Phase Transitions in Hierarchical Models: The Role of Coupling Coefficients

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Abstract

We investigate the role of coupling coefficients in determining the existence of phase transitions for Hierarquical models of Dyson type in arbitrary dimensions. In particular, we show that, when the random spin variables display discrete symmetry, and the coupling constants are chosen so that asymptotically they match the behavior of the usual Laplacean, a phase transition is not expected to occur in 2 dimensions. We show this fact in two different ways. First we develop a heuristic argument based on the balance of energy and entropy of the model. The rigorous proof is then obtained by the generalization to several dimensions of a theorem due to Dyson that states the non-existence of phase-transitions when the coupling constants of the model are bounded.

Key Words: Classical spin systems, Hierarchical models, Ising model, Phase Transitions

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1 Introduction.

Hierarchical models (HM) were introduced in physics literature by Dyson [1] for the study of ferromagnetic spin systems with long range interactions in one dimension. A few years latter, Baker [2] rediscovered them and pointed out that, for this kind of system, the Renormalization Group transformations are exact.

The basic idea to construct HM is to integrate the short distance interactions of the Hamiltonian in order to extract the long distance asymptotic behavior of the model. This is done by splitting the spin variables into two parts. The first one, generated by the so-called block-spin transformation, will describe the effective interaction between mean-spins defined over blocks of a predefined size. The other one, orthogonal to the first, is produced by the fluctuation transformation and will be used to describe the interactions inside each block.

However being a strong simplification of the original model, and even not displaying some important symmetry characteristics, such as the invariance under translations of the Hamiltonian, HM are believed to preserve all relevant information concerning the infrared behavior of the original models. In this sense, they are frequently used to describe situations close to the critical temperature, in various statistical mechanics models, where this kind of effects dominate the global system.

More recently, Paiva and Perez [3] developed a Hierarchical Laplacean which generalizes Dyson's HM for any dimension and block size. Although their main concern was the study of random walks in a random environment, their formulation, being general, permits one to build any of the usual statistical mechanics spin systems, in a hierarchical version. The coupling parameters in their Hierarchical Laplacean are chosen so that its large distance behavior is equal to the one showed by the usual lattice Laplacean.

We are going to show in this paper that, if the coupling parameters are chosen in such a way, the Ising Hierarchical Model does not have a phase transition in two dimensions. We do this in two different ways. The first one is heuristic but in some sense clarifies the point we have in mind. We develop an energy-entropy argument for this model, calculating the energy cost of inserting, in a system that finds itself in a pure phase, a region of reversed magnetization. It turns out that, for one and two dimenions, this energy cost is finite, so that the pure phase is unstable and long range order is not expected to occur for any positive temperature (for details about this kind of argument, the reader is refered to [4]). The second way of proving this fact is rigorous, since we extend a theorem by Dyson [1], that asserts the absence of a phase transition when the coupling parameters are bounded, to the case of any dimension and block size (Dyson's result is restricted to one dimension and block size equals to 2). We then find that, for the particular choice of coupling coefficients of Paiva and Perez, $d = 2$ is exactly the limiting dimension, so that for $d \leq 2$ a phase transition is not expected. It is important to remark that our proof lies strongly on the use of Griffiths inequalities [5] which are not available for models in which the spin variables display continuous symmetry. It would be very interesting to obtain similar results in such a case.

2 The Hierarchical Laplacean

Consider a $d-$dimensional region $\Lambda \subset \mathbf{Z}^d$. Without loss of generality we assume the number of point inside Λ to be L^{dN}. Λ is now divided in blocks of volume L^d. A block of first hierarchy is defined by:

$$B_x^{(1)} = L^{-\frac{d}{2}} \sum_{y \in B(Lx,L)} \delta_y,$$

where $B(Lx, L)$ means the block of center Lx and side L. A second hierarchy block by:

$$B_x^{(2)} = L^{-\frac{d}{2}} \sum_{y \in B(Lx,L)} B_y^{(1)},$$

and so on.

The Hierarchical Laplacean is given by:

$$-\Delta_h = \sum_{x \in \mathbf{Z}^d} \sum_{n \geq 1} L^{-2n} Q_x^{(n)},$$

where:

$$Q_x^{(n)} = \sum_{y \in B(Lx,L)} \mid B_y^{(n-1)} \rangle \langle B_y^{(n-1)} \mid - \mid B_x^{(n)} \rangle \langle B_x^{(n)} \mid .$$

The first term of the difference is the block operator, that projects on functions which have support on $B(L^n x, L^n)$. The $Q_x^{(n)}$ is the fluctuation operator.

