QUASI-GEOSTROPHIC TYPE EQUATIONS WITH WEAK INITIAL DATA

Jiahong Wu
School of Mathematics
The Institute for Advanced Study
Princeton, NJ 08540

Abstract

We study the initial value problem for the quasi-geostrophic type equations

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + (-\Delta)^{\lambda} \theta = 0, \quad \text{on} \quad \mathbb{R}^n \times (0, \infty),$$

$$\theta(x, 0) = \theta_0(x), \quad x \in \mathbb{R}^n$$

where λ is a fixed parameter and $u = (u_j)$ is divergence free and determined from θ through the Riesz transform $u_j = \pm \mathcal{R}_{\pi(j)} \theta$ ($\pi(j)$ being a permutation of j, $j = 1, 2, \cdots, n$). The initial data θ_0 is taken in the Sobolev space of negative indices $\dot{L}_{r,p}$ (see the text for the definition). We prove local well-posedness if $\frac{1}{2} < \lambda \leq 1$ and

$$1 < p < \infty, \quad \frac{n}{p} \leq 2\lambda - 1, \quad r = \frac{n}{p} - (2\lambda - 1) (\leq 0)$$

The solution is global if θ_0 is sufficiently small.

1 Introduction

In this paper we study the initial value problem (IVP) of the dissipative quasi-geostrophic type (QGS) equations

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + (-\Delta)^{\lambda} \theta = 0, \quad \text{on} \quad \mathbb{R}^n \times (0, \infty), \tag{1.1}$$

$$\theta(x, 0) = \theta_0(x), \quad x \in \mathbb{R}^n \tag{1.2}$$

where $\lambda (0 \leq \lambda \leq 1)$ is a fixed parameter and the velocity $u = (u_1, u_2, \cdots, u_n)$ is divergence free and determined from θ by

$$u_j = \pm \mathcal{R}_{\pi(j)} \theta, \qquad \pi(j) \text{ is a permutation of } j,\ j = 1, 2, \cdots, n \tag{1.3}$$

where u_j may take either $+$ or $-$ sign and $\mathcal{R}_j = \partial_j (-\Delta)^{-\frac{1}{2}}$ are the Riesz transforms. Here Riesz potential operator $(-\Delta)^{\alpha}$ is defined through the Fourier transform:

$$\widehat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x) dx$$

$$(\widehat{(-\Delta)^{\alpha} f})(\xi) = (2\pi |\xi|)^{2\alpha} \widehat{f}(\xi)$$

A particular important special case of (1.1) is the 2-D dissipative quasi-geostrophic equations in which the velocity $u = (u_1, u_2)$ can also be defined through the stream function ψ:

$$u = (u_1, u_2) = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1} \right), \quad (-\Delta)^{\frac{1}{2}} \psi = -\theta \tag{1.4}$$

The 2-D QGS equations are derived from more general quasi-geostrophic approximations for flow in rapidly rotating 3-D half space, which in some important cases reduce to the evolution equation for the temperature on the 2-D boundary given in (1.1), (1.2),(1.4) ([12],[2]). The scalar θ represents the potential temperature and u is the fluid velocity. These equations have been under active investigation because of mathematical importance and potential applications in meteorology and oceanography ([12],[2], [1],[6]). As pointed out in [2], the non-dissipative 2-D QGS equations are strikingly analogous to the 3-D Euler equations and thus serve as a simple model in seeking possible singular solutions.

We are mainly interested in the well-posedness result for initial data θ_0 in weighted Lebesgue spaces, $\theta_0 \in \dot{L}_{r,p}(\mathbb{R}^n)$ (defined below). By well-posedness we mean existence, uniqueness and persistence (i.e. the solution describes a continuous curve belonging to the same space as does the initial data) and continuous dependence on the data.

Here the homogeneous Lebesgue space $\dot{L}_{s,q}(\mathbb{R}^n)$ consists of all v such that

$$(-\Delta)^{\frac{s}{2}} v \in L^q, \quad s \in \mathbb{R}, \quad 1 \leq q < \infty$$

and the standard norm is given by

$$\|v\|_{s,q} = \|(-\Delta)^{\frac{s}{2}} v\|_{L^q}$$

These spaces are also called the spaces of Riesz potentials. Kato and Ponce [10] consider the Navier-Stokes equations with initial data in this type of spaces.

