POINCARÉ–INVARIANT MARKOV PROCESSES AND GAUSSIAN RANDOM FIELDS ON RELATIVISTIC PHASE SPACE

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Abstract. We give a complete characterization, including a Lévy–Itô decomposition, of Poincaré–invariant Markov processes on $\mathbb{H}^1_+ \times \mathbb{M}^2$, the relativistic phase space in 1+1 space–time dimensions. Then, by means of such processes, we construct Poincaré–invariant Gaussian random fields, and we prove a "no–go" theorem for the random fields corresponding to Brownian motions on $\mathbb{H}^1_+ \times \mathbb{M}^2$.

1. Introduction

It is a well known fact that there do not exist Poincaré–invariant Markov processes on Minkowski space (see [Ha], [Du]). Here we consider Poincaré–invariant Markov processes in relativistic phase space. This space was proposed by E.Nelson (see [N2]), under the name "clock space", as the right arena where to tackle the problem of the construction of Relativistic Quantum Fields by stochastic methods without using analytic continuation in time, i.e. without using the Euclidean approach, which is not applicable in the case of curved space–times.

Let us denote by $\mathbb{M}^d$, $d \geq 2$, the d–dimensional Minkowski space and by $\tilde{\mathbb{M}}^d := \mathbb{H}^{d-1}_+ \times \mathbb{M}^d$ the relativistic phase space, where

$$\mathbb{H}^{d-1}_+ := \left\{ (x_0, \cdots, x_{d-1}) \in \mathbb{R}^d \ : \ x_0^2 - \sum_{k=1}^{d-1} x_k^2 = 1,\ x_0 \geq 1 \right\} .$$

In [N2] the relativistic phase space $\tilde{\mathbb{M}}^d$ is named clock space since every "clock" $(u,x) \in \tilde{\mathbb{M}}^d$ determines a splitting $(\mathbb{M}_u \oplus \mathbb{M}_u^\perp) + x$ of $\mathbb{M}^d$ in "space" and "time". Indeed (u,x) determines the $(d-1)$–dimensional, oriented, spacelike, affine subspace $\mathbb{M}_{(u,x)} := \mathbb{M}_u + x$, $\mathbb{M}_u := \{u\}^\perp$. Here $\perp$ indicates the orthogonal complement w.r.t. the standard Euclidean metric on $\mathbb{R}^d$. From now on we will use the identification $\mathbb{H}^{d-1}_+ \simeq \mathbb{R}^{d-1}$, so that $\tilde{\mathbb{M}}^d \simeq \mathbb{R}^{d-1} \times \mathbb{R}^d \equiv \mathbb{R}^{2d-1}$. For any proper Lorentz transformation Λ we will denote by ϕ_Λ the corresponding transformation on $\mathbb{R}^{d-1}$, i.e. we let Λ act on $\mathbb{H}^{d-1}_+$ as an invariant subset of $\mathbb{M}^d$, and then we use the above identification $\mathbb{H}^d_+ \simeq \mathbb{R}^{d-1}$. With

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such a notation one can check that any Poincaré–invariant Riemannian metric $\tilde{g}$ on $\tilde{\mathbb{M}}^d$ is of the kind

$$\tilde{g}(u,x) \equiv \tilde{g}(u) = \tilde{\Lambda}(u) \circ \tilde{g}_0 \circ \tilde{\Lambda}(u), \tag{1.1}$$

