BELL'S INEQUALITIES AND ALGEBRAIC STRUCTURE[1]

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December 17, 1996

Abstract. We provide an overview of the connections between Bell's inequalities and algebraic structure.

1. Introduction

Motivated by the desire to bring into the realm of testable hypotheses at least some of the important matters concerning the interpretation of quantum mechanics evoked in the controversy surrounding the Einstein-Podolsky-Rosen paradox [18][5], Bell discovered the first example [3][4] of a family of inequalities which are now generally called Bell's inequalities. These inequalities provide an upper bound on the strength of correlations between systems which are no longer interacting but have interacted in the past. Stated briefly, Bell showed that if the correlation experiments can be modelled by a single classical probability measure, then the strength of the correlations must satisfy a bound which is violated by certain quantum mechanical predictions (and, as has been verified experimentally, by nature - for a review of this original application of Bell's inequalities and the experiments performed, see [14][15]). Hence, it was established that in quantum mechanics and in nature there are correlations which cannot be modelled by "local hidden-variable theories". Though there are many other applications of Bell's inequalities besides the one due to Bell (see *e.g.* [26][42][40]), in this paper we shall concentrate on the relation between Bell's inequalities and algebraic structure which unexpectedly emerged in our study of Bell's inequalities in the context of quantum field theory.

First, we need to specify the mathematical context more precisely. As suggested by Ludwig's approach to statistical theories [24] (see, in particular, [41]), the minimal amount of structure required to model correlation experiments involving two subsystems (the only situation we shall discuss here) is a so-called correlation duality $(\hat{p}, \mathcal{A}, \mathcal{B})$, composed of two order unit

[1]This paper is an extended version of a talk given at the International Conference on Operator Algebras and Quantum Field Theory, held at the Accademia dei Lincei, Rome, in July, 1996.

spaces $\mathcal{A}$ and $\mathcal{B}$ (real vector spaces with a vector ordering $\geq$ and a unit 1) and a bilinear function $\hat{p} : \mathcal{A} \times \mathcal{B} \mapsto \mathrm{I\!R}$ such that $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $A, B \geq 0$ imply $\hat{p}(A,B) \geq 0$ and such that $\hat{p}(1,1) = 1$. The function $\hat{p}$ models the preparation of the ensemble of systems. The observables of the subsystem corresponding, for example, to $\mathcal{A}$ are represented by partitions $\{A_i \mid i \in I\}$ of the unit in $\mathcal{A}$: $\sum_i A_i = 1$ with $A_i \geq 0$ for each $i \in I$. Each $i \in I$ corresponds to a possible outcome of the experiment. The probability (relative frequency) of the joint occurrence of $i \in I$ and $j \in J$ in the two subsystems, respectively, is then given by $\hat{p}(A_i, B_j)$.

In quantum theory this structure is supplemented with additional assumptions, of which we only mention in this introduction that the order unit spaces $\mathcal{A}$ and $\mathcal{B}$ are posited to be the Hermitian part of subalgebras (again denoted by $\mathcal{A}$ and $\mathcal{B}$) of a unital C^*-algebra $\mathcal{C}$ and that they commute elementwise. Moreover, the bilinear function $\hat{p}$ is given by a state ϕ on $\mathcal{C}$: $\hat{p}(A,B) \equiv \phi(AB)$.

In [32][34] we defined the *maximal Bell correlation* $\beta(\hat{p}, \mathcal{A}, \mathcal{B})$ in a correlation duality $(\hat{p}, \mathcal{A}, \mathcal{B})$ to be

$$\begin{aligned}\beta(\hat{p}, \mathcal{A}, \mathcal{B}) &\equiv \frac{1}{2} \sup \big(\hat{p}(A_1, B_1) + \hat{p}(A_1, B_2) + \hat{p}(A_2, B_1) - \hat{p}(A_2, B_2)\big) \\ &= \frac{1}{2} \sup \big(\hat{p}(A_1, B_1 + B_2) + \hat{p}(A_2, B_1 - B_2)\big)\,,\end{aligned}$$

where the supremum is taken over all elements $A_i \in \mathcal{A}$, $B_i \in \mathcal{B}$ satisfying $-1 \leq A_i, B_i \leq 1$, $i = 1, 2$. In the case where $\mathcal{A}$ and $\mathcal{B}$ are the Hermitian parts of C^*-algebras $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$ and $\hat{p}$ is given by a state ϕ, then we shall write instead

$$\beta(\phi, \mathcal{A}, \mathcal{B}) = \frac{1}{2} \sup \phi(A_1(B_1 + B_2) + A_2(B_1 - B_2))\,.$$

In [34], the following bounds for $\beta(\hat{p}, \mathcal{A}, \mathcal{B})$ were proven.

Theorem 1.1. [32][34] *(a) If $(\hat{p}, \mathcal{A}, \mathcal{B})$ is an arbitrary correlation duality, then*

$$\beta(\hat{p}, \mathcal{A}, \mathcal{B}) \leq 2\,. \tag{1.1}$$

(b) If $\mathcal{A}$ (or $\mathcal{B}$) is the Hermitian part of some C^*-algebra, then

$$\beta(\hat{p}, \mathcal{A}, \mathcal{B}) \leq \sqrt{2}\,, \tag{1.2}$$

for every admissible $\hat{p}$ as described above.

(c) If $\mathcal{A}$ (or $\mathcal{B}$) is the Hermitian part of some abelian C^*-algebra, then

$$\beta(\hat{p}, \mathcal{A}, \mathcal{B}) \leq 1\,, \tag{1.3}$$

for every admissible $\hat{p}$ as described above.

The estimate (1.2) was first given in [12] and subsequently rediscovered by a number of researchers. In fact, it has been shown [23] that if $\mathcal{A}$ and

$\mathcal{B}$ are merely distributive real algebras with identity, then the inequality $\beta(\phi, \mathcal{A}, \mathcal{B}) \leq \sqrt{2}$ must hold for any state ϕ. Hence, the bound (1.2) is also satisfied by Jordan algebras and distributive Segal algebras. It is easy to construct correlation dualities $(\hat{p}, \mathcal{A}, \mathcal{B})$ saturating the bound (1.1), but it can also be saturated for suitable choice of $(\hat{p}, \mathcal{A}, \mathcal{B})$ with $\mathcal{A}$ and $\mathcal{B}$ nondistributive real algebras (see *e.g.* [22]). We shall see below that the bound (1.2) can also be saturated.

