A non-Gaussian fixed point for ϕ^4 in $4 - \epsilon$ dimensions - I.

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Abstract

We consider the ϕ^4 quantum field theory in four dimensions. The Gaussian part of the measure is modified to simulate $4 - \epsilon$ dimensions where ϵ is small and positive. We give a renormalization group analysis for the infrared behavior of the resulting model. We find that the Gaussian fixed point is unstable but that there is a hyperbolic non-Gaussian fixed point a distance $\mathcal{O}(\epsilon)$ away. In a neighborhood of this fixed point we construct the stable manifold.

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Contents

1 Introduction

We consider a Euclidean ϕ^4 quantum field theory in d dimensions as given by functional integrals of the form

$$\frac{\int [...] e^{-V(\phi)} d\mu_v(\phi)}{\int e^{-V(\phi)} d\mu_v(\phi)} \tag{1}$$

The integral is over some collection of functions ϕ on $\mathbf{R^d}$ and μ_v is a Gaussian measure with covariance $v = (-\Delta)^{-1}$. Up to terms that can be absorbed by adjusting v the potential V has the form

$$V(\phi) = \lambda \int :\phi^4:_v + \mu \int :\phi^2:_v \tag{2}$$

where λ is a coupling constant and μ is a mass. It is a basic problem of quantum field theory to establish the existence of such integrals and study their properties. The problem is also relevant to the study of critical properties in statistical mechanical systems, in which case ϕ is interpreted as an order parameter.

We are interested in the case when λ and μ are small. Furthermore we focus on the IR (infrared, long distance) behavior of the model and so insert an UV (ultraviolet, short distance) regularization, imposed as a regularization of the covariance v. With this modification the model is well-defined in finite volume and one seeks to take the infinite volume limit.

We attack the problem by the renormalization group (RG) method. One successively integrates out short distance modes and then rescales to obtain a new effective potential. The new potential is a lot more complicated. However the leading terms are of the form (2) with new coupling constants (λ', μ'). One repeats the transformation and studies the flow of the theory. An attracting fixed point is supposed to encapsulate the long distance behavior of all models that flow to it.

The massless Gaussian measure ($\lambda = 0, \mu = 0$) is a fixed point for all d. Suppose $d > 4$. In a space of general potentials one finds that the linearization around this fixed point has μ growing ("relevant") and has λ shrinking ("irrelevant"). One has a hyperbolic fixed point as in figure 1. If one selects $\mu = \mu(\lambda)$ to lie on the stable manifold then the fixed point is strictly attracting. These are the critical field theories.

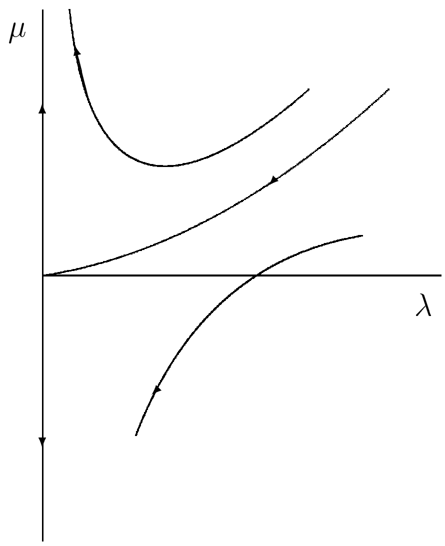

FIGURE 1: *RG* FLOW FOR $d = 4$

If $d = 4$ then the λ term is stable ("marginal"). However this only refers to the linearization, and an analysis of higher order terms shows that λ still flows to zero, so the qualitative picture is the same as for $d > 4$. A rigorous treatment for $d = 4$ (in a lattice model) has been given by Gawedzki and Kupiainen [GK85], [GK86]. Other treatments can be found in [FMRS87],[MP85],[MP89].

For $d < 4$ one finds that λ becomes a second relevant variable and the analysis becomes cloudy.

To gain insight into this question one can study the flow equations in $4-\epsilon$ dimensions not just for integer ϵ, but also for ϵ small and positive. Expanding in powers of ϵ and keeping the lowest orders one predicts that there is a second fixed point which lies at $\lambda = \mathcal{O}(\epsilon)$ and hence is non-Gaussian. (see figure 2.) The stable manifold around this fixed point is again supposed to correspond to critical theories. The predictions one gets for these theories (e.g. for critical exponents) turn out to be pretty good not just for small ϵ but also for ϵ integer. This was one of the early successes of the RG approach (see [WF72],[WK74]).

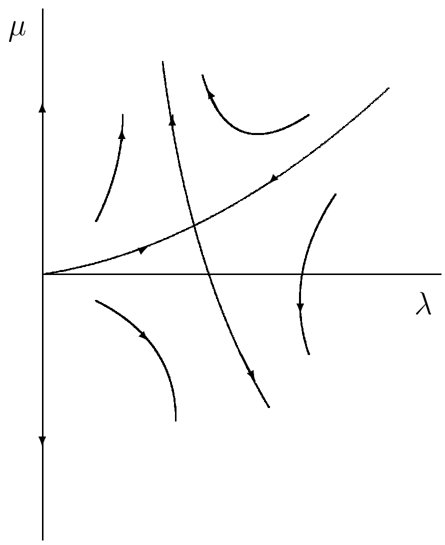

FIGURE 2: *RG* FLOW FOR $d = 4 - \epsilon$

Our goal in this paper is to give a rigorous version of this. We start with the theory in $d = 4$ but modify the Gaussian measure so that its covariance has scaling behavior appropriate for $4 - \epsilon$ dimensions. This is taken as the definition of the theory in $4 - \epsilon$ dimensions. Assuming ϵ is small we analyze the complete RG flow including all the remainder terms. We prove the existence of the second fixed point and the associated stable manifold corresponding to critical field theories.

The analysis in the present paper is carried out at the level of polymer activities, and in infinite volume. In a subsequent paper [BDH97] we make the connection with finite volume. We also analyze the long distance behavior of the two point function for the critical theories and find that the non-Gaussian fixed point gives rise to an anomalous decay rate.

Our analysis uses RG tools created for other models ([BY90], [DH91], [DH93], [BDH94b], [BDH94a], [BDH95]). We particularly rely on a technical companion paper [BDH96] which contains a number of innovations and has fresh proofs of all the basic theorems.

The model is admittedly a bit artificial. However we hope that these techniques can be adapted to study a more realistic model such as N-component ϕ^4 in $d = 3$ with $1/N$ as a small parameter. This is a favorite subject for more heuristic treatments. We are also confident that our methods can be adapted rather easily to give a fresh treatment of the $d = 4$ results.

We note that there is previous work on the existence of non-Gaussian fixed points. For hierarchical ϕ^4 models these include a $d = 4-\epsilon$ version and a $d = 3, N-$component version [GK83], a computer assisted proof of existence for d=3, N=1 [KW86],[KW91], and a proof for the infinitesimal RG [Fel87]. The analysis by Felder [Fel85] of planar graph ϕ^4 in $4 + \epsilon$ dimensions is particularly close in spirit to this paper.

In the rest of the introduction we define the model, define the RG transformations, and preview our results.

1.1 The model

We shall formulate the model both on $\mathbf{R}^4$ and on tori $\Lambda_N = \mathbf{R}^4/(L^N\mathbf{Z})^4$ of side L^N. Working for the moment on $\mathbf{R}^4$, the Gaussian measure μ_v is taken to have mean zero and covariance

$$\begin{aligned} v(x-y) &= \int_1^\infty d\alpha\ \alpha^{\epsilon/2-2}\ e^{-|x-y|^2/4\alpha} \\ &= \int_{\mathbf{R}^4} dp\ p^{-2-\epsilon}\ \Gamma(1+\epsilon/2,p^2)\ e^{ip\cdot(x-y)} \end{aligned} \tag{3}$$

where the incomplete Gamma function is $\Gamma(a,x) = \int_x^\infty\ e^{-t}\ t^{a-1}\ dt$. One can see from this that v is the kernel of an ultra–violet regularization of the operator $(-\Delta)^{-1-\epsilon/2}$. This means it scales like $|x|^{-2+\epsilon}$ for large values of $|x|$, and thus can be taken as a definition of a (regularized) inverse Laplacian in $4-\epsilon$ dimensions. The analysis we present actually holds for a wider class of covariances, with $e^{-|x-y|^2/4\alpha}$ replaced by more general functions $f(|x-y|/\sqrt{\alpha})$.

For discrete RG problems such as ours, the covariance is written as a sum

$$v(x-y) = \sum_{j=0}^\infty L^{-(2-\epsilon)j}C(L^{-j}(x-y)) \tag{4}$$

where each term is a rescaling with scale factor $L > 1$ of the "fluctuation" covariance

$$C(x) = \int_1^{L^2}\ d\alpha\ \alpha^{\epsilon/2-2}\ e^{-x^2/4\alpha} \tag{5}$$

When we work on a torus Λ_N, we define v_N by summing over periods $z \in (L^N\mathbf{Z})^4$, and truncating the sum over scales. This yields

$$v_N(x-y) = \sum_{j=0}^{N} L^{-(2-\epsilon)j} C_j(L^{-j}(x-y)) \tag{6}$$

where

$$C_j(x) = \sum_{z \in (L^{N-j}\mathbf{Z})^4} C(x+z)$$

Note that C_j is almost independent of j when $N-j$ is large.

Let $\mu = \mu_{v_N}$ be a Gaussian measure with mean zero and covariance v_N. Since v_N is smooth this measure is realized on a Sobolev space of $\mathcal{C}^3$ functions on Λ_N. The model is now the study of the measure

$$e^{-V(\phi)} d\mu_v(\phi) = \exp(-V(\Lambda_N, \phi, v; \lambda, \zeta, \mu)) d\mu_{v_N}(\phi) \tag{7}$$

where the potential is

$$V(\Lambda, \phi, v; \lambda, \zeta, \mu) = \lambda \int_\Lambda :\phi^4:_v + \zeta \int_\Lambda :(\partial\phi)^2:_v + \mu \int_\Lambda :\phi^2:_v \tag{8}$$

Here we have allowed for an adjustment in the field strength by including the ζ term. The measure is well defined for $N < \infty$, but our interest is in the limit(s) as $N \to \infty$.

We also need the potential localized in subsets X of Λ_N. In this case we find it convenient to allow two versions of the field strength and for $\zeta = (\zeta_1, \zeta_2)$ define

$$\begin{aligned} V(X, \phi, v; \lambda, \zeta, \mu) &= \\ & \lambda \int_X :\phi^4:_v + \zeta_1 \int_X :(\partial\phi)^2:_v + \zeta_2 \int_X :\phi(-\Delta)\phi:_v + \mu \int_X :\phi^2:_v \end{aligned} \tag{9}$$

When $X = \Lambda_N$ we can integrate by parts and recover the previous version with field strength $\zeta_1 + \zeta_2$.

1.2 RG transformations

Let $\mathcal{Z}(\phi) = e^{-V(\phi)}$. We want to consider integrals of the form $\int \mathcal{Z}(\phi) d\mu_v(\phi)$ or more generally convolutions

$$(\mu_v * \mathcal{Z})(\phi) = \int d\mu_v(\zeta)\ \mathcal{Z}(\phi + \zeta) \tag{10}$$

Now from (4) or (6) the covariance $v = v_N$ has the decomposition

$$v_N(x-y) = \sum_{k=0}^{N} \hat{C}_k(x-y) \tag{11}$$

where $\hat{C}_k(x) = L^{-(2-\epsilon)k} C_k(L^{-k}x)$ and $C_k = C$ in infinite volume. The RG idea is that one can then write μ_v as an iterated convolution

$$\mu_v * \mathcal{Z} = \mu_{\hat{C}_N} * \mu_{\hat{C}_{N-1}} * \ldots * \mu_{\hat{C}_1} * \mu_{\hat{C}_0} * \mathcal{Z} \tag{12}$$

Or if we define

$$\hat{\mathcal{Z}}_j = \mu_{\hat{C}_{j-1}} * \ldots * \mu_{\hat{C}_1} * \mu_{\hat{C}_0} * \mathcal{Z} \tag{13}$$

then we have a sequence on densities $\hat{\mathcal{Z}}_0 = \mathcal{Z}, \hat{\mathcal{Z}}_1, \hat{\mathcal{Z}}_2, \ldots$ related by

$$\hat{\mathcal{Z}}_{j+1} = \mu_{\hat{C}_j} * \hat{\mathcal{Z}}_j \tag{14}$$

We want to study $\hat{\mathcal{Z}}_N$ as $N \to \infty$.

We scale the density $\hat{\mathcal{Z}}_j$ on Λ_N to $\mathcal{Z}_j$ on Λ_{N-j} by

$$\mathcal{Z}_j(\phi) = \hat{\mathcal{Z}}_j(\phi_{L^{-j}}) \tag{15}$$

where for any $r > 0$ the rescaled field $\phi_r(x)$ is defined to be

$$\phi_r(x) = r^{1-\epsilon/2}\phi(rx) \tag{16}$$

Then the densities are related by

$$\mathcal{Z}_{j+1}(\phi) = (\mu_{C_j} * \mathcal{Z}_j)(\phi_{L^{-1}}) \tag{17}$$

This is the basic RG transformation.

Each fluctuation convolution is well defined and controllable because of the smoothness and exponential decay of the covariances C_j. Each step is almost independent of j for a large torus, and is actually independent of j for infinite volume. The result of all this is that the study of the original integral $\mu_v * \mathcal{Z}$ is replaced by the study of an (almost) autonomous discrete flow in the space of densities. The problem is to follow the flow.

1.3 The RG flow

We now sketch our treatment of the RG flow, referring to subsequent sections for precise statements and proofs.

To control the flow it is important to keep track of the local structure of the densities $\mathcal{Z}_j$. To begin with we have the exponential of a local functional. This form is not preserved but something like it is true. We find that we can always write the densities in the form

$$\mathcal{Z}_j(\phi) = (\mathcal{E}xp\ A_j)(\Lambda_{N-j}, \phi) \equiv \sum_{\{X_i\}} \prod_i A_j(X_i, \phi) \tag{18}$$

Here the $\mathcal{E}$xponential is defined by the sum which is over all partitions $\{X_i\}$ of the volume Λ into unions of unit blocks called polymers. The functionals $A_j(X, \phi)$ are called polymer activities, have tree decay in X, and only depend on the restriction of ϕ to X. The expansion is called a polymer expansion. Our formalism using these polymer expansions as a tool for controlling the RG was initiated by Brydges and Yau [BY90]. Similar ideas but adapted towards numerical computations were developed by Mack and Pordt [MP89]. We review the results we need in section 2.

The RG mapping $\mathcal{Z}_j \to \mathcal{Z}_{j+1}$ of densities is now replaced by a mapping $A_j \to A_{j+1}$ of polymer activities. At the level of polymer activities the RG transformation also makes sense in infinite volume, and is independent of j. In this paper we only study this infinite volume transformation. The problem of whether this infinite volume transformation correctly represents the infinite volume limit of the model is deferred to a paper in preparation [BDH97].

In sections 3,4 we will show that the activities A_j can be written in the form

$$A_j(X,\phi) = B_j(X,\phi,v;\lambda_j,\zeta_j,\mu_j,w_j) + R_j(X,\phi) \tag{19}$$

Here $B_j(X,\phi)$ is an explicit functional of certain effective coupling constants $(\lambda_j,\zeta_j,\mu_j)$ and a function w_j on $\Lambda_{N-j}\times\Lambda_{N-j}$. The leading contribution comes when $X=\Delta=$ a unit block. We have

$$B_j(\Delta,\phi,v;\lambda_j,\zeta_j,\mu_j,w_j) = \exp(-V(\Delta,\phi,v;\lambda_j,\zeta_j,\mu_j)) + \dots \tag{20}$$

with further contributions computed in second order perturbation theory. The term R_j is a remainder term incorporating all higher orders of perturbation theory.

The polymer activities A_j are thus parametrized by variables $(\lambda_j,\zeta_j,\mu_j,R_j,w_j)$. These variables transform nicely under the RG. Assuming roughly that

$$\lambda_j = \mathcal{O}(\epsilon), (\zeta_1)_j = \mathcal{O}(\epsilon), (\zeta_2)_j = \mathcal{O}(\epsilon^2), \mu_j = \mathcal{O}(\epsilon^2), R_j = \mathcal{O}(\epsilon^3)$$

and that ϵ is sufficiently small we find a flow equation of the form:

$$\begin{aligned}
\lambda_{j+1} &= L^{2\epsilon}\left(\lambda_j - a(w_j)\lambda_j^2 + \mathcal{O}(\epsilon^3)\right) \\
(\zeta_i)_{j+1} &= L^{\epsilon}\left((\zeta_i)_j - b_i(w_j)\lambda_j^2 + \mathcal{O}(\epsilon^3)\right) \\
\mu_{j+1} &= L^{2+\epsilon}\left(\mu_j - c(w_j)\lambda_j^2 - d(w_j)\lambda_j(\zeta_{1,j}+\zeta_{2,j}) + \mathcal{O}(\epsilon^3)\right) \\
R_{j+1} &= \mathcal{O}(\epsilon^3) \\
w_{j+1}(x) &= L^{2-\epsilon}\left(w_j(Lx) + C(Lx)\right)
\end{aligned} \tag{21}$$

with $b_2 = 0$. The initial conditions should have $R_0 = 0, w_0 = 0$. The function w_j can be explicitly computed and converges to a limit w_∞ as $j \to \infty$. We have nevertheless included it as a dynamical variable to make the flow autonomous in infinite volume.

The powers of L in the first three equations are bigger by a factor of L^ϵ than we would expect if we were really in $4-\epsilon$ dimensions. This is due to the fact that our spatial integrals are in 4 rather than $4-\epsilon$ dimensions.

With generic initial data we cannot iterate the RG mapping indefinitely for we have three expanding variables and would soon leave the region of definition of the mapping. Some further analysis is needed.

The origin is an unstable fixed point. To find the second fixed point replace $a(w_j)$ by its limiting value $a(w_\infty)$ and ignore the higher order terms in the λ equation. One finds an approximate fixed point at

$$\bar{\lambda} = \frac{L^{2\epsilon}-1}{L^{2\epsilon}a(w_\infty)} = \mathcal{O}(\epsilon) \tag{22}$$

Our further analysis is carried out in a neighborhood of this approximate fixed point. The deviation $\tilde{\lambda} = \lambda - \bar{\lambda}$ replaces λ and satisfies

$$\tilde{\lambda}_{j+1} = (2-L^\epsilon)\tilde{\lambda}_j + \mathcal{O}(\epsilon^3) \tag{23}$$

Note that $(2-L^\epsilon) < 1$ so that $\tilde{\lambda}_j$ is a contracting variable.

In the new variables $(\tilde{\lambda}, \zeta, \mu, R, w)$ we have a mapping on a neighborhood of the origin in a Banach space for which the linearization has two expanding directions (ζ, μ) and the rest contracting $(\tilde{\lambda}, R, w)$, and for which the nonlinear part is very small. By a version of the stable manifold theorem there is a hyperbolic fixed point in this neighborhood, and associated with this fixed point is a stable manifold of dimension 2 and an unstable manifold of codimension 2. The stable manifold is given as the graph of a C^∞ function $\zeta = \zeta(\tilde{\lambda}, R, w), \mu = \mu(\tilde{\lambda}, R, w)$, or specializing to the initial values $R = 0, w = 0$ it has the form $\zeta = \zeta(\tilde{\lambda}), \mu = \mu(\tilde{\lambda})$. Densities corresponding to points on this manifold flow to the density of the fixed point. These are the critical field theories. The details of this argument are given in section 5.