The L^{-2n} prefactor is introduced to obtain the standard Green's function decay with distance, as we are going to see. Using the fact that:

$$\sum_{x\in\mathbf{Z}^d}\sum_{n\geq 1} Q_x^{(n)} = \mathbf{1},$$

we can immediatly show that the associated Green's function for this operator is given by:

$$G(x,y;z) = (-\Delta_h - z)^{-1}(x,y) =$$
$$= \left(-\delta_x, \sum_{n\geq 1}\sum_{w\in\mathbf{Z}^d} \frac{Q_w^{(n)}}{L^{-2n}-z}\delta_y\right).$$

Developing this expression, we have:

$$G(0,x;z) = (L^d-1)\sum_{n\geq N(0,x)} L^{-dn}(L^{-2n}-z)^{-1} + \delta_{0,x}(L^{-2}-z)^{-1},$$

where $N(0,x)$ is the smallest hierarchy that contains both 0 and x. If we now introduce the hierarchial distance $\mid x-y\mid_h$ defined by:

$$\mid x-y\mid_h = \begin{cases} L^{N(x,y)}\,, if \quad x\neq y \\ \\ 0 \qquad\quad , if \quad x=y \end{cases}$$

and make the remark that $\mid x-y\mid\to\infty$ ($\mid . \mid$is the usual euclidean distance) implies $\mid x-y\mid_h\to\infty$, we are lead to, for $d>2$, taking the limit $z\to 0$:

$$-\Delta_h^{-1} = \frac{1-L^{-2}}{L^{-2}-L^{-d}}L^{-2N}(\mid x\mid_h)^{-d} + \delta_{0,x}L^2.$$

For $d=1$, we have:

$$-\Delta_h^{-1} = (L-1)\mid x\mid_h,$$

and, for $d=2$:

$$-\Delta_h^{-1} = (L^2-1)\log_L \mid x\mid_h,$$

so that in every case the asymptotic decay of the hierarchical Laplacean is equal to the one of the usual Laplacean, namely $|x|^{2-d}$. This Hierarchical Laplacean is an extension of Dyson's first Hierarchical Model in the sense that taking one-dimensional spin arrays with $L=2$ we recover the energy expression in [1].

3 Energy-Entropy Argument for The Hierarquical Model

Suppose now that, at each point $x \in \Lambda$, is defined an Ising random variable $\varphi(x)$ that can assume values on the set $\{-1,+1\}$. To develop our energy-entropy argument, we need to study the energy cost of inserting a 'bubble' of reversed spins in a purely magnetized phase.

The quadratic form corresponding to the energy of this system can be easily calculated:

$$H_N = (\varphi, -\Delta_h \varphi) =$$

$$= \sum_{n=0}^{N} L^{-n(d+2)}(L^{-2} - 1 + \delta_{n,0}) \sum_{x \in Z^d} [\varphi_B^{(n)}(x)]^2. \tag{1}$$

The block-spin variables $\varphi_B^{(n)}(x)$ are defined as follows:

$$\varphi_B^{(0)}(x) = \varphi(x),$$

$$\varphi_B^{(n)}(x) = \sum_{y \in B(Lx,L)} \varphi(y).$$

For the totally magnetized system, the energy will be, when $N \to \infty$:

$$(\varphi, -\Delta_h \varphi)_0 = \sum_{n=0}^{\infty} L^{-n(d+2)}(L^{-2} - 1 + \delta_{n,0}) \sum_{x \in Z^d} L^{2dn}$$

The energy cost for the introduction of a whole k hierarchy of reversed spins will be given by:

$$\Delta E = (\varphi, -\Delta_h \varphi)_+ - (\varphi, -\Delta_h \varphi)_0,$$

where $(\varphi, -\Delta_h \varphi)_+$ is the model energy with the reversed hierarchy. We have also:

$$(\varphi, -\Delta_h \varphi)_+ = (\varphi, -\Delta_h \varphi)_+^{n \le k} + (\varphi, -\Delta_h \varphi)_+^{n > k},$$

$$(\varphi, -\Delta_h \varphi)_0 = (\varphi, -\Delta_h \varphi)_0^{n \le k} + (\varphi, -\Delta_h \varphi)_0^{n > k},$$

but $(\varphi, -\Delta_h \varphi)_+^{n \le k} = (\varphi, -\Delta_h \varphi)_0^{n \le k}$. In view of these properties, we obtain:

$$\Delta E = \sum_{n>k}^{\infty} \frac{L^{-2n} L^{-2}}{L^{dn}} \left\{ \left[L^{dn} - 2L^{dk} \right]^2 - L^{2dn} \right\}.$$

