g_*)$ as a rounded cone with its cap cut off, the missing cap representing the forbidden core region where the approximation breaks down. Since g is independent of ϑ, $(\mathcal{M}_*, g_*)$ is rotationally symmetric.

This picture is very reminiscent of the geodesic approximation, where the low energy dynamics of two vortices is approximated by geodesic motion on the moduli space of static solutions. Samols [6] regards physical space as the complex plane and defines the position of each vortex to be $z_1, z_2 \in \mathbb{C}$ where the Higgs field vanishes. The degree 2 moduli space is then $\mathbb{C} \times \mathcal{M}$ where $\mathbb{C}$ is the space of centre of mass positions and $\mathcal{M}$ is the space of relative positions, on which is induced a nontrivial metric g_s by the kinetic energy functional (5). Defining relative coordinates (σ, ϑ) such that

$$\sigma e^{i\vartheta} = \frac{1}{2}(z_1 - z_2), \tag{82}$$

rotation and parity symmetries are sufficient to restrict g_s to the form $g_s = f_1(\sigma)d\sigma^2 + f_2(\sigma)\sigma^2 d\vartheta^2$. However, using properties of the Bogomol'nyi equations, Samols was able to prove that g_s must be Hermitian in $\sigma e^{i\vartheta}$, and this provides the extra constraint that $f_1 \equiv f_2$, so

$$g_s = F_s^2(\sigma)(d\sigma^2 + \sigma^2 d\vartheta^2). \tag{83}$$

Numerical computation of F_s reveals that $(\mathcal{M}, g_s)$ is a rounded cone of strictly positive curvature.

Given this similarity, and the precedent set by Manton's rederivation of the asymptotic Atiyah-Hitchin metric by point particle techniques [2], we are led to suggest that g_* is the asymptotic form of g_s. We would like to identify the radial coordinate r with 2σ, but given that $H \neq G$, g_* cannot be Hermitian with respect to $re^{i\vartheta}$, so such an identification is impossible. It follows that the vortex positions $\mathbf{y}$ and $\mathbf{z}$ do not coincide with zeros of the Higgs field. Coincidence is recovered asymptotically, but more slowly than the asymptotic convergence of g_* to the trivial flat metric. In previous applications of the method of linear retarded potentials, this has not happened: asymptotic coincidence of coordinates is faster than asymptotic flatness of the metric. The essentially new feature here is that the linear theory is massive, so it may be this which is responsible. Defining soliton position is always somewhat arbitrary because solitons can really only be considered independent particles when infinitely remote from one another. Since we defined vortex position in terms of the asymptotics of ϕ, in the case of degree 2 configurations we should expect there to be a discrepency between this and the usual definition, exponentially small in the vortex separation.

This does not disqualify g_* from being the asymptotic form of g_s (in fact, $4g_s$ since [6] uses a different normalization): we can always construct a radial coordinate in terms of which g_* does take the Hermitian form. If one defines the function

$$s(r) = s_d \exp\left[\int_{r_d}^{r} \frac{d\lambda}{\lambda}\sqrt{\frac{G(\lambda)}{H(\lambda)}}\right], \tag{84}$$

the transformation $r \mapsto s(r)$ is manifestly a bijection. In terms of (s, ϑ),

$$g_* = 4F^2(s)(ds^2 + s^2 d\vartheta^2) \tag{85}$$

where $F(s(r)) = \frac{1}{2}r\sqrt{H(r)}/s(r)$. To identify s with σ, we fix s_d such that $\lim_{r\to\infty} s(r)/r = 1/2$. It is then easily shown that $s(r)/r$ is a strictly increasing function on $[r_d, \infty)$ with minimum $s_d/r_d < 1/2$ and supremum $1/2$ (these observations follow from the fact that $G(\lambda) > H(\lambda)$ for all $\lambda > 0$, and are independent of the value of q). It follows that a given point $(r, \vartheta) \in \mathcal{M}_*$ represents a two-vortex configuration with inter-zero distance less than r, the difference vanishing exponentially at large r. Note that if we choose $r_d = r_s$ there is a $(\lambda - r_s)^{-\frac{1}{2}}$ singularity in the integrand of (84) because $H(r_s) = 0$. Given the specific formula (75) for Λ, we must evaluate this integral numerically, so to simplify matters we choose $r_d > r_s$, that is, we exclude a slightly larger cap from $\mathcal{M}_*$ than is strictly necessary. Choosing $r_d = 3$ one finds that $s_d \approx 1.22$. Figure 5 presents a plot of F compared with Samols' profile function F_s, from which it seems plausible that g_* is the asymptotic form of g_s. Since both functions are known only numerically, there is no rigorous test of this.

