Off-diagonal Long Range Order

and Superconductive Electrodynamics

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Abstract

We present a general, model independent, quantum statistical derivation of superconductive electrodynamics from the assumptions of off-diagonal long range order (ODLRO), local gauge covariance and thermodynamic stability. On this basis, we obtain the Meissner and Josephson effects, the quantisation of trapped magnetic flux, and the metastability of supercurrents. A key to these results is that the macroscopic wave function, specified by the ODLRO condition, enjoys the rigidity property that London[1,2] envisaged for the microstate of a superconductor.

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I.Introduction

The object of this article is to provide a model-independent derivation of the electromagnetic properties of superconductors from their order structure, the essential imput being the assumptions of off-diagonal long range order, gauge covariance and thermodynamic stability. Thus, our approach to the subject is centred on very general principles, and is therefore at the opposite pole from that provided by the computational techniques of many-body theory.

In order to explain the need for such an approach, let us first recall that, in the case of metallic superconductivity, the electron pairing hypothesis[3,4] has been amply substantiated, at an empirical level, by the the measured values of the magnetic flux quantisation[5] and of the Josephson tunnelling frequency[6]. Furthermore, the widely accepted microscopic theory of superconductivity, devised by Bardeen, Cooper and Schrieffer (BCS)[7] on the basis of this hypothesis, provides an accurate picture of the thermodynamics[8−10] of superconductors, especially of their second order phase transitions. On the other hand, the electrodynamics of this theory is seriously flawed in that, firstly, it violates the principle of local gauge covariance[11,12], and consequently does not even admit a precise definition of the local current density; and, secondly, there is no indication that its ansatz for current-carrying states have the metastability properties[13,14] of supercurrents. In fact, the violation of the gauge principle by the BCS theory arises from its *truncation* of a fully gauge invariant model (Fröhlich's electron-phonon system[15]), that retains only the interactions that give rise to the electron pairing. Attempts[16,17] to remedy the situation by taking some account of the residual interactions have led to derivations of the Meissner effect that are only *approximately* gauge covariant. Since exact gauge covariance is required for a consistent electrodynamics, this is no solution of the problem. As regards ceramic, i.e. high T_c, superconductors, the microscopic theory is less developed, and has certainly not led to an electrodynamics, even though various interesting ideas concerning its underlying quantum mechanisms have been proposed, e.g. in Refs. 18-23.

Thus, there is a need for a quantum-based gauge covariant electrodynamics of the superconducting phase. Since, at an empirical level, this electrodynamics has such sharply defined *qualitative* characteristics, common to the vast variety of superconducting materials, a corresponding quantum theory should surely isolate their origins in general, qualitative terms. We remark here that the traditional techniques of many-body theory[24,25] are unsuited to this purpose, since these are designed for essentially approximative calculations rather than precise classifications.

In view of these considerations, we have adopted a different approach[26−28] to superconductive electrodynamics, based on the hypothesis of *off-diagonal long range order* (ODLRO), which encapsulates the qualitative features of the electron pairing in a gauge covariant way. This hypothesis was proposed by Yang[29] as a characterisation of the structure of the superconducting phase, following a similar proposal by O.Penrose[30,31] in connection with the theory of superfluid Helium. Formally, the ODLRO condition may be expressed in terms of the relevant quantised matter field ψ by the formulae

$$\lim_{|y|\to\infty}\left[\langle\psi(x)\psi^\star(x+y)\rangle-\Phi(x)\Phi^\star(x+y)\right]=0\ for\ bosons, \tag{1.1a}$$

and

$$\lim_{|y|\to\infty}\left[\langle\psi_\uparrow(x_1)\psi_\downarrow(x_2)\psi_\downarrow^\star(x_2'+y)\psi_\uparrow^\star(x_1'+y)\rangle-\Phi(x_1,x_2)\Phi^\star(x_1+y,x_2+y)\right]$$
$$=0\ for\ fermions, \tag{1.1b}$$

where, in both cases, Φ is a complex-valued function, often termed the *macroscopic wave function*, such that $\Phi(x+y)$, or $\Phi(x_1+y,x_2+y)$, does not tend to zero as $|y|\to\infty$. In the bosonic case, ODLRO is simply a generalisation of Bose- Einstein condensation[30,31] to interacting systems.

We note here that the ODLRO condition is fulfilled not only by the BCS ansatz, but also some of those for ceramic superconductivity[19,21], as well as that of Feynman[32] for superfluid Helium (cf Ref. 31). It is also satisfied in the low temperature phases of a number of tractable, gauge covariant models, namely the ideal Bose gas[33], the hard sphere Bose fluid on a lattice[34], and, at least at zero temperature, the Hubbard model with *attractive* interaction between electrons on the same site[35].