where $\tilde{\Lambda}(u) := T_u\phi_{\Lambda(u)} \otimes \Lambda(u)$, $\Lambda(u)$ denotes the unique Lorentz transformation such that $\phi_{\Lambda(u)}(u) = 0$, $T_u\phi_{\Lambda(u)}$ is the tangent map at u of $\phi_{\Lambda(u)}$, and $\tilde{g}_0$ is any constant Riemannian metric on $\mathbb{R}^{2d-1} \equiv \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^{d-1}$ invariant under the transformations of the kind $A \otimes 1 \otimes A$, $A \in SO(d-1)$. Therefore, considering the Laplace–Beltrami operators corresponding to such Riemannian metrics, one can construct a lot of Poincaré–invariant Brownian motions on $\tilde{\mathbb{M}}^d$. This was the main motivation in [N2]: even if Minkowski metric is indefinite, and so it is unfriendly to probabilists, on clock space there is a natural Poincaré–invariant Riemannian (positive definite) metric (in [N2] only the metric given by the choice $\tilde{g}_0 = \mathrm{I}$ was considered). Using then the Green funtions of such a kind of Markov processes one can define Poincaré–invariant Gaussian random fields on $\tilde{\mathbb{M}}^2$ indexed by finite energy measures, and so it is possible to study a problem proposed (in a different way) by Nelson in [N2]. The problem consists of trying to determine a Poincaré–invariant Gaussian random field X_μ such that, if X^u_ν denotes the Nelson free field (see [N1]) in the reference frame that uses the "clock" (u,x) (by traslational invariance X^u_ν does not depend on the choice of the origin x), then $X_{\delta_u \otimes \nu} \stackrel{d}{=} X^u_\nu$, where $\stackrel{d}{=}$ means " with the same distributions". In Theorem 3.2, we obtain a sort of "no–go" theorem for Brownian motions on $\tilde{\mathbb{M}}^2$: there is no Poincaré–invariant Riemannian metric $\tilde{g}$ on $\tilde{\mathbb{M}}^2$ such that the Gaussian random field given by the exponentially killed Brownian motion corresponding to the Laplace–Beltrami operator $-\Delta_{\tilde{g}}$ satisfies the above requirements. This result tells us that the nice Riemannian structures of $\tilde{\mathbb{M}}^2$ are of no use, in that Brownian motions on $\tilde{\mathbb{M}}^d$ are not the processes to be considered in order to define the right (as regards Quantum Free Fields) Gaussian Random Fields on $\tilde{\mathbb{M}}^d$. Luckily $\tilde{\mathbb{M}}^d$ is a far more interesting object than a Riemannian manifold: since

$$\tilde{\mathbb{M}}^d \simeq ISO(1,d-1)/SO(d-1) \ ,$$

it is an homogeneous space, and so there is a lot of room left for trying to solve the above Nelson's problem. Indeed, since G–invariant Markovian semigroups on homogeneous spaces of a Lie group G are nothing but convolution semigroups of measures, and their generators have been classified in a fundamental paper by Hunt in 1956 (see [Hu]), one can give a fairly complete characterization of Poincaré–invariant Markov processes on $\tilde{\mathbb{M}}^d$. In the case $d = 2$, $\tilde{\mathbb{M}}^2 \simeq ISO(1,1)$, i.e. $\tilde{\mathbb{M}}^2$ is nothing but the connected component of identity of the inhomogeneous Lorentz group, and for this case we give a Lévy–Khinchine formula for the generators (see thm. 2.1), and a Lévy–Itô decomposition for the processes (see thm. 2.2).

2. Markov Processes on Relativistic Phase Space

As above let $\tilde{\mathbb{M}}^d = \mathbb{H}^{d-1}_+ \times \mathbb{M}^d$. Since $\mathbb{H}^{d-1}_+$ is in one to one correspondence with the set of Lorentz boosts (remember that $\mathbb{H}^{d-1}_+ \simeq D = \{ \ v \in \mathbb{R}^{d-1} \ : \ \|v\| < 1 \ \}$), we have

$$\tilde{\mathbb{M}}^d \simeq ISO(1,d-1)/SO(d-1).$$

G–invariant Markovian semigroups on homogeneous spaces of a Lie group G are nothing but convolution semigroups of measures, and their generators have been classified in [Hu]. The simplest situation occurs when $d = 2$. From now on we will restrict ourselves to this case. Therefore $\tilde{\mathbb{M}}^2 \simeq ISO(1,1) \simeq \mathbb{R}^3$, and $\mathbb{R}^3$ is a Lie group with the composition law given by

$$(u, x) \cdot (v, y) := (u + v, \Lambda_u y + x) \ ,$$

where Λ_u denotes the Lorentz boost corresponding to u. By working out Hunt's results (see [Hu, thm.5.1] and [He, §4.1, §4.2]) we obtain the following Lévy–Khinchine type result:

Theorem 2.1. *Let $\{\mu_t\}_{t>0}$ be a convolution semigroup of measures on $\tilde{\mathbb{M}}^2 \simeq ISO(1,1)$ $\simeq \mathbb{R}^3$. Then*

$$\begin{aligned} T_t f(u,x) :=& f * \check{\mu}_t(u,v) \\ \equiv& \int_{\mathbb{R}^3} f((u,x)\cdot(v,y))d\mu_t(v,y) \\ =& \int_{\mathbb{R}^3} f((u+v, x+\Lambda_u y))d\mu_t(v,y) \end{aligned}$$

is a Poincaré–invariant Markovian semigroup on $\overline{C}_\infty(\mathbb{R}^3) := C_\infty(\mathbb{R}^3) \oplus \mathbb{R}$, where $C_\infty(\mathbb{R}^3)$ denotes the space of continuous functions vanishing at infinity. It has a generator

$$A \ : \ D \ \to \ \overline{C}_\infty(\mathbb{R}^3)$$

with $\overline{C}^2_\infty(ISO(1,1)) \subset D$, where $\overline{C}^2_\infty(ISO(1,1))$ denotes the set of functions $f \in \overline{C}_\infty(\mathbb{R}^3)$ such that $f \in C^2(\mathbb{R}^3)$ and $X_i f$, $X_j X_k f \in \overline{C}_\infty(\mathbb{R}^3)$, $\{X_0, X_1, X_2\}$ being the basis of the left–invariant Lie algebra of $ISO(1,1)$ defined by

$$X_0 f(u, x_1, x_2) = \partial_u f(u, x_1, x_2),$$

$$\begin{aligned} X_1 f(u,x_1,x_2) &= \mathrm{cosh} u \partial_{x_1} f(u,x_1,x_2) + \mathrm{sinh} u \partial_{x_2} f(u,x_1,x_2), \\ X_2 f(u,x_1,x_2) &= \mathrm{sinh} u \partial_{x_1} f(u,x_1,x_2) + \mathrm{cosh} u \partial_{x_2} f(u,x_1,x_2). \end{aligned}$$

On $\overline{C}^2_\infty(ISO(1,1))$ A has the form

$$\begin{aligned} Af(u,x) =& \sum_{0\le j\le 2} a_{ij} X_i X_j f(u,x) + \sum_{0\le j\le 2} b_j X_j f(u,x) \\ &+ \int_{\mathbb{R}^3} \left(f(u+v, x+\Lambda_u y) - f(u,x) - \sum_{0\le j\le 2} X_j f(u,x)\sigma_j(v,y) \right) d\nu(v,y), \end{aligned}$$

where

i) $b \in \mathbb{R}^3$;

ii) the matrix (a_{ij}) is symmetric and positive semi–definite;

iii) $\{\sigma_0, \sigma_1, \sigma_2\}$ is the system of canonical coordinates in $\overline{C}^2_\infty(ISO(1,1))$ near the identity 0 defined by $\sigma_j(\tilde{x}) \equiv \sigma_j(u,x) := \dfrac{\tilde{x}_j}{1 + u^2 + e^{u^2}|x|^2}$;

iv) ν is a positive measure not charging 0 *such that*

$$\int_{\mathbb{R}^3} \phi(\tilde{x}) d\nu(\tilde{x}) < +\infty,$$

$\phi > 0$ being a Hunt function, i.e. $\phi \in \overline{C}^2_\infty(ISO(1,1))$, $\phi \geq \delta > 0$ away from 0, and $\phi(\tilde{x}) \simeq \sum_{0 \leq j \leq 2} \sigma_j^2(\tilde{x})$ near 0. Conversely, given an operator A defined as above on $\overline{C}^2_\infty(ISO(1,1))$, there exists exactly one convolution semigroup of measures $\{\mu_t\}_{t>0}$ on $\overline{C}_\infty(\mathbb{R}^3)$ with corresponding semigroup $T_t f(u,x) := f * \check{\mu}_t(u,v)$, and infinitesimal generator A' such that $A' = A$ on $\overline{C}^2_\infty(ISO(1,1))$.