We prove that if $\frac{1}{2} < \lambda \leq 1$ and $\theta_0 \in \dot{L}_{r,p}$ with r, p satisfying

$$1 < p < \infty, \quad \frac{n}{p} \leq 2\lambda - 1, \quad r = \frac{n}{p} - (2\lambda - 1)(\leq 0)$$

then the IVP (1.1), (1.3), (1.2) is locally well-posed. The solution is global if θ_0 is sufficiently small. The detailed statements are given in Theorem 2.2 of the next section.

Although there is a large literature on quasi-geostrophic equations ([12], [1], [6], [2]), not many rigorous mathematical results concerning the solutions have been obtained. In [2] Constantin-Majda-Tabak prove finite time existence results for smooth data and develop mathematical criteria characterizing blowup for the 2-D non-dissipative QGS equation. In [13] Resnick obtain solutions of 2-D QGS equations with L^2 data on periodic domain by using Galerkin approximation. In a previous paper [15], the vanishing dissipation limits and Gevrey class regularity [3] for the 2-D dissipative QGS equations are obtained. In this paper we consider the IVP of the general n-D QGS type equations (defined by (1.1), (1.3, (1.2)) with initial data in Sobolev spaces of negative indices and establish local well-posedness results. For sufficiently small initial data, the solution is global. By taking $n = 2$ and $p = 2$, the well-posedness reduces to the L^2 results in 2-D.

The main result is presented in the next section and its proof is based on the contraction-mapping principle.

2 Well-posedness

We'll need to use the spaces of weighted continuous functions in time, which have been introduced by Kato, Ponce and others in solving the Navier-Stokes equations ([8],[10],[11]).

Definition 2.1 *Suppose $T > 0$ and $\alpha \geq 0$ are real numbers. The spaces $C_{\alpha,s,q}$ and $\dot{C}_{\alpha,s,q}$ are defined as*

$$C_{\alpha,s,q} \equiv \{f \in C((0,T), \dot{L}_{s,q}), \quad \|f\|_{\alpha,s,q} < \infty\}$$

where the norm is given by

$$\|f\|_{\alpha,s,q} = \sup\{t^{\alpha}\|f\|_{s,q}, \quad t \in (0,T)\}$$

$\dot{C}_{\alpha,s,q}$ is a subspace of $C_{\alpha,s,q}$:

$$\dot{C}_{\alpha,s,q} \equiv \{f \in C_{\alpha,s,q}, \quad \lim_{t\to 0} t^{\alpha}\|f\|_{s,q} = 0\}$$

When $\alpha = 0$, $\bar{C}_{s,q}$ are used for $BC([0,T), \dot{L}_{s,q})$.

These spaces are important in uniqueness and local existence problems ([8],[10],[11]). $f \in C_{\alpha,s,q}$ (resp. $f \in \dot{C}_{\alpha,s,q}$) implies that $\|f(t)\|_{s,q} = O(t^{-\alpha})$ (resp. $o(t^{-\alpha})$).

The main result of this section is the well-posedness theorem that states

Theorem 2.2 *Assume that $\lambda > 1/2$ and $\theta_0 \in \dot{L}_{r,p}$ with r, p satisfying*

$$1 < p < \infty, \quad \frac{n}{p} \leq 2\lambda - 1, \quad r = \frac{n}{p} - (2\lambda - 1)(\leq 0) \tag{2.1}$$

Then there exists $T = T(\theta_0)$ and a unique solution $\theta(t)$ of the IVP (1.1),(1.3), (1.2) in the time interval $[0,T)$ satisfying

$$\theta \in Y_T \equiv \left(\cap_{p\leq q<\infty} \bar{C}_{\frac{n}{q}-(2\lambda-1),q}\right) \cap \left(\cap_{p\leq q<\infty} \cap_{s>\frac{n}{q}-(2\lambda-1)} \dot{C}_{\left(s-\frac{n}{q}+(2\lambda-1)\right)/(2\lambda),s,q}\right)$$

In particular,

$$\theta \in BC([0,T), \dot{L}_{r,p}) \cap (\cap_{s>r} C((0,T), \dot{L}_{s,p}))$$

Furthermore, for some neighborhood V of θ_0, the mapping

$$\mathfrak{P} : V \longmapsto Y_T : \quad \theta_0 \longmapsto \theta$$

is Lipschitz.