Our project, then, is to derive the principal electrodynamic properties of superconductors from precisely specified assumptions of ODLRO, gauge covariance and thermodynamic stability. In fact, progress towards this objective has already been achieved in Refs. 26-28 and 36, which have provided derivations of the Meissner effect[26], the quantisation of trapped magnetic flux[27,36], and the Josephson effect[27], as well as a sketched approach to the theory of persistent currents[28].

The present article will be devoted to a review and further development of this work. Our essential aim will be to provide a coherent account of the way in which the electrodynamics of superconductors, including the metastabilty of the supercurrents, stems from their order structure. We shall keep the mathematics quite simple, formulating the theory within the standard second quantisation framework of condensed matter physics. We remark here that one may put the whole theory onto a completely rigorous basis by recasting it, as in Ref. 27, within the framework of operator algebraic statistical mechanics.

We shall organise the presentation of the theory as follows. We start, in Sec. II, with a general formulation of our gauge-covariant model of matter in interaction with a classical magnetic field. This formulation covers the cases of both continuous and lattice systems. A key result here is that the dynamics of the model, in the presence of a uniform magnetic induction B, is covariant with respect to the *regauged space translations*[37,26]

$$\psi(x)\to\psi(x+a)\exp\Big(\frac{-ie(B\times a).x}{2\hbar c}\Big), \tag{1.2}$$

where a is an arbitrary displacement and the sinusoidal factor arises from the corresponding regauging of the magnetic vector potential.

In Sec. III, we employ this result to derive the Meissner effect, as in Ref. 26. The essential point is that the formula (1.2) for regauged space translations, together with the assumption of ODLRO, leads to the conclusion that *either* Φ *or* B must vanish. The requirement of thermodynamic stability of the ordered phase then implies that the latter condition prevails, for sufficiently weak applied fields. Thus, we have a Meissner effect, since normally diamagnetic systems can accommodate non-zero uniform induction. The key to this result is that the macroscopic wave function exhibits a rigidity, in the face of the applied field, of the kind envisaged by London[1] at a more microscopic level. It is worth remarking here that the principle of local gauge covariance, which was an obstacle to the previous theories, is an essential ingredient of our argument leading to the Meissner effect!

In Sec. IV, we extend that argument, along the lines of Ref. 36, to derive the quantisation of trapped magnetic flux in multiply-connected systems.

In Sec. V, we formulate the theory of persistent currents in a multiply-connected body, such as a ring. Here, the supercurrents are none other than the currents that implement the Meissner effect by screening the trapped magnetic flux from the interior of the body[2], and the essential problem is that of the stabilisation of both the trapped flux and its screening current. In fact, this is a problem of *metastability*, since the current-carrying state has higher free energy than that of true thermal equilibrium at the same temperature. Thus, adopting our earlier characterisation[14] of metastable states by thermodynamic stability against strictly local, rather than global, disturbances, we formulate the condition for the persistence of currents in terms of a variational principle, that is subject to the constraint of the flux quantisation. In this way, we characterise the phenomenon of superconductivity itself, i.e. the persistence of currents, by a relatively simple thermodynamic assumption, together with that of ODLRO. Furthermore, we show that this phenomenon is intimately related to a superselection rule, that forbids locally induced transitions between states with different flux quantum numbers.

In Sec. VI, we formulate the Josephson effect and derive its tunnelling frequency form the properties of the macroscopic wave function.

We conclude, in Sec. VII, with a brief discussion of open problems of the theory.

II. The General Model

Our model, Σ, is an infinitely extended system, consisting of electrons and possibly another species of particles, e.g. phonons or ions, in a space, X, which may be either a Euclidean continuum or a lattice. Points in X will usually be denoted by x, sometimes by y, a or b. We shall denote the electronic component of Σ by Σ_{el}, and the other component, if any, by Σ'. We shall assume that the dynamics of the model is covariant w.r.t. gauge transformations of both first and second kind, space translations and time reversals. These assumptions represent general demands of quantum mechanics and electromagnetism, and are fulfilled by particular models, such as those of Fröhlich[15] and Hubbard[38], on which theories of metallic and high T_c superconductivity, respectively, are based.

For simplicity, we shall generally employ a notation appropriate to the case where X is a continuum. This may easily be translated into the corresponding one for lattice case by standard procedures employed in gauge theories[39].