Initially, the purpose of these inequalities was to serve as necessary conditions for their respective hypotheses. In particular, considering (1.2), a correlation experiment in the laboratory yielding

$$\hat{p}(A_1, B_1) + \hat{p}(A_1, B_2) + \hat{p}(A_2, B_1) - \hat{p}(A_2, B_2) > \sqrt{2}$$

could not thus be described by quantum theory, as pointed out by Cirel'son [12].[2] The bound (1.3) is equivalent [34] to the CHSH version [13] of Bell's inequality. There are many different proofs of this inequality in the literature - at least one for each metatheoretical framework for the discussion of such correlation experiments. In [33] we proved a stronger result than that given here, namely, that if the correlation duality is quasi-classical[3], then (1.3) must hold for every admissible $\hat{p}$. But for our present purposes, the special case indicated above will suffice, since it makes clear that if the given correlation experiment can be modelled by a classical theory (thereby yielding abelian observable algebras), then Bell's inequality (1.3) must hold. Any correlation experiment yielding

$$\hat{p}(A_1, B_1) + \hat{p}(A_1, B_2) + \hat{p}(A_2, B_1) - \hat{p}(A_2, B_2) > 1$$

could therefore not be modelled by a classical theory (at least a classical theory providing a correlation duality). And since quantum theory predicts (see below) and nature confirms the existence of such correlations violating Bell's inequality, the reader may begin to grasp the significance of Bell's original intention.

Henceforth, we shall restrict our attention to the case where $\mathcal{A}$ and $\mathcal{B}$ are (Hermitian parts of) C^*-subalgebras of a unital C^*-algebra $\mathcal{C}$, each containing the identity $\mathbb{1} \in \mathcal{C}$. In Section 2 we shall examine the properties of the maximal Bell correlation $\beta(\phi, \mathcal{A}, \mathcal{B})$ and shall use it to define an algebraic invariant $\beta(\mathcal{A}, \mathcal{B})$ of the (isomorphism class of the) pair $(\mathcal{A}, \mathcal{B})$. If $\mathcal{A}$ and $\mathcal{B}$ are realized as a pair of commuting von Neumann algebras on a separable Hilbert space $\mathcal{H}$, it would follow that $\mathcal{A}$ is contained in $\mathcal{B}'$, the commutant of $\mathcal{B}$. Therefore, defining $\beta(\mathcal{A}, \mathcal{B})$ for $(\mathcal{A}, \mathcal{B})$ is equivalent to defining an algebraic invariant of the (isomorphism class of the) inclusion $\mathcal{A} \subset \mathcal{B}'$ of von Neumann algebras - an invariant which is quite distinct from that of Jones and others (see, *e.g.* [27]). We shall also explain in Section 2 which algebraic structural properties are associated with the maximal possible value of the invariant $\beta(\mathcal{A}, \mathcal{B})$. There were some surprises here. In Section 3 we shall report on results obtained in the special context of algebraic quantum field theory, which, among other things, establish that $\beta(\mathcal{A}, \mathcal{B})$ takes on infinitely many distinct values for suitable choices of $(\mathcal{A}, \mathcal{B})$.

[2] This has not been observed in the laboratory.

[3] See [41] for a definition of quasi-classical correlation duality.

2. Maximal Bell Correlations and Algebraic Invariants

As shall be made clear later, it is most useful to assume that the algebra $\mathcal{C}$ above is the algebra $\mathcal{B}(\mathcal{H})$ of all linear bounded operators on a separable Hilbert space $\mathcal{H}$, and that $\mathcal{A}$ and $\mathcal{B}$ are commuting von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$. In order to state assertions succinctly, we set for such algebras $\mathcal{A},\mathcal{B} \subset \mathcal{B}(\mathcal{H})$

$$\mathcal{T}(\mathcal{A},\mathcal{B}) \equiv \Big\{ \frac{1}{2}\big(A_1(B_1+B_2)+A_2(B_1-B_2)\big) \Big| \; A_i = A_i^* \in \mathcal{A},\; B_i = B_i^* \in \mathcal{B},\; -\mathbb{1} \leq A_i, B_i \leq \mathbb{1} \Big\} .$$

The set $\mathcal{T}(\mathcal{A},\mathcal{B})$ contains all the observables relevant for testing violations of Bell's inequalities in the (independent) subsystems modelled by the (mutually commuting) pair of algebras of observables $(\mathcal{A},\mathcal{B})$. The $\sigma\big(\mathcal{B}(\mathcal{H}),\mathcal{B}(\mathcal{H})_*\big)$-closed convex hull of $\mathcal{T}(\mathcal{A},\mathcal{B})$ will be denoted by $\overline{\mathcal{T}}(\mathcal{A},\mathcal{B})$. Since in such expressions we can replace A_i by λA_i with $-1 \leq \lambda \leq 1$, the image $\phi\big(\mathcal{T}(\mathcal{A},\mathcal{B})\big) \subset \mathbb{R}$ of $\mathcal{T}(\mathcal{A},\mathcal{B})$ under any state $\phi \in \mathcal{B}(\mathcal{H})^*$ is a symmetric interval[4] around zero. This interval is therefore characterized by the maximal Bell correlation

$$\beta(\phi,\mathcal{A},\mathcal{B}) = \sup\{\phi(T) \,|\, T \in \mathcal{T}(\mathcal{A},\mathcal{B})\} .$$

Of course, if the state ϕ is normal on $\mathcal{B}(\mathcal{H})$, then we also have

$$\beta(\phi,\mathcal{A},\mathcal{B}) = \sup\{\phi(T) \,|\, T \in \overline{\mathcal{T}}(\mathcal{A},\mathcal{B})\} .$$

Since $\mathbb{1} \in \mathcal{T}(\mathcal{A},\mathcal{B})$ (set $A_i = B_i = \mathbb{1}$), we have $\beta(\phi,\mathcal{A},\mathcal{B}) \geq 1$. As mentioned above, any value larger than 1 is a *violation* of Bell's inequalities in the given state, detectable by observables A_i, B_i in the subalgebras $\mathcal{A},\mathcal{B}$. On the other hand, by Theorem 1.1, the largest possible value for $\beta(\phi,\mathcal{A},\mathcal{B})$ in any state is $\sqrt{2}$, and if this value is attained we speak of a *maximal violation* of Bell's inequalities by the pair of algebras $(\mathcal{A},\mathcal{B})$ in the state ϕ.