2 Review

In order to make the present paper reasonably self–contained, we include here a concise review of the definitions and results for a single RG transformation, adapted to the problem at hand. The reader wishing all the details in a more general setting is directed to the original paper [BDH96].

2.1 Polymer expansions

At the jth RG step, we have a *base space* $\Lambda = \Lambda_{N-j}$ which is a 4–dimensional torus $\mathbf{R}^4/L^{N-j}\mathbf{Z}^4$ of side L^{N-j}. A *polymer* X is a possibly empty union of blocks of Λ where a *block* , Δ, is an open *unit* hypercube in Λ centered on a point of the lattice $\mathbf{Z}^4/L^{N-j}\mathbf{Z}^4$. *Polymer activities* are complex valued functions $K(X,\phi)$ defined on polymers X and fields ϕ (in fact, $K(X)$ will depend only on the restriction $\phi|_X$). We include the empty set as a polymer and assume, unless cautioned otherwise, that $K(\emptyset) = 0$.

As we have mentioned, we shall regard the densities $\mathcal{Z}_j$ as a polymer exponential of a polymer activity $A_j(X, \phi)$

$$\mathcal{Z}_j = \mathcal{E}xp(A_j) = \mathcal{I} + A_j + \frac{1}{2!} A_j \circ A_j + ...$$

The polymer exponential is that associated to the following commutative product on the space of polymer functions $A(X), B(X), \ldots$ [Rue69, GMLMS71]

$$(A \circ B)(X) = \sum_{Y \subset X} A(Y) B(X \setminus Y)$$

Here $\mathcal{I}(\emptyset) = 1$ and otherwise $\mathcal{I}(X) = 0$.

A local density can be written as the polymer expansion

$$\mathcal{E}xp(\Box e^{-V})(\Lambda)$$

where

$$\Box(X) = \begin{cases} 1 \text{ if } X \text{ is a unit block} \\ 0 \text{ otherwise,} \end{cases} \tag{24}$$

and $V(X, \phi)$ is the local potential (2). In our problem, the densities at every scale are nearly local, which means that they can all be written in the form

$$A = \Box e^{-V} + K$$

where the activities $K(X, \phi)$ are small in a sense we now describe.

2.2 Polymer norms

Our polymer activities $K(X, \phi)$ will have certain decay properties depending on the "size" of X, certain growth and decay behaviour depending on the value of ϕ and its derivatives, and finally analyticity in the variable ϕ. This will be summarized by finiteness of a certain type of norm for K.

First we suppose that for any X, $K(X, \phi)$ is defined for $\phi \in \mathcal{C}^3(\bar{X})$, the Banach space of thrice differentiable functions on $\bar{X}$ with norm

$$\|f\| = \max_{|\beta| \leq 3} \sup_x |\partial^\beta f(x)|.$$

The closure $\bar{X}$ here just means we assume that the partial derivatives all have continuous boundary values. All ϕ–derivatives of K are assumed to exist: these are symmetric multi–linear functionals defined by

$$\frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_n} K(X, \phi + \sum s_i f_i)|_{s=0} \quad = \quad K_n(X, \phi; f_1, \cdots, f_n).$$

We further impose that $K(X,\phi)$ should be Frechet-analytic in ϕ in a complex strip around the real space $\mathcal{C}^3(\Lambda)$.

The size of the derivative $K_n(X,\phi)$ is measured by the norm

$$\|K_n(X,\phi)\| = \sup\{|K_n(X,\phi;f_1,\ldots,f_n)| : f_j \in \mathcal{C}^3(\bar{X}),\ \|f_j\|_{\mathcal{C}^3(X)} \leq 1\}. \tag{25}$$

for $n > 0$ and $\|K_0(X,\phi)\| = |K_0(X,\phi)|$.

Actually, we need a localized version of this norm, and therefore we consider derivatives restricted to neighborhoods

$$\tilde{\Delta} = \{x : \mathrm{dist}(x,\Delta) < 1/4\} \tag{26}$$

of blocks Δ. Let $\Delta^{\times n} = (\Delta_1,\ldots,\Delta_n)$ be an n-tuple of blocks and define

$$\begin{aligned} & \|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}} \\ = \ & \sup\{|K_n(f_1,\ldots,f_n)| : f_j \in \mathcal{C}^3(\bar{X}),\ \|f_j\|_{\mathcal{C}^3(X)} \leq 1,\ \mathrm{supp} f_j \subset \tilde{\Delta}_j \cap \bar{X}\} \end{aligned} \tag{27}$$

A connection between the natural norm (25) and the localized version is given if we select a smooth partition of unity χ_Δ indexed by unit blocks Δ such that $\mathrm{supp}\chi_\Delta \subset \tilde{\Delta}$. We assume that each χ_Δ is a translate of a fixed function χ. We define $\|\chi\|$ as the best constant such that

$$\|\chi_\Delta f\| \leq \|\chi\|\ \|f\| \tag{28}$$

Then we have

$$\|K_n(X,\phi)\| \leq \|\chi\|^n \sum_{\Delta^{\times n}} \|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}} \tag{29}$$

The growth of $\|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}}$ in ϕ will be controlled by a *large field regulator* which is some variation of the standard choice $G = G(\kappa) = G(\kappa, X, \phi)$ where

$$G(\kappa, X, \phi) = \exp(\kappa \|\phi\|^2_{X,2,\sigma}) \tag{30}$$

Here

$$\|\phi\|^2_{X,a,b} = \sum_{a \leq |\beta| \leq b} \|\partial^\beta \phi\|^2_X. \tag{31}$$

and $\|\phi\|_X$ is the $L^2(X)$ norm. We take $\sigma = 6$ which is large enough so that this norm can be used in Sobolev inequalities for any low order derivative $\partial^\alpha\phi$. For any such G, we define a norm on derivatives $K_n(X,\phi)$ by

$$\|K_n(X)\|_G = \sum_{\Delta^{\times n}} \sup_{\phi \in \mathcal{C}^3} \|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}} G^{-1}(X,\phi) \tag{32}$$

To control decay in the "size" of X we introduce *large set regulators* $\Gamma_p(X)$ which are defined in dimension $d = 4$ by

$$\begin{aligned} \Gamma_p(X) &= 2^{p|X|}\Gamma(X) \\ \Gamma(X) &= L^{(d+2)|X|}\Theta(X) \\ \Theta(X) &= \inf_\tau \prod_{b \in \tau} \theta(|b|) \end{aligned} \tag{33}$$

The *volume* $|X|$ of X is the number of blocks in X. The infimum is over trees τ composed of bonds b connecting the centers of the blocks in X, and the length $|b|$ of a bond $b = xy$ is defined to be the ℓ^∞-metric $\sup_{1\le j\le d}|x_j - y_j|$. θ is a polynomially increasing function chosen so that $\theta(s) = 1$ for $s = 0, 1$ and

$$\theta(\{s/L\}) \ge L^{-d-1}\theta(s), \qquad s = 2, 3, \dots \tag{34}$$

where $\{x\}$ denotes the smallest integer greater than or equal to x. For any polymer function $K(X)$ we define

$$\|K\|_\Gamma = \sup_\Delta \sum_{X\supset\Delta} |K(X)|\Gamma(X) \tag{35}$$

Now we have the ingredients with which to assemble our norms. For analyticity in a strip of width $h > 0$, our preferred choice is

$$\begin{aligned} \|K(X)\|_{G,h} &= \sum_n \frac{h^n}{n!}\|K_n(X)\|_G \\ \|K\|_{G,h,\Gamma} &= \| \, \|K(\cdot)\|_{G,h}\|_\Gamma \end{aligned} \tag{36}$$

However, sometimes we change the order and take

$$\begin{aligned} \|K_n\|_{G,\Gamma} &= \| \, \|K_n(\cdot)\|_G\|_\Gamma \\ \|K\|_{G,\Gamma,h} &= \sum_n \frac{h^n}{n!}\|K_n\|_{G,\Gamma} \end{aligned} \tag{37}$$

There is also a limiting case of the norms $\|K\|_{G,\Gamma,h}$ in which G^{-1} is concentrated at $\phi = 0$. These are called *kernel norms* and are defined by

$$\begin{aligned} |K_n(X,0)| &= \sum_{\Delta^{\times n}} \|K_n(X,0)\|_{\tilde{\Delta}^{\times n}} \\ |K|_{h,\Gamma} &= \| \sum_n \frac{h^n}{n!}|K_n(\cdot,0)| \, \|_\Gamma \\ |K|_{\Gamma,h} &= \sum_n \frac{h^n}{n!}\| \, |K_n(\cdot,0)| \, \|_\Gamma . \end{aligned} \tag{38}$$

2.3 Bounds on e^{-V}

We now state a bound on $\|e^{-V(X)}\|_{G,h}$ for the ϕ^4 potential in 4–dimensions:

$$V(X) = V(X, \phi, v; \lambda, \zeta, \mu) \tag{39}$$

given in equation (9) with the regulator $G = G(\kappa)G_0^{-1}(\kappa_0)$ where

$$G_0(\kappa_0, X, \phi) = \exp(\kappa_0\|\phi\|_X^2) \tag{40}$$

and $G(\kappa)$ is defined in (30).

The bound is proved under the following hypotheses:

1. $\mathrm{Re}(\lambda)h^4$ is positive and bounded by a sufficiently small constant,
2. $\mathrm{Im}(\lambda)/\mathrm{Re}(\lambda)$ is bounded by a constant.
3. $|\mu|h^2 \leq \mathrm{Re}(\lambda)h^4, \quad |\zeta|h^2 \leq \mathrm{Re}(\lambda)h^4, \quad \kappa_0 h^2 \leq \mathrm{Re}(\lambda)h^4$
4. $h^{-2}v(0),\ h^{-2}\partial^2 v(0),\ h^{-2}\kappa^{-1},\ h^{-2}\kappa_0^{-1}$ are all bounded by constants.

In the above, constants are independent of L and $|\zeta| = \max(|\zeta_1|, |\zeta_2|)$.

Theorem 1 *Under the above hypotheses for any polymer X:*

$$\|e^{-V(X)}\|_{G(\kappa)G_0^{-1}(\kappa_0),h} \leq 2^{|X|}; \quad |e^{-V(X)}|_h \leq 2^{|X|}. \tag{41}$$

If X is a subset of a unit block Δ, then

$$\|e^{-V(X)}\|_{G(\kappa,\Delta)G_0^{-1}(\kappa_0,\Delta),h} \leq 2; \quad |e^{-V(X)}|_h \leq 2; \tag{42}$$

Remark We will need a generalization of this in which the fields $:\phi^4(x):, :\phi^2(x):$, etc. in V are multiplied by functions of x. The results still hold provided the functions are pointwise bounded above and below by suitable constants.

We define $P(X,\phi)$ to be a polynomial of degree r if derivatives of higher order than r vanish. The following result will also be needed when we come to control the perturbative part of the analysis

Lemma 1 *Consider the regulator $G(\kappa)G_0^{-1}(\kappa_0)$ above with $\kappa = \mathcal{O}(\lambda^{1/2}), h = \mathcal{O}(\lambda^{-1/4})$. For any polynomial P of degree r there is a constant $\mathcal{O}(1)$ (depending on r) such that*

$$\|Pe^{-V}\|_{G(\kappa)G_0^{-1}(\kappa_0),h,\Gamma} \leq \mathcal{O}(1)|P|_{h,\Gamma_1} \tag{43}$$

2.4 Results on a single RG map

The RG map is the composition of fluctuation, extraction and scaling. We discuss each of these steps in turn. The jth fluctuation step is the map induced on polymer activities by Gaussian convolution with respect to the measure with covariance $C = C_j(x-y)$ given in equation (6). These covariances are smooth functions, invariant under the symmetries of the torus, which decay rapidly in the separation $|x-y|$. Control of the fluctuation map in general depends on smoothness of the fluctuation covariance C and finiteness of the following norm:

$$\|C\|_\theta = 3^d \sup_{\Delta_1} \sum_{\Delta_2} C(\Delta_1,\Delta_2)\theta\left(d(\Delta_1,\Delta_2)\right) \tag{44}$$

$$C(\Delta_1,\Delta_2) = \|\chi_{\Delta_1} C \chi_{\Delta_2}\|_{\mathcal{C}^6} \tag{45}$$

Here $\theta(s)$ is the function given by (34), and $\chi_\Delta(x)$ is the "bump" function chosen earlier. For theorem 3 we also require the condition on C:

$$\sup_{0\leq|\beta|\leq 12} |\partial^\beta C(0)| \leq \mathcal{O}(1) \tag{46}$$

The main fluctuation theorem refers to norms which involve a one–parameter family of regulators

$$G_t(X,\phi) = \left(2^{|X|}G(2\kappa, X, \phi)\right)^t G(\kappa, X, \phi)^{1-t} \tag{47}$$

which has been constructed to satisfy

$$\mu_{(t-s)C} * G_s(X,\phi) \leq G_t(X,\phi)$$

when κ is small enough depending on a norm of C.

Theorem 2 *For any polymer activity A and any $t \in [0,1]$, there is a unique polymer activity $A(t)$ so that*

$$\mu_{tC} * (\mathcal{E}xpA) = \mathcal{E}xp(A(t)) \tag{48}$$

The map $\bar{\mathcal{F}}_t(A) \equiv A(t)$ is analytic. If $h' < h$ and

$$\|A\|_{G(0),\Gamma,h} \leq D \equiv \frac{(h-h')^2}{16\|C\|_\theta} \tag{49}$$

then

$$\|\bar{\mathcal{F}}_t(A)\|_{G(t),\Gamma,h'} \leq \|A\|_{G(0),\Gamma,h} \tag{50}$$

It turns out that the family $\bar{\mathcal{F}}_t(A) = A(t)$ solves the following flow equation

$$E(A(t)) \equiv \frac{\partial}{\partial t}A(t) - \Delta_C A(t) - \frac{1}{2}\mathcal{B}_C(A(t), A(t)) = 0 \tag{51}$$

where the functional Laplacian Δ_C is the operator

$$\Delta_C A(\phi) = \frac{1}{2}\int \; d\mu_C(\zeta) \; A_2(\phi;\zeta,\zeta)$$

and $\mathcal{B}_C$ is a bilinear operator on activities:

$$\mathcal{B}_C(A,B)(\phi) = \int \; d\mu_C(\zeta) \; A_1(\phi;\zeta) \circ B_1(\phi;\zeta)$$

We will improve the estimates in the above theorem by constructing approximate solutions to this flow equation using a perturbative analysis. Given any family $t \to B(t)$, we measure how well it matches an exact evolution by the *error* $E(B(t))$. The following theorem tracks the growth of the *remainder* $R(t)$ defined by

$$A(t) = B(t) + R(t) \tag{52}$$

Theorem 3 *1. Let $B(t)$ be a continuously differentiable function of $t \in [0,1]$. Suppose $h > h'$ and $\|R(0)\|_{G(0),\Gamma,h}$, $\sup_{0\le t\le 1} \|B(t)\|_{G(t),\Gamma,h} \le \frac{1}{4}D$ where D is defined in (49). Then*

$$\|R(t)\|_{G(t),\Gamma,h'} \le 2(\|R(0)\|_{G(0),\Gamma,h} + t \sup_{s\le t} \|E(B(s))\|_{G(s),\Gamma,h}) \tag{53}$$

2. Suppose further that $\|R(0)\|_{G(0),\Gamma,h}$, $\sup_{0\le t\le 1} \|B(t)\|_{G(t),\Gamma,h} \le h'/(2\|C\|_\theta)$ and $h' \ge 2$. Then for any $M \ge 0$,

$$\begin{aligned} |R(t)|_{\Gamma_{-1},1/2} \;&\le\; \mathcal{O}(1)(|R(0)|_{\Gamma_{-1},1} + (h')^{-M}\|R(0)\|_{G(0),\Gamma,h}) \\ &+\; \mathcal{O}(1)\sup_{s\le t}(|E(B(s))|_{\Gamma_{-1},1} + (h')^{-M}\|E(B(s))\|_{G(s),\Gamma,h}) \end{aligned} \tag{54}$$

where $\mathcal{O}(1)$ depends on M.

Now consider the extraction step. Suppose that the polymer activity has the form $A = \Box e^{-V} + K$. The extraction step consists in removing from $K(X)$ ϕ–independent terms $F_0(X)$ and ϕ–dependent terms $F_1(X,\phi)$ which are both assumed to satisfy a certain *localization* property: $F(X,\phi)$ has the decomposition

$$F(X,\phi) = \sum_{\Delta\subset X} F(X,\Delta,\phi) \tag{55}$$

where Δ is summed over open blocks, and $F(X,\Delta,\phi)$ has the ϕ dependence localized in Δ, i.e. $F(X,\Delta,\phi)$ is a functional on $\mathcal{C}^3(\bar{\Delta})$. The extraction step replaces the potential V by a potential $V(F)$ defined on a unit block Δ by

$$(V(F))(\Delta) = V(\Delta) - \sum_{Y\supset\Delta} F(Y,\Delta) \tag{56}$$

The following theorem gives the essential properties of the extraction step. The bounds are obtained when we have the following *stability* of V relative to the perturbation F_1: there are positive numbers $f(X)$ independent of ϕ and a regulator G such that for all Δ

$$\|\exp\left\{-V(\Delta) - \sum_{X\supset\Delta} z(X)F_1(X,\Delta)\right\}\|_{G,h} \le 2 \tag{57}$$

for all complex $z(X)$ with $|z(X)|f(X) \le 2$. The variation of Theorem 1 allowing variable coefficients is used to verify this condition.

Theorem 4 *If K is a polymer activity and $F_0(X), F_1(X,\phi)$ satisfy the localization hypothesis (55), then there exists a new polymer activity $\mathcal{E}(K,F_0,F_1)$ so that:*

$$\mathcal{E}xp(\Box e^{-V} + K)(\Lambda) = e^{\sum_X F_0(X)}\mathcal{E}xp(\Box e^{-V(F_1)} + \mathcal{E}(K,F_0,F_1))(\Lambda), \tag{58}$$

where the linearization $\mathcal{E}_1$ of $\mathcal{E}$ in K, F_0, F_1 is

$$\mathcal{E}_1(K, F_0, F_1) = K - (F_0 + F_1)e^{-V}. \tag{59}$$

If in addition F_1 satisfies stability hypothesis (57), $\|f\|_{\Gamma_4}$, and $\|K\|_{G,\Gamma_2,h}$ are sufficiently small, and $\sum_{Y\supset\Delta} |F_0(X,\Delta)| \le \log 2$ then $\mathcal{E}$ is jointly analytic in K, F_0, F_1 and there is $\mathcal{O}(1)$ such that

$$\begin{aligned} \|\mathcal{E}(K,F_0,F_1)\|_{G,h,\Gamma} &\le \mathcal{O}(1)(\|K\|_{G,h,\Gamma_2} + \|f\|_{\Gamma_4}); \quad (60)\\ |\mathcal{E}(K,F_0,F_1)|_{h,\Gamma} &\le \mathcal{O}(1)(|K|_{h,\Gamma_2} + \|f\|_{\Gamma_4}). \quad (61) \end{aligned}$$

Finally $\mathcal{E}_{\ge 2} = \mathcal{E} - \mathcal{E}_1$ satisfies

$$\begin{aligned} \|\mathcal{E}_{\ge 2}(K,F_0,F_1)\|_{G,h,\Gamma} &\le \mathcal{O}(1)\|K\|_{G,h,\Gamma_2}\|f\|_{\Gamma_4}\\ |\mathcal{E}_{\ge 2}(K,F_0,F_1)|_{h,\Gamma} &\le \mathcal{O}(1)|K|_{h,\Gamma_2}\|f\|_{\Gamma_4}. \end{aligned}$$

Note the distinguished role of Λ in equation (58): the equation does not generally hold for other polymers X.