We shall describe the electronic subsystem Σ_{el} in terms of a quantised field,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_{-1} \end{pmatrix} \equiv \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \tag{2.1}$$

which satisfies the canonical anticommutation relations

$$[\psi_s(x), \psi^\star_{s'}(x')]_+ = \delta_{ss'}\delta(x - x');\ [\psi_s(x), \psi_{s'}(x')]_+ = 0 \tag{2.2}$$

We shall sometimes use the symbol $\psi^\#$ to denote either ψ or $\psi^\star$.

The observables[40] of Σ_{el} are formed from the polynomials in ψ and $\psi^\star$ that are invariant under gauge transformations of the first kind, $\psi \rightarrow \psi e^{i\alpha}$, with α constant. Thus, they are generated algebraically by operators of the form

$$\psi^\star_{s_1}(x_1)..\ .\psi^\star_{s_n}(x_n)\psi_{s_{n+1}}(x_{n+1})..\ .\psi_{s_{2n}}(x_{2n}).$$

We shall be concerned with the properties of the the system in the presence of a classical electromagnetic field (E, B), represented by a scalar potential ϕ and a vector potential, A : thus, $E = -\nabla\phi - c^{-1}\partial A/\partial t$ and $B = curl A$. We assume that the dynamics of the model is covariant w.r.t. gauge transformations of the second kind, as given by the formula

$$A \rightarrow A + \nabla\chi,\ \phi \rightarrow \phi - c^{-1}\frac{\partial\chi}{\partial t},\ \psi \rightarrow \psi \exp(\frac{ie\chi}{\hbar c}) \qquad (2.3),$$

where χ is an arbitrary function of position and time. We assume that the observables representing the position-dependent densities of electronic charge, current and magnetic polarisation in the presence of this field are given by the standard formulae

$$\rho = -e\psi^\star\psi;\ j(x) = \frac{ie}{2}(\psi^\star\nabla\psi - (\nabla\psi^\star)\psi) + \frac{e}{c}A\psi^\star\psi;\ and\ m = \frac{e\hbar}{mc}\psi^\star\sigma\psi, \tag{2.4}$$

where $-e$ is the electronic charge and σ is the spin vector, whose components $(\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. We note here that ρ, j and m are invariant w.r.t. all gauge transformations.

We assume that the formal Hamiltonian of the model is of the form

$$H = \int(\hbar\nabla\psi^\star + \frac{ie}{c}A\psi^\star).(\hbar\nabla\psi - \frac{ie}{c}A)dx - \int(B.m + \rho\phi)dx + V_{el} + H_{int} + H', \tag{2.5}$$

where V_{el} is the potential energy of the interelectronic interactions, H' is the Hamiltonian for Σ' and H_{int} is the energy of interaction between Σ_{el} and Σ'. We assume that these three contributions to H are all independent of the potentials ϕ and A, and thus that H is gauge invariant. Further, we stipulate that the magnetic interactions between the electrons, that stem from the sources $j(x)$ and $m(x)$

according to Maxwell's equations, are not incorporated into either V_{el} or H_{int}, but are represented by the dependence of B on these sources (cf. Comment at the end of this Section). In the theory that follows, we shall confine our considerations to situations where B is static, $E = 0$, and, except in Sec. 6, we shall take it that $\phi = 0$ and χ is time-independent.

We shall assume that the dynamics is covariant w.r.t. space translations and time reversals. The transformations of the fields ψ, A, corresponding to space translations $a(\in X)$, are given by

$$A(x)\to(x+a),\ \psi(x)\to\psi(x+a). \tag{2.6}$$

The operation of time reversal serves to transform ψ_s, $\psi_s^\star$ and A to $\psi_{-s}^\star$, ψ_{-s} and $-A$, respectively, and to invert the order of the terms in the operator products. Thus, its effective action on the electronic observables and vector potential is given by

$$\psi_{s_1}^{\#}(x_1).\,.\psi_{s_n}^{\#}(x_n)\to\psi_{-s_n}^{\#\star}(x_n).\,.\psi_{-s_1}^{\#\star}(x_1);\ A\to-A. \tag{2.7}$$

Specialising now to the case where the magnetic induction B is uniform and so may be represented by the vector potential $A(x) = \frac{1}{2}B\times x$, and choosing $\chi(x) = -(B\times x).a$, we have the relation $A(x) + \nabla\chi(x) = \chi(x-a)$. Hence, by (2.3) and (2.6), the dynamics of Σ is covariant w.r.t. the transformation[37,26]

$$\psi(x)\to\psi_a(x)\equiv\psi(x+a)\exp(\frac{-ie(B\times x).a)}{e\hbar c}),\ A(x)\to A(x), \tag{2.8}$$

together with the transformation of the $\Sigma'-$ observables corresponding to space translations. Since (2.6) consists of a space translation, compensated by a gauge transformation in such a way as to leave A unchanged, we term it a *regauged space translation.*