The main object of this section is to model critical vortex scattering by solving the geodesic problem on $(\mathcal{M}_*, g_*)$. We could use g_* in Hermitian form, but this would introduce an extra layer of numerical approximation, so it is better to work with g_* in the original coordinates (r, ϑ). The scattering problem is defined in terms of asymptotic parameters in any case: impact paramter b and impact speed v_∞, both defined where the vortices are infinitely remote from one another and there is no ambiguity in the term "vortex position." In fact, the geometry of geodesics is independent of initial velocity, as may be seen by rescaling t in the geodesic equation (78), so deflection angle is independent of v_∞. (It should be emphasized that this is a property of the approximation, not the abelian Higgs model itself. In fact, numerical simulations [13] show that scattering *is* approximately speed independent at critical coupling for low to moderate v_∞, but this breaks down at very high speeds.) So the scattering data $\Theta(b)$ provide a physically interesting, coordinate independent characterization of the metric structure on $\mathcal{M}_*$.

Without loss of generality, we can solve the initial value problem $r(0) = r_0$, $\vartheta(0) = 0$, $\dot{r} = 0$, $\dot{\vartheta} = \omega_0$, parametrized by (r_0, J), where $J = r^2 H(r)\dot{\vartheta}$ is the conserved momentum conjugate to ϑ. It is easily shown that $\lim_{t\to\infty}\vartheta$ is

$$\vartheta_\infty = \int_{r_0}^{\infty} \frac{dr}{r^2 H(r)} \left[\frac{1}{G(r)}\left(\frac{1}{r_0^2 H(r_0)} - \frac{1}{r^2 H(r)}\right)\right]^{-\frac{1}{2}}. \tag{86}$$

By equating energy at $r = r_0$ and as $r \to \infty$, one finds that $b(r_0) = \frac{1}{2} r_0 \sqrt{H(r_0)}$, so the absence of $J = 2v_\infty b$ in (86) implies that ϑ_∞ is independent of v_∞ as claimed. Approximate evaluation of the function $\vartheta_\infty(r_0)$ is performed using the same algorithm as was used for type II vortex scattering, summarized in equation (41). Figure 6 shows b plotted against $\Theta(b(r_0)) := \pi - 2\vartheta_\infty(r_0)$, the deflection angle, in comparison with Samols' scattering data, obtained using the geodesic approximation. (A comparison of the geodesic approximation with numerical simulations of the full field equations is made in [6].) The fit is remarkably good for moderate to large b, but deteriorates as b becomes small. This is to be expected since small b collisions probe the small r region of $\mathcal{M}_*$. That $\Theta(b)$ is a decreasing function is a corollary of the fact that the curvature is strictly positive [7].

9 Conclusion

In this paper we have presented a point source formalism for long range vortex dynamics. We used this framework to rederive the static intervortex potential from a new perspective and to calculate the velocity dependent interaction of critically coupled vortices. Reinterpreting the latter geometrically led to a conjectured formula for the asymptotic metric on the two vortex moduli space. We solved the scattering problem for $\mu^2 = 2$ and $\mu^2 = 1$ vortices and found reasonable agreement with numerical simulations, despite the simplicity of the model. It is worth pointing out that, compared with other studies of vortex dynamics [6, 13, 16], the present work required only very lightweight numerical work. It would be straightforward to apply the method to other situattions of interest: to derive the asymptotic forces between a static vortex-antivortex pair, or higher winding conglomerations (in the type I regime), or larger collections of vortices for example.