Now consider the scaling step. The scaled field is

$$\phi_{L^{-1}}(x) = L^{-(1-\epsilon/2)}\phi(x/L) \tag{62}$$

and functionals scale by

$$K_{L^{-1}}(X,\phi) = K(LX, \phi_{L^{-1}}) \tag{63}$$

The scaling map on polymer activities $\mathcal{S}(K) = \mathcal{S}(K,V)$ is defined by the equation

$$\mathcal{E}xp(\Box e^{-V} + K)(LX, \phi_{L^{-1}}) = \mathcal{E}xp\left((\Box e^{-V})_{L^{-1}} + \mathcal{S}(K)\right)(X,\phi) \tag{64}$$

from which one derives the explicit formula

$$\begin{aligned} \mathcal{S}(K)(Z,\phi) &= \sum_{\{X_j\}\to LZ} \exp(-V(LZ\setminus X, \phi_{L^{-1}})) \prod_j K(X_j, \phi_{L^{-1}})\\ &= \sum_{\{X_j\}\to LZ} \exp(-V_{L^{-1}}(Z\setminus L^{-1}X, \phi)) \prod_j K_{L^{-1}}(L^{-1}X_j, \phi) \quad (65) \end{aligned}$$

Here the sum is over disjoint 1–polymers $\{X_j\}$ with union X such that the L-block closures $\bar{X}_j^L$ are "overlap connected" (see [BDH96]) and have union LZ.

From theorem 1, we will verify that in our model, V satisfies the following stability bound: for all L^{-1}–scale polymers $X \subset$ some block Δ

$$\|(e^{-V})_{L^{-1}}(X)\|_{G,h} \le 2. \tag{66}$$

for $G = G(\kappa)$ with κ small enough. This bound is needed for the main result on scaling.

Define $\dim(\phi) = 1 - \epsilon/2$ and

$$\begin{aligned} h_L &= L^{-\dim\phi}h\\ a &= 2^4\|\chi\| \end{aligned} \tag{67}$$

where $\|\chi\|$ is the norm (28) of the partition of unity bump function defined earlier.

Theorem 5 *Let V satisfy the stability assumption (66) and suppose $\|K\|_{G_L, ah_L, \Gamma_{-5}}$ is sufficiently small. Then*

$$\begin{aligned} \|\mathcal{S}(K)\|_{G,h,\Gamma} &\leq \mathcal{O}(1)L^4 \|K\|_{G_L, ah_L, \Gamma_{-5}} \\ |\mathcal{S}(K)|_{h,\Gamma} &\leq \mathcal{O}(1)L^4 |K|_{ah_L, \Gamma_{-5}} \end{aligned} \tag{68}$$

We also need a sharper estimate on the linearization $\mathcal{S}_1$ of $\mathcal{S}$

$$\begin{aligned} \mathcal{S}_1(K)\,(Z,\phi)) &= \sum_{X:\bar{X}^L = LZ} (e^{-V})(LZ \setminus X, \phi_{L^{-1}}) K(X, \phi_{L^{-1}}) \\ &= \sum_{X:\bar{X}^L = LZ} (e^{-V})_{L^{-1}}(Z \setminus L^{-1}X, \phi) K_{L^{-1}}(L^{-1}X, \phi) \end{aligned} \tag{69}$$

The new estimate needs the stronger bound: for L^{-1} scale polymers X contained in any block Δ:

$$\|(e^{-V})_{L^{-1}}(X)\|_{g,h} \leq 2. \tag{70}$$

where

$$\begin{aligned} g(X,\phi) &= G_0^{-1}(\kappa_0, X, \phi) G(\kappa/2, X, \phi) \\ &= \exp\left[-\kappa_0 \|\phi\|_X^2 + \kappa/2 \|\partial\phi\|_{X,2,\sigma}^2\right] \end{aligned} \tag{71}$$

The *scaling dimension* of a polymer activity K is defined by

Definition 1

$$\begin{aligned} \dim(K_n) &= r_n + n(1-\epsilon/2); \\ \dim(K) &= \inf_n \dim(K_n) \end{aligned} \tag{72}$$

where the infimum is taken over n such that $K_n(X,0) \neq 0$. Here r_n is defined to be the largest integer satisfying $r_n \leq 3$ and $K_n(X, \phi = 0; p^{\times n}) = 0$ whenever $p^{\times n}$ is an n–tuple of polynomials of total degree less than r_n.

Roughly r_n gives the number of derivatives in K_n.

Theorem 6 *Let V satisfy (70).*

1. *If $K(X)$ is supported on large sets, then*

$$\begin{aligned} \|\mathcal{S}_1(K)\|_{G,h,\Gamma} &\leq \mathcal{O}(1)L^{-1} \|K\|_{G_L, ah_L, \Gamma_{-5}} \\ |\mathcal{S}_1(K)|_{h,\Gamma} &\leq \mathcal{O}(1)L^{-1} |K|_{ah_L, \Gamma_{-5}} \end{aligned} \tag{73}$$

2. *If $K(X)$ is supported on small sets, and in addition $\kappa_0 h^2 \geq \mathcal{O}(1)$ and $\kappa h^2 \geq \mathcal{O}(1)$, then*

$$\begin{aligned} \|\mathcal{S}_1(K)\|_{G,h,\Gamma} &\leq \mathcal{O}(1)L^{4-\dim(K)} \|K\|_{G_L, h/2, \Gamma_{-5}} \\ |\mathcal{S}_1(K)|_{h,\Gamma} &\leq \mathcal{O}(1)L^{4-\dim(K)} |K|_{h/2, \Gamma_{-5}} \end{aligned} \tag{74}$$

2.5 Infinite volume

We have explained the RG transformation on finite tori with side L^N. We now observe that every polymer formula makes sense when evaluated on finite polymers in the infinite volume $\mathbf{R}^4$. Indeed the only explicit volume dependence is in the fluctuation step where we would replace C_j by C. (There is one exception to this claim which is the extraction formula (58). But this formula only motivates the definition of the extraction map, and is not needed in the definition in the sense that the formula for $\mathcal{E}$ in [BDH96] is well defined for infinite volume.) Furthermore, every bound we have stated is uniform for large volumes. Correspondingly there is an infinite volume version of each of these results.

In the next two sections we treat finite volume and infinite volume in parallel. In the final section we treat only the infinite volume flow. We leave it to the companion paper [BDH97] to discuss the relationship between the infinite volume flow and the finite volume flow.

3 The specific RG map

A single RG transformation has been defined, and the important estimates recorded. However we have not yet capitalized on the freedom in the extraction step. Our strategy now is to specify the extractions F so that in the rescaling step we have activities K with $\dim K > 4$. Choosing the correct extractions requires second order perturbation theory and control over the higher order remainders.

We make the ansatz that after any number of iterations the polymer activity A can be expressed in the following form:

$$A = A(\vec{\lambda}, R, w) = B(\vec{\lambda}, v, w) + R \tag{75}$$

where

$$B(\vec{\lambda}, v, w) = \left(\square + Q(\vec{\lambda}, v, w)\right) e^{-V(\vec{\lambda}, v)} \tag{76}$$

Here $\vec{\lambda}$ are effective coupling constants

$$\vec{\lambda} = (\lambda, \zeta, \mu) = (\lambda, \zeta_1, \zeta_2, \mu), \tag{77}$$

$w = w(x-y)$ is a kernel and the remainder $R = R(X, \phi)$ is a polymer activity. The potential $V = V(\vec{\lambda}, v)$ is defined in (9). The term $Q = Q(\vec{\lambda}, v, w)$ is a polynomial in the field ϕ, and is the regular part of the contribution of second order perturbation theory. The exact definition of Q is given below.

Once Q is defined our goal is to exhibit a mapping $(\vec{\lambda}, R, w) \to (\vec{\lambda}', R', w')$ such that the transform of the activity $A = A(\vec{\lambda}, R, w)$ is $A' = A(\vec{\lambda}', R', w')$, and to demonstrate that the remainder stays small.

3.1 Definition and evolution of Q

The discussion of second order perturbation theory for the ϕ_3^4 model given in sections 3 and 7 of [BDH95] can be largely repeated for our present model. Apart from the change in dimension, and the introduction of the ϵ parameter, the other extra ingredient here is the inclusion of the wave function coupling constants ζ_1, ζ_2 and corresponding terms in Q involving $\partial\phi$. The present version also has the improvement that leading order extractions are not restricted to small sets. This means that our treatment of the RG flow agrees with the standard perturbative treatment to second order in λ.

Let $v_t = v - tC$. Recall that the fluctuation step is an evolution of polymer activities $A(t)$ such that $\mu_{v_t} * \mathcal{E}xp\,(A(t))$ is constant in t. Then $A(t)$ solves

$$E(A) \equiv \frac{\partial A}{\partial t} - \Delta_C A - \frac{1}{2}\mathcal{B}_C(A, A) = 0 \tag{78}$$

We consider approximations $B(t)$ to $A(t)$. We will say that $B(t)$ is a solution at order $\mathcal{O}(\vec{\lambda})$ if every term in $E(B)$ is order $\mathcal{O}(\vec{\lambda}^2)$.

Lemma 2 *The family $B(t) = \Box \exp\left(-V(\vec{\lambda}, v_t)\right)$ solves (78) at order $\mathcal{O}(\vec{\lambda})$.*

Proof The $\mathcal{O}(\vec{\lambda})$ terms in $E(B)$ are $(\partial/\partial t - \Delta_C)V$. This vanishes because of the following "martingale" property of Wick ordering

$$\left(\frac{\partial}{\partial t} - \Delta_C\right) :\phi^n:_{v_t} = 0 \tag{79}$$

$\Box$

We define Q to improve this and get an $\mathcal{O}(\vec{\lambda}^2)$ solution. Actually we find we do not have to take all $\vec{\lambda}^2$ terms. We will be interested in coupling constants of roughly the following sizes: $\lambda = \mathcal{O}(\epsilon), \zeta_1 = \mathcal{O}(\epsilon), \zeta_2 = \mathcal{O}(\epsilon^2), \mu = \mathcal{O}(\epsilon^2)$. It is almost sufficient to restrict ourselves to the $\mathcal{O}(\epsilon^2)$ terms , i.e. $\lambda^2, \lambda\zeta_1, \zeta_1^2$. This is not quite true since ζ_2 has to be a bit larger than $\mathcal{O}(\epsilon^2)$ and so we include also terms $\lambda\zeta_2, \zeta_1\zeta_2$. The excluded terms are then $\lambda\mu, \zeta_1\mu, \zeta_2\mu, \mu^2, \zeta_2^2$.

The functional Q is made up of monomials of the form

$$\mathcal{Q}^{(m,n)}(v, w; X, \phi) := \int_{\tilde{X}} :\phi(x)^m:_v\, w(x,y) :\phi(y)^n:_v\, dx dy \tag{80}$$

where

$$\tilde{X} = \begin{cases} \Delta \times \Delta & X = \Delta \\ \Delta \times \Delta' \cup \Delta' \times \Delta & X = \Delta \cup \Delta' \\ \emptyset & \text{otherwise} \end{cases} \tag{81}$$

Also a primed index indicates the field ϕ is replaced by $\partial\phi$ (and still Wick-ordered if appropriate) and a doubly primed index indicates that ϕ is replaced by $(-\Delta\phi)$. By convention $:\phi^0: = 1$.

Now define

$$\begin{aligned}
\mathcal{Q}(\vec{\lambda}, v, w) \;\; := \;\; & \lambda^2 \left[8\mathcal{Q}^{(3,3)}(v,w) + 36\mathcal{Q}^{(2,2)}(v,w^2) + 48\mathcal{Q}^{(1,1)}(w^3) + 12\mathcal{Q}^{(0,0)}(w^4)\right] \\
& + \lambda\zeta_1 \left[8\mathcal{Q}^{(3,1')}(v, \partial w) + 12\mathcal{Q}^{(2,0)}(v, (\partial w)^2)\right] \\
& + \zeta_1^2 \left[2\mathcal{Q}^{(1',1')}(\partial\partial w) + \mathcal{Q}^{(0,0)}\left((\partial\partial w)^2\right)\right] \\
& + \lambda\zeta_2 \left[4\mathcal{Q}^{(3,1)}(v, -\Delta w) + 4\mathcal{Q}^{(3,1'')}(v,w) + 12\mathcal{Q}^{(2,0)}\left(v, w(-\Delta w)\right)\right] \\
& + 2\zeta_1\zeta_2 \left[\mathcal{Q}^{(1',1)}(-\Delta\partial w) + \mathcal{Q}^{(1',1'')}(\partial w) + \mathcal{Q}^{(0,0)}\left(\partial w(-\Delta\partial w)\right)\right] \qquad (82)
\end{aligned}$$

Lemma 3 *Let $w_t = w + tC$. Then*

$$B(t) = \left(\square + \mathcal{Q}(\vec{\lambda}, v_t, w_t)\right) e^{-V(\vec{\lambda}, v_t)}$$

solves (78) at order $\mathcal{O}(\vec{\lambda}^2)$, except for terms of order $\lambda\mu, \zeta_1\mu, \zeta_2\mu, \mu^2, \zeta_2^2$.

Proof The $\mathcal{O}(\vec{\lambda}^2)$ terms in $E(B)$ are

$$e^{-V(\vec{\lambda}, v_t)}\left(\left(\frac{\partial}{\partial t} - \Delta_C\right)\mathcal{Q}(\vec{\lambda}, v_t, w_t) - J(t)\right) \qquad (83)$$

where

$$J(t; X, \phi) = \begin{cases} \frac{1}{2}\int_{\tilde{X}} C(x-y)\frac{\partial V(t)}{\partial\phi(x)}\frac{\partial V(t)}{\partial\phi(y)}dx\,dy & |X| \le 2 \\ 0 & \text{otherwise} \end{cases} \qquad (84)$$

But $\mathcal{Q}$ is defined exactly so that the terms in (83) of order $\lambda^2, \lambda\zeta_1, \zeta_1^2, \lambda\zeta_2, \zeta_1\zeta_2$ vanish. This is a calculation using identities like

$$\left(\frac{\partial}{\partial t} - \Delta_C\right)\mathcal{Q}^{(m,n)}(v_t, w_t) = \mathcal{Q}^{(m,n)}(v_t, w_t') - mn\mathcal{Q}^{(m-1,n-1)}(v_t, w_t C) \qquad (85)$$

(See also equation (144) in [BDH95]). □

This is not quite what we want because we will be repeatedly extracting local terms from $\mathcal{Q}(\vec{\lambda}, v_t, w_t)$. This is handled by the following variation

Lemma 4 *Let $P_t(X, \phi)$ be any (integrated) v_t-Wick ordered polynomial in ϕ and derivatives of ϕ whose coefficients are $\mathcal{O}(\vec{\lambda}^2)$ and t-independent. Then*

$$B(t) = \left(\square + [\mathcal{Q}(\vec{\lambda}, v_t, w_t) - P_t]\right) e^{-V(\vec{\lambda}, v_t)}$$

solves (78) to order $\vec{\lambda}^2$ except for terms of order $\lambda\mu, \zeta_1\mu, \zeta_2\mu, \mu^2, \zeta_2^2$.

Proof $(\partial/\partial t - \Delta_C)P_t = 0$ by (79). $\square$

Now we consider what extractions we want to make. To avoid unbounded growth of $\mathcal{Q}$ in the scaling step we want to extract from it a local term $\mathcal{Q}_{sing}$. The basic criterion is that

$$\dim[\mathcal{Q} - \mathcal{Q}_{sing}] > 4 \tag{86}$$

It is sufficient to arrange this for small sets, but we actually do it for all sets.

We choose $\mathcal{Q}_{sing} = \mathcal{Q}_{sing}(\lambda, v, w, v(0))$ where

$$\begin{aligned}
&\mathcal{Q}_{sing}\left(\vec{\lambda}, v, w, v(0)\right) \\
&= \mathcal{Q}^{(4,0)}\left(v, 36\lambda^2 w^2 + 4\lambda\zeta_2(-\Delta w)\right) \\
&\quad +\mathcal{Q}^{(2',0)}\left(v, 48\lambda^2 w^3 T + 2\zeta_1^2 \partial\partial w + 2\zeta_1\zeta_2 \ell(-\Delta \partial w)\right) \\
&\quad +\mathcal{Q}^{(2,0)}\left(v, 144\lambda^2 v(0) w^2 + 48\lambda^2 w^3 + 12\zeta_1\lambda(\partial w)^2 + 12\zeta_2\lambda\left(v(0)+w\right)(-\Delta w)\right) \\
&\quad +\mathcal{Q}^{(1+1',0)}\left(2\zeta_1\zeta_2(-\Delta\partial w)\right) \\
&\quad +\mathcal{Q}^{(0,0)}(...)
\end{aligned} \tag{87}$$

where the argument of $\mathcal{Q}^{(0,0)}$ is

$$\begin{aligned}
(...) \quad = \quad & \lambda^2\left(72 v(0)^2 w^2 + 48(v(0) - (\partial\partial v)(0)T)w^3 + 12 w^4\right) \\
& +\zeta_1^2\left(2(\partial\partial v)(0)\partial\partial w + (\partial\partial w)^2\right) \\
& +2\zeta_1\zeta_2\left((\partial\partial v)(0)\ell + 2\partial w)(-\Delta\partial w)\right)
\end{aligned} \tag{88}$$

Here the superscript $1+1'$ indicates the field $\phi\partial\phi =: \phi\partial\phi :_v$ and ℓ, T denote the kernels

$$\begin{aligned}
\ell_a(x-y) &= -(x-y)_a \\
T_{ab}(x,y) &= -\frac{1}{2}(x-y)_a(x-y)_b
\end{aligned} \tag{89}$$

(Actually $\mathcal{Q}_{sing}\left(\vec{\lambda}, v, w, v(0)\right)$ also depends on $(\partial\partial v)(0)$ but we have suppressed it from the notation). The demonstration that this satisfies the extraction criterion (86) comes a little later.