We summarize some fundamental properties of maximal Bell correlations in the following lemma.

Lemma 2.1. [34][39] *Let $\mathcal{A},\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ be mutually commuting von Neumann algebras and $\phi \in \mathcal{B}(\mathcal{H})^*$ be a state. Then all of the following assertions are true.*

1. $1 \leq \beta(\phi,\mathcal{A},\mathcal{B}) \leq \sqrt{2}$.
2. *The map $\phi \mapsto \beta(\phi,\mathcal{A},\mathcal{B})$ is convex.*
3. *The map $\phi \mapsto \beta(\phi,\mathcal{A},\mathcal{B})$ is lower semicontinuous in the $\sigma(\mathcal{B}(\mathcal{H})^*,\mathcal{B}(\mathcal{H}))$ topology.*
4. $|\beta(\phi,\mathcal{A},\mathcal{B}) - \beta(\psi,\mathcal{A},\mathcal{B})| \leq \sqrt{2}\,\|\phi-\psi\|$, *so, in particular, $\beta(\phi,\mathcal{A},\mathcal{B})$ is norm continuous in the state ϕ.*
5. *If either $\mathcal{A}$ or $\mathcal{B}$ is abelian, then $\beta(\phi,\mathcal{A},\mathcal{B}) = 1$ for all states $\phi \in \mathcal{B}(\mathcal{H})^*$. If both of the algebras are nonabelian, there exists a normal state $\phi \in \mathcal{B}(\mathcal{H})_*$ such that $\beta(\phi,\mathcal{A},\mathcal{B}) = \sqrt{2}$.*
6. *If ϕ is a convex combination of product states, then $\beta(\phi,\mathcal{A},\mathcal{B}) = 1$.*
7. *There exists a (possibly non-normal) state ϕ with $\beta(\phi,\mathcal{A},\mathcal{B}) = 1$.*

[4] In particular, $\phi(\mathcal{T}(\mathcal{A},\mathcal{B}))$ is a connected set.

Note that Lemma 2.1 (5) asserts that for *any* choice of a pair $(\mathcal{A}, \mathcal{B})$ of nonabelian, mutually commuting von Neumann algebras, there exists a normal state ϕ such that the upper bound in (1.2) is attained, *i.e.* Bell's inequality is maximally violated. As explained in [38], though the basic idea goes all the way back to Bell, if two commuting algebras $\mathcal{A}$ and $\mathcal{B}$ contain copies of the set $M_2(\mathbb{C})$ of two-by-two complex matrices, they maximally violate Bell's inequality in some normal state. And such copies of $M_2(\mathbb{C})$ can always be found in nonabelian algebras. It is interesting that the presence of copies of $M_2(\mathbb{C})$ is also *necessary* for maximal violation of Bell's inequalities.

Proposition 2.2. [34] *Let $(\mathcal{A}, \mathcal{B})$ be a pair of commuting subalgebras of a unital C^*-algebra $\mathcal{C}$. If $A_i \in \mathcal{A}$, $B_j \in \mathcal{B}$, are selfadjoint and bounded in norm by* 1, *and if ϕ is a state on $\mathcal{C}$ with its restrictions to both $\mathcal{A}$ and $\mathcal{B}$ faithful, then the equality*

$$\frac{1}{2}\phi((A_1(B_1+B_2)+A_2(B_1-B_2)))=\sqrt{2}$$

entails that $A_i^2 = \mathbb{1}$, $i = 1, 2$, and $A_1A_2 + A_2A_1 = 0$ (similarly for the B_j). Therefore, A_1, A_2 and $A_3 \equiv -\frac{i}{2}[A_1, A_2]$ form a realization of the Pauli spin matrices in $\mathcal{A}$ and hence generate a copy of $M_2(\mathbb{C})$ in $\mathcal{A}$ (similarly for the B_j in $\mathcal{B}$). Moreover, the A_i, resp. the B_j, are contained in the centralizer of $\mathcal{A}$ in ϕ, resp. centralizer of $\mathcal{B}$ in ϕ.

This has consequences for experimental physics: if one wishes to measure a maximal violation of Bell's inequalities, one must observe quantities which can be modelled by Pauli spin matrices.

By Lemma 2.1 (4) and the norm-continuity of the map $\lambda \mapsto \lambda\phi + (1-\lambda)\psi$, it is clear that the range of the convex functional $\beta(\cdot, \mathcal{A}, \mathcal{B})$ on the state space is an interval. According to Lemma 2.1 (5), this interval contains $\sqrt{2}$ in all cases of interest, *i.e.* where neither algebra is abelian. Considering the range of $\beta(\cdot, \mathcal{A}, \mathcal{B})$ over *all* states $\phi \in \mathcal{B}(\mathcal{H})^*$, we see from Lemma 2.1 (7) that the lower endpoint will always be attained and be equal to 1. However, restricted to the *normal* state space the infimum may be strictly between 1 and $\sqrt{2}$ (and this indeed occurs - see below). We summarize such facts in the next lemma.

Lemma 2.3. [39] *Let $\mathcal{A}, \mathcal{B} \subset \mathcal{B}(\mathcal{H})$ be commuting nonabelian von Neumann algebras.*

1. *The range of the map $\phi \to \beta(\phi, \mathcal{A}, \mathcal{B})$ as ϕ runs through the set of all states on $\mathcal{B}(\mathcal{H})$ is the closed interval $[1, \sqrt{2}]$.*
2. *The range of the map $\phi \to \beta(\phi, \mathcal{A}, \mathcal{B})$ as ϕ runs through the set of all normal states on $\mathcal{B}(\mathcal{H})$ is the interval $\{c, \sqrt{2}]$,[5] where $c \in [1, \sqrt{2}]$.*
3. *If $c < \sqrt{2}$, then there exists a norm dense set of normal states ψ such that $\beta(\psi, \mathcal{A}, \mathcal{B}) < \sqrt{2}$.*

[5]The interval may or may not contain the point c, and it may be degenerate, *i.e.* equal to the singleton set $\left\{\sqrt{2}\right\}$.