After its many successes in Bogomol'nyi field theories (Yang-Mills-Higgs, abelian Higgs and sigma models most notably) attempts are now being made to generalize the geodesic approximation in the absence of a saturable Bogomol'nyi bound. The idea is that a moduli space of physically relevant configurations is proposed, usually on rather *ad hoc* grounds, and the metric and potential on this space (restrictions of the kinetic and potential energy functionals of the field theory) are calculated. Generically, this metric must be evaluated numerically from first principles, that is, by calculating the L^2 inner products of every (unordered) pair of tangent vectors, at each configuration. This is a very intensive procedure. The computational cost would be significantly reduced if the need for such numerical work could be contained within a relatively small core region of moduli space where the solitons are close together – if, for example, the asymptotic form of the metric could be found analytically from a point source approximation, a technique employed in [17] in the context of the Skyrme model without pion mass. The present work develops in a simple setting a possible way of doing this when the linearized theory is massive. However, one should note that the mismatch of moduli space coordinates encountered in the abelian Higgs model could cause major problems if it occurs generically. We were able to construct a coordinate transformation quite easily, but this was on a two dimensional manifold with rotational symmetry. For higher dimensional moduli spaces (if the field theory is defined on $\mathbb{R}^{3+1}$, or the solitons have orientations or internal degrees of freedom for example) the transformation may be far more complicated.

Appendix

In this appendix we present a detailed derivation of the velocity dependent interaction Lagrangians L_ψ and L_A for critically coupled point vortices. It is convenient to define the functions

$$\Upsilon(\lambda) := \lambda K_0'(\lambda), \qquad \Lambda(\lambda) := K_0(\lambda) - \frac{2K_0'(\lambda)}{\lambda}. \tag{87}$$

Substituting the scalar sources (50) for vortex 1 at $\mathbf{y}(t)$ and vortex 2 at $\mathbf{z}(t)$ into L_ψ, (68), one obtains

$$\begin{aligned} L_\psi &= \frac{q^2}{2\pi}\int d^2\mathbf{x}\, d^2\mathbf{x}'\, K_0(|\mathbf{x}'-\mathbf{x}|)\left[1-\frac{1}{2}(|\dot{\mathbf{y}}|^2+|\dot{\mathbf{z}}|^2)\right]\delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{x}'-\mathbf{z}) \\ &\quad -\frac{q^2}{4\pi}\int d^2\mathbf{x}\, d^2\mathbf{x}'\, \Upsilon(|\mathbf{x}-\mathbf{x}'|)\dot{\mathbf{y}}\cdot\nabla_y\delta(\mathbf{x}-\mathbf{y})\dot{\mathbf{z}}\nabla_z\delta(\mathbf{x}'-\mathbf{z})+\cdots \\ &= \frac{q^2}{2\pi}\left[1-\frac{1}{2}(|\dot{\mathbf{y}}|^2+|\dot{\mathbf{z}}|^2)\right]K_0(|\mathbf{y}-\mathbf{z}|)1\frac{q^2}{4\pi}(\dot{\mathbf{y}}\cdot\nabla_y)(\dot{\mathbf{z}}\cdot\nabla_z)\Upsilon(|\mathbf{y}-\mathbf{z}|). \end{aligned} \tag{88}$$

The magnetic interaction Lagrangian is considerably more complicated. First we note that (69) can be split up as follows,

$$L_A = -\frac{1}{2\pi}(S_1 - S_2) + \frac{1}{4\pi}S_3 \tag{89}$$

where

$$\begin{aligned} S_1 &= \int d^2\mathbf{x}\, d^2\mathbf{x}'\, K_0(|\mathbf{x}-\mathbf{x}'|) j^0_{(1)}(t,\mathbf{x}) j^0_{(2)}(t,\mathbf{x}') \\ S_2 &= \int d^2\mathbf{x}\, d^2\mathbf{x}'\, K_0(|\mathbf{x}-\mathbf{x}'|)\mathbf{j}_{(1)}(t,\mathbf{x})\cdot\mathbf{j}_{(2)}(t,\mathbf{x}') \\ S_3 &= \int d^2\mathbf{x}\, d^2\mathbf{x}'\, \Upsilon(|\mathbf{x}-\mathbf{x}'|)\partial_t j^\mu_{(1)}(t,\mathbf{x})\partial_t j^{(2)}_\mu(t,\mathbf{x}'). \end{aligned} \tag{90}$$