Now we define the Q that appears in our ansatz (76). It is defined by

$$Q(\vec{\lambda}, v, w) = \mathcal{Q}_{reg}(\vec{\lambda}, v, w) \equiv \mathcal{Q}(\vec{\lambda}, v, w) - \mathcal{Q}_{sing}\left(\vec{\lambda}, v, w, v(0)\right) \tag{90}$$

The objective is to show that $\mathcal{Q}_{reg}$ is RG covariant,

$$\begin{aligned}
Q(\vec{\lambda}, v, w) - \mathcal{Q}_{sing}\left(\vec{\lambda}, v, w, v(0)\right) &\xrightarrow{\mathcal{F}} Q(\vec{\lambda}, v^{\#}, w^{\#}) - \mathcal{Q}_{sing}\left(\vec{\lambda}, v^{\#}, w, v(0)\right) \\
&\xrightarrow{\mathcal{E}} Q(\vec{\lambda}, v^{\#}, w^{\#}) - \mathcal{Q}_{sing}\left(\vec{\lambda}, v^{\#}, w^{\#}, v^{\#}(0)\right) \\
&\xrightarrow{\mathcal{S}} Q(\vec{\lambda}, v, w) - \mathcal{Q}_{sing}\left(\vec{\lambda}, v, w, v(0)\right)
\end{aligned} \tag{91}$$

where $v^{\#} = v_1$ and $w^{\#} = w_1$. Introduce

$$Q(t) = \mathcal{Q}(\vec{\lambda}, v_t, w_t) - \mathcal{Q}_{sing}\left(\vec{\lambda}, v_t, w, v(0)\right) \tag{92}$$

This satisfies $Q(0) = Q$ and solves (78) at order $\mathcal{O}(\vec{\lambda}^2)$ (with exceptions) by Lemma 4. At $t = 1$ it equals $Q(\vec{\lambda}, v^{\#}, w^{\#}) - \mathcal{Q}_{sing}\left(\vec{\lambda}, v^{\#}, w, v(0)\right)$ so we have the first step in (91). We also note that if we define

$$F_Q(t) = \mathcal{Q}_{sing}\left(\vec{\lambda}, v_t, w_t, v_t(0)\right) - \mathcal{Q}_{sing}\left(\vec{\lambda}, v_t, w, v(0)\right) \tag{93}$$

then we have the decomposition into regular and singular parts:

$$Q(t) = \mathcal{Q}_{reg}(\vec{\lambda}, v_t, w_t) + F_Q(t) \tag{94}$$

Extracting $F_Q(1)$ will give the second step in (91). The third step follows from the following scaling property:

Lemma 5

$$\mathcal{S}_1\left(\mathcal{Q}_{reg}(\vec{\lambda}, v, w)e^{-V(\vec{\lambda},v)}\right) = \mathcal{Q}_{reg}(\vec{\lambda}_L, v_L, w_L)e^{-V(\vec{\lambda}_L,v_L)}$$

where kernels scale by

$$\begin{aligned} v_L(x-y) &= L^{2-\epsilon} v\left(L(x-y)\right) \\ w_L(x-y) &= L^{2-\epsilon} w\left(L(x-y)\right) \end{aligned} \tag{95}$$

and the coupling constants scale by

$$\vec{\lambda}_L = (L^{2\epsilon}\lambda, L^{\epsilon}\zeta_1, L^{\epsilon}\zeta_2, L^{2+\epsilon}\mu)$$

Proof $\mathcal{S}_1$ is linear so it suffices to prove that

$$\begin{aligned} &\mathcal{S}_1\left(\mathcal{Q}(\vec{\lambda}, v, w)e^{-V(\vec{\lambda},v)}\right) = \mathcal{Q}(\vec{\lambda}_L, v_L, w_L)e^{-V(\vec{\lambda}_L,v_L)} \\ &\mathcal{S}_1\left(\mathcal{Q}_{sing}\left(\vec{\lambda}, v, w, v(0)\right)e^{-V(\vec{\lambda},v)}\right) = \mathcal{Q}_{sing}\left(\vec{\lambda}_L, v_L, w_L, v_L(0)\right)e^{-V(\vec{\lambda}_L,v_L)} \end{aligned}$$

These are all straightforward computations. For example we have

$$\sum_{X:\bar{X}^L=LZ} \mathcal{Q}^{(m,n)}(v, w; X, \phi_{L^{-1}}) = L^{d_{m,n}}\mathcal{Q}^{(m,n)}(v_L, w_L; Z, \phi). \tag{96}$$

where

$$d_{m,n} = 8 - (1-\epsilon/2)(m+n+2) - \{\text{number of primed indices}\} \tag{97}$$

$\square$

We shall now verify that the extraction criterion (86) is satisfied with our choice of $\mathcal{Q}_{sing}$. For this we need to exhibit cancellations between $\mathcal{Q}$ and $\mathcal{Q}_{sing}$. We rewrite (87) to parallel equation (82) by defining at (X,ϕ)

$$\begin{aligned}\mathcal{Q}^{(2,2)}_{sing}(v,w^2) &= \int_{\tilde{X}} :\phi(x)^2:_v^2 w(x-y)^2 dxdy \\ &= \mathcal{Q}^{(4,0)}(v,w^2)+4\mathcal{Q}^{(2,0)}(v,v(0)w^2)+2\mathcal{Q}^{(0,0)}(v(0)^2w^2)\end{aligned} \tag{98}$$

$$\begin{aligned}\mathcal{Q}^{(3,1)}_{sing}(v,-\Delta w) &= \int_{\tilde{X}} :\phi(x)^3:_v \phi(x)(-\Delta w)(x-y)dxdy \\ &= \mathcal{Q}^{(4,0)}(v,-\Delta w)+3v(0)\mathcal{Q}^{(2,0)}(v,-\Delta w)\end{aligned} \tag{99}$$

$$\begin{aligned}\mathcal{Q}^{(1,1)}_{sing}(w^3) &= \frac{1}{2}\int_{\tilde{X}} [\phi(x)^2+\phi(y)^2-[(x-y)\cdot\partial\phi(x)]^2]w(x-y)^3dxdy \\ &= \mathcal{Q}^{(2,0)}(v,w^3)-\mathcal{Q}^{(2',0)}(v,w^3T)+\mathcal{Q}^{(0,0)}\Big((v(0)-(\partial\partial v)(0)T)w^3\Big)\end{aligned} \tag{100}$$

$$\begin{aligned}\mathcal{Q}^{(1',1')}_{sing}(\partial\partial w) &= \int_{\tilde{X}} (\partial_a\phi(x))(\partial_b\phi(x))(\partial_a\partial_b w)(x-y)dxdy \\ &= \mathcal{Q}^{(2',0)}(v,\partial\partial w)+\mathcal{Q}^{(0,0)}((\partial\partial v)(0)\partial\partial w)\end{aligned} \tag{101}$$

$$\begin{aligned}\mathcal{Q}^{(1',1)}_{sing}(-\Delta\partial w) &= \int_{\tilde{X}} \partial\phi(x)[\phi(x)+(y-x)\partial\phi(x)](-\Delta\partial w)(x-y)dxdy \\ &= \mathcal{Q}^{(1+1',0)}(-\Delta\partial w)+\mathcal{Q}^{(2',0)}(\ell(-\Delta\partial w))+\mathcal{Q}^{(0,0)}((\partial\partial v)(0)\ell(-\Delta\partial w))\end{aligned} \tag{102}$$

We also define

$$\mathcal{Q}^{(2,0)}_{sing}=\mathcal{Q}^{(2,0)} \qquad \mathcal{Q}^{(0,0)}_{sing}=\mathcal{Q}^{(0,0)} \tag{103}$$

with the remaining singular parts equal to zero.

Now defining $\mathcal{Q}^{(m,n)}_{reg}=\mathcal{Q}^{(m,n)}-\mathcal{Q}^{(m,n)}_{sing}$ we find at (X,ϕ)

$$\begin{aligned}\mathcal{Q}^{(2,2)}_{reg}(v,w^2) &= -\frac{1}{2}\int_{\tilde{X}} (:\phi(x)^2:_v - :\phi(y)^2:_v)^2 w(x-y)^2\,dxdy \\ \mathcal{Q}^{(3,1)}_{reg}(v,-\Delta w) &= \int_{\tilde{X}} :\phi(x)^3:_v (-\Delta w)(x-y)\,(\phi(y)-\phi(x))\,dxdy \\ \mathcal{Q}^{(1,1)}_{reg}(w^3) &= -\frac{1}{2}\int_{\tilde{X}} [(\phi(x)-\phi(y))^2-[(x-y)\cdot\partial\phi(x)]^2]w(x-y)^3dxdy \\ \mathcal{Q}^{(1',1')}_{reg}(\partial\partial w) &= -\frac{1}{2}\int_{\tilde{X}} (\partial_a\phi(x)-\partial_a\phi(y))\,(\partial_a\partial_b w)(x-y)\,(\partial_b\phi(x)-\partial_b\phi(y))\,dxdy \\ \mathcal{Q}^{(1',1)}_{reg}(-\Delta\partial w) &= \int_{\tilde{X}} \partial\phi(x)[\phi(y)-\phi(x)-(y-x)\partial\phi(x)](-\Delta\partial w)(x-y)dxdy\end{aligned} \tag{104}$$

We also have

$$\mathcal{Q}_{reg}^{(3,3)} = \mathcal{Q}^{(3,3)} \quad \mathcal{Q}_{reg}^{(3,1')} = \mathcal{Q}^{(3,1')} \quad \mathcal{Q}_{reg}^{(3,1'')} = \mathcal{Q}^{(3,1'')} \quad \mathcal{Q}_{reg}^{(1',1'')} = \mathcal{Q}^{(1',1'')} \tag{105}$$

with the remaining regular parts equal to zero. A direct check shows that every non–zero $\mathcal{Q}_{reg}^{(m,n)}$ has dim > 4.

Collecting terms we have

$$\begin{aligned}\mathcal{Q}_{reg}(\vec{\lambda}, v, w) \;=\; & \lambda^2 \left[8\mathcal{Q}^{(3,3)}(v,w) + 36\mathcal{Q}_{reg}^{(2,2)}(v,w^2) + 48\mathcal{Q}_{reg}^{(1,1)}(w^3)\right] \\ & +8\lambda\zeta_1\mathcal{Q}^{(3,1')}(v,\partial w) + 2\zeta_1^2\mathcal{Q}_{reg}^{(1',1')}(\partial\partial w) \\ & +\lambda\zeta_2\left[4\mathcal{Q}_{reg}^{(3,1)}(v,-\Delta w)4 + \mathcal{Q}^{(3,1'')}(v,w)\right] \\ & +\zeta_1\zeta_2\left[2\mathcal{Q}_{reg}^{(1',1)}(-\Delta\partial w) + 2\mathcal{Q}^{(1',1'')}(\partial w)\right]\end{aligned} \tag{106}$$

3.2 Fluctuation

We summarize what happens when the fluctuation step is applied to the ansatz (75):

$$\begin{aligned} A &= B + R \\ B &= (\Box + Q)e^{-V}. \end{aligned} \tag{107}$$

The fluctuation step constructs a new activity $A^{\#}$ such that $\mu_{v^{\#}} * \mathcal{E}xp(A^{\#}) = \mu_v * \mathcal{E}xp(A)$ where

$$v^{\#} = v - C.$$

We define $A^{\#} = A(1)$ where $A(t)$ is determined by the requirement that for $v_t = v - tC$ the quantity $\mu_{v_t} * \mathcal{E}xp\,(A(t))$ is constant in t and therefore equal to $\mu_v * \mathcal{E}xp(A)$. Let $B(t)$ be the approximate solution from the previous section and define $R(t)$ to be the remainder. Thus we have

$$\begin{aligned} A(t) &= B(t) + R(t) \\ B(t) &= (\Box + Q(t))\, e^{-V(t)}. \end{aligned} \tag{108}$$

where $Q(t)$ is given by (92) and $V(t) = V(\vec{\lambda}, v_t)$. Then defining $B^{\#} = B(1)$ and $R^{\#} = R(1)$ we have

$$\begin{aligned} A^{\#} &= B^{\#} + R^{\#} \\ B^{\#} &= (\Box + Q^{\#})e^{-V^{\#}} \end{aligned} \tag{109}$$

where

$$\begin{aligned} V^{\#} &= V(1) = V(\vec{\lambda}, v^{\#}) \\ Q^{\#} &= Q(1) = \mathcal{Q}_{reg}(\vec{\lambda}, v^{\#}, w^{\#}) + F_Q \\ F_Q &= F_Q(1) = \mathcal{Q}_{sing}\left(\vec{\lambda}, v^{\#}, w^{\#}, v^{\#}(0)\right) - \mathcal{Q}_{sing}\left(\vec{\lambda}, v^{\#}, w, v(0)\right) \\ w^{\#} &= w_1 = w + C. \end{aligned} \tag{110}$$

We also define $K^{\#}$ by
$$A^{\#} = \Box e^{-V^{\#}} + K^{\#}$$
so that
$$K^{\#} = Q^{\#} e^{-V^{\#}} + R^{\#}.$$

3.3 Extraction

By theorem 4, the result of extracting a quantity $F = F_0 + F_1$ from $K^{\#}$ is a new activity of the form
$$A^* = \Box e^{-V^*} + K^* \tag{111}$$
where
$$\begin{aligned} V^*(\Delta) &= V^{\#}(\Delta) - \sum_{X \supset \Delta} F_1(X, \Delta) \\ K^* &= \mathcal{E}(K^{\#}, F) \\ &= K^{\#} - F e^{-V^*} + \mathcal{E}_{\geq 2}(K^{\#}, F) \end{aligned} \tag{112}$$
Here F_1 is the non-constant part of F, and $F_1(X, \Delta)$ is related to $F_1(X)$ by the local decomposition:
$$F_1(X) = \sum_{\Delta \subset X} F_1(X, \Delta)$$

We choose F of the form
$$F = F_Q + F_R \tag{113}$$
where F_Q is the singular part of $Q^{\#}$ defined above, and F_R is an extraction from $R^{\#}$ defined below. Then we have
$$\begin{aligned} K^* &= Q^{\#} e^{-V^{\#}} - F_Q e^{-V^*} + R^{\#} - F_R e^{-V^*} + \mathcal{E}_2(K^{\#}, F) \\ &= \mathcal{Q}_{reg}(\vec{\lambda}, v^{\#}, w^{\#}) e^{-V^*} + R^{**} \end{aligned} \tag{114}$$
where
$$R^{**} = (R^{\#} - F_R e^{-V^*}) + Q^{\#} \left(e^{-V^{\#}} - e^{-V^*} \right) + \mathcal{E}_{\geq 2}(K^{\#}, F) \tag{115}$$
The local decomposition of $F_{Q,1}$ is obtained as follows. The functional $F_{Q,1}$ is a sum of monomials of the form
$$F(X) = \int_{\tilde{X}} :\phi^n(x): f(x-y) dx dy \tag{116}$$
For each such monomial the local decomposition holds with $F(X, \Delta)$ defined for $X = \Delta_1 \cup \Delta_2$ by :
$$F(X, \Delta) = \begin{cases} \int_{\Delta_1 \times \Delta_2} :\phi^n(x): f(x-y) dx dy & \text{if } \Delta = \Delta_1 \\ \int_{\Delta_2 \times \Delta_1} :\phi^n(x): f(x-y) dx dy & \text{if } \Delta = \Delta_2 \\ 0 & \text{otherwise} \end{cases} \tag{117}$$

The next lemma is important because it shows that F_Q is responsible for the flow of the coupling constants $\vec{\lambda}$ and is the origin of the explicit formulas for this flow.

Lemma 6

$$\sum_{X\supset\Delta} F_{Q,1}(X,\Delta) = V(\Delta, \delta\vec{\lambda}_Q, v^{\#})$$

where

$$\begin{aligned}
\delta\lambda_Q &= a(w)\lambda^2 = 36\lambda^2 \int_\Lambda \left[w^{\#2} - w^2\right] \\
\delta\zeta_{1,Q} &= b_1(w)\lambda^2 = 48\lambda^2 \int_\Lambda \left[w^{\#3}T - w^3 T\right] \\
\delta\zeta_{2,Q} &= b_2(w)\lambda^2 = 0 \\
\delta\mu_Q &= c(w)\lambda^2 - d(w)\lambda(\zeta_1+\zeta_2) \\
&= 144\lambda^2 \int_\Lambda \left[v^{\#}(0)w^{\#2} - v(0)w^2\right] \\
&\quad + 48\lambda^2 \int_\Lambda \left[w^{\#3} - w^3\right] \\
&\quad + 12(\zeta_1+\zeta_2)\lambda \int_\Lambda \left[(\partial w^{\#})^2 - (\partial w)^2\right]
\end{aligned} \tag{118}$$

where T was defined in (89).

Proof By (116)

$$\sum_{X\supset\Delta} F(X,\Delta) = \left(\int_\Delta : \phi^n(x) : dx\right)\left(\int_\Lambda f(x-y)dy\right) \tag{119}$$

which has the right form to be one of the monomials in $V(\Delta, \delta\vec{\lambda}_Q, v^{\#})$. The formulas for $\delta\vec{\lambda}_Q$ were obtained from (87), (110). Note that many contributions to $\delta\vec{\lambda}_Q$ are zero since we integrate a total derivative. For example the $\mathcal{O}(\zeta_1^2, \zeta_1\zeta_2)$ changes in the field strength vanish since

$$2\zeta_1^2 \int_\Lambda (\partial_a\partial_b w^{\#} - \partial_a\partial_b w) = 0 \tag{120}$$

and since an integration by parts gives

$$\begin{aligned}
&\zeta_1\zeta_2 \int_\Lambda (y-x)_a(-\Delta\partial_a w)(x-y)dy \\
&\qquad = 4\zeta_1\zeta_2 \int_\Lambda (-\Delta w)(x-y)dy \\
&\qquad = 0
\end{aligned} \tag{121}$$

We have also used

$$12\lambda\zeta_2 \int_\Lambda \left[w^{\#}(-\Delta w^{\#}) - w(-\Delta w)\right] = 12\lambda\zeta_2 \int_\Lambda \left[(\partial w^{\#})^2 - (\partial w)^2\right]$$

to obtain the $\lambda\zeta_2$ contribution to $\delta\mu_Q$.

□

Construction of F_R: $F_R(X,\phi)$ is zero if X is not a small set; otherwise it is a local polynomial of the form

$$\begin{aligned} F_R(X,\phi) &= \alpha_0(X)|X| + \alpha_{2,0}(X)\int_X \phi^2 + \sum_a \alpha_{2,1}(X,a)\int_X \phi(\partial_a\phi) \\ &+ \sum_{a,b} \alpha_{2,2}(X,a,b)\int_X (\partial_a\phi)(\partial_b\phi) + \sum_{a,b}\alpha'_{2,2}(X,a,b)\int_X \phi(\partial_a\partial_b\phi) \\ &+ \alpha_4(X)\int_X \phi^4 \end{aligned} \tag{122}$$

with coefficients α determined by the next lemma. Let us introduce the notation

$$F_n = (F_R)_n(X,0) \qquad R_n = (R^{\#})_n(X,0) \qquad V_n = V_n^{\#}(X,0) \tag{123}$$

Lemma 7 *There is a unique choice of coefficients α in $F_R(X,\phi)$ such that*

$$\begin{aligned} &\dim(R^{\#} - F_R e^{-V^{\#}}) > 4 \\ &\sum_{X\supset\Delta} F_{R,1}(X,\Delta) = V(\Delta,\delta\vec{\lambda}_R, v^{\#}) \end{aligned} \tag{124}$$

Furthermore $\alpha_{m,n}(X)$ and $\delta\vec{\lambda}_R$ are polynomials in $e^{V_0}R_n(p^{\times n})$ and $V_n(p^{\times n})$ where $n = 0,2,4$ and $p = 1, x_a, x_b, x_a x_b$. The x-origin can be chosen arbitrarily for each X. The polynomials have $\mathcal{O}(1)$ coefficients and are linear in R_n.