The number c appearing in Lemma 2.3 is an invariant of the (isomorphism class of the unordered) pair $(\mathcal{A},\mathcal{B})$ of von Neumann algebras. We therefore define, for any pair $(\mathcal{A},\mathcal{B})$ of commuting von Neumann algebras on a Hilbert space $\mathcal{H}$, the *Bell correlation invariant* of the pair to be this number:

$$\beta(\mathcal{A},\mathcal{B}) \equiv \inf \{\beta(\phi,\mathcal{A},\mathcal{B}) \,|\, \phi \text{ a normal state on } \mathcal{B}(\mathcal{H})\} \, .$$

A variant of this which will be of interest is

$$\beta_*(\mathcal{A},\mathcal{B}) = \sup \left\{\lambda \in \mathbb{R} \,\middle|\, \exists T \in \overline{\mathcal{T}}(\mathcal{A},\mathcal{B}) \text{ with } T \geq \lambda \mathbb{1}\right\} \, .$$

Thus $\beta(\mathcal{A},\mathcal{B}) \geq \lambda$ means that for any normal state ϕ there is an operator $T \in \overline{\mathcal{T}}(\mathcal{A},\mathcal{B})$ (or, equivalently, an operator $T \in \mathcal{T}(\mathcal{A},\mathcal{B})$) with $\phi(T) \geq \lambda$. If this operator can be chosen independently of ϕ, then we have $\beta_*(\mathcal{A},\mathcal{B}) \geq \lambda$. From this discussion it is clear that

$$\beta_*(\mathcal{A},\mathcal{B}) \leq \beta(\mathcal{A},\mathcal{B}) \leq \beta(\phi,\mathcal{A},\mathcal{B}) \, ,$$

for all normal states $\phi \in \mathcal{B}(\mathcal{H})_*$. We emphasize that a consequence of Lemma 2.3 is that if $(\mathcal{A},\mathcal{B})$ is any pair of mutually commuting, nonabelian von Neumann algebras, then for *any* value $r \in (\beta(\mathcal{A},\mathcal{B}),\sqrt{2}]$ there exists a normal state $\phi \in (\mathcal{A} \vee \mathcal{B})_*$ such that $\beta(\phi,\mathcal{A},\mathcal{B}) = r$, where $\mathcal{A} \vee \mathcal{B}$ denotes the subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\mathcal{A}$ and $\mathcal{B}$. The following result was communicated to us by Shulman, and it establishes that $\beta(\mathcal{A},\mathcal{B})$ and $\beta_*(\mathcal{A},\mathcal{B})$ are alternative ways to compute the same algebraic invariant.

Proposition 2.4. [30] *For any pair $(\mathcal{A},\mathcal{B})$ of commuting von Neumann algebras, one has $\beta(\mathcal{A},\mathcal{B}) = \beta_*(\mathcal{A},\mathcal{B})$.*

Proof. It is easy to see that the definition entails

$$\beta_*(\mathcal{A},\mathcal{B}) = \sup\{\lambda(T) \mid T \in \overline{\mathcal{T}}(\mathcal{A},\mathcal{B})\} \, ,$$

where $\lambda(T) \equiv \sup\{\lambda \in \mathbb{R} \mid T \geq \lambda \cdot 1\}$. It follows that

$$\lambda(T) = \inf\{\phi(T) \mid \phi \in \mathcal{S}(\mathcal{B}(\mathcal{H}))\} = \inf\{\phi(T) \mid \phi \in \mathcal{S}_*(\mathcal{B}(\mathcal{H}))\} \, ,$$

where $\mathcal{S}(\mathcal{B}(\mathcal{H}))$, resp. $\mathcal{S}_*(\mathcal{B}(\mathcal{H}))$, is the set of all states, resp. normal states, on $\mathcal{B}(\mathcal{H})$. Hence, since $\overline{\mathcal{T}}(\mathcal{A},\mathcal{B})$ is *-weakly compact and the function $(\phi,T) \mapsto \phi(T)$ is continuous on $\mathcal{S}(\mathcal{B}(\mathcal{H})) \times \overline{\mathcal{T}}(\mathcal{A},\mathcal{B})$, one has

$$\beta_*(\mathcal{A},\mathcal{B}) = \sup_{T \in \overline{\mathcal{T}}(\mathcal{A},\mathcal{B})} \inf_{\phi \in \mathcal{S}_*(\mathcal{B}(\mathcal{H}))} \phi(T) = \inf_{\phi \in \mathcal{S}_*(\mathcal{B}(\mathcal{H}))} \sup_{T \in \overline{\mathcal{T}}(\mathcal{A},\mathcal{B})} \phi(T) \, ,$$

using the generalization of Ky Fan's minimax result in Prop. 1 of [6]. Since $\sup_{T \in \overline{\mathcal{T}}(\mathcal{A},\mathcal{B})} \phi(T) = \sup_{T \in \mathcal{T}(\mathcal{A},\mathcal{B})} \phi(T)$, one finds

$$\beta_*(\mathcal{A},\mathcal{B}) = \inf_{\phi \in \mathcal{S}_*(\mathcal{B}(\mathcal{H}))} \beta(\phi,\mathcal{A},\mathcal{B}) = \beta(\mathcal{A},\mathcal{B}) \, ,$$

as asserted.

□

Hence we may conclude from the comment above that if $\beta(\mathcal{A},\mathcal{B}) \geq \lambda$, then there exists an element $T \in \overline{\mathcal{T}}(\mathcal{A},\mathcal{B})$ such that $\phi(T) \geq \lambda$ for *all* normal states ϕ. (For the maximal possible value, $\lambda = \sqrt{2}$, this was established earlier [36].)

Some basic properties of this invariant are given next.