Source 1 is given explicitly in (59) and source 2 is identical, but with $\mathbf{y}\to\mathbf{z}$, $\dot{\mathbf{y}}\to\dot{\mathbf{z}}$. Note that since $\mu^2 = 1$, we set $m \equiv q$. Substituting these sources into S_1 and S_2 yields

$$\begin{aligned} S_1 &= q^2\int d^2\mathbf{x}\, d^2\mathbf{x}'\, K_0(|\mathbf{x}-\mathbf{x}'|)(\widehat{\mathbf{k}}\times\dot{\mathbf{y}})\cdot\nabla_y\delta(\mathbf{x}-\mathbf{y})(\widehat{\mathbf{k}}\times\dot{\mathbf{z}})\cdot\nabla_z\delta(\mathbf{x}'-\mathbf{z}) \\ &= q^2(\widehat{\mathbf{k}}\times\dot{\mathbf{y}})\cdot\nabla_y(\widehat{\mathbf{k}}\times\dot{\mathbf{z}})\cdot\nabla_z K_0(|\mathbf{y}-\mathbf{z}|) \\ &= -q^2\dot{\mathbf{y}}\cdot\dot{\mathbf{z}}\frac{K_0'(|\mathbf{y}-\mathbf{z}|)}{|\mathbf{y}-\mathbf{z}|} - q^2\frac{(\widehat{\mathbf{k}}\times\dot{\mathbf{y}})\cdot(\mathbf{y}-\mathbf{z})(\widehat{\mathbf{k}}\times\dot{\mathbf{z}})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|^2}\Lambda(|\mathbf{y}-\mathbf{z}|), \end{aligned} \tag{91}$$

$$\begin{aligned} S_2 &= q^2\int d^2\mathbf{x}\, d^2\mathbf{x}'\, K_0(|\mathbf{x}-\mathbf{x}'|) \\ &\left[\left(1-\frac{|\dot{\mathbf{y}}|^2}{2}\right)\widehat{\mathbf{k}}\times\nabla_y\delta(\mathbf{x}-\mathbf{y}) - \frac{1}{2}\dot{\mathbf{y}}(\widehat{\mathbf{k}}\times\dot{\mathbf{y}})\cdot\nabla_y\delta(\mathbf{x}-\mathbf{y}) - \frac{1}{2}(\widehat{\mathbf{k}}\times\dot{\mathbf{y}})\dot{\mathbf{y}}\cdot\nabla_y\delta(\mathbf{x}-\mathbf{y})\right] \\ \bullet &\left[\left(1-\frac{|\dot{\mathbf{z}}|^2}{2}\right)\widehat{\mathbf{k}}\times\nabla_z\delta(\mathbf{x}'-\mathbf{z}) - \frac{1}{2}\dot{\mathbf{z}}(\widehat{\mathbf{k}}\times\dot{\mathbf{z}})\cdot\nabla_z\delta(\mathbf{x}'-\mathbf{z}) - \frac{1}{2}(\widehat{\mathbf{k}}\times\dot{\mathbf{z}})\dot{\mathbf{z}}\cdot\nabla_z\delta(\mathbf{x}'-\mathbf{z})\right] \end{aligned}$$