Proof Recall that dim was defined in definition (1). The first four non-vanishing derivatives of $R^{\#} - F_R\exp(-V^{\#})$ evaluated at $\phi = 0$ are

$$\begin{aligned} J_0 &= R_0 - F_0 e^{-V_0} \\ J_2 &= R_2 - (F_2 - F_0V_2)\, e^{-V_0} \\ J_4 &= R_4 - \Big(F_4 - 6\mathrm{Sym}(F_2V_2) + 3F_0\mathrm{Sym}(V_2^2) - F_0V_4\Big)\, e^{-V_0} \end{aligned} \tag{125}$$

Referring to the definition of dimension we find that the dim > 4 condition in (124) is equivalent to the vanishing of these derivatives when evaluated on the constant 1 and monomials x_a, $x_a x_b$ as follows

$$\begin{aligned} &J_0 = 0 \\ &J_2(1^{\times 2}) = 0, \quad J_2(1,x_a) = 0, \quad J_2(x_a,x_b) = 0, \quad J_2(1,x_ax_b) = 0 \\ &J_4(1^{\times 4}) = 0 \end{aligned} \tag{126}$$

By using these equations in the order given we determine recursively the coefficients α in (122) in the order

$$\alpha_0(X),\ \alpha_{2,0}(X),\ \alpha_{2,1}(X,a),\ \alpha_{2,2}(X,a,b),\ \alpha'_{2,2}(X,a,b),\ \alpha_4(X)$$

and easily see that α is a polynomial (linear) in $e^{V_0}R_n$ and V_n evaluated on $1, x_a, x_b, x_a x_b$. For example we have

$$
\begin{aligned}
\alpha_0(X) &= \frac{1}{|X|} e^{V_0}(X) R_0(X) \\
\alpha_{2,0}(X) &= \frac{1}{2|X|} e^{V_0}(X) \Big(R_2(X; 1^{\times 2}) + R_0(X) V_2(X; 1^{\times 2}) \Big) \\
\alpha_{2,1}(X,a) &= e^{V_0}(X) \Big[R_2(X; 1, x_a) + R_0(X) V_2(X; 1, x_a) \\
&\quad - \frac{1}{|X|} \Big(R_2(X; 1^{\times 2}) + R_0(X) V_2(X; 1^{\times 2}) \Big) \int_X x_a \Big]
\end{aligned}
\tag{127}
$$

Now F_R is defined. Define $F_{R,1}(X, \Delta)$ by dropping the $\alpha_0(X)$ term and restricting the integrations in (122) to Δ. It remains to establish (124). Consider

$$
\sum_{\Delta} \left(\sum_{X \supset \Delta} \alpha_{2,1}(X,a) \right) \int_{\Delta} \phi(\partial_a \phi) \tag{128}
$$

Each iteration of the renormalization group map preserves the symmetry of the initial data under reflections that map the lattice of unit blocks to itself. As a consequence of (127) with x-origin chosen at the center of Δ one can show that

$$
\alpha_{2,1}(r_a X, a) = -\alpha_{2,1}(X, a) \tag{129}
$$

where r_a is the reflection in the a-direction through Δ. Hence the sum above is zero and there is no term of this form in the potential. Similarly we conclude that

$$
\begin{aligned}
\sum_{X \supset \Delta} \alpha_{2,2}(X,a,b) &= \alpha_{2,2} \delta_{a,b} \\
\sum_{X \supset \Delta} \alpha'_{2,2}(X,a,b) &= \alpha'_{2,2} \delta_{a,b}
\end{aligned}
\tag{130}
$$

and these contribute to $\zeta_{1,R},\ \zeta_{2,R}$ respectively. From these considerations we find that the changes in the coupling constants are given by (taking into account the Wick ordering)

$$
\begin{aligned}
\delta\lambda_R &= \sum_{X \supset \Delta} \alpha_4(X) \\
\delta\zeta_R &= (\alpha_{2,2}, -\alpha'_{2,2}) \\
\delta\mu_R &= \sum_{X \supset \Delta} \Big(\alpha_{2,0}(X) + 6 v^{\#}(0) \alpha_4(X) \Big)
\end{aligned}
\tag{131}
$$

□

We introduce the notation

$$
\begin{aligned}
\delta\lambda_R &= -r(\vec{\lambda}, w, R) \\
\delta\zeta_{i,R} &= -s_i(\vec{\lambda}, w, R) \\
\delta\mu_R &= -t(\vec{\lambda}, w, R)
\end{aligned}
\tag{132}
$$

Note that r, s, t are linear functions of $R^{\#}$, which in turn is a function of $(\vec{\lambda}, w, R)$.

The new potential is now $V^* = V(\vec{\lambda}^*, v^{\#})$ where $\vec{\lambda}^* = \vec{\lambda} - \delta\vec{\lambda}_Q - \delta\vec{\lambda}_R$. Thus we have $\vec{\lambda}^*$ given by

$$\begin{aligned}
\lambda^* &= \lambda - a(w)\lambda^2 + r(\vec{\lambda}, w, R) \\
\zeta_i^* &= \zeta_i - b_i(w)\lambda^2 + s_i(\vec{\lambda}, w, R) \\
\mu^* &= \mu - c(w)\lambda^2 - d(w)\lambda(\zeta_1 + \zeta_2) + t(\vec{\lambda}, w, R)
\end{aligned} \tag{133}$$

Referring to (114) we see that the coupling constants $\vec{\lambda}^*$ in the potential do not match the coupling constants $\vec{\lambda}$ in $Q^{**} = \mathcal{Q}_{reg}(\vec{\lambda}, v^{\#}, w^{\#})$. We make a further adjustment by defining

$$Q^* = \mathcal{Q}_{reg}(\vec{\lambda}^*, v^{\#}, w^{\#}) \tag{134}$$

and have

$$K^* = Q^* e^{-V^*} + R^* \tag{135}$$

where

$$R^* = (Q^{**} - Q^*)e^{-V^*} + R^{**} \tag{136}$$

3.4 Scaling

After scaling we have a new activity

$$A' = \Box e^{-V'} + K' \tag{137}$$

where $K' = \mathcal{S}(K^*)$. By definition of $\mathcal{S}$

$$V' = V(\vec{\lambda}^*, v^{\#})_{L^{-1}} = V(\vec{\lambda}^*_L, v^{\#}_L) = V(\vec{\lambda}', v') \tag{138}$$

where $v' = v^{\#}_L(x - y)$ and

$$\vec{\lambda}' = \vec{\lambda}^*_L = (L^{2\epsilon}\lambda^*, L^{\epsilon}\zeta^*, L^{2+\epsilon}\mu^*) \tag{139}$$

Also

$$K' = Q' e^{-V'} + R' \tag{140}$$

where

$$\begin{aligned}
Q' e^{-V'} &= \mathcal{S}_1(Q^* e^{-V^*}) \\
R' &= \mathcal{S}_1(R^*) + \mathcal{S}_{\geq 2}(K^*)
\end{aligned} \tag{141}$$

$\mathcal{S}_1$ is the linearization of $\mathcal{S}$ and $\mathcal{S}_{\geq 2} = \mathcal{S} - \mathcal{S}_1$. From lemma 5

$$Q' = \mathcal{Q}_{reg}(\vec{\lambda}', v', w') = Q(\vec{\lambda}', v', w') \tag{142}$$

with $w' = w^{\#}_L$ which means that the form of our ansatz (75) has been preserved as we wanted.

3.5 Summary

Starting with the polymer activity (75) we have found that one RG step transforms the ansatz to

$$A' = A(\vec{\lambda}', R', w') = \left(\square + Q(\vec{\lambda}', v', w')\right) e^{-V(\vec{\lambda}', v')} + R' \tag{143}$$

The new coupling constants are obtained from (133),(139) and are given by

$$\begin{aligned} \lambda' &= L^{2\epsilon}\left(\lambda - a(w)\lambda^2 + r(\vec{\lambda}, w, R)\right) \\ \zeta_i' &= L^{\epsilon}\left(\zeta_i - b_i(w)\lambda^2 + s_i(\vec{\lambda}, w, R)\right) \\ \mu' &= L^{2+\epsilon}\left(\mu - c(w)\lambda^2 - d(w)\lambda(\zeta_1+\zeta_2) + t(\vec{\lambda}, w, R)\right) \end{aligned} \tag{144}$$

The new remainder R' is given as a function

$$R' = U(\vec{\lambda}, R, w) \tag{145}$$

by equations (109),(115),(136),(141). Finally the new kernel is

$$w'(x-y) = L^{2-\epsilon}\left(w\left(L(x-y)\right) + C\left(L(x-y)\right)\right) \tag{146}$$

In infinite volume $v' = v$ and everything is reduced to this single mapping in the variables $\vec{\lambda}, R, w$.

4 Estimates

We observe the convention that $\mathcal{O}(1)$ stands for a constant independent of ϵ and L and that C stands for a constant which may depend on L but is independent of ϵ. The value of C may vary from line to line. This is consistent with the usage in section 2.

4.1 Estimates on $w, a(w), b(w), ...$

We begin with some estimates on w. Recall that $w' = w_L + C_L$ where $w_L(x) = L^{2-\epsilon}w(Lx)$. Starting with $w_0 = 0$ we get a sequence w_j defined by $w_{j+1} = (w_j)_L + C_L$ or

$$w_j(x) = \sum_{l=1}^{j} L^{(2-\epsilon)l} C(L^l x) \tag{147}$$

This can also be written

$$w_j(x) = \int_{L^{-2j}}^{1} d\alpha\ \alpha^{\epsilon/2-2} e^{-x^2/4\alpha} \tag{148}$$

For $x \neq 0$ this also makes sense for $j = \infty$. We sometimes write $w_\infty = w_*$.

The basic properties of C and w_j are contained in:

Lemma 8 *For any multi-index β there is a constant $\mathcal{O}(1)$ such that for all x and* $1 \leq j \leq \infty$

$$\begin{aligned} |(\partial^\beta C)(x)| &\leq \mathcal{O}(1) e^{-2|x|/L} \\ |(\partial^\beta w_j)(x)| &\leq \mathcal{O}(1) e^{-\frac{3}{2}|x|} |x|^{-2-|\beta|+\epsilon} \end{aligned} \tag{149}$$

Proof We have

$$|C(x)| \leq \mathcal{O}(1) \int_1^{L^2} d\alpha \alpha^{\epsilon/2-2} e^{-2|x|/\sqrt{\alpha}} \leq \mathcal{O}(1) e^{-2|x|/L} \tag{150}$$

and also

$$\begin{aligned} w_j(x) &\leq \mathcal{O}(1) \int_{L^{-2j}}^1 d\alpha \ \alpha^{\epsilon/2-2} \ e^{-\frac{3}{2}|x| - \frac{1}{2}|x|/\sqrt{\alpha}} \\ &\leq \mathcal{O}(1) e^{-\frac{3}{2}|x|} \int_0^\infty d\alpha \ \alpha^{\epsilon/2-2} \ e^{-\frac{1}{2}|x|/\sqrt{\alpha}} \\ &= \mathcal{O}(1) e^{-\frac{3}{2}|x|} \ |x|^{-2+\epsilon} \end{aligned}$$

The bounds for the derivatives follow similarly.

□

We will find it useful to cast this in a Banach space setting. Let $\mathcal{W}$ be the Banach space of complex-valued functions on $\mathbf{R^4}$ for which the following norm is finite:

$$\|w\| = \sup_{|\beta| \leq 3} \sup_x (|x|^{9/4+|\beta|} |\partial^\beta w(x)| e^{|x|}) \tag{151}$$

By the lemma, the sequence w_j starting at $w_0 = 0$ is in $\mathcal{W}$ and there is a constant k independent of L such that for $0 \leq j \leq \infty$

$$\|w_j\| \leq k/2 \tag{152}$$

The point of taking the power $9/4$ here is to make a norm in which the mapping $w \to w'$ is a strong contraction. It is easy to show that for any $w, \tilde{w} \in \mathcal{W}$

$$\|w' - \tilde{w}'\| \leq L^{-1/4} \|w - \tilde{w}\| \tag{153}$$

Thus, starting from any $w_0 = w$ with $\|w\| \leq k/2$ we generate a sequence w_j converging to the unique fixed point w_* and satisfying

$$\begin{aligned} \|w_j - w_*\| &\leq L^{-j/4} \|w - w_*\| \leq L^{-j/4} k \\ \|w_j\| &\leq k \end{aligned} \tag{154}$$

Hereafter we consider general $w \in \mathcal{W}$ with $\|w\| \leq k$.

Lemma 9 *There is a constant C such that for $w \in \mathcal{W}$ and $\|w\| \leq k$*

$$|a(w)|, |b_1(w)|, |c(w)|, |d(w)| \leq C \tag{155}$$

Furthermore there is a constant k' independent of L such that

$$|a(w_1) - a(w_2)| \leq k' \|w_1 - w_2\| \tag{156}$$

and similarly for b, c, d.

Proof We have, from lemma 6,

$$\begin{aligned}
a(w) &= 36 \int (2wC + C^2) \\
b_1(w) &= 48 \int (3w^2C + 3wC^2 + C^3)T \\
c(w) &= 48 \int \left(6v^{\#}(0)wC + 3v^{\#}(0)C^2 + 3\left(C - C(0)\right)w^2 + 3wC^2 + C^3\right) \\
d(w) &= 12 \int \left(2\partial w \partial C + (\partial C)^2\right)
\end{aligned} \tag{157}$$

The issue is the singularity at $x = 0$. First consider $a(w)$. For $w \in \mathcal{W}$ we have

$$|w(x)| \leq |x|^{-9/4} e^{-|x|} \|w\| \tag{158}$$

Therefore w is integrable and since $C \leq \mathcal{O}(1)$ we have that wC is integrable and $\int wC \leq \mathcal{O}(1)$. Since $|C(x)| \leq \mathcal{O}(1)e^{-2|x|/L}$ we also have that C^2 is integrable (but the bound depends on L). For the Lipschitz bound only the first term contributes and so we can use the bound (158) for $w_1 - w_2$ to get a bound independent of L.

For the bound on $b_1(w)$ we use that $|T(x)| \leq \mathcal{O}(|x|^2)$ and

$$|x|^2 |w(x)|^2 \leq |x|^{-5/2} e^{-2|x|} \|w\|^2 \tag{159}$$

to bound the first term and get the result. The Lipschitz bound follows similarly.

For the bound on $c(w)$ we use $|C(x) - C(0)| \leq \mathcal{O}(|x|)$ to obtain the factor of x needed to integrate w^2. We also use $|v^{\#}(0)| \leq \mathcal{O}(1)$.

Finally $d(w)$ is bounded since

$$|\partial w(x)| \leq |x|^{-13/4} e^{-|x|} \|w\| \tag{160}$$

is integrable.

□

In fact we need a sharper bound on $a(w)$.

Lemma 10 *Let L be sufficiently large and $\epsilon < \epsilon(L)$. Then there exist positive constants c_1, c_2 independent of L such that for real $w \in \mathcal{W}$ and $\|w\| \leq k$*

$$c_1 \log L \leq a(w) \leq c_2 \log L \tag{161}$$

Proof As noted the first term in $a(w)$ is bounded independently of L. Thus it suffices to show that

$$\mathcal{O}(1)\log L \le \int C(x)^2 dx \le \mathcal{O}(1)\log L \tag{162}$$

This expression is written as

$$\int C^2(x)dx = \int_1^{L^2} d\alpha \int_1^{L^2} d\beta\ (\alpha\beta)^{\epsilon/2-2} \int e^{-x^2/4\alpha} e^{-x^2/4\beta} dx$$

To obtain an upper bound, we write this as twice the integral subject to $\beta \le \alpha$, and bound $e^{-x^2/4\alpha}$ by $\mathcal{O}(1)$. Then provided $\epsilon \log L \le 1$

$$\begin{aligned}
\int C^2(x)dx &\le 2\int_1^{L^2} d\alpha \int_1^{\alpha} d\beta\ (\alpha\beta)^{\epsilon/2-2} \int e^{-x^2/4\beta} d^4x \\
&\le \mathcal{O}(1) \int_1^{L^2} d\alpha \int_1^{\alpha} d\beta\ (\alpha\beta)^{\epsilon/2-2}\beta^2 \\
&\le \mathcal{O}(1) \int_1^{L^2} d\alpha\ \alpha^{\epsilon-1} \\
&\le \mathcal{O}(1)\log L
\end{aligned}$$

The integral is also bounded from below by the integral over the subregion $\alpha \in (2, L), \beta \in (\alpha/2, \alpha)$. On this region we can use the lower bound $e^{-x^2/4\alpha} \ge e^{-x^2/4\beta}$ Then provided $\epsilon \log L \le 1$ we find

$$\begin{aligned}
\int C^2(x)dx &\ge 2\int_2^{L^2} d\alpha \int_{\alpha/2}^{\alpha} d\beta\ (\alpha\beta)^{\epsilon/2-2} \int \left(e^{-x^2/4\beta}\right)^2 dx \\
&\ge \mathcal{O}(1) \int_2^{L^2} d\alpha \int_{\alpha/2}^{\alpha} d\beta\ (\alpha\beta)^{\epsilon/2-2}\beta^2 \\
&\ge \mathcal{O}(1) \int_2^{L^2} d\alpha\ \alpha^{\epsilon-1} \\
&\ge \mathcal{O}(1)\log L
\end{aligned}$$

□

4.2 The domain $\mathcal{D}$

In the rest of this section we give estimates on the various quantities which enter our flow equations. The control of this flow is relegated to the next section.

To get useful estimates we must carefully specify a domain $\mathcal{D}$ for the coupling constants $\vec{\lambda} = (\lambda, \zeta, \mu)$, the remainder R, and the kernel w.

For the coupling constant we consider a region defined by the inequalities

$$|\lambda - \bar{\lambda}| < \epsilon^{\alpha} \quad |\zeta_1| < \epsilon^{1-\delta} \quad |\zeta_2| < \epsilon^{2-\Delta-\delta} \quad |\mu| < \epsilon^{2-2\delta} \tag{163}$$

Here δ, Δ are small positive numbers with $\Delta > 4\delta$. This domain is supposed to contain the fixed point we are searching for. We will choose $\bar{\lambda} > 0$ to be a low order approximation to the fixed point satisying $\bar{\lambda} = \mathcal{O}(\epsilon)$ and we take $1 < \alpha < 2$ so that $\lambda = \mathcal{O}(\epsilon)$. We also allow $\alpha = 1$ but then impose the condition

$$|\lambda - \bar{\lambda}| < d\epsilon \tag{164}$$

with d sufficiently small so that $\mathrm{Re}(\lambda) > 0$. The choice of the integral powers of ϵ for ζ_1, μ is also motivated by a low order calculation of the fixed point. It turns out to be convenient to reduce these powers slightly, which gives a slightly larger domain.

For the remainder R we consider the norms

$$\begin{aligned} \|\cdot\| &= \|\cdot\|_{G,h,\Gamma} \\ |\cdot| &= |\cdot|_{1,\Gamma} \end{aligned} \tag{165}$$

as defined in section 2. We take $G = G(\kappa, X, \phi)$ defined in (30) with $\kappa = \epsilon^{1/2}$. We also choose $h = \epsilon^{-1/4}$. Now define

$$|\!|\!|\cdot|\!|\!| = \max\{\epsilon^2\|\cdot\|, |\cdot|\} \tag{166}$$

We consider remainders R which satisfy the bound

$$|\!|\!|R|\!|\!| \leq \epsilon^{3-\Delta} \tag{167}$$

Finally w should satisfy

$$\|w\| \leq k \tag{168}$$

The region of complex valued $(\vec{\lambda}, R, w)$ satisfying (163), (167), (168) will be called $\mathcal{D}$.