Remark 2.3 *If $\|\theta_0\|_{r,p}$ is small enough, then we can take $T = \infty$.*

We prove this theorem by the method of integral equations and contraction-mapping arguments. Following standard practice ([4], [5], [7], [10]), we write the QGS equation (1.1) into the integral form:

$$\theta = K\theta_0(t) + G(u,\theta)(t) \equiv e^{-\Lambda^{2\lambda}t}\theta_0 + \int_0^t e^{-\Lambda^{2\lambda}(t-\tau)}(u\cdot\nabla\theta)(\tau)d\tau \tag{2.2}$$

where $K(t) = e^{-\Lambda^{2\lambda}t}$ is the solution operator of the linear equation

$$\partial_t\theta + \Lambda^{2\lambda}\theta = 0, \quad \text{with} \quad \Lambda = (-\Delta)^{\frac{1}{2}}$$

We observe that $u\cdot\nabla\theta = \sum_j u_j\partial_j\theta = \nabla\cdot(u\theta)$ provided that $\nabla\cdot u = 0$. This provides an alternative expression for G:

$$G(u,\theta)(t) = G(u\theta)(t) = \int_0^t \nabla\cdot e^{-\Lambda^{2\lambda}(t-\tau)}(u\theta)(\tau)d\tau$$

We shall solve (2.2) in the spaces of weighted continuous functions in time introduced in the beginning of this section. To this end we need estimates for the operators K and G acting between these spaces. These are established in the two propositions that follow.

Proposition 2.4 *(i) For $1 \le q < \infty$ and $s \in \mathbb{R}$, the operator K maps continuously from $\dot{L}_{s,q}$ into $\dot{C}_{s,q} \equiv BC([0,\infty), \dot{L}_{s,q})$.*

(ii) If q_1, q_2, s_1, s_2 and α_2 satisfy

$$q_1 \le q_2, \quad s_1 \le s_2,$$

$$\alpha_2 = \frac{1}{2\lambda}(s_2 - s_1) + \frac{1}{2\lambda}\left(\frac{n}{q_1} - \frac{n}{q_2}\right)$$

Then K maps continuously from $\dot{L}_{s_1,q_1}$ to $\dot{C}_{\alpha_2,s_2,q_2}$.

Proof. To prove (i), it suffices to prove that for some constant C

$$\|K\phi(t)\|_{L^q} \le C\|\phi\|_{L^q}, \quad \text{for any } t \in [0,\infty)$$

which can be established by using the Young's inequality

$$\|K\phi(t)\|_{L^q} \le \|K(t)\|_{L^1}\|\phi\|_{L^q}$$

and the fact that

$$\widehat{K}(t)(\xi) = e^{-|2\pi\xi|^{2\lambda}t}, \quad \|K(t)\|_{L^1} = \widehat{K}(t)(0) = 1$$

To prove assertion (ii), we first note that the operator $(-\Delta)^{\frac{s_0}{2}}K(t)$ has the property

$$\|(-\Delta)^{\frac{s_0}{2}}K(t)\|_{L^q(\mathbb{R}^n)} \le Ct^{\frac{1}{2\lambda}\left(-s_0-n(1-\frac{1}{q})\right)} \tag{2.3}$$

where $s_0 \ge 0$, $q \in [1, \infty)$ and C is a constant. The proof of this property is similar to that for the heat operator ([4], [5], [10]). To show (ii), it is enough to show for some constant C

$$\sup_{t\in[0,T)} t^{\alpha_2}\|(-\Delta)^{\frac{s_0}{2}}K\phi(t)\|_{L^{q_2}} \le C\|\phi\|_{L^{q_1}}$$

with $s_0 = s_2 - s_1 \ge 0$. This can again be proved by using the property (2.3) and Young's inequality

$$\|(-\Delta)^{\frac{s_0}{2}}K\phi(t)\|_{L^{q_2}} \le C\|(-\Delta)^{\frac{s_0}{2}}K(t)\|_{L^q}\|\phi\|_{L^{q_1}}$$

with $\frac{1}{q} = 1 - \left(\frac{1}{q_1} - \frac{1}{q_2}\right)$.