One can check that if in the above decomposition $b = 0$, $\nu = 0$, and a is positive definite, then A is the Laplace–Beltrami operator corresponding to the Poincaré–invariant Riemannian metric $\tilde{\Lambda}_{-u} a^{-1} \tilde{\Lambda}_{-u}$, $\tilde{\Lambda}_u := 1 \otimes \Lambda_u$.

By using the results contained in [AK, thm.3.1] (see also [Ku] and [Ho]), one then obtains the following characterization for the Markov processes corresponding to the Markovian generators given in the previous theorem:

Theorem 2.2. *Let $\tilde{X}_t$ be the Markov process on $\tilde{\mathbb{M}}^2$ corresponding to the Markovian generator A given in thm.2.1. Then $\tilde{X}_t$ is a Lévy process on the Lie group $ISO(1,1)$, i.e.*

i) it has independent increments: for any sequence $t_0, ..., t_n$, $0 = t_0 < t_1 < ... < t_n$, the random variables $\{\ (\tilde{X}_{t_{k-1}})^{-1} \cdot \tilde{X}_{t_k},\ k = 1, ..., n\ \}$ are independent;

ii) it is continuous in probability: $(\tilde{X}_s)^{-1} \cdot \tilde{X}_t$ converges in law to 0 as $s \to t$;

iii) it is time homogeneous: the law of $(\tilde{X}_s)^{-1} \cdot \tilde{X}_t$ equals that of $(\tilde{X}_{s+h})^{-1} \cdot \tilde{X}_{t+h}$ for any $t > s$ and $h > 0$.

Moreover the cádlág process $\tilde{X}_t$ admits the following Lévy–Itô decomposition:

$$\begin{aligned} \tilde{X}_t = \tilde{X}_0 &+ \int_0^t \tilde{\Lambda}_{X^0_{s-}} dB^a_s + \int_0^t \tilde{\Lambda}_{X^0_{s-}} b\, ds \\ &+ \int_0^{t+} \tilde{\Lambda}_{X^0_{s-}} \int_{\mathbb{R}^3} \tilde{x}\ dN_\nu(s, \tilde{x}) - \sigma(\tilde{x}) d\nu(\tilde{x}) ds, \end{aligned}$$

where B^a_t is the Brownian motion on $\mathbb{R}^3$ with zero mean and covariance $a_{ij}t$, $N_\nu((s,t] \times E)$ is the Poisson random measure on $[0, +\infty) \times \mathbb{R}^3$ with intensity $(t-s)\nu(E)$, and X^0_t is the $\mathbb{H}^1_+ (\simeq \mathbb{R})$ component of $\tilde{X}_t$.

Let us now consider any Lévy process X^0_t on $\mathbb{R}$, so that

$$X^0_t = X^0_0 + B^0_s + b^0 t + \int_{\mathbb{R}} u\ dN_{\nu^0}((0,t], u) - \frac{u}{1+u^2}\ t\ d\nu^0(u),$$

where B^0_t is the Brownian motion on $\mathbb{R}$ with zero mean and variance $a^0 t$, $b^0 \in \mathbb{R}$, and $N_{\nu^0}((0,t] \times I)$ is the Poisson random measure on $[0, +\infty) \times \mathbb{R}$ with intesity $(t-s)\nu^0(I)$, ν^0 a positive measure not charging 0 such that $\int_{\mathbb{R}} |x|^2 \wedge 1\ d\nu^0(x) < +\infty$. Using X^0_t we can define a Lorentz–invariant process on $\mathbb{H}^1_+$ by $V_t := (\cosh X^0_t, \sinh X^0_t)$. We can interpret V_t as a velocity process. Indeed $\mathbb{H}^1_+$ is the velocity space in relativity theory. We can therefore define the position process X_t on $\mathbb{M}^2$ by