Then we can prove the following result:

Theorem 7 *Let L be sufficiently large and ϵ sufficiently small (depending on L). Then r, s, t, U are analytic functions of $(\vec{\lambda}, R, w)$ on $\mathcal{D}$ and are bounded there by*

$$\begin{aligned} |r|, |s|, |t| &\leq \mathcal{O}(1)\epsilon^{3-\Delta} \\ |\!|\!|U|\!|\!| &\leq \mathcal{O}(L^{-1+2\epsilon})\epsilon^{3-\Delta} \end{aligned} \tag{169}$$

4.3 Estimates on Q

Recall our convention on $\mathcal{O}(1)$ and C. We always assume that the hypotheses of theorem 7 are satisfied.

Lemma 11 *$\mathcal{Q}_{reg}(\vec{\lambda}, v, w)$ satisfies*

$$\|\mathcal{Q}_{reg}e^{-V}\| \leq C\epsilon^{1/2-2\delta} \tag{170}$$

$$|\mathcal{Q}_{reg}e^{-V}| \leq C\epsilon^{2-2\delta} \tag{171}$$

Proof By lemma 1 we have

$$\begin{aligned} \|\mathcal{Q}_{reg} e^{-V}\| &\leq \mathcal{O}(1) |\mathcal{Q}_{reg}|_{h,\Gamma_1} \\ |\mathcal{Q}_{reg} e^{-V}| &\leq \mathcal{O}(1) |\mathcal{Q}_{reg}|_{1,\Gamma_1} \end{aligned} \tag{172}$$

Now we bound the right side of these inequalities. The main point is that the kernel of $\mathcal{Q}_{reg}$ has integrable short distance singularities.

We examine $\mathcal{Q}_{reg}$ term by term. We first claim that

$$\lambda^2 |\mathcal{Q}^{(3,3)}|_{h,\Gamma_1} \leq C\epsilon^2 h^6 \|w\| \tag{173}$$

Then the two choices $h = \epsilon^{-1/4}$ and $h = 1$ lead to the bound we want.

To see this suppose we compute the sixth functional derivative of $:\phi(x)^3::\phi(y)^3:$ at $\phi = 0$ by substituting $\phi = \sum_j \alpha_j f_j$, performing derivatives with respect to all α, and setting them all to zero. The result is

$$6!\ \mathrm{Sym}\, (f_1(x) f_2(x) f_3(x) f_4(y) f_5(y) f_6(y))$$

This is bounded by a constant if $\|f_\alpha\|_{C^3} \leq 1$. This leads to the estimate

$$\begin{aligned} \|(\mathcal{Q}^{(3,3)})_6(\Delta \cup \Delta', 0)\| &\leq \mathcal{O}(1) \int_{\Delta\times\Delta'} |w(x-y)| dx dy \\ &\leq \mathcal{O}(1) \|w\| \int_{\Delta\times\Delta'} |x-y|^{-9/4} e^{-|x-y|} dx dy \\ &\leq \mathcal{O}(1) \|w\| e^{-d(\Delta,\Delta')} \end{aligned} \tag{174}$$

The norm restricted to $\tilde{\Delta}^{\times 6}$ has the same bound. The norm $|(\mathcal{Q}^{(3,3)})_6(\Delta\cup\Delta', 0)|$ defined in (38) also has the same bound since the sum over $\tilde{\Delta}^{\times 6}$ has $\mathcal{O}(1)$ non-vanishing terms. (At most $\mathcal{O}(1)$ block neighborhoods $\tilde{\Delta}$ intersect $\Delta \cup \Delta'$.) The same bound holds for lower derivatives and higher derivatives vanish. Taking the h-norm gives a factor h^6 which combined with the factor ϵ^2 from the λ^2 gives the factor $\epsilon^2 h^6$. Finally the Γ-norm is estimated by

$$\sum_{\Delta'} \Gamma(\Delta \cup \Delta') e^{-d(\Delta,\Delta')} \leq C \tag{175}$$

and (173) is established.

Next we have

$$\lambda^2 |\mathcal{Q}^{(2,2)}_{reg}|_{h,\Gamma_1} \leq C\epsilon^2 h^4 \|w\|^2 \tag{176}$$

For this we evaluate the fourth functional derivative at $\phi = 0$ of

$$\left(:\phi^2(x): - :\phi^2(y):\right)^2 = \left(\phi^2(x) - \phi^2(y)\right)^2$$

with the result.

$$4!\ \mathrm{Sym}\, (f_1(x) f_2(x) - f_1(y) f_2(y)) (f_3(x) f_4(x) - f_3(y) f_4(y))$$

This is bounded in absolute value by $\mathcal{O}(1)|x-y|^2$ for test functions in $\mathcal{C}^3$. Therefore $|\lambda^2 \mathcal{Q}^{(2,2)}_{reg}|_{h,\gamma\Gamma}$ is bounded by $\mathcal{O}(1)\epsilon^2 h^4$ times the Γ norm of

$$\begin{aligned} \int_{\Delta\times\Delta'} |x-y|^2 |w(x-y)|^2 dxdy &\leq \|w\|^2 \int_{\Delta\times\Delta'} |x-y|^{-5/2} e^{-2|x-y|} dxdy \\ &\leq \mathcal{O}(1)\|w\|^2 e^{-d(\Delta,\Delta')} \end{aligned} \tag{177}$$

which gives the bound we claimed.

To show

$$\lambda^2 |\mathcal{Q}^{(1,1)}_{reg}|_{h,\Gamma_1} \leq C\epsilon^2 h^2 \|w\|^3 \tag{178}$$

we refer to (100) and use

$$\begin{aligned} [\phi(x)-\phi(y)]^2 - [(x-y)\cdot(\partial\phi)(x)]^2 &= \\ [\phi(x)-\phi(y)-(x-y)\cdot(\partial\phi)(x)]\,&[\phi(x)-\phi(y)+(x-y)\cdot(\partial\phi)(x)] \end{aligned} \tag{179}$$

to show that the second functional derivative is

$$2 \text{ Sym}\,[f_1(x)-f_1(y)-(x-y)\cdot(\partial f_1)(x)]\,[f_2(x)-f_2(y)+(x-y)\cdot(\partial f_2)(x)]$$

This leads to a factor $\mathcal{O}(1)(x-y)^3$ so that $\lambda^2|\mathcal{Q}^{(1,1)}_{reg}|_{h,\gamma\Gamma}$ is bounded by $\mathcal{O}(1)\epsilon^2 h^2$ times the Γ norm of

$$\begin{aligned} \int_{\Delta\times\Delta'} |x-y|^3 |w(x-y)|^3 dxdy &\leq \|w\|^3 \int_{\Delta\times\Delta'} |x-y|^{-15/4} e^{-3|x-y|} dxdy \\ &\leq \mathcal{O}(1)\|w\|^3 e^{-d(\Delta,\Delta')} \end{aligned} \tag{180}$$

To show

$$\zeta_1^2 |\mathcal{Q}^{(1',1')}_{reg}|_{h,\Gamma_1} \leq C\epsilon^{2-2\delta} h^2 \|w\|^2 \tag{181}$$

we refer to (101). Here the derivative contains

$$\text{Sym }\,[\partial_a f_1(x) - \partial_a f_1(y)]\,[\partial_b f_2(x) - \partial_b f_2(y)] \tag{182}$$

which leads to a factor $|x-y|^2$. Then we use

$$\begin{aligned} \int_{\Delta\times\Delta'} |x-y|^2 |(\partial\partial w)(x-y)| dxdy &\leq \|w\| \int_{\Delta\times\Delta'} |x-y|^{-9/4} e^{-|x-y|} dxdy \\ &\leq \mathcal{O}(1)\|w\| e^{-d(\Delta,\Delta')} \end{aligned} \tag{183}$$

The other terms are bounded similarly.

□

Lemma 12

$$\begin{aligned} \|F_Q e^{-V^\#}\| &\leq C\epsilon^{1/2-2\delta} \\ |F_Q e^{-V^\#}| &\leq C\epsilon^{2-2\delta} \end{aligned} \tag{184}$$

Proof We have

$$
\begin{aligned}
F_Q &= \mathcal{Q}_{sing}\left(\vec{\lambda}, v^{\#}, w^{\#}, v^{\#}(0)\right) - \mathcal{Q}_{sing}\left(\vec{\lambda}, v^{\#}, w, v(0)\right) \\
&= \mathcal{Q}^{(4,0)}\left(v^{\#}, 36\lambda^2(2wC + C^2)\right) \\
&\quad + \mathcal{Q}^{(2',0)}\left(v^{\#}, 48\lambda^2(3w^2C + 3wC^2 + C^3)T + 2\zeta_1^2 \partial\partial C\right) \\
&\quad + \mathcal{Q}^{(2,0)}\Big(v^{\#}, 48\lambda^2(3v^{\#}(0)(2wC + C^2) + 3\left(C - C(0)\right) w^2 + 3wC^2 + C^3) \\
&\quad + 12\zeta_1\lambda(2\partial C \partial w + 2(\partial C)^2)\Big) \\
&\quad + \mathcal{Q}^{(0,0)}(\ldots) \\
&\quad + \mathcal{O}(\lambda\zeta_2) + \mathcal{O}(\zeta_1\zeta_2)
\end{aligned} \tag{185}
$$

where the argument of $\mathcal{Q}^{(0,0)}$ is

$$
\begin{aligned}
\ldots &= \lambda^2\Big[48w^3(C - C_2) + 72w^2\left(2v_2^{\#}C - 2v(0)^{\#}C(0) + C^2 - C(0)^2\right) \\
&\quad + 48w\left(C^3 + 3v_2^{\#}C^2 + 3v^{\#}(0)^2C\right) + 12\left(C^4 + 4v_2^{\#}C^3 + 6v^{\#}(0)^2C^2\right)\Big] \\
&\quad + \zeta_1^2\Big[2(\partial\partial v^{\#})(0)(\partial\partial C) + 2\left((\partial\partial C) - (\partial\partial C)(0)\right)\partial\partial w + (\partial\partial C)^2\Big]
\end{aligned}
$$

Here we have introduced the notation

$$
v_2(x) = v(0) - (\partial\partial v)(0)T(x) = v(0) + \frac{1}{2}(\partial_a\partial_b v)(0)x_a x_b \tag{186}
$$

Since reflection symmetry implies $(\partial v)(0) = 0$ this is the expansion of v around $x = 0$ to second order. Now every term above has a kernel which is integrable. Note particularly that w^3 which has a singularity $\mathcal{O}(|x|^{-27/4})$ is multiplied by $C - C_2$ which is $\mathcal{O}(|x|^3)$ for an overall bound $\mathcal{O}(|x|^{-15/4})$. Also $|\partial\partial w| = \mathcal{O}(|x|^{-17/4})$ is multiplied by $(\partial\partial C) - (\partial\partial C)(0)$ which is $\mathcal{O}(|x|)$ for an overall bound $\mathcal{O}(|x|^{-13/4})$. Finally all the $w^2 = \mathcal{O}(|x|^{-9/2})$ are multiplied by functions which are $\mathcal{O}(|x|)$ for a bound $\mathcal{O}(|x|^{-7/2})$. The terms $\mathcal{O}(\lambda\zeta_2)$ and $\mathcal{O}(\zeta_1\zeta_2)$, which we have not written out, are treated similarly.

Because the kernels are locally integrable and exponentially decreasing, bounds of the type we have been discussing lead to the claimed estimate. We omit the details.

Finally, the bounds on α_Q are straightforward.

□

4.4 Estimates on r, s, t, U

We introduce the norms

$$
\begin{aligned}
\|\cdot\|_{\#} &= \|\cdot\|_{G^{\#},h^{\#},\Gamma^{\#}} \\
|\cdot|_{\#} &= |\cdot|_{1/2,\Gamma^{\#}}
\end{aligned} \tag{187}
$$

where $G^{\#} = G^{\#}(\kappa, X, \phi)$ equals $G(2\kappa, X, \phi)2^{-|X|}$ where G_t was defined in (47), $\Gamma^{\#} = \Gamma_{-1}$, and $h^{\#} = h/2$.

Note that our bound $\|R\| < \epsilon^{3-\Delta}$ implies that

$$\begin{aligned} \|R\| &\leq \epsilon^{1-\Delta} \\ |R| &\leq \epsilon^{3-\Delta} \end{aligned} \tag{188}$$

Lemma 13 $R^{\#} = R^{\#}(\vec{\lambda}, R, w)$ *is analytic on* $\mathcal{D}$ *and satisfies there*

$$\begin{aligned} \|R^{\#}\|_{\#} &\leq \mathcal{O}(1)\epsilon^{1-\Delta} \\ |R^{\#}|_{\#} &\leq \mathcal{O}(1)\epsilon^{3-\Delta} \end{aligned} \tag{189}$$

Remark Together with a weaker bound on $Q^{\#}e^{-V^{\#}}$ from lemma 11 we obtain

$$\begin{aligned} \|K^{\#}\|_{\#} &\leq C\epsilon^{1/2-2\delta} \\ |K^{\#}|_{\#} &\leq C\epsilon^{2-2\delta} \end{aligned} \tag{190}$$

Proof Refer to (108,109) and theorem 3. With $B(t) = (\Box + Q(t))\, e^{-V(t)}$ and $E(t) = E\left(B(t)\right)$ we obtain

$$\begin{aligned} \|R^{\#}\|_{\#} &\leq 2(\|R\| + \sup_t \|E(t)\|_{G_t,h,\Gamma}) \\ |R^{\#}|_{\#} &\leq \mathcal{O}(1)\left(|R| + \sup_t |E(t)| + \mathcal{O}(\epsilon^3)\right) \end{aligned} \tag{191}$$

taking into account that $G(1)(X) = 2^{|X|}G^{\#}(X)$, transfering the factor $2^{|X|}$ to the Γ to get $\Gamma_{-1} = \Gamma^{\#}$ and choosing $M = 12$.

Taking into account (188) we see that it suffices to show that

$$\begin{aligned} \|E(t)\|_{G_t,h,\Gamma} &\leq C\epsilon^{1-3\delta} \\ |E(t)|_{1/2,\Gamma} &\leq C\epsilon^{3-3\delta} \end{aligned} \tag{192}$$

Working from the definitions (78) for $E(t)$ and (84) for J, one finds

$$E(t) = \left(\frac{\partial}{\partial t} - \Delta_C + J\right)\Box\, e^{-V(t)} \tag{193}$$

$$+\; Q(t)\cdot\left(\frac{\partial}{\partial t} - \Delta_C\right) e^{-V(t)} \tag{194}$$

$$+\; C\left(\frac{\partial Q(t)}{\partial\phi}, \frac{\partial V(t)}{\partial\phi}\right) e^{-V(t)} \tag{195}$$

$$+\; \left[\left(\frac{\partial}{\partial t} - \Delta_C\right) Q(t) - J\right] e^{-V(t)} \tag{196}$$

$$+\; \left[-\frac{1}{2}\mathcal{B}_C\left(\Box e^{-V(t)}, \Box e^{-V(t)}\right) + J(1-\Box)e^{-V(t)}\right] \tag{197}$$

$$-\; \mathcal{B}_C\left(\Box e^{-V(t)}, Q(t)e^{-V(t)}\right) - \frac{1}{2}\mathcal{B}_C\left(Q(t)e^{-V(t)}, Q(t)e^{-V(t)}\right) \tag{198}$$

We estimate each of these terms using lemma 1 and theorem 1. Thus it suffices to bound the monomial parts of these terms in the norms $|\cdot|_{h,\Gamma}$ with either $h=\epsilon^{-1/4}$ or $h=1/2$. We use that for a monomial of degree p:

$$|\cdot|_{h,\Gamma} \leq Ch^p \times |\text{coupling constants}| \tag{199}$$

This is based on the fact that all kernels are integrable and contribute a constant C and that each field ϕ in the polynomial contributes a factor h. The largest monomial in Q has degree 6 and is $\mathcal{O}(\lambda^2)=\mathcal{O}(\epsilon^2)$. The largest monomial in V has degree 4 and is $\mathcal{O}(\lambda)=\mathcal{O}(\epsilon)$. The largest coupling constant is $\zeta_1=\mathcal{O}(\epsilon^{1-\delta})$.

Terms (193),(197) are both zero by the definition of J. The definition of Q is designed so that terms contained in (196) have coefficients $\lambda\mu,\zeta_1\mu,\zeta_2\mu,\mu^2,\zeta_2^2$ and so this term is bounded by $C\epsilon^{1-3\delta},C\epsilon^{3-3\delta}$ for the choices $h=\epsilon^{-1/4}$ and $h=1/2$ respectively. (Terms $\lambda\zeta_2,\zeta_1\zeta_2$ would not quite have been small enough).

Term (194) is equal to

$$Q(t)C\left(\frac{\partial V(t)}{\partial\phi},\frac{\partial V(t)}{\partial\phi}\right)e^{-V(t)} \tag{200}$$

We find that this term is $C\epsilon,C\epsilon^{4-4\delta}$. Term (195) has the estimate $C\epsilon,C\epsilon^{3-3\delta}$. Term (198) is $C\epsilon,C\epsilon^{3-3\delta}$.

□

Next we introduce norms

$$\begin{aligned} \|\cdot\|_* &= \|\cdot\|_{G^*,h^*,\Gamma^*} \\ |\cdot|_* &= |\cdot|_{1/2,\Gamma^*} \end{aligned} \tag{201}$$

where $G^*=G^\#,\Gamma^*=(\Gamma^\#)_{-4}=\Gamma_{-5},h^*=h^\#$. Recall how dimension was defined in definition 1.

Lemma 14 *R^* is analytic on $\mathcal{D}$ and can be written in the form $R^*=R_1^*+R_2^*$ where* $\dim(R_1^*(X))>4$ *for X small and*

$$\begin{aligned} \|R_1^*\|_* &\leq \mathcal{O}(1)\epsilon^{1-\Delta} \\ |R_1^*|_* &\leq \mathcal{O}(1)\epsilon^{3-\Delta} \\ \|R_2^*\|_* &\leq C\epsilon^{1-4\delta} \\ |R_2^*|_* &\leq C\epsilon^{3-4\delta} \end{aligned} \tag{202}$$

Remark Combining this with a bound on $Q^*e^{-V^*}$ gives:

$$\begin{aligned} \|K^*\|_* &\leq C\epsilon^{1/2-2\delta} \\ |K^*|_* &\leq C\epsilon^{2-2\delta} \end{aligned} \tag{203}$$

Proof From section 3.3

$$\begin{aligned} R_1^* &= R^\# - F_R e^{-V^*} \\ R_2^* &= \mathcal{E}_{\geq 2}(K^\#, F) \\ &+ Q^\#(e^{-V^\#} - e^{-V^*}) + (Q^{**} - Q^*)e^{-V^*} \end{aligned} \tag{204}$$

Then R_1^* has dimension greater than 4 by construction.

Our bound $|R^\#|_\# \leq \mathcal{O}(1)\epsilon^{3-\Delta}$ implies that $|F_R|_\# \leq \mathcal{O}(1)\epsilon^{3-\Delta}$, hence that

$$|(\alpha_R)_{n,m}(X)| \leq \mathcal{O}(1)\epsilon^{3-\Delta}\Gamma_\#(X)^{-1} \tag{205}$$

for $(n, m) = (0, 0), (2, 0), ...$ and hence that

$$\begin{aligned} \|F_R e^{-V^*}\|_* &\leq \mathcal{O}(1)\epsilon^{2-\Delta} \\ |F_R e^{-V^*}|_* &\leq \mathcal{O}(1)\epsilon^{3-\Delta} \end{aligned} \tag{206}$$

(Each field reduces the $\|\cdot\|_*$ norm by $\epsilon^{-1/4}$ and the maximum number of fields in F_R is 4.) These are sufficient to give the bounds on R_1^*.