$$
\begin{aligned}
&= q^2 \int d^2\mathbf{x}\, d^2\mathbf{x}'\, K_0(|\mathbf{x}-\mathbf{x}'|) \left[1 - \frac{1}{2}(|\dot{\mathbf{y}}|^2 + |\dot{\mathbf{z}}|^2)\right] \nabla_y \delta(\mathbf{x}-\mathbf{y}) \cdot \nabla_z \delta(\mathbf{x}'-\mathbf{z}) \\
&\quad -\frac{q^2}{2} \int d^2\mathbf{x}\, d^2\mathbf{x}'\, K_0(|\mathbf{x}-\mathbf{x}'|) [(\widehat{\mathbf{k}} \times \nabla_y \delta(\mathbf{x}-\mathbf{y})) \cdot \dot{\mathbf{z}} (\widehat{\mathbf{k}} \times \dot{\mathbf{z}}) \cdot \nabla_z \delta(\mathbf{x}'-\mathbf{z}) + (\mathbf{y} \leftrightarrow \mathbf{z})] \\
&\quad -\frac{q^2}{2} \int d^2\mathbf{x}\, d^2\mathbf{x}'\, K_0(|\mathbf{x}-\mathbf{x}'|) [(\widehat{\mathbf{k}} \times \nabla_y \delta(\mathbf{x}-\mathbf{y})) \cdot (\widehat{\mathbf{k}} \times \dot{\mathbf{z}}) \dot{\mathbf{z}} \cdot \nabla_z \delta(\mathbf{x}-\mathbf{y}) + (\mathbf{y} \leftrightarrow \mathbf{z})] \\
&\quad + \cdots \\
&= q^2 \left[1 - \frac{1}{2}(|\dot{\mathbf{y}}|^2 + |\dot{\mathbf{z}}|^2)\right] \nabla_y \cdot \nabla_z K_0(|\mathbf{y}-\mathbf{z}|) \\
&\quad + \frac{q^2}{2} [(\widehat{\mathbf{k}} \times \dot{\mathbf{z}}) \cdot \nabla_y (\widehat{\mathbf{k}} \times \dot{\mathbf{z}}) \cdot \nabla_z K_0(|\mathbf{y}-\mathbf{z}|) + (\mathbf{y} \leftrightarrow \mathbf{z})] \\
&\quad - \frac{q^2}{2} [\dot{\mathbf{z}} \cdot \nabla_y \dot{\mathbf{z}} \cdot \nabla_z K_0(|\mathbf{y}-\mathbf{z}|) + (\mathbf{y} \leftrightarrow \mathbf{z})] \\
&= -q^2 \left[1 - \frac{1}{2}(|\dot{\mathbf{y}}|^2 + |\dot{\mathbf{z}}|^2)\right] K_0(|\mathbf{y}-\mathbf{z}|) \\
&\quad + \frac{q^2}{2} \left\{ \left[\left(\frac{\dot{\mathbf{z}} \cdot (\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|} \right)^2 - \left(\frac{(\widehat{\mathbf{k}} \times \dot{\mathbf{z}}) \cdot (\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|} \right)^2 \right] + [\mathbf{y} \leftrightarrow \mathbf{z}] \right\} \Lambda(|\mathbf{y}-\mathbf{z}|), \qquad (92)
\end{aligned}
$$

where use has been made of Bessel's equation and the formula ($\boldsymbol{\beta}$ is independent of $\mathbf{y}$)

$$
\boldsymbol{\alpha} \cdot \nabla_y \boldsymbol{\beta} \cdot \nabla_z K_0(|\mathbf{y}-\mathbf{z}|) = -\boldsymbol{\alpha} \cdot \boldsymbol{\beta} \frac{K_0'(|\mathbf{y}-\mathbf{z}|)}{|\mathbf{y}-\mathbf{z}|} - \frac{\boldsymbol{\alpha} \cdot (\mathbf{y}-\mathbf{z}) \boldsymbol{\beta} \cdot (\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|^2} \Lambda(|\mathbf{y}-\mathbf{z}|). \qquad (93)
$$

Note that $(\mathbf{y} \leftrightarrow \mathbf{z})$ denotes repeated terms with $(\mathbf{y}, \dot{\mathbf{y}})$ interchanged with $(\mathbf{z}, \dot{\mathbf{z}})$. Finally, we turn to S_3:

$$
\begin{aligned}
S_3 &= -\int d^2\mathbf{x}\, d^2\mathbf{x}'\, \Upsilon(|\mathbf{x}-\mathbf{x}'|) \partial_t \mathbf{j}_{(1)}(t, \mathbf{x}) \cdot \mathbf{j}_{(2)}(t, \mathbf{x}') + \cdots \\
&= -q^2 \int d^2\mathbf{x}\, d^2\mathbf{x}'\, \Upsilon(|\mathbf{x}-\mathbf{x}'|) \nabla_y (\dot{\mathbf{y}} \cdot \delta(\mathbf{x}-\mathbf{y})) \cdot \nabla_z (\dot{\mathbf{z}} \cdot \delta(\mathbf{x}'-\mathbf{z})) \\
&= -q^2 (\dot{\mathbf{y}} \cdot \nabla_y)(\dot{\mathbf{z}} \cdot \nabla_z) \nabla_y \cdot \nabla_z \Upsilon(|\mathbf{y}-\mathbf{z}|) \\
&= q^2 (\dot{\mathbf{y}} \cdot \nabla_y)(\dot{\mathbf{z}} \cdot \nabla_z) \Upsilon(|\mathbf{y}-\mathbf{z}|) \\
&\quad -2q^2 \left[(\dot{\mathbf{y}} \cdot \dot{\mathbf{z}}) \frac{K_0'(|\mathbf{y}-\mathbf{z}|)}{|\mathbf{y}-\mathbf{z}|} + \frac{\dot{\mathbf{y}} \cdot (\mathbf{y}-\mathbf{z}) \dot{\mathbf{z}} \cdot (\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|^2} \Lambda(|\mathbf{y}-\mathbf{z}|) \right]. \qquad (94)
\end{aligned}
$$