We denote by Q_t the time-translate, in Heisenberg representation, of an arbitrary observable, Q, of Σ. A dynamical characterisation of thermal equilibrium states of the system at inverse temperature β is given by the Kubo- Martin-Schwinger (KMS) condition[41,42], i.e.,

$$\langle Q_tQ'\rangle_A = \langle Q'Q_{t+i\hbar\beta}\rangle_A, \tag{2.9}$$

for arbitrary observables Q and Q', where the angular brackets denote expectation value and the suffix A indicates its dependence on A. Most importantly, this condition is valid even for infinite systems[43], where the traditional Gibbs canonical formulation is not directly applicable. In the case where B is uniform, the covariance of the dynamics w.r.t regauged space translations ensures that the model supports equilibrium states that are invariant under these translations.

Turning now to the thermodynamics of the model, we define $f(B,T)$ to be its global free energy density at temperature T, under the action of the uniform induction, B. This function can be formulated by standard statistical mechanical methods[44] in terms of the microscopic description of Σ, with B taken to be a given control variable.

On the other hand, this induction is *not* given, a priori, in the physical situation where Σ is subjected to a magnetic field, H_{ex}, due to a fixed source of current density J_{ex}. For, in this situation, the induction, B, and the equilibrium state of Σ co-determine one another.

To formulate the thermodynamics of Σ under these conditions, we have to incorporate the energy of the field, B, and of its interaction with the source, J_{ex}, into the picture. Thus, we have to consider the system under consideration to comprise the composite, $\tilde{\Sigma}$, of Σ and the field B. The contribution to the internal energy of $\tilde{\Sigma}$ due to the induction, B, and its coupling to the source is then

$$\int (\frac{1}{2}B^2 - c^{-1}A.J_{ex})dx,$$

and, in view of the Maxwell equation $curl H_{ex} = c^{-1}J_{ex}$, this reduces to

$$\int (\frac{1}{2}B^2 - H_{ex}.B)dx. \qquad (2.10)$$

Thus, in the case where B and H_{ex} are uniform, the free energy density of $\tilde{\Sigma}$ is

$$\phi(B,T,H_{ex}) = f(B,T) + \frac{1}{2}B^2 - B.H_{ex}. \qquad (2.11)$$

The equilibrium value of B, corresponding to given H_{ex} and T, is then obtained by minimising ϕ, and the resultant Gibbs free energy density is

$$\tilde{f}(H_{ex},T) = min_B \phi(B,T,H_{ex}). \qquad (2.12)$$

Comment. This formulation of the thermodynamics of $\tilde{\Sigma}$ is, of course, semi-classical, since the field B is treated as classical. In this picture, the field energy (2.10) stems from the magnetic interactions of the currents and spins in Σ both with one another and with the external source. For a discussion of the problem of a fully quantum formulation, see Ref. 27, Sec.4.

III. ODLRO and the Meissner Effect.

In order to formulate ODLRO, we introduce the *pair field*

$$\Psi(x_1,x_2)=\psi_{\uparrow}(x_1)\psi_{\downarrow}(x_2). \tag{3.1}$$

We then say that a state of Σ possesses the property of ODLRO if there is a *classical* two-point field $\Phi(x_1,x_2)$ such that[29]

$$\lim_{|y|\to\infty}\left[\langle\Psi(x_1,x_2)\Psi^{\star}(x_1'+y,x_2'+y)\rangle_A-\Phi(x_1,x_2)\Phi^{\star}(x_1'+y,x_2'+y)\right]=0, \tag{3.2}$$

and, further, $\Phi(x_1+y,x_2+y)$, does not tend to zero as $|y|\to\infty$. In this case, Φ is termed the *macroscopic wave function* of the state.

Note. Although Ψ is not an an observable, in the sense specified in Sec. II, the quantity in angular brackets in the ODLRO condition (3.2) is.

In order to relate ODLRO to superconductive electrodynamics, we note that the essential distinction between normal diamagnetism and the Meissner effect is that the former can support a uniform, static, non-zero magnetic induction and the latter cannot. Thus, we base our derivation of the Meissner effect on considerations of the response of a state possessing ODLRO to the action of a uniform magnetic field.

The following two Propositions, which we shall prove at the end of this Section, provide us with the key to the relationship between ODLRO and the electromagnetic properties of our model.