$$X_t := X_0 + \int_0^t V_s ds.$$

By thm. 2.1, one can check that $\tilde{X}_t := (X_t^0, X_t)$ is a Poincaré–invariant Markov process on $\tilde{\mathbb{M}}^2$. Such a class of processes coincides with the one constructed in [Du]. We call the processes in this class "Dudley processes". In 1+1 space–time dimensions we have therefore that Dudley processes are in one to one correspondence with Lévy processes on $\mathbb{R}$. Since we are in a relativistic framework the real parameter t has to be interpreted as the proper time. We can go to relative time considering the random time–change given by the additive functional

$$A_t := \int_0^t \cosh X_s^0 \, ds.$$

Let $\check{X}_t^0$ be the corresponding time change of the Lévy process X_t^0. Then we can interpret $\tanh \check{X}_t^0$ as the spatial velocity process corresponding to relative time, and so

$$Y_t := Y_0 + \int_0^t \tanh \check{X}_s^0 ds \tag{2.2}$$

has to be interpreted as the spatial position process paramentrized by relative time. Suppose now that the characteristic exponent of the infinitely divisible law corresponding to X_t^0 is a function of velocity of light c. It would be interesting to study the non–relativistic limit of such a class of processes. For example one can pose the following:

Problem 2.3. *For which, if any, Lévy processes X_t^0 on $\mathbb{R}$, does the process defined by (2.2) converge, in some sense, to Brownian motion as $c \to \infty$?*

3. Gaussian Random Fields, the Nelson Free Field, and a "no–go" theorem for Laplace–Beltrami operators on $\tilde{\mathbb{M}}^2$

By [Dy] (see also [R]) there is a standard way to associate to a symmetric Markov process a Gaussian Random field: let $p_t(x,y)$ be the transition density of the given Markov process (with state space S), and define the Green function of the corresponding exponentially killed (with killing rate $\alpha > 0$) Markov process by

$$G_\alpha(x,y) := \int_0^\infty e^{-\alpha t} p_t(x,y) \, dt \ .$$

Let $\mathcal{M}_E$ be the set of measures with finite energy, i.e. the set of Radon signed measures such that

$$\int_{S\times S} G_\alpha(x,y) d|\mu|(x) d|\mu|(y) < \infty \ ,$$

and define, for any $\mu, \nu \in \mathcal{M}_E$,

$$\langle \mu, \nu \rangle_E := \int_{S\times S} G_\alpha(x,y) d\mu(x) d\nu(y) \ .$$

Then $(\mathcal{M}_E, \langle\cdot,\cdot\rangle_E)$ is a pre–Hilbert space, and we will denote its completion (the so–called Green space) by $(\mathcal{E}, \langle\cdot,\cdot\rangle_E)$. It is known (see e.g. [S, thm.I.6]) that there exists a unique Gaussian random field X_μ indexed by elements in $\mathcal{E}$ with zero mean and covariance $E(X_\mu X_\nu) = \langle \mu, \nu \rangle_E$. The same construction holds also in the case of Markov processes not having absolutely continuous transition functions (see [R]). If for example we apply this procedure to the exponentially killed Brownian motion on $\mathbb{R}^d$, i.e. to the Markov process with Green function K_α having Fourier transform given by $\hat{K}_\alpha(\lambda) = (\|\lambda\|^2 + \alpha)^{-1}$, then we obtain the Nelson free field indexed by $H^{-1}(\mathbb{R}^d)$ (see [N1], [S]). Therefore, by any symmetric Poincaré–invariant Markov process constructed in the previous section, we can define a Poincaré–invariant Gaussian Random Field on the relativistic phase space $\tilde{\mathbb{M}}^2$. Now we want to study the following