By theorem 4

$$\begin{aligned} \|\mathcal{E}_{\geq 2}(K^\#, F)\|_* &\leq \mathcal{O}(1)\|f\|_{\Gamma^\#}\|K^\#\|_\# \\ |\mathcal{E}_{\geq 2}(K^\#, F)|_* &\leq \mathcal{O}(1)\|f\|_{\Gamma^\#}|K^\#|_\# \end{aligned} \tag{207}$$

where we take for some constant C_f

$$f(X) = \begin{cases} C_f\epsilon^{1-\delta} & X \text{ small} \\ \theta(X)^{-2}C_f\epsilon^{1-\delta} & X \text{ large}, |X| = 2 \\ 0 & X \text{ large}, |X| \geq 3 \end{cases} \tag{208}$$

To see that this f satisfies the stability hypothesis (57) of the theorem: The $:\phi^4:$ contribution to F_Q has the form $\int_\Delta \alpha(X, \Delta, x){:}\,\phi^4(x){:}\,dx$ where $|X| \leq 2$. If $X = \Delta \cup \Delta'$ then from lemma 12 $\sup_x |\alpha(X, \Delta, x)| \leq C\epsilon^{2-\delta}e^{-d(\Delta,\Delta')}$. Then we have that

$$\begin{aligned} \sum_{|X|\leq 2, X\supset\Delta} \sup_x |\alpha(X, \Delta, x)|2/f(X) &\leq C\epsilon/C_f \sum_{\Delta'} \theta(\Delta, \Delta')^2 e^{-d(\Delta,\Delta')} \\ &\leq C\epsilon/C_f \end{aligned} \tag{209}$$

By taking C_f sufficiently large this is bounded by a small multiple of ϵ so that stability holds by theorem 1: the $\lambda{:}\,\phi^4{:}$ part of V remains dominant. The same arguments show that the other terms in F_Q are also compatible with stability.

The contributions to F_R have the form $\alpha(X) \int_X p(\phi(x))\, dx$ where α is supported on small sets and $|\alpha(X)| \leq \mathcal{O}(\epsilon^{3-\Delta})$. Then we have summing over X small only

$$\sum_{X\supset\Delta} |\alpha(X)|2/f(X) \leq C\epsilon^{2-\Delta+\delta}/C_f \tag{210}$$

Again the sum of these is less than a small multiple of ϵ.

With this choice of f we have

$$\|f\|_{\Gamma^{\#}} \leq C\epsilon^{1-\delta} \tag{211}$$

and combined with the bound (190) on $K^{\#}$ this gives

$$\begin{aligned} \|\mathcal{E}_{\geq 2}(K^{\#}, F)\|_* &\leq C\epsilon^{1-4\delta} \\ |\mathcal{E}_{\geq 2}(K^{\#}, F)|_* &\leq C\epsilon^{3-4\delta} \end{aligned} \tag{212}$$

Using

$$e^{-V^{\#}} - e^{-V^*} = -\int_0^1 e^{-tV^{\#}-(1-t)V^*}(V^{\#} - V^*)dt \tag{213}$$

one also shows

$$\begin{aligned} \|Q^*(e^{-V^{\#}} - e^{-V^*})\|_* + \|(Q^{**} - Q^*)e^{-V^*}\|_* &\leq C\epsilon^{1-4\delta} \\ |Q^*(e^{-V^{\#}} - e^{-V^*})|_* + |(Q^{**} - Q^*)e^{-V^*}|_* &\leq C\epsilon^{3-4\delta} \end{aligned} \tag{214}$$

to complete the proof.

$\square$

Lemma 15 *R' is analytic on $\mathcal{D}$ and satisfies there*

$$\begin{aligned} \|R'\| &\leq \mathcal{O}(L^{-1+2\epsilon})\epsilon^{1-\Delta} \\ |R'| &\leq \mathcal{O}(L^{-1+2\epsilon})\epsilon^{3-\Delta} \end{aligned} \tag{215}$$

Proof From section 3.4:

$$R' = \mathcal{S}_1(R_1^*) + \mathcal{S}_1(R_2^*) + \mathcal{S}_{\geq 2}(K^*) \tag{216}$$

By theorem 5 a crude bound is

$$\begin{aligned} \|\mathcal{S}(K^*)\| &\leq \mathcal{O}(1)L^4\|K^*\|_{G_L, ah_L, \Gamma_{-5}} \\ &\leq \mathcal{O}(1)L^4\|K^*\|_* \\ &\leq C\epsilon^{1/2-2\delta} \\ |\mathcal{S}(K^*)| &\leq C\epsilon^{2-2\delta} \end{aligned} \tag{217}$$

Here we have used $G_L^{-1} \leq (G^*)^{-1}$ and $ah_L \leq h^*$.

For $\mathcal{S}_1(R_1^*)$ we use theorem 6, break the estimate into large and small sets and find:

$$\begin{aligned} \|\mathcal{S}_1(R_1^*)\| &\leq \mathcal{O}(L^{-1+2\epsilon})\|R_1^*\|_* \\ &\leq \mathcal{O}(L^{-1+2\epsilon})\epsilon^{1-\Delta} \\ |\mathcal{S}_1(R_1^*)| &\leq \mathcal{O}(L^{-1+2\epsilon})\epsilon^{3-\Delta} \end{aligned} \tag{218}$$

Here we have used $\dim(R^*) \geq 5 - 2\epsilon$. Indeed from equation (125) and the definition of dimension we find that $\dim(R_2^*) = 2\dim(\phi) + 3 = 5 - \epsilon$, $\dim(R_4^*) = 4\dim(\phi) + 1 = 5 - 2\epsilon$ and $\dim(R_6^*) = 6\dim(\phi) = 6 - 3\epsilon$.

We also have

$$\begin{aligned} \|\mathcal{S}_1(R_2^*)\| &\leq C\|R_2^*\|_* \leq C\epsilon^{1-4\delta} \\ |\mathcal{S}_1(R_2^*)| &\leq C|R_2^*|_* \leq C\epsilon^{3-4\delta} \end{aligned} \tag{219}$$

which is sufficient because $\Delta > 4\delta$ and ϵ is chosen after L. Finally by Cauchy bounds and (217) we find

$$\begin{aligned} \|\mathcal{S}_{\geq 2}(K^*)\| &\leq C\epsilon^{1-4\delta} \\ |\mathcal{S}_{\geq 2}(K^*)| &\leq C\epsilon^{4-4\delta} \end{aligned} \tag{220}$$

to complete the proof.

□

Proof of Theorem 7 That $r(\vec{\lambda}, R, w), s(\vec{\lambda}, R, w), t(\vec{\lambda}, R, w)$ are analytic in $\mathcal{D}$ and satisfy there

$$|r|, |s|, |t| \leq \mathcal{O}(1)\epsilon^{3-\Delta} \tag{221}$$

follows from the same properties for the $\alpha_{n,m}(X)$ in (122). See lemma 7 and note that the bounds follow from (167). The bound on $R' = U(\vec{\lambda}, R, w)$ is contained in the previous lemma.

□

5 Critical Models

5.1 The problem

We now restrict to infinite volume so that the RG flow is the iteration of a fixed mapping. In this section we prove the existence of critical theories by showing that one can pick $\mu_0 = \mu_0(\lambda_0), \zeta_0 = \zeta_0(\lambda_0)$ so that the RG flow tends to a fixed point. This result is part of the stable manifold theorem, but we have not been able to find a version in the literature which is exactly applicable to our problem, so we have to give an independent proof, at least in part.

So far we have proved that the renormalization group transformation can be iterated as long as the parameters $(\vec{\lambda}_j, R_j, w_j)$ that define the theory remain in the domain $\mathcal{D}$ and the flow after j iterations is given by the equations

$$\begin{aligned} \lambda_{j+1} &= L^{2\epsilon}[\lambda_j - a(w_j)\lambda_j^2 + r(\vec{\lambda}_j, R_j, w_j)] \\ \zeta_{j+1} &= L^{\epsilon}[\zeta_j - b(w_j)\lambda_j^2 + s(\vec{\lambda}_j, R_j, w_j)] \\ \mu_{j+1} &= L^{2+\epsilon}[\mu_j - c(w_j)\lambda_j^2 - d(w_j)\lambda_j(\zeta_{1,j} + \zeta_{2,j}) + t(\vec{\lambda}_j, R_j, w_j)] \end{aligned}$$

$$
\begin{aligned}
R_{j+1} &= U(\vec{\lambda}_j, R_j, w_j) \\
w_{j+1} &= (w_j)_L + C_L
\end{aligned}
\tag{222}
$$

The starting point is $(\vec{\lambda}_0, 0, 0)$ with $\vec{\lambda}_0 = (\lambda_0, \zeta_0, \mu_0)$, but we consider the more general case $(\vec{\lambda}_0, R_0, w_0)$.

If we ignore the higher order term r and replace w_j by its limiting value w_* we find the approximate fixed point for λ

$$
\bar{\lambda} = \frac{L^{2\epsilon} - 1}{L^{2\epsilon} a(w_*)}
\tag{223}
$$

In Section 4.2 we required that $\bar{\lambda} = \mathcal{O}(\epsilon)$. This is valid since for ϵ small we have $L^{2\epsilon} \sim 1 + 2\epsilon \log L + \dots$ and we have seen that $a(w_*) \sim \mathcal{O}(\log L)$. (We continue to assume that L is chosen large followed by ϵ small, that $\mathcal{O}(1), \mathcal{O}(\epsilon)$, etc. are L independent constants, and that C is a possibly L dependent constant which may change from line to line.)

We now rewrite the equations replacing λ_j by the deviation $\tilde{\lambda}_j = \lambda_j - \bar{\lambda}$. With

$$
\xi_j = (\tilde{\lambda}_j, \zeta_j, \mu_j, R_j, w_j)
\tag{224}
$$

we have

$$
\begin{aligned}
\tilde{\lambda}_{j+1} &= (2 - L^{2\epsilon})\tilde{\lambda}_j + \tilde{r}(\xi_j) \\
\zeta_{j+1} &= L^{\epsilon}\zeta_j + \tilde{s}(\xi_j) \\
\mu_{j+1} &= L^{2+\epsilon}\mu_j + \tilde{t}(\xi_j) \\
R_{j+1} &= U(\xi_j) \\
w_{j+1} &= (w_j)_L + C_L
\end{aligned}
\tag{225}
$$

where we have defined

$$
\begin{aligned}
\tilde{r}(\xi) &= L^{2\epsilon}[-a(w_*)\tilde{\lambda}^2 + (a(w_*) - a(w))\,\lambda^2 + r(\xi)] \\
\tilde{s}_i(\xi) &= L^{\epsilon}[-b_i(w)\lambda^2 + s_i(\xi)] \\
\tilde{t}(\xi) &= L^{2+\epsilon}[-c(w)\lambda^2 - d(w)\lambda(\zeta_1 + \zeta_2) + t(\xi)]
\end{aligned}
\tag{226}
$$

In this equation $r(\xi) = r(\bar{\lambda} + \tilde{\lambda}, \zeta, \mu, R, w)$, etc. Now $\xi_0 = (\tilde{\lambda}_0, \zeta_0, \mu_0, R_0, w_0)$ is the initial point. Note that the shift has changed an expanding variable λ into a contracting variable $\tilde{\lambda}$. The equations (225) are the iteration of a single mapping which we call f so that

$$
\xi_{j+1} = f(\xi_j)
\tag{227}
$$

As a norm on ξ we take

$$
\|\xi\| = \sup(\epsilon^{-\alpha}|\tilde{\lambda}|, \epsilon^{-1+\delta}|\zeta_1|, \epsilon^{-2+\Delta+\delta}|\zeta_2|, \epsilon^{-2+2\delta}|\mu|, \epsilon^{-3+\Delta}\|\!|R|\!\|, k^{-1}\|w\|)
\tag{228}
$$

where $\Delta > 4\delta > 0$ and $1 < \alpha$ and k appeared in (154). This norm defines the Banach space E. The unit ball $E(1)$ is the image of the domain $\mathcal{D}$ under the shift $\tilde{\lambda} = \lambda - \bar{\lambda}$.

In section 5.3 to come, we will need to replace the factor $\epsilon^{-\alpha}|\tilde{\lambda}|$ by $(d\epsilon)^{-1}|\tilde{\lambda}|$ with d sufficiently small. The parameter d will be chosen independently of L, ϵ. This will define a Banach space F.

We collect the bounds that we will need. Assuming $\xi \in E(1)$ we have from the previous section:

$$\begin{aligned} |r(\xi)|, |s_i(\xi)|, |t(\xi)| &\leq \mathcal{O}(1)\epsilon^{3-\Delta} \\ \|U(\xi)\| &\leq \mathcal{O}(L^{-1+2\epsilon})\epsilon^{3-\Delta} \end{aligned} \tag{229}$$

It follows that (recall $b_2 = 0$)

$$\begin{aligned} |\tilde{r}(\xi)| &\leq \mathcal{O}(\log L)\epsilon^{2\alpha} + \mathcal{O}(1)\|w - w_*\|\epsilon^2 + \mathcal{O}(1)\epsilon^{3-\Delta} \\ |\tilde{s}_1(\xi)| &\leq C\epsilon^2 \;\; ; \;\; |\tilde{s}_2(\xi)| \leq \mathcal{O}(1)\epsilon^{3-\Delta} \\ |\tilde{t}(\xi)| &\leq C\epsilon^{2-\delta} \end{aligned} \tag{230}$$

Recall that r is analytic in $E(1)$. Therefore we can use the Cauchy representation for $r\left(\xi' + z(\xi - \xi')\right)$ as a function of the complex variable z with $|z| < \mathcal{O}(\|\xi - \xi'\|^{-1})$ to obtain Lipschitz bounds: for $\xi, \xi' \in E(1/2)$:

$$\begin{aligned} &|r(\xi) - r(\xi')|, |s_i(\xi) - s_i(\xi')|, |t(\xi) - t(\xi')| \leq \mathcal{O}(1)\epsilon^{3-\Delta}\|\xi - \xi'\| \\ &\|U(\xi) - U(\xi')\| \leq \mathcal{O}(L^{-1+2\epsilon})\epsilon^{3-\Delta}\|\xi - \xi'\| \end{aligned} \tag{231}$$

By Lemma 9 this implies the bounds:

$$\begin{aligned} |\tilde{r}(\xi) - \tilde{r}(\xi')| &\leq \left(\mathcal{O}(\log L)\epsilon^{\alpha} + \mathcal{O}(1)\|w - w_*\|\epsilon\right)|\tilde{\lambda} - \tilde{\lambda}'| \\ &\quad + \mathcal{O}(1)\|w - w'\|\epsilon^2 + \mathcal{O}(1)\epsilon^{3-\Delta}\|\xi - \xi'\| \end{aligned} \tag{232}$$

and the same bound with ϵ^{α} replaced by $d\epsilon$. We also have

$$\begin{aligned} |\tilde{s}_1(\xi) - \tilde{s}_1(\xi')| &\leq C\left(|\tilde{\lambda} - \tilde{\lambda}'|\epsilon + \|w - w'\|\epsilon^2 + \epsilon^{3-\Delta}\|\xi - \xi'\|\right) \\ |\tilde{s}_2(\xi) - \tilde{s}_2(\xi')| &\leq \mathcal{O}(1)\epsilon^{3-\Delta}\|\xi - \xi'\| \\ |\tilde{t}(\xi) - \tilde{t}(\xi')| &\leq C\Bigg(|\tilde{\lambda} - \tilde{\lambda}'|\epsilon^{1-\delta} + |\zeta - \zeta'|\epsilon \\ &\qquad + \|w - w'\|\epsilon^{2-\delta} + \epsilon^{3-\Delta}\|\xi - \xi'\|\Bigg) \end{aligned} \tag{233}$$

5.2 Existence of global solutions

We first want to prove the existence of bounded global solutions, that is sequences $\xi_j \in E(1/2)$ defined for all $j = 0, 1, \ldots$ and satisfying $\xi_{j+1} = f(\xi_j)$ for $j = 0, 1, \ldots$. We start by looking for finite sequences ξ_j with $0 \le j \le k$ with specified initial conditions λ_0, R_0 and specified final conditions $\zeta_k = \zeta_f, \mu_k = \mu_f$. It is straightforward to check that a sequence ξ_j is a solution if and only if it satisfies the "integral" equations:

$$\begin{aligned}
\tilde{\lambda}_j &= (2-L^{2\epsilon})^j(\tilde{\lambda}_0) + \sum_{l=0}^{j-1}(2-L^{2\epsilon})^{j-l-1}\tilde{r}(\xi_l); \quad j = 1,\ldots,k \\
\zeta_j &= L^{-\epsilon(k-j)}\zeta_f - \sum_{l=j}^{k-1} L^{-\epsilon(l+1-j)}\tilde{s}(\xi_l); \quad j = 0,\ldots,k-1 \\
\mu_j &= L^{-(2+\epsilon)(k-j)}\mu_f - \sum_{l=j}^{k-1} L^{-(2+\epsilon)(l+1-j)}\tilde{t}(\xi_l); \quad j = 0,\ldots k-1 \\
R_j &= U(\xi_{j-1}); \quad j = 1,\ldots,k
\end{aligned} \tag{234}$$

The imposition of a mixture of initial and final conditions has given equivalent "integral" equations in which only exponentially decaying factors appear. This is a standard device in the theory of hyperbolic dynamical systems.

To obtain a global solution it is natural to consider the formal limit as $k \to \infty$:

$$\begin{aligned}
\tilde{\lambda}_j &= (2-L^{2\epsilon})^j(\tilde{\lambda}_0) + \sum_{l=0}^{j-1}(2-L^{2\epsilon})^{j-l-1}\tilde{r}(\xi_l); \quad j \ge 1 \\
\zeta_j &= -\sum_{l=j}^{\infty} L^{-\epsilon(l+1-j)}\tilde{s}(\xi_l); \quad j \ge 0 \\
\mu_j &= -\sum_{l=j}^{\infty} L^{-(2+\epsilon)(l+1-j)}\tilde{t}(\xi_l); \quad j \ge 0 \\
R_j &= U(\xi_{j-1}); \quad j \ge 1
\end{aligned} \tag{235}$$

When $\xi_j \in E(1/2)$ these sums converge (see below) and one can check that any solution of these equations is a bounded solution of $\xi_{j+1} = f(\xi_j)$. Thus we focus on solving this equation. Notice that the equations are now independent of ζ_f, μ_f.