Changing variables, $n-k=j$:

$$\Delta E = \sum_{j=0}^{\infty} \frac{L^{-2(j+k)}L^{-2}}{L^{d(j+k)}} \left\{ \left[L^{d(j+k)} - 2L^{dk} \right]^2 - L^{2d(j+k)} \right\} =$$

$$= L^{k(d-2)} \sum_{j=0}^{\infty} 4L^{-j(d+2)}L^{-2}\{1 - L^{dj}\} = L^{k(d-2)}\Gamma(d),$$

where $L^{k(d-2)}$ is the volume of the reversed block in $d-2$ dimensions. The conclusion is that the Hierarchical Model defined by (1) seems not to have a phase transition for dimension less than three, since for one and two dimensions the energy cost of inserting a region of reversed magnetization in a pure phase is finite. This fact is related to the chosen decay of the Hierarchical Laplacean with distance, as we are going to see next.

4 Absence of Phase Transitions for Bounded Coupling Parameters

We begin by rewriting the expression of the Hamiltonian (1) as

$$H_N = \sum_{n=0}^{N} L^{-2dn} J_n \sum_{x\in\Lambda} [\varphi_B^{(n)}(x)]^2, \tag{2}$$

where

$$J_n = L^{n(d-2)}(L^{-2} - 1 + \delta_{n,0}) \tag{3}$$

defines the coupling parameters of the model. The interaction of a spin with all other spins in Λ is given by

$$I_N = \sum_{n=1}^{N} 2J_n L^{-2dn}(L^{dn} - 1). \tag{4}$$

It is also convenient to define the function

$$f_N(n) = L^{-2dn}\langle\ [\varphi_B^{(n)}]^2\ \rangle_N. \tag{5}$$

so that $f_N(0) = 1$. Here, the expectation value is taken over the system confined in a finite volume. The spontaneous magnetization in the thermodynamical limit can then be represented as

$$m^2 = \lim_{n\to\infty} f(n), \tag{6}$$

where $f(n) = \lim_{N\to\infty} f_N(n)$.

Theorem: *If the J_n are bounded, $m^2 = 0$ in the limit $N \to \infty$.*

Proof

As in [1], we need only prove the theorem for $J_n = 1$. Let now p_N be the probability, for finite N, that two spins, $\varphi(x)$ and $\varphi(x')$, such that both x and x' lie in the same first hierarchy block, are parallel. By Griffiths' inequalities, this probability does not decrease if all spins in the system, except $\varphi(x)$ and $\varphi(x')$ are locked parallel by infinite couplings. We have then

$$p_N \le \frac{1}{1 + e^{-2\beta I_N}}.$$

Using the fact that $p_N < \lim_{n\to\infty} p_N$ (since I_N is monotonically increasing in N), we have

$$p_N < p = \frac{1}{1 + e^{-2\beta I}} = g < 1, \tag{7}$$

where, since $J_n = 1$, we have

$$I = \lim_{N\to\infty} I_N = 2L^{-2}\left(\frac{1}{L^d - 1} - \frac{1}{L^{2d} - 1}\right).$$

Since all the spins contained in a given block are equivalent, we have

$$f_N(n) = L^{-2dn} L^{d(n-1)} \frac{L^d(L^d - 1)}{2} \langle\, [\varphi(x) + \varphi(x')]\varphi_B^{(n)}(x)\, \rangle_N$$

so that

$$f_N(n) = L^{-dn}(L^d - 1) p_N \langle\, \varphi_B^{(n)}(x)\, \rangle_{N,lock}$$

We now stress that the above average is taken over the model where the two spins $\varphi(x)$ and $\varphi(x')$ are locked parallel.

Again, by Griffiths inequalities, the above average will not decrease if the spins contained in each first hierarchy block are all locked parallel. This is equivalent to a hierarchical model of the same kind, with the same previous coupling parameters but with a number of spins $L^{d(N-1)}$. We have then

$$f_N(n) \le L^{-d(n-1)}(L^d - 1) p_N \langle\, \varphi_B^{(n-1)}(x)\, \rangle_{N-1} = p_N f_{N-1}(n-1). \tag{8}$$

Iterating (8), we obtain

$$f_N(n) < g f_{N-1}(n-1) < g^n f_{N-n}(0) = g^n,$$

so that we have, in the thermodynamical limit

$$f(n) \leq g^n,$$

where, by (7), $g < 1$. Using then the definition of the spontaneous magnetization (6), we finally get

$$m^2 = \lim_{n\to\infty} f(n) \leq \lim_{n\to\infty} g^n = 0.$$

Corollary: *The model defined by* (2) *has zero spontaneous magnetization at any finite temperature in two dimensions*

Proof

Imediate, since we see that, for $d = 2$, the coupling coefficients J_n, defined by (3), are bounded.

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