Problem 3.1. *Does there exist, on $\tilde{\mathbb{M}}^2$, a symmetric, Poincaré–invariant Markov process such that, if G_α denotes the Green funtion of the corresponding exponentially killed process, and if K_α denotes the free kernel on $\mathbb{M}^2$, then, for any $\alpha > 0$, and for a.e. u,x,y,*

$$G_\alpha(u,u,x-y) = K_\alpha(\Lambda_u^{-1}(x-y)) \ ?$$

This problem was proposed (in a different way) in [N2]. An affirmative answer to this question would knit together the Nelson Free Fields for the various refence frames into a Poincaré invariant Gaussian Random Field in $\tilde{\mathbb{M}}^2$, i.e. if X_μ denotes the random field corresponding to G and if X_ν^u denotes the Nelson free field in the reference frame that uses the "clock" u, then $X_{\delta_u \otimes \nu} \stackrel{d}{=} X_\nu^u$, where $\stackrel{d}{=}$ means " with the same distributions". Let us remind that, by [GR], even if the Nelson free field was originally constructed as an object living on the Euclidean space $\mathbb{R}^d$, it can be interpreted as living on the physical Minkowski space–time $\mathbb{M}^d$. However it is not a Poincaré–invariant field: its construction is frame–dependent in that it depends on the way $\mathbb{M}^d$ is splitted in "space" and "time".

Of course the most natural symmetric kernel G_α on $\tilde{\mathbb{M}}^2$ to look at is the one corresponding to $(-\Delta_{\tilde{g}} + \alpha)^{-1}$, where $\Delta_{\tilde{g}}$ is the Laplace–Beltrami operator associated to the Poincaré–invariant Riemannian metric $\tilde{g}$. Unfortunately one has the following "no–go" result:

Theorem 3.2. *There is no Poincaré–invariant Riemannian metric $\tilde{g}$ on $\tilde{\mathbb{M}}_2$ such that*

$$G_\alpha(u,u,x-y) = K_\alpha(\Lambda_u^{-1}(x-y)),$$

where $G_\alpha(u,v,x-y)$ is the kernel of $(-\Delta_{\tilde{g}} + \alpha)^{-1}$, and $K_\alpha(x-y)$ is the free kernel on $\mathbb{M}^2$.

Proof. For notational convenience we will take $\alpha = 1$. By thm. 2.1 and by (1.1), if

$$\tilde{g}_0^{-1} = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_{11} & a_{12} \\ a_2 & a_{12} & a_{22} \end{pmatrix},$$

then

$$\begin{aligned}\Delta_{\tilde g} f = &(a_2 \mathrm{cosh}u + a_1 \mathrm{sinh}u)(\partial_t f + 2\partial^2_{ux} f)\\ &+ (a_1 \mathrm{cosh}u + a_2 \mathrm{sinh}u)(\partial_x f + 2\partial^2_{ut} f)\\ &+ a_0 \partial^2_{uu} f\\ &+ 2(a_{12}(\cosh^2 u + \sinh^2 u) + (a_{11} + a_{22})\mathrm{cosh}u\mathrm{sinh}u)\partial^2_{tx} f\\ &+ (a_{11}\cosh^2 u + 2a_{12}\mathrm{cosh}u\mathrm{sinh}u + a_{22}\sinh^2 u)\partial^2_{tt} f\\ &+ (a_{11}\sinh^2 u + 2a_{12}\mathrm{cosh}u\mathrm{sinh}u + a_{22}\cosh^2 u)\partial^2_{xx} f.\end{aligned}$$