Proposition 2.5. [39] *Let $\mathcal{A},\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ be commuting von Neumann algebras.*

1. *If $\mathcal{A}_1 \subset \mathcal{A}_2$ and $\mathcal{B}_1 \subset \mathcal{B}_2$, then $\beta(\mathcal{A}_1,\mathcal{B}_1) \leq \beta(\mathcal{A}_2,\mathcal{B}_2)$.*
2. *Let $\mathcal{A} = \bigoplus_{i\in I} \mathcal{A}_i$, and $\mathcal{B} = \bigoplus_{i\in I} \mathcal{B}_i$, with I an arbitrary index set. Then $\beta(\mathcal{A},\mathcal{B}) = \inf_{i\in I}\beta(\mathcal{A}_i,\mathcal{B}_i)$.*
3. *If the pair $(\mathcal{A},\mathcal{B})$ is split, i.e. if there is a type I factor $\mathcal{M}$ with $\mathcal{A} \subset \mathcal{M} \subset \mathcal{B}'$, then $\beta(\mathcal{A},\mathcal{B}) = 1$.*
4. *If $\beta(\mathcal{A},\mathcal{B}) > 1$, then both $\mathcal{A}$ and $\mathcal{B}$ are nonabelian. Moreover, if $\beta(\mathcal{A},\mathcal{B}) > 1$, there exist no normal states in the weak*-closure of the convex hull of the product states across $(\mathcal{A},\mathcal{B})$, i.e. all normal states on $\mathcal{A} \vee \mathcal{B}$ are "entangled".*

In the terminology of [37], if $\beta(\mathcal{A},\mathcal{B})$ takes the value $\sqrt{2}$, we say that the pair $(\mathcal{A},\mathcal{B})$ is *maximally correlated*, since $\beta(\mathcal{A},\mathcal{B}) = \sqrt{2}$ entails that Bell's inequalities are *maximally* violated in *every* normal state across $(\mathcal{A},\mathcal{B})$. A very useful fact is that a pair $(\mathcal{A},\mathcal{B})$ is maximally correlated if and only if there is maximal violation of Bell's inequalities in at least one faithful normal state on $\mathcal{A} \vee \mathcal{B}$.

Proposition 2.6. [37] *Let $(\mathcal{A},\mathcal{B})$ be a pair of commuting von Neumann algebras acting on a separable Hilbert space $\mathcal{H}$. Then the following are equivalent.*

1. *$(\mathcal{A},\mathcal{B})$ is maximally correlated.*
2. *There exists a faithful state $\phi \in (\mathcal{A} \vee \mathcal{B})_*$ such that $\beta(\phi,\mathcal{A},\mathcal{B}) = \sqrt{2}$.*

Thus, in order to check if a pair is maximally correlated, it suffices to check the value of $\beta(\phi,\mathcal{A},\mathcal{B})$ in any conveniently chosen faithful normal state ϕ. A structural characterization of maximally correlated pairs of von Neumann algebras was given in [37].

Theorem 2.7. [37] *Let $(\mathcal{A},\mathcal{B})$ be a pair of commuting von Neumann algebras acting on a separable Hilbert space $\mathcal{H}$. Then the following are equivalent.*

1. *The pair $(\mathcal{A},\mathcal{B})$ is maximally correlated.*
2. *There exists a type I factor $\mathcal{M} \subset \mathcal{A} \vee \mathcal{B}$ such that $\mathcal{A} \cap \mathcal{M}$ and $\mathcal{B} \cap \mathcal{M}$ are spatially isomorphic to the hyperfinite type II_1-factor $\mathcal{R}_1$ and are relative commutants of each other in $\mathcal{M}$.*
3. *Up to unitary transformation, $\mathcal{H} = \mathcal{H}_1 \otimes \tilde{\mathcal{H}}$, where $\mathcal{H}_1$ is the standard representation space of $\mathcal{R}_1$, and there exist von Neumann algebras $\tilde{\mathcal{A}} \subset \tilde{\mathcal{B}}' \subset \mathcal{B}(\tilde{\mathcal{H}})$ such that $\mathcal{A} = \mathcal{R}_1 \otimes \tilde{\mathcal{A}}$ and $\mathcal{B} = \mathcal{R}_1' \otimes \tilde{\mathcal{B}}$.*

In the special case $\mathcal{B} = \mathcal{A}'$, we set $\beta(\mathcal{A}) \equiv \beta(\mathcal{A},\mathcal{A}')$, which yields a new invariant of the von Neumann algebra $\mathcal{A}$ itself. Surprisingly, in this special case the property that $(\mathcal{A},\mathcal{A}')$ is maximally correlated (*i.e.* $\beta(\mathcal{A},\mathcal{B}) = \sqrt{2}$) can be restated in terms of the algebraic properties L_λ, resp. L'_λ, which were isolated and studied some 25 years ago by Powers [28][29], resp. by Araki [1], as part of the program to classify von Neumann algebras. For the reader's convenience, we first recall the definitions of these properties.

Let $\mathcal{A}$ be a C^*-algebra with unit $\mathbb{1}$. Then $N \in \mathcal{A}$ is called a I_2-generator if $N^2 = 0$ and $NN^* + N^*N = \mathbb{1}$. Let $V_\mathcal{A}$ denote the set of I_2-generators in $\mathcal{A}$. Clearly, if N is contained in $V_{\mathcal{A}}$, then N^*N and NN^* are nonzero complementary projections, *i.e.* their sum is $\mathbb{I}$ and their product is zero. Moreover, the C^*-algebra generated by N is isomorphic to $M_2(\mathbb{C})$ and contains the unit $\mathbb{I}$ of $\mathcal{A}$. Conversely, if $\mathcal{A}$ contains a copy of $M_2(\mathbb{C})$ containing $\mathbb{I}$, then $V_{\mathcal{A}} \neq \emptyset$. Note that if $A_i \in \mathcal{A}$ satisfies $A_i^* = A_i$, $A_i^2 = \mathbb{I}$ and $A_1A_2 + A_2A_1 = 0$ (which, by Prop. 2.2, is the case if A_1, A_2 are maximal violators of Bell's inequalities in some faithful state on $\mathcal{A}$), then $N \equiv \frac{1}{2}(A_1 + iA_2)$ is an element of $V_{\mathcal{A}}$.