It remains to substitute (91), (92) and (94) into (89) and collect terms. Note that the first two terms in S_2 and $\frac{1}{2}S_3$ combined are $-2\pi L_\psi$.

$$
\begin{aligned}
L_A &= \frac{1}{2\pi} \left(S_2 + \frac{S_3}{2} - S_1 \right) \\
&= -L_\psi + \frac{q^2}{4\pi} \left[\left(\frac{\dot{\mathbf{z}} \cdot (\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|} \right)^2 + \left(\frac{\dot{\mathbf{y}} \cdot (\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|} \right)^2 \right. \\
&\quad - \left(\frac{(\widehat{\mathbf{k}} \times \dot{\mathbf{z}}) \cdot (\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|} \right)^2 - \left(\frac{(\widehat{\mathbf{k}} \times \dot{\mathbf{y}}) \cdot (\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|} \right)^2 \\
&\quad \left. -2 \frac{\dot{\mathbf{y}} \cdot (\mathbf{y}-\mathbf{z}) \dot{\mathbf{z}} \cdot (\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|^2} + 2 \frac{(\widehat{\mathbf{k}} \times \dot{\mathbf{y}}) \cdot (\mathbf{y}-\mathbf{z}) (\widehat{\mathbf{k}} \times \dot{\mathbf{z}}) \cdot (\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|^2} \right] \Lambda(|\mathbf{y}-\mathbf{z}|)
\end{aligned}
$$

$$= \quad -L_\psi + \frac{q^2}{4\pi}\left[\left(\frac{(\dot{\mathbf{y}}-\dot{\mathbf{z}})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|}\right)^2 - \left(\frac{(\widehat{\mathbf{k}}\times(\dot{\mathbf{y}}-\dot{\mathbf{z}}))\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|}\right)^2\right]\Lambda(|\mathbf{y}-\mathbf{z}|). \quad (95)$$

Hence $L_{\rm int} = L_A + L_\psi$ is the expression quoted, (76).

Acknowledgments: I would like to thank Bernd Schroers for many valuable conversations, and Trevor Samols for reproducing his data on critical vortex scattering and the profile function. This work developed partly while I was a research student at the University of Durham, England, financially supported by the UK Particle Physics and Astronomy Research Council.

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Figure captions

Figure 1: The potential function $U(r)$ for $\mu^2 = 0.4$ and $\mu^2 = 2.0$.

Figure 2: The geometry of vortex scattering.

Figure 3: The scattering of $\mu^2 = 2$ (type II) vortices: deflection angle Θ versus impact parameter b at four different impact speeds. The solid curves were produced using the point particle approximation, the crosses by numerical simulation of the full field equations [13].

Figure 4: A moving current loop. The unit vector $\widehat{\mathbf{k}}$ is directed out of the page. The left hand picture shows the senses of circulation of the charge carriers, while the right hand picture depicts the electric charge density as seen in the laboratory frame.

Figure 5: The metric profile function $F(s)$ of the metric g_* in Hermitian form (solid curve) compared with Samols' [6] numerically determined function (dashed curve).

Figure 6: Critical vortex scattering: deflection angle Θ versus impact parameter b. The results of the point particle approximation (solid curve) should be compared with the results of the numerically implemented geodesic approximation (dashed curve), which is in good agreement with numerical simulations of the full field equations [6].