By partial Fourier transform on $\mathbb{R}^2$, $-\Delta_{\tilde g}$ trasforms to the (parameter–dependent, $\lambda \in \mathbb{R}^2$) ordinary differential operator on $\mathbb{R}$

$$\begin{aligned}D_{\tilde g}(\lambda) f(u) = &-a_0 f''(u)\\ &+ 2i\left((a_1 \mathrm{cosh}u + a_2 \mathrm{sinh}u)\lambda_1 + (a_2 \mathrm{cosh}u + a_1 \mathrm{sinh}u)\lambda_2\right) f'(u)\\ &+ \Big(2(a_{12}(\cosh^2 u + \sinh^2 u) + (a_{11} + a_{22})\mathrm{cosh}u\mathrm{sinh}u)\lambda_1\lambda_2\\ &\quad + (a_{11}\cosh^2 u + 2a_{12}\mathrm{cosh}u\mathrm{sinh}u + a_{22}\sinh^2 u)\lambda_1^2\\ &\quad + (a_{11}\sinh^2 u + 2a_{12}\mathrm{cosh}u\mathrm{sinh}u + a_{22}\cosh^2 u)\lambda_2^2\\ &\quad + i\left((a_2 \mathrm{cosh}u + a_1 \mathrm{sinh}u)\lambda_1 + (a_1 \mathrm{cosh}u + a_2 \mathrm{sinh}u)\lambda_2\right)\Big) f(u),\end{aligned}$$

and the thesis says that there exists no Poincaré–invariant Riemannian metric $\tilde g$ on $\tilde{\mathbb{M}}^2$ such that

$$G^\lambda(u,u) = \frac{1}{\|\Lambda_u \lambda\|^2 + 1},$$

where $G^\lambda(u,v) = 2\pi \hat G(u,v,\lambda)$ is the kernel of $(D_{\tilde g}(\lambda) + 1)^{-1}$. By Poincaré invariance we can always assume, without loss of generality, that $\lambda_1\lambda_2 = 0, \lambda_1^2 + \lambda_2^2 \neq 0$. Therefore it will suffice to prove that there exist no Poincaré–invariant Riemannian metric $\tilde g$ on $\tilde{\mathbb{M}}^2$ such that

$$G^\lambda(u,u) = \frac{1}{\lambda^2 \mathrm{cosh}2u + 1}.$$

where $G^\lambda(u,v)$ is now the kernel of $(D_{\tilde g}(\lambda) + 1)^{-1}$,

$$D_{\tilde g}(\lambda) f(u) = -a_0 f''(u) + 2i\lambda F(u) f'(u) + (i\lambda F'(u) + \lambda^2 G(u)) f(u),$$

with $\lambda \in \mathbb{R}$, and

$$\begin{aligned}F(u) :=& a_1 \mathrm{cosh}u + a_2 \mathrm{sinh}u,\\ G(u) :=& a_{11}\cosh^2 u + 2a_{12}\mathrm{cosh}u\mathrm{sinh}u + a_{22}\sinh^2 u.\end{aligned}$$

Alternatively $D_{\tilde g}(\lambda)$ can be written as

$$D_{\tilde g}(\lambda) f(u) = -\left(p_1 f'(u)\right)' + (q_0(u) f(u))' - q_0^*(u) f(u)' + p_0(u) f(u),$$

with

$$p_1 = a_0, \qquad q_0 = i\lambda F, \qquad p_0 = \lambda^2 G,$$

so that we are in the same setting as in [W]. Let us now define the unitary operator

$$U \;:\; L^2(\mathbb{R}) \;\to\; L^2(\mathbb{R})$$

$$Uf(u) := f(u) \exp -\frac{1}{2}\int_0^u \frac{2i\lambda F(s)}{a_0} ds = f(u) \exp \frac{i\lambda}{a_0}\left(a_2 - \frac{F'(u)}{2}\right),$$

so that

$$U^{-1} D_{\tilde g}(\lambda) U f = -a_0 f'' + \lambda^2 \left(G - \frac{F^2}{a_0}\right) f.$$