Definition. [29][1] *A von Neumann algebra $\mathcal{A}$ is said to have property L_λ (resp. L'_λ) with $\lambda \in [0,1/2]$ if for every $\epsilon > 0$ and any normal state ϕ on $\mathcal{A}$ (resp. finite family $\{\phi_i\}_{i=1}^n$ of normal states on $\mathcal{A}$), there exists an $N \in V_{\mathcal{A}}$ such that for any $A \in \mathcal{A}$,*

$$|\lambda\phi(AN) - (1-\lambda)\phi(NA)| \leq \epsilon\|A\| \tag{2.1}$$

(resp. for any $A \in \mathcal{A}$ and $i = 1, \ldots, n$

$$|\lambda\phi_i(AN) - (1-\lambda)\phi_i(NA)| \leq \epsilon\|A\|).$$

An alternative characterization of property L'_λ has been made in terms of the asymptotic ratio set [2][1] of the algebra $\mathcal{A}$. The asymptotic ratio set $r_\infty(\mathcal{A})$ of a von Neumann algebra $\mathcal{A}$ is the set of all $\alpha \in [0,1]$ such that $\mathcal{A}$ is W^*-isomorphic to $\mathcal{A}\overline{\otimes}\mathcal{R}_\alpha$, where $\{\mathcal{R}_\alpha\}_{\alpha\in[0,1]}$ is the family of hyperfinite factors constructed by von Neumann [25], Powers [28] and Araki and Woods [2]. It is known that property L'_λ is strictly stronger than property L_λ [1], that property L'_λ implies property $L'_{1/2}$ for any λ [1][2], and that property L'_λ for $\mathcal{A}$ is equivalent to $\frac{\lambda}{1-\lambda} \in r_\infty(\mathcal{A})$ [1]. Using Prop. 2.2 one easily sees that if $A_1, A_2 \in \mathcal{A}$ are maximal violators of Bell's inequalities in the normal state ϕ on $\mathcal{A} \vee \mathcal{B}$, where $\mathcal{B} \subset \mathcal{A}'$, then $N \equiv \frac{1}{2}(A_1 + iA_2) \in V_{\mathcal{A}}$ satisfies (2.1) with $\epsilon = 0$ and $\lambda = 1/2$. We can now state our structure result for maximally correlated pairs $(\mathcal{A}, \mathcal{A}')$.

Theorem 2.8. *Let $\mathcal{A}$ be a von Neumann algebra in a separable Hilbert space $\mathcal{H}$ with cyclic and separating vector. Then the following conditions are equivalent* [36].

1. $\beta(\mathcal{A}) = \sqrt{2}$.
2. *There exists a sequence $\{T_n\} \subset \mathcal{T}(\mathcal{A}, \mathcal{A}')$ such that $T_n \to \sqrt{2} \cdot \mathbb{I}$ in the σ-weak topology, i.e. $\beta_*(\mathcal{A}) = \sqrt{2}$.*
3. *$\mathcal{A} \simeq \mathcal{A} \otimes \mathcal{R}_1$, where $\mathcal{R}_1$ is the hyperfinite type II_1 factor, i.e. $\mathcal{A}$ is strongly stable.*
4. *$\mathcal{A}$ has property $L'_{1/2}$.*

Moreover, the following are equivalent to each other [37].

5. *$\mathcal{A}$ has property $L_{1/2}$.*
6. *For any vector state $\omega(A) = \langle\Omega, A\Omega\rangle$, $\Omega \in \mathcal{H}$, one has $\beta(\omega, \mathcal{A}, \mathcal{A}') = \sqrt{2}$.*

Note that this result and Prop. 2.5 (1) entail that if $(\mathcal{A}, \mathcal{B})$ is maximally correlated, then both $\mathcal{A}$ and $\mathcal{B}$ are strongly stable and have property $L'_{1/2}$.

On the one hand, there are many algebras having property $L'_{1/2}$, and, on the other, if $\mathcal{A}$ is a type I factor, then $\beta(\mathcal{A}) = 1$, by Proposition 2.5 (3). Though one therefore has examples of von Neumann algebras with $\beta(\mathcal{A})$ taking values at the endpoints of the admissible interval $[1, \sqrt{2}]$, at present it is not known whether there exist factors such that $1 < \beta(\mathcal{A}) < \sqrt{2}$. It would be interesting to find such factors, since Theorem 2.8 would entail that $\mathcal{A}$ could not have property L'_λ for any λ, *i.e.* $\beta(\mathcal{A})$ would then be established as a nontrivial invariant of von Neumann algebras extending the family L'_λ. However, in the more general setting where $\mathcal{B} \neq \mathcal{A}'$, we shall see that there are indeed examples of pairs $(\mathcal{A}, \mathcal{B})$ such that $1 < \beta(\mathcal{A}, \mathcal{B}) < \sqrt{2}$.

3. Quantum Field Theory and the Bell Correlation Invariant

The original motivation of this work was to study Bell's inequalities in the context of quantum field theory. But the results in this special case (as viewed from the standpoint of operator algebra theory) can also be used to show that the invariant $\beta(\mathcal{A}, \mathcal{B})$ defined above can attain values other than 1 and $\sqrt{2}$. We shall therefore specialize now to the context of algebraic quantum field theory (see *e.g.* [21]).

The basic object in AQFT is a net of C^*-algebras $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ associating to open subsets[6] $\mathcal{O}$ of Minkowski space unital C^*-algebras $\mathcal{A}(\mathcal{O})$ satisfying the following basic axioms, which are naturally motivated by the interpretation of $\mathcal{A}(\mathcal{O})$ as the algebra generated by all observables which can be measured in the spacetime region $\mathcal{O}$:

1. *Isotony*: If $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$; hence, the inductive limit C^*-algebra $\mathcal{A} \equiv \bigvee_{\mathcal{O}\in\mathcal{R}} \mathcal{A}(\mathcal{O})$ exists.

2. *Locality*: If $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$, then every element of $\mathcal{A}(\mathcal{O}_1)$ commutes with every element of $\mathcal{A}(\mathcal{O}_2)$.

3. *Poincaré Covariance*: There is a faithful representation $\mathcal{P}_+^\uparrow \ni \lambda \mapsto \alpha_\lambda \in \mathrm{Aut}\mathcal{A}$ of the identity component $\mathcal{P}_+^\uparrow$ of the Poincaré group in the automorphism group $\mathrm{Aut}\mathcal{A}$ of $\mathcal{A}$ such that $\alpha_\lambda(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O}_\lambda)$, where $\mathcal{O}_\lambda$ is the image under λ of the region $\mathcal{O}$.