Let

$$\underline{\xi} = (\xi_0, \xi_1, \xi_2, ...)$$

Then the above equation can be written in the form

$$\underline{\xi} = F(\underline{\xi})$$

where $F(\underline{\xi}) = (F_0(\underline{\xi}), F_1(\underline{\xi}), \ldots)$ and

$$F_j(\underline{\xi}) = \left(F_j^{\lambda}(\underline{\xi}), F_j^{\zeta_1}(\underline{\xi}), F_j^{\zeta_2}(\underline{\xi}), F_j^{\mu}(\underline{\xi}), F_j^{R}(\underline{\xi})\right) \tag{236}$$

is the right side of the flow equation augmented by the initial data $F_0^{\lambda}(\underline{\xi}) = \lambda_0,\ F_0^{R}(\underline{\xi}) = R_0$. We regard this as a fixed point equation in the Banach space $\underline{E}$ of all sequences

$\underline{\xi}$ with norm $\|\underline{\xi}\| = \sup_j \|\xi_j\|$. The ball $\underline{E}(1)$ is all sequences with entries in $E(1)$. Since the w_j component of ξ_j must be the solution of $w_{j+1} = (w_j)_L + C_L$ starting with w_0, we insert this solution w_j into $\underline{\xi}$ and only consider the first four components $\xi_j = (\tilde{\lambda}_j, \zeta_j, \mu_j, R_j, w_j)$ to be unknowns in the fixed point equation.

Theorem 8 *Let $1 < \alpha < 2 - \Delta$ and let λ_0, R_0, w_0 be given such that $(\tilde{\lambda}_0, 0, 0, R_0, w_0) \in E(1/4)$. Let the w_j components of $\underline{\xi}$ be fixed as described above. Then*

1. *If $\underline{\xi} \in \underline{E}(1/2)$ then $F(\xi) \in \underline{E}(1/2)$.*
2. *For $\underline{\xi}, \underline{\xi}'$ in $\underline{E}(1/2)$ we have $\|F(\underline{\xi}) - F(\underline{\xi}')\| \leq \|\underline{\xi} - \underline{\xi}'\|/2$*
3. *For any $\underline{\xi}^{(0)}$ in $\underline{E}(1/2)$ the iterates $\underline{\xi}^{(1)} = F(\underline{\xi}^{(0)}), \underline{\xi}^{(2)} = F(\underline{\xi}^{(1)})$, etc. converge to a limit $\underline{\xi} = \lim_{n\to\infty} \underline{\xi}^{(n)}$ in $\underline{E}(1/2)$ which satisfies $\underline{\xi} = F(\underline{\xi})$ and hence $\xi_{j+1} = f(\xi_j)$.*

Proof (1.) Note that because we have fixed w_j, by (154) the bound (230) can be written

$$|\tilde{r}(\xi_j)| \leq \mathcal{O}(\log L)\epsilon^{2\alpha} + \mathcal{O}(L^{-j/4})\epsilon^2 + \mathcal{O}(1)\epsilon^{3-\Delta} \tag{237}$$

Then we have for $j \geq 1$:

$$\begin{aligned} |F_j^\lambda| &\leq (2 - L^{2\epsilon})^j |\tilde{\lambda}_0| + \sum_{l=0}^{j-1} (2 - L^{2\epsilon})^{j-l-1} |\tilde{r}(\xi_l)| \\ &\leq (2 - L^{2\epsilon})^j (\epsilon^\alpha/4) + \sum_{l=0}^{j-1} (2 - L^{2\epsilon})^{j-l-1} \mathcal{O}(1)(\epsilon^{2\alpha} + L^{-l/4}\epsilon^2 + \epsilon^{3-\Delta}) \\ &\leq \epsilon^\alpha/4 + \mathcal{O}(1)(\epsilon^{2\alpha-1} + \epsilon^2 + \epsilon^{2-\Delta}) \\ &\leq \epsilon^\alpha/2. \end{aligned} \tag{238}$$

Next we have

$$\begin{aligned} |F_j^{\zeta_1}| &\leq \sum_{l=j}^{\infty} L^{-\epsilon(l+1-j)} |\tilde{s}_1(\xi_l)| \\ &\leq \sum_{l=j}^{\infty} L^{-\epsilon(l+1-j)} C\epsilon^2 \\ &\leq C\epsilon \\ &\leq \epsilon^{1-\delta}/2 \end{aligned} \tag{239}$$

and similarly

$$|F_j^{\zeta_2}| \leq \epsilon^{2-\Delta-\delta}/2 \tag{240}$$

Next we have

$$\begin{aligned} |F_j^\mu| &\leq \sum_{l=j}^\infty L^{-(2+\epsilon)(l+1-j)} |\tilde{t}(\xi_l)| \\ &\leq \sum_{l=j}^\infty L^{-(2+\epsilon)(l+1-j)} C\epsilon^{2-\delta} \\ &\leq C\epsilon^{2-\delta} \\ &\leq \epsilon^{2-2\delta}/2 \end{aligned} \tag{241}$$

Finally we have

$$\|F_j^R\| = \|U(\xi_{j-1})\| \leq \mathcal{O}(L^{-1+2\epsilon})\epsilon^{3-\Delta} \leq \frac{1}{2}\epsilon^{3-\Delta} \tag{242}$$

(2.) We denote $\delta\underline{\xi} = \underline{\xi} - \underline{\xi}'$ and $\delta F = F(\underline{\xi}) - F(\underline{\xi}')$, etc. Since w_j is fixed there is no variation in w and the estimates (232), (233) become

$$\begin{aligned} |\delta\tilde{r}(\xi_j)| &\leq (\mathcal{O}(\log L)\epsilon^\alpha + \mathcal{O}(L^{-j/4})\epsilon)|\delta\tilde{\lambda}_j| + \mathcal{O}(1)\epsilon^{3-\Delta}\|\delta\xi_j\| \\ |\delta\tilde{s}_1(\xi_j)| &\leq C(\epsilon|\delta\tilde{\lambda}_j| + \epsilon^{3-\Delta}\|\delta\xi_j\|) \\ &\leq C\epsilon^{1+\alpha}\|\delta\xi_j\| \\ |\delta\tilde{t}(\xi_j)| &\leq C(|\delta\tilde{\lambda}|\epsilon^{1-\delta} + |\delta\zeta|\epsilon + \epsilon^{3-\Delta}\|\delta\xi_j\|) \\ &\leq C\epsilon^{2-\delta}\|\delta\xi_j\| \end{aligned} \tag{243}$$

Then we have

$$\begin{aligned} \epsilon^{-\alpha}|\delta F_j^\lambda| &\leq \sum_{l=0}^{j-1}(2-L^{2\epsilon})^{j-l-1}\epsilon^{-\alpha}|\delta\tilde{r}(\xi_l)| \\ &\leq \sum_{l=0}^{j-1}(2-L^{2\epsilon})^{j-l-1}(\mathcal{O}(\log L)\epsilon^\alpha + \mathcal{O}(L^{-j/4})\epsilon + \mathcal{O}(1)\epsilon^{3-\Delta-\alpha})\|\delta\xi_l\| \\ &\leq (\mathcal{O}(\log L)\epsilon^{\alpha-1} + \mathcal{O}(1)\epsilon + \mathcal{O}(1)\epsilon^{2-\Delta-\alpha})\|\delta\underline{\xi}\| \\ &\leq \|\delta\underline{\xi}\|/2 \end{aligned} \tag{244}$$

Next we have

$$\begin{aligned} \epsilon^{-1+\delta}|\delta F_j^{\zeta_1}| &\leq \sum_{l=j}^\infty L^{-\epsilon(l+1-j)}\epsilon^{-1+\delta}|\delta\tilde{s}_1(\xi_l)| \\ &\leq \sum_{l=j}^\infty L^{-\epsilon(l+1-j)}C\epsilon^{\alpha+\delta}\|\delta\xi_l\| \\ &\leq C\epsilon^{\alpha-1+\delta}\|\delta\underline{\xi}\| \\ &\leq \|\delta\underline{\xi}\|/2. \end{aligned} \tag{245}$$

and similarly

$$\epsilon^{-2+\Delta+\delta}|\delta F_j^{\zeta_2}| \leq \|\delta\underline{\xi}\|/2 \tag{246}$$

Next we have

$$
\begin{aligned}
\epsilon^{-2+2\delta}|\delta F_j^{\mu}| &\leq \sum_{l=j}^{\infty} L^{-(2+\epsilon)(l+1-j)} \epsilon^{-2+2\delta} |\delta \tilde{t}(\xi_l)| \\
&\leq \sum_{l=j}^{\infty} L^{-(2+\epsilon)(l+1-j)} C\epsilon^{\delta} \|\delta \underline{\xi}\| \\
&\leq \|\delta\underline{\xi}\|/2 \qquad (247)
\end{aligned}
$$

Finally by (231) we have

$$
\epsilon^{-3+\Delta}\|\delta F_j^R\| = \epsilon^{-3+\Delta}\|\delta U(\xi_{j-1})\| \leq \mathcal{O}(L^{-1+2\epsilon})\|\delta\underline{\xi}\| \leq \frac{1}{2}\|\delta\underline{\xi}\| \qquad (248)
$$

Combining all these gives

$$
\|\delta F\| \leq 1/2\|\delta\xi\|
$$

(3.) This is standard.

□

Remark Thus we have shown that for each (λ_0, R_0, w_0) in $E(1/4)$ there exists (ζ_0, μ_0) (given by (235)) such that the trajectory ξ_j starting at that $\xi_0 = (\lambda_0, \zeta_0, \mu_0, R_0, w_0)$ stays in $E(1/2)$ for all j. In the next section we will see that (μ_0, ζ_0) is unique.

5.3 The stable manifold and the fixed point

The existence results of the previous section refer to the Banach space E for which the norm $\|\xi\|$ is given by (228) with weight factor $\epsilon^{-\alpha}|\tilde{\lambda}|$ for fixed $1 < \alpha < 2 - \Delta$. For uniqueness we consider the Banach space F with norm $\|\xi\|$ still given by (228) but now with weight factor $(d\epsilon)^{-1}|\tilde{\lambda}|$. Since $\epsilon^{\alpha} < d\epsilon$ the norm in F is dominated by the norm in E and so $E(\frac{1}{2}) \subset F(\frac{1}{2})$, etc.

We now return to regarding w as a dynamical variable so that the problem is again to study the iterations of the mapping f.

The Banach space F can be regarded as a product $F_1 \times F_2$ where F_1 is all triples $(\tilde{\lambda}, R, w)$ and F_2 is all pairs (ζ, μ). Then F_1 corresponds to contracting data and F_2 corresponds to expanding data. Let p_i be the projection onto F_i, let $\xi_i = p_i\xi$, and let $f_i = p_i \circ f$.

We follow the analysis of Shub, Irwin ([Shu87], p51).

Lemma 16 *Suppose* $\|\xi_2 - \xi_2'\| > \|\xi_1 - \xi_1'\|$ *so that* $\|\xi - \xi'\| = \|\xi_2 - \xi_2'\|$. *Then*

$$
\begin{aligned}
\|f_1(\xi) - f_1(\xi')\| &\leq (1-\epsilon)\|\xi - \xi'\| \\
\|f_2(\xi) - f_2(\xi')\| &\geq (1+\epsilon)\|\xi - \xi'\| \qquad (249)
\end{aligned}
$$

Remark Note that with ϵ^α replaced by $d\epsilon$ the bounds (232), (233) can be written

$$\begin{aligned}
(d\epsilon)^{-1}|\tilde{r}(\xi) - \tilde{r}(\xi')| &\leq \Big(\mathcal{O}(\log L)d + \mathcal{O}(1)d^{-1}\Big)\,\epsilon\|\xi - \xi'\| \\
\epsilon^{-1+\delta}|\tilde{s}_1(\xi) - \tilde{s}_1(\xi')| &\leq C\epsilon^{1+\delta}\|\xi - \xi'\| \\
\epsilon^{-2+\Delta+\delta}|\tilde{s}_2(\xi) - \tilde{s}_2(\xi')| &\leq C\epsilon^{1+\delta}\|\xi - \xi'\| \\
\epsilon^{-2+2\delta}|\tilde{t}(\xi) - \tilde{t}(\xi')| &\leq C\epsilon^{\delta}\|\xi - \xi'\|
\end{aligned} \tag{250}$$

Proof We estimate

$$f_1 = (f_\lambda, f_R, f_w) = \Big((2 - L^\epsilon)\tilde{\lambda} + \tilde{r},\ U,\ w_L + C_L\Big) \tag{251}$$

We have

$$\begin{aligned}
(d\epsilon)^{-1}|f_\lambda(\xi) - f_\lambda(\xi')| &\leq \Big((2 - L^{2\epsilon}) + (\mathcal{O}(\log L)d + \mathcal{O}(1)d^{-1})\epsilon)\Big)\,\|\xi - \xi'\| \\
&\leq \Big(1 + \epsilon(-2\log L + \mathcal{O}(1)d\log L + \mathcal{O}(1)d^{-1})\Big)\,\|\xi - \xi'\| \\
&\leq (1 - \epsilon)\|\xi - \xi'\|
\end{aligned} \tag{252}$$

where we first choose d small enough so that $-2 + \mathcal{O}(1)d < 0$ and then choose L large enough for that $(-2 + \mathcal{O}(1)d^{-1})\log L + \mathcal{O}(1) < -1$.

We also have

$$\epsilon^{-3+\Delta}|||f_R(\xi) - f_R(\xi')||| \leq \mathcal{O}(L^{-1+2\epsilon})\|\xi - \xi'\| \leq (1 - \epsilon)\|\xi - \xi'\| \tag{253}$$

and

$$\begin{aligned}
k^{-1}\|f_w(\xi) - f_w(\xi')\| &= k^{-1}\|w_L - w'_L\| \\
&\leq L^{-1/4}\|\xi - \xi'\| \\
&\leq (1 - \epsilon)\|\xi - \xi'\|
\end{aligned} \tag{254}$$

This completes the proof of the first bound.

For the second bound we estimate

$$f_2 = (f_\zeta, f_\mu) = (L^\epsilon\zeta + \tilde{s},\ L^{2+\epsilon}\mu + \tilde{t}) \tag{255}$$

Note that one of $\epsilon^{-1+\delta}|\zeta_1 - \zeta'_1|$ and $\epsilon^{-2+\Delta+\delta}|\zeta_2 - \zeta'_2|$ and $\epsilon^{-2+2\delta}|\mu - \mu'|$ must be equal to $\|\xi_2 - \xi'_2\| = \|\xi - \xi'\|$. In the first case we have

$$\begin{aligned}
\|f_2(\xi) - f_2(\xi')\| &\geq \epsilon^{-1+\delta}|f_{\zeta_1}(\xi) - f_{\zeta_1}(\xi')| \\
&\geq L^\epsilon\epsilon^{-1+\delta}|\zeta_1 - \zeta'_1| - \epsilon^{-1+\delta}\|\tilde{s}_1(\xi) - \tilde{s}_1(\xi')\| \\
&\geq (L^\epsilon - C\epsilon^{1+\delta})\|\xi - \xi'\| \\
&\geq (1 + \epsilon)\|\xi - \xi'\|
\end{aligned} \tag{256}$$

The second case is similar, and in the last case we have

$$
\begin{aligned}
\|f_2(\xi) - f_2(\xi')\| &\geq \epsilon^{-2+2\delta}|f_\mu(\xi) - f_\mu(\xi')| \\
&\geq L^{2+\epsilon}\epsilon^{-2+2\delta}|\mu - \mu'| - \epsilon^{-2+2\delta}\|\tilde{t}(\xi) - \tilde{t}(\xi')\| \\
&\geq (L^{2+\epsilon} - C\epsilon^{\delta})\|\xi - \xi'\| \\
&\geq (1+\epsilon)\|\xi - \xi'\|
\end{aligned} \tag{257}
$$

□

Now we can state our main result. Consider the RG transformation f on F and we define the stable manifold to be

$$
M = \{\xi \in E(1/2) : \xi_j = f^j(\xi) \in F(1/2) \text{ for all } j = 1, 2, ...\} \tag{258}
$$

where f^j is the j-fold composition of f with itself.

Theorem 9 *The stable manifold M is the graph of a function $\xi_2 = h(\xi_1)$ which is Lipschitz continuous with Lipschitz constant 1. The RG mapping f is a contraction on M and thus has a unique fixed point $\xi^* = (\xi_1^*, \xi_2^*)$ where $\xi_2^* = h(\xi_1^*)$.*

Proof Suppose ξ, ξ' are two points on M. We claim that

$$
\|\xi_2 - \xi_2'\| \leq \|\xi_1 - \xi_1'\| \tag{259}
$$

If not then $\|\xi_2 - \xi_2'\| > \|\xi_1 - \xi_1'\|$ and the previous lemma is applicable. It follows that

$$
\begin{aligned}
&\|f_2(\xi) - f_2(\xi')\| \geq (1+\epsilon)\|\xi - \xi'\| \\
&> (1-\epsilon)\|\xi - \xi'\| \geq \|f_1(\xi) - f_1(\xi')\|
\end{aligned} \tag{260}
$$

whence $\|f(\xi) - f(\xi')\| \geq (1+\epsilon)\|\xi - \xi'\|$. Now apply the lemma again to $f(\xi), f(\xi')$ and get

$$
\begin{aligned}
&\|f_2(f(\xi)) - f_2(f(\xi'))\| \geq (1+\epsilon)\|f(\xi) - f(\xi')\| \\
&> (1-\epsilon)\|f(\xi) - f(\xi')\| \geq \|f_1(f(\xi)) - f_1(f(\xi'))\|
\end{aligned} \tag{261}
$$

whence $\|f^2(\xi) - f^2(\xi')\| \geq (1+\epsilon)^2\|\xi - \xi'\|$. Continuing in this fashion we get

$$
\|f^n(\xi) - f^n(\xi')\| \geq (1+\epsilon)^n\|\xi - \xi'\|
$$

which gives a contradiction as $n \to \infty$.

Because of the inequality we have that $\xi_1 = \xi_1'$ implies that $\xi_2 = \xi_2'$. Thus there is at most one ξ_2 for each ξ_1 on M. Recall from the remark at the end of Section 5.2 that we already know that there is at least one ξ_2 and hence there is exactly one ξ_2. Thus M is the graph of a function $\xi_2 = h(\xi_1)$ and the inequality says that $\|(h(\xi_1) - h(\xi_1')\| \leq$

$\|\xi_1 - \xi_1'\|$. On M we have $\|\xi - \xi'\| = \|\xi_1 - \xi_1'\|$ and hence $\|f(\xi) - f(\xi')\| = \|f_1(\xi) - f_1(\xi')\|$. We have already seen that $\|f_1(\xi) - f_1(\xi')\| \leq (1-\epsilon)\|\xi - \xi'\|$ and so f is a contraction on M.

$\square$

Remarks To express M in terms of the original variables, write h as a pair of functions

$$\begin{aligned} \zeta_0 &= h_\zeta(\tilde{\lambda}_0, R_0, w_0) \\ \mu_0 &= h_\mu(\tilde{\lambda}_0, R_0, w_0) \end{aligned} \tag{262}$$

The case of interest is $R_0 = 0, w_0 = 0$ which gives our functions $\mu_0 = \mu_0(\lambda_0), \zeta_0 = \zeta_0(\lambda_0)$. When $R_0 = 0, w_0 = 0$ the point $(\zeta_{0,1}(\lambda_0), \zeta_{0,2}(\lambda_0))$ describes the same density as the point $(\zeta_{0,1}(\lambda_0) + \zeta_{0,2}(\lambda_0), 0)$ so we can get on the stable manifold with a potential of the form (8). Note that the fixed point has the form $\xi^* = (\tilde{\lambda}^*, \zeta^*, \mu^*, R^*, w^*)$ where $\zeta^* = h_\zeta(\lambda^*, R^*, w^*), \mu^* = h_\mu(\lambda^*, R^*, w^*)$.

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