Now we give estimates for the operator G :

$$G(g)(t) = \int_0^t \nabla \cdot K(t-\tau)g(\tau)d\tau$$

Proposition 2.5 *If $q_1, q_2, s_1, s_2, \alpha_1$ and α_2 satisfy*

$$q_1 \le q_2,$$

$$s_1 - 1 \le s_2 < s_1 + 2\lambda - 1 - \left(\frac{n}{q_1} - \frac{n}{q_2}\right)$$

$$\alpha_1 < 1, \quad \alpha_2 = \alpha_1 - 1 + \frac{1}{2\lambda}\left[s_2 - s_1 + 1 + \frac{n}{q_1} - \frac{n}{q_2}\right]$$

Then G maps continuously from $\dot{C}_{\alpha_1,s_1,q_1}$ to $\dot{C}_{\alpha_2,s_2,q_2}$.

Proof. Let $g \in \dot{C}_{\alpha_1,s_1,q_1}$. Clearly,

$$\|G(g)\|_{\alpha_2,s_2,q_2} = \sup_{t\in[0,T)} t^{\alpha_2} \int_0^t \|(-\Delta)^{\frac{(1+s_0)}{2}} K(t-\tau) \left((-\Delta)^{\frac{s_1}{2}} g(\tau)\right) \|_{L^{q_2}} d\tau$$

where $s_0 = s_2 - s_1$. Using Young's inequality

$$\|G(g)\|_{\alpha_2,s_2,q_2} \le \sup_{t\in[0,T)} t^{\alpha_2} \int_0^t \|(-\Delta)^{\frac{(1+s_0)}{2}} K(t-\tau)\|_{L^q} \| \left((-\Delta)^{\frac{s_1}{2}} g(\tau)\right) \|_{L^{q_1}} d\tau$$

with $\frac{1}{q} = 1 - \left(\frac{1}{q_1} - \frac{1}{q_2}\right)$. If $s_0 + 1 \ge 0$, we can use the property (2.3) of operator K and obtain

$$\begin{aligned}
\|G(g)\|_{\alpha_2,s_2,q_2} &\le C\|g\|_{\alpha_1,s_1,q_1} \sup_{t\in[0,T)} t^{\alpha_2} \int_0^t (t-\tau)^{-\frac{1}{2\lambda}\left(s_0+1+n(1-\frac{1}{q})\right)} \tau^{-\alpha_1} d\tau \\
&\le C\|g\|_{\alpha_1,s_1,q_1} \sup_{t\in[0,T)} t^{\alpha_2-\alpha_1+1-\frac{1}{2\lambda}\left(s_0+1+n(1-\frac{1}{q})\right)} \\
&\qquad \times B\left(1 - \frac{1}{2\lambda}\left[s_0 + 1 + n(1-\frac{1}{q})\right], 1-\alpha_1\right)
\end{aligned}$$

where C is a constant and $B(a,b)$ is the Beta function

$$B(a,b) = \int_0^1 (1-x)^{a-1} x^{b-1} dx$$

By noticing that $B(a,b)$ is finite when $a > 0$, $b > 0$ and that

$$s_0 = s_2 - s_1, \quad 1 - \frac{1}{q} = \frac{1}{q_1} - \frac{1}{q_2}$$

we obtain

$$\|G(g)\|_{\alpha_2,s_2,q_2} \le C\|g\|_{\alpha_1,s_1,q_1}$$

if the indices satisfy

$$0 \le s_2 - s_1 + 1 < 2\lambda - \frac{n}{q_1} - \frac{n}{q_2}$$

$$\alpha_1 < 1, \quad \alpha_2 = \alpha_1 - 1 + \frac{1}{2\lambda}\left[s_2 - s_1 + 1 + \frac{n}{q_1} - \frac{n}{q_2}\right]$$

To prove Theorem 2.2, we'll also need the following singular integral operator estimate and its proof can be found in [14].

Lemma 2.6 *For $u = (u_j)$ with $u_j = \pm\mathcal{R}_{\pi(j)}\theta(\, j = 1, 2, \cdots, n)$ where $\mathcal{R}_j$ are the Riesz transforms, we have the estimate*

$$\|u\|_{L^q} \le C_q \|\theta\|_{L^q}, \quad 1 < q < \infty$$

where C_q is a constant depending on q.

Proof of Theorem 2.2. We distinguish between two cases: $r < 0$ and $r = 0$. For $r < 0$, we define

$$X = \bar{C}_{r,p} \cap \dot{C}_{-\frac{r}{2\lambda},0,p}$$

with norm for $\theta \in X$ given by

$$\|\theta\|_X = \|\theta - K\theta_0\|_{0,r,p} + \|\theta\|_{-\frac{r}{2\lambda},0,p}$$

and the complete metric space X_R to be the closed ball in X of radius R. Consider the operator $\mathcal{A}(\theta, \theta_0) : X_R \times V \longmapsto X$

$$\mathcal{A}(\theta, \theta_0)(t) = K\theta_0(t) + G(u\theta)(t), \quad 0 < t < T$$

where V is some neighborhood of θ_0 in $\dot{L}_{r,p}$ and T will be chosen.