In addition, the set of interesting states (or representations) on the net is selected by some further general considerations, for example the requirement that the representation be Poincaré, or at least translation, covariant. Among these are vacuum representations $(\mathcal{H}, \pi, U(\mathbb{R}^4), \Omega)$. In particular, in the representation space $\mathcal{H}$ there exist a unit vector Ω with corresponding vector state ω_0 on $\mathcal{A}$ and a strongly continuous unitary representation $U(\mathbb{R}^4)$ of the translation subgroup of the Poincaré group, whose joint generators satisfy the spectrum condition, which leaves Ω invariant, and which satisfies[7]

$$U(x)\pi(A)U(x)^{-1} = \pi(\alpha_x(A)), \qquad \text{for all}\, x \in \mathbb{R}^4\,,\ A \in \mathcal{A}\,.$$

When restricting one's attention to a single representation, it is convenient and customary to identify the algebras $\mathcal{A}(\mathcal{O})$ with the von Neumann algebras $\pi(\mathcal{A}(\mathcal{O}))''$ on the representation space $\mathcal{H}$.

[6] The index set $\mathcal{R}$ can be a proper directed subset of the collection of all open subsets of Minkowski space.

[7] For a purely algebraic characterization of vacuum states on Minkowski space, see [11].

A striking fact which emerged in our study of Bell's inequalities is that in quantum field theory, as opposed to nonrelativistic quantum theory - where the observable algebras for subsystems are typically type I, so Prop. 2.5 (3) yields $\beta(\mathcal{A},\mathcal{B}) = 1$ - there are many spacetime regions with observable algebras evincing maximal violation of Bell's inequalities in every normal state, *i.e.* maximal violation independent of how the system has been prepared. In the following, wedge regions in Minkowski space are Poincaré transforms of the unbounded region $\{x \in \mathbb{R}^4 \mid x^1 > |x^0|\}$, and double cones are the nonempty interior of the bounded intersection of the forward, resp. backward, lightcone of two timelike separated points. For a given open set $\mathcal{O} \subset \mathbb{R}^4$, $\mathcal{O}'$ denotes the interior of the set of all points in $\mathbb{R}^4$ which are spacelike separated from all points of $\mathcal{O}$.

Theorem 3.1. [32][34][35][36][37] *For any net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ of local algebras satisfying the above assumptions the following is true.*

1. *In any vacuum representation, in any superselection sector occurring in the Doplicher, Haag, Roberts theory of superselection structure* [16], *and in any massive particle representation as described by Buchholz and Fredenhagen* [8], *one has $\beta(\mathcal{A}(W)) = \sqrt{2}$, for all wedge regions $W \subset \mathbb{R}^4$. Hence, if wedge duality is satisfied (i.e. if $\mathcal{A}(W)' = \mathcal{A}(W')$ for all wedges W), then $(\mathcal{A}(W), \mathcal{A}(W')$ is maximally correlated.*
2. *In any dilatation-invariant vacuum representation satisfying wedge duality, the pair $(\mathcal{A}(W_1), \mathcal{A}(W_2))$ is maximally correlated for any spacelike separated wedges W_1, W_2, independent of the distance of separation.*
3. *In any free field theory, in any locally Fock field theory (e.g. $P(\phi)_2$, $Yukawa_2$, etc.), and in any dilatation-invariant theory satisfying wedge duality, the pair $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ is maximally correlated for any pair $(\mathcal{O}_1, \mathcal{O}_2)$ of tangent double cones in Minkowski space, hence $\beta(\mathcal{A}(\mathcal{O})) = \sqrt{2}$ for any double cone $\mathcal{O}$.*

One sees that quantum field theory predicts maximal violation of Bell's inequalities, quite independently of dynamics or even preparation. We remark that, although such tangent algebras are maximally correlated and therefore are not split, they still manifest very strong independence properties - in particular, arbitrary normal states on the subalgebras $\mathcal{A}(\mathcal{O}_1)$, $\mathcal{A}(\mathcal{O}_2)$ have simultaneous normal extensions to $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$ [19].

Going beyond the cases mentioned in Theorem 3.1, it is known that in many concrete quantum field models [7][17][31] (or under general, physically motivated assumptions [9][10]), pairs of algebras $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ associated with strictly spacelike separated double cones are split. From Lemma 2.3 (2) we may conclude that under such circumstances, for any $r \in [1, \sqrt{2}]$ there exists a normal state ϕ such that $\beta(\phi, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) = r$. Thus, in principle, every possible degree of violation of Bell's inequalities can be attained by suitable preparation on strictly spacelike separated double cone algebras, no matter how far apart their localizations are separated. However, in such cases one also has $\beta(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) = 1$ (Prop. 2.5 (3)). On the other hand, pairs of algebras $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ associated with spacelike separated wedges are not split [7], allowing the *a priori* possibility that $1 < \beta(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) < \sqrt{2}$ for such algebras, a possibility which we shall see realized below.

In [34] it was shown that in any irreducible vacuum sector with a strictly positive mass gap $m > 0$ the maximal Bell correlation in the vacuum state ω_0 satisfies an upper bound which decreases exponentially down to 1:

$$\beta(\omega_0, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) \leq 1 + 2e^{-md(\mathcal{O}_1,\mathcal{O}_2)}\,, \tag{3.1}$$

where $d(\mathcal{O}_1, \mathcal{O}_2)$ is the maximal timelike distance $\mathcal{O}_1$ can be translated before $\mathcal{O}_1 \not\subset \mathcal{O}_2'$. Note that, in general, this is not the same as the smallest spacelike distance $d'(\mathcal{O}_1, \mathcal{O}_2) \equiv \inf \sqrt{-(x_1 - x_2)^2}$ between the regions, although for wedges these two quantities coincide. The estimate (3.1) is a useful bound for large $d(\mathcal{O}_1, \mathcal{O}_2)$, but is clearly too crude for small distances[8]. The refinement of the bound (3.1) for small distances is based on the following proposition.