By the definition of F and G we have

$$a_0 G - F^2 = a_0 \tilde g^{-1}((x_0, x), (x_0, x)),$$

with

$$x_0 = \frac{1}{a_0}(g_1 \mathrm{cosh} u + a_2 \mathrm{sinh} u), \qquad x = (-\mathrm{cosh} u, -\mathrm{sinh} u).$$

and, by positive definiteness of $\tilde g$, we have $a_0 G - F^2 > 0$. Therefore, by [W, thm.6.3] the differential operator

$$L := U^{-1} D_{\tilde g}(\lambda) U \;:\; D \;\to\; L^2(\mathbb{R})$$

$$D = \{ f \in L^2(\mathbb{R}) \;:\; f, f' \in AC_{loc}(\mathbb{R}),\; Lf \in L^2(\mathbb{R}) \},$$

is self–adjoint, and so $D_{\tilde g}(\lambda)$ is self–adjoint on

$$\{ f \in L^2(\mathbb{R}) \;:\; f, a_0 f' - i\lambda F f \in AC_{loc}(\mathbb{R}),\; D_{\tilde g}(\lambda) f \in L^2(\mathbb{R}) \}.$$

Therefore, by [W, thms. 7.2 and 7.3], there exists a fundamental system f_-, f_+ (univocally defined up to multiplication by constants) of solutions of the equation

$$D_{\tilde g}(\lambda) f = -f \tag{3.1}$$

such that

$$f_- \in L^2(-\infty, 0) \cap (L^2(0, +\infty))^c, \qquad f_+ \in L^2(0, +\infty) \cap (L^2(-\infty, 0))^c,$$

and

$$G^\lambda(u, v) = \begin{cases} W(f_+, f_-)^{-1} f_+(u) f_-^*(v) & v \le u \\ \\ W(f_+, f_-)^{-1} f_-(u) f_+^*(v) & u < v \end{cases},$$

where $W(f_+, f_-)$ denotes the Wronskian. By the definition of U, the solutions f of equation (3.1) are $U^{-1}h$, where the h's satisfy

$$h'' - Qh = 0, \qquad Q = \frac{\lambda^2 G + 1}{a_0} - \frac{\lambda^2 F^2}{a_0^2}\,. \tag{3.2}$$

Since we proved that $a_0 G - F^2 > 0$, we have $Q > 0$, and so Q has no turning points. Moreover the functions $(Q')^2 Q^{-5/2}$, $Q'' Q^{-3/2}$ are in $L^1(\mathbb{R})$. Therefore, by the theory of asympotic integration of differential equations (see e.g. [KC, chap.I, §6]), equation (3.2) has two linearily independent solutions h_-, h_+ such that, if $u \gg 1$,

$$h_-(u) = \frac{1}{Q^{1/4}(u)} \exp \int_0^u Q^{1/2}(s) ds \ (1 + \mathrm{o}(1)),$$
$$h_+(u) = \frac{1}{Q^{1/4}(u)} \exp - \int_0^u Q^{1/2}(s) ds \ (1 + \mathrm{o}(1)).$$

This implies, for the two independent solutions of $D_{\tilde{g}}(\lambda) f = -f$, if $u \gg 1$,

$$|f_-(u)| \sim \frac{1}{Q^{1/4}(u)} \exp \int_0^u Q^{1/2}(s) ds,$$
$$|f_+(u)| \sim \frac{1}{Q^{1/4}(u)} \exp - \int_0^u Q^{1/2}(s) ds.$$

Therefore we have, if $u \gg 1$,

$$f_+(u) f_-^*(u) \sim \frac{1}{Q^{1/2}(u)} \sim c(\lambda) e^{-u}.$$

whereas

$$\frac{1}{\lambda^2 \cosh 2u + 1} \sim \frac{1}{\lambda^2} e^{-2u},$$

and the proof is done.

The above theorem tells us that there is no hope to solve problem 3.1 by using a diffusion process. The Poincaré–invariant processes solving problem 3.1 (if any) must have jumps. By [Dy] and [R] this also tells us that the corresponding Gaussian random fields will not be Markov w.r.t. all open sets in $\tilde{\mathbb{M}}^2$.

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