Using Proposition 2.4 by substituting $s = r, q = p$ in (i) and

$$q_1 = q_2 = p, \quad s_1 = r, \quad s_2 = 0, \quad \alpha_2 = -\frac{r}{2\lambda}$$

in (ii), we find that $K\tilde{\theta}_0(t) \in X_R$ for $\tilde{\theta}_0 \in V$ if T is taken small enough and V is chosen properly.

To estimate G, we use Proposition 2.5 with

$$q_1 = \frac{p}{2}, \quad q_2 = p, \quad s_1 = 0, \quad s_2 = l + r, \quad \alpha_1 = -\frac{r}{\lambda}, \quad \alpha_2 = \frac{l}{2\lambda}$$

to obtain for some constant c

$$\|G(u\theta)\|_{\frac{l}{2\lambda},l+r,p} \le c\|u\theta\|_{-\frac{r}{\lambda},0,\frac{p}{2}} \le c\|u\|_{-\frac{r}{2\lambda},0,p}\|\theta\|_{-\frac{r}{2\lambda},0,p}$$

for $l \in [0, -2r)$. To estimate u in terms of θ, we use Lemma 2.6, i.e. for $1 < p < \infty$,

$$\|u\|_{L^p} \le C_p \|\theta\|_{L^p}$$

and eventually we obtain

$$\|G(u\theta)\|_{\frac{l}{2\lambda},l+r,p} \le cC_p\|\theta\|^2_{-\frac{r}{2\lambda},0,p} \le cC_pR^2$$

We should notice here that the restrictions (2.1) on r, p are necessary in order to apply Propositions 2.4, 2.5 and Lemma 2.6.

Furthermore,

$$\|\mathcal{A}(\theta,\theta_0) - \mathcal{A}(\tilde{\theta},\theta_0)\|_X = \|G(u\theta) - G(\tilde{u}\tilde{\theta})\|_X$$

where $\tilde{u} = (\tilde{u}_j)$ with $\tilde{u}_j = \pm\mathcal{R}_{\pi(j)}\tilde{\theta}(j = 1, 2, \cdots, n)$. Using Proposition 2.5 again,

$$\|\mathcal{A}(\theta,\theta_0) - \mathcal{A}(\tilde{\theta},\theta_0)\|_X \le \|G((\tilde{u}-u)\tilde{\theta})\|_X + \|G(u(\theta-\tilde{\theta}))\|_X$$

$$\le c\left(\|\tilde{u}-u\|_X\|\tilde{\theta}\|_X + \|\theta-\tilde{\theta}\|_X\|u\|_X\right)$$

Since $(\tilde{u}-u)_j = \pm\mathcal{R}_{\pi(j)}(\tilde{\theta}-\theta)$, Lemma 2.6 implies that

$$\|u\|_X \le C_p\|\theta\|_X, \quad \|\tilde{u}-u\|_X \le C_p\|\tilde{\theta}-\theta\|_X$$

Therefore for constant $C = cC_p$,

$$\|\mathcal{A}(\theta,\theta_0) - \mathcal{A}(\tilde{\theta},\theta_0)\|_X \le C(\|\tilde{\theta}\|_X + \|\theta\|_X)\|\tilde{\theta}-\theta\|_X$$

Our above estimates show that if we choose T to be small and R appropriately, then $\mathcal{A}$ maps X_R into itself and is a contraction. Consequently there exists a unique fixed point $\theta \in X_R$: $\theta = \mathfrak{P}(\theta_0)$ satisfying $\theta = \mathcal{A}(\theta,\theta_0)$. It is easy to see from these estimates that the uniqueness can be extended to all R' by further reducing the the time interval and thus to the whole X.