Proposition 3.2. [39] *Let ω be a state, defined on an algebra containing two commuting C^*-algebras, $\mathcal{A}$ and $\mathcal{B}$, and satisfying*

$$|\omega(AB) - \omega(A)\omega(B)| \leq \Gamma \cdot \big(\omega(A^*A)\omega(AA^*)\omega(B^*B)\omega(BB^*)\big)^{1/4}\,,$$

for some constant $0 \leq \Gamma \leq 1$, and all $A \in \mathcal{A}$, $B \in \mathcal{B}$, with $\|A\| \leq 1$, $\|B\| \leq 1$. Then

$$\beta(\omega, \mathcal{A}, \mathcal{B}) \leq \min\Big\{\sqrt{2}\ \frac{6 + 4\sqrt{2} + \Gamma}{7 + 4\sqrt{2}}\ ,\ 1 + \sqrt{2}\,\Gamma\Big\}\,.$$

Combined with the cluster bound available in vacuum representations with a mass gap [20], Prop. 3.2 yields the following result, which provides sharp short-distance bounds on the vacuum Bell correlation.

Corollary 3.3. [39] *Let $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ be a local net in an irreducible vacuum representation $(\mathcal{H}, \pi, U(\mathrm{I\!R}^4), \Omega)$ with a mass gap $m > 0$. Then, for any pair $(\mathcal{O}_1, \mathcal{O}_2)$ of spacelike separated regions,*

$$\beta(\omega_0, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) \leq \sqrt{2} - \frac{\sqrt{2}}{7 + 4\sqrt{2}}(1 - e^{-md(\mathcal{O}_1,\mathcal{O}_2)})\,.$$

Therefore, in a vacuum representation with a mass gap and for any two convex spacetime regions $\mathcal{O}_1$ and $\mathcal{O}_2$ with a nonzero spacelike distance between them, Corollary 3.3 yields the bound $\beta(\omega_0, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) < \sqrt{2}$, *i.e.* one has $\beta(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) < \sqrt{2}$. (Note that in *massless* theories, such strictly spacelike separated wedge algebras *are*, in fact, maximally correlated [34].)

We remark that a consequence of Corollary 3.3 is that for a norm dense set of vector states, the maximal Bell correlation across two strictly spacelike separated wedge algebras lies strictly between 1 and $\sqrt{2}$, at least for suitably

[8] Note that it would be very interesting to obtain a lower bound on the quantity $\beta(\omega_0, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ which would decrease to 1 exponentially (with the same exponent m). This would entail that $\beta(\omega_0, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ contains information about the lowest mass in the theory as well as metric information about the underlying space-time.

small (but nonzero) spacelike separation [39]. But it is still *a priori* possible that $\beta(\mathcal{A}(W_a), \mathcal{A}(W')) = 1$. This possibility has been eliminated, as follows. First of all, the following lemma was proven, which gives useful sufficient conditions under which the Bell correlation invariant coincides with the maximal Bell correlation in a particular normal state.

Lemma 3.4. [39] *Let $(\mathcal{A}, \mathcal{B})$ be a pair of commuting von Neumann algebras on a Hilbert space $\mathcal{H}$. Consider an automorphism $\alpha \in \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ such that $\alpha(\mathcal{A}) = \mathcal{A}$ and $\alpha(\mathcal{B}) = \mathcal{B}$ and such that for all $A \in \tilde{\mathcal{A}} \vee \tilde{\mathcal{B}}$ (where $\tilde{\mathcal{A}}$ is dense in $\mathcal{A}$ and $\tilde{\mathcal{B}}$ is dense in $\mathcal{B}$) one has $\alpha^n(A) \to \omega_0(A) \cdot \mathbb{1}$ in the σ-weak topology of $\mathcal{B}(\mathcal{H})$ as $n \to \infty$ for some normal state $\omega_0 \in \mathcal{B}(\mathcal{H})_*$. Then*

$$\beta_*(\mathcal{A}, \mathcal{B}) = \beta(\mathcal{A}, \mathcal{B}) = \beta(\omega_0, \mathcal{A}, \mathcal{B}) .$$

This yields the following corollary in algebraic quantum field theory. We shall call a spacelike cylinder any open spacetime region $\mathcal{O}$ such that for some spacelike direction $\vec{a}$ it is true that $\mathcal{O} = \mathcal{O} + t\vec{a}$ for all $t \in \mathbb{R}$. Of course, such regions are necessarily unbounded, and wedges are examples of spacelike cylinders.

Corollary 3.5. [39] *Let $\mathcal{O}_1$, $\mathcal{O}_2$ be parallel spacelike cylinders in three or more spacetime dimensions with $\mathcal{O}_1 \subset \mathcal{O}_2'$. Then in any irreducible vacuum representation $(\mathcal{H}, \pi, U(\mathbb{R}^4), \Omega)$ of a local net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{R}}$ one has the equality*

$$\beta(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) = \beta(\omega_0, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) ,$$

where ω_0 is the (unique) normal vacuum state on the representation.

Taken in conjunction with the vacuum cluster bound, Corollary 3.3, this entails the following theorem.

Theorem 3.6. [39] *Consider the function*

$$b(a) = \beta(\mathcal{A}(W_1), \mathcal{A}(W_2)) ,$$

where $a \geq 0$ is the spacelike separation of the wedges W_1, W_2. In an irreducible vacuum representation with a mass gap $m > 0$, the function b is lower semicontinuous and nonincreasing, and $1 \leq b(a) \leq \sqrt{2}$. The upper bound is an equality if and only if $a = 0$. Consequently, the quantity $\beta(\mathcal{A}(W_1), \mathcal{A}(W_2))$ takes infinitely many different values (depending on the spacelike separation of the wedges), with an accumulation point at $\sqrt{2}$.

Hence, the invariant $\beta(\mathcal{A}, \mathcal{B})$ can distinguish between infinitely many different isomorphism classes of pairs $(\mathcal{A}, \mathcal{B})$. At this point it is not known whether $\beta(\cdot, \cdot)$ can take any value between 1 and $\sqrt{2}$.

Acknowledgements. *This paper grew out of a very pleasant collaboration with Reinhard Werner. Part of this manuscript was written while the author was visiting the University of Rome under a grant from the CNR in the summer of 1996. For this opportunity he would like to thank his many colleagues at the University of Rome, particularly Profs. Claudio D'Antoni, John E. Roberts, and Laszlo Zsido.*

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