To prove the Lipschitz continuity of $\mathfrak{P}$ on V, let $\theta = \mathfrak{P}(\theta_0)$ and $\zeta = \mathfrak{P}(\zeta_0)$ for $\theta_0, \zeta_0 \in V$. Then

$$\|\theta-\zeta\|_X = \|\mathcal{A}(\theta,\theta_0) - \mathcal{A}(\zeta,\zeta_0)\|_X$$

$$\le \|\mathcal{A}(\theta,\theta_0) - \mathcal{A}(\zeta,\theta_0)\|_X + \|\mathcal{A}(\zeta,\theta_0) - \mathcal{A}(\zeta,\zeta_0)\|_X$$

$$\le \gamma\|\theta-\zeta\|_X + \|K(\theta_0-\zeta_0)\|_X$$

Since $\mathcal{A}$ is a contraction, $\gamma < 1$. Therefore the asserted property is obtained by applying Proposition 2.4 to the second term of the last inequality.

To show that θ is in the asserted class Y_T (defined in Theorem 2.2), we notice that

$$\theta = \mathcal{A}(\theta, \theta_0) \equiv K\theta_0 + G(u\theta)$$

We apply Proposition 2.4 twice to $K\theta_0$ to show that

$$K\theta_0 \in \bar{C}_{\frac{n}{q}-(2\lambda-1),q}, \qquad K\theta_0 \in \dot{C}_{\left(s-\frac{n}{q}+(2\lambda-1)\right)/(2\lambda),s,q}$$

for any $p \leq q < \infty$ and $s > \frac{n}{q} - (2\lambda - 1)$. To show the second part

$$G(u\theta) \in \bar{C}_{\frac{n}{q}-(2\lambda-1),q}, \quad p \leq q < \infty \tag{2.4}$$

we use Proposition 2.5 with

$$q_1 = \frac{p}{2}, \quad q_2 = q, \quad s_1 = 0, \quad s_2 = \frac{n}{q} - (2\lambda - 1), \quad \alpha_1 = -\frac{r}{\lambda}, \quad \alpha_2 = 0$$

and obtain

$$\|G(u\theta)\|_{0,\frac{n}{q}-(2\lambda-1),q} \leq C\|u\theta\|_{-\frac{r}{\lambda},0,\frac{p}{2}} \leq C\|u\|_{-\frac{r}{2\lambda},0,p}\|\theta\|_{-\frac{r}{2\lambda},0,p}$$

The asserted property (2.4) is established after we apply Lemma 2.6 to u.

Once again we apply Proposition 2.5 with

$$q_1 = \frac{p}{2}, \quad q_2 = q, \quad s_1 = 0, \quad s_2 = s,$$

$$\alpha_1 = -\frac{r}{\lambda}, \quad \alpha_2 = \frac{1}{2\lambda}\left[s - \left(\frac{n}{q} - (2\lambda - 1)\right)\right]$$

to show that

$$G(u\theta) \in \dot{C}_{\left(s-\frac{n}{q}+(2\lambda-1)\right)/(2\lambda),s,q}, \quad \text{for } s > \tfrac{n}{q} - (2\lambda - 1), \tag{2.5}$$

but s should also satisfy

$$s < 2\lambda - 1 - \left(\frac{2n}{p} - \frac{n}{q}\right)$$

as required by Proposition 2.5. For large s, (2.5) can be shown by an induction process (see an analogous argument in [8]).

We now deal with the case $r = 0$. We define

$$X = \bar{C}_{0,p} \cap \dot{C}_{\frac{1}{4},0,\frac{4\lambda-2}{3\lambda-2}p}$$

with the norm

$$\|\theta\|_X = \|\theta - K\theta_0\|_{0,0,p} + \|\theta\|_{\frac{1}{4},0,\frac{4\lambda-2}{3\lambda-2}p}$$

For $\theta \in X_R$, we have by Proposition 2.5,

$$\|G(u\theta)\|_X = \|G(u\theta)\|_{0,0,p} + \|G(u\theta)\|_{\frac{1}{4},0,\frac{4\lambda-2}{3\lambda-2}p}$$

$$\leq c\|u\theta\|_{\frac{1}{2},0,\frac{2\lambda-1}{3\lambda-2}p} \leq c\|u\|_{\frac{1}{4},0,\frac{4\lambda-2}{3\lambda-2}p}\|\theta\|_{\frac{1}{4},0,\frac{4\lambda-2}{3\lambda-2}p}$$

Here c is a constant which may depends on the indices λ, p, n. By using Lemma 2.6 again, we obtain for some C

$$\|G(u\theta)\|_X \leq C\|\theta\|_X^2 \leq CR^2$$

Once the above estimates have been established, the rest of the proof in this case is similar to that described in the case $r < 0$.

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