Proposition 3.1. *(1) The ODLRO condition (3.2) defines the macroscopic wave function Φ, up to a phase factor, i.e., if Φ_1 and Φ_2 are two such functions which satisfy this condition for the same state, then $\Phi_1=e^{i\alpha}\Phi_2$, where α is some real constant.*

(2) In the case where $A=0$ and where the state is invariant with respect to time reversals, the function Φ is real-valued, up to a constant phase factor.

Proposition 3.2. *If the system is in a translationally invariant state possessing the property of ODLRO, then*

(1) in the case where X is a continuum, the magnetic induction B vanishes, and

(2) in the case where X is a lattice, B is restricted to the discrete set of values $\{B^{(n)}\}$, given by the condition that $B^{(n)}.(a\times b)$ is an integral multiple of $\pi e/\hbar c$,

for any a, b in X. Thus, the vectors $B^{(n)}$ are the sites of an associated lattice, $\mathcal{B}$, defined by this condition.

Comments. (1) Prop. 3.2 demonstrates that, in the continuum case, translationally invariant (including equilibrium) states with ODLRO do not admit uniform magnetic fields, i.e., they exhibit the Meissner effect.

(2) In the case where X is a lattice, this conclusion must be modified by the possibility that ODLRO might be compatible with a non-zero quantised induction $B^{(n)}$.

(3) The question of whether ODLRO, with or without the quantised magnetic induction $B^{(n)}$, prevails is a thermodynamic one.

(4) The removal of ODLRO by even infinitessimal changes in B from the value 0 or, in the lattice case, $B^{(n)}$ suggests the advent of a phase transition.

(5) In the case of lattice systems, the non-zero $B^{(n)}$'s are of the order of $\hbar c/el^2$, where l is the spacing of the lattice X. Thus, for typical values of l, e.g. 10^{-8}cm., they are of the order of 10^9G, which is not only many orders of magnitude larger than any known critical fields for superconductors, but also much larger than the internal fields in ferromagnets.

We base our treatment of the thermodynamics and phase structure of the system, which is evidently needed in view of Comment (3), on the following assumptions.

(III.1) *(a) The equilibrium state of Σ corresponding to any given (A, T) is unique and therefore translationally invariant.*

(b) Similarly, the equilibrium state of $\tilde{\Sigma}$ corresponding to given (H_{ex}, T) is unique and translationally invariant.

Note. This assumption precludes the applicability of our treatment to the mixed phase of type II superconductors, since this breaks the space translational symmetry of the system.

The next assumption, prompted by the above Comment (5), excludes ODLRO states with non-zero quantised induction $B^{(n)}$ from the theory, and thus puts the results of Prop. 3.2 for continuous and lattice systems onto the same footing.

(III.2) *Even in the case of lattice systems, the equilibrium states of $\tilde{\Sigma}$ that possess the property of ODLRO carry no magnetic induction.*

The next assumption follows the line suggested by the above Comment (4).

(III.3) *(a) For $B = 0$, the system Σ undergoes a phase transition at a temperature T_c, such that, for $T < T_c$ only, it exhibits the property of ODLRO.*

(b) For $B \neq 0$, on the other hand, the thermodynamic potential $f(B, T)$ of the model is a smooth function of both its arguments, which tends to a definite limit as $B \rightarrow 0$.

In view of (III.2) and (III.3), we define the free energy densities of Σ for the ODLRO and normal phases at temperatures below T_c to be

$$f_{\mathcal{O}}(T) = f(0, T) \tag{3.3}$$

and

$$f_{\mathcal{N}}(B, T) = f(B, T) \ for \ B \neq 0, \tag{3.4}$$

respectively. Correspondingly, in view of equns. (2.11), (2.12) and (3.5), we define the Gibbs potential $\tilde{f}_{\mathcal{N}}$ for the normal phase of $\tilde{\Sigma}$ by the formula

$$\tilde{f}_{\mathcal{N}}(H_{ex}, T) = min_B \big(f_{\mathcal{N}}(B, T) + \frac{1}{2}B^2 - H_{ex}.B \big). \tag{3.5}$$

We note that, by (III.3b), the potential $f_{\mathcal{N}}$ may be extended to $B = 0$ by continuity, according to the formula

$$f_{\mathcal{N}}(0, T) = \lim_{B \rightarrow 0} f_{\mathcal{N}}(B, T). \tag{3.6}$$

Our next assumption serves to relate the phase structure of the partial system Σ to that of $\tilde{\Sigma}$.

(III.4) *For $T < T_c$ and $H_{ex} = 0$, the equilibrium state of $\tilde{\Sigma}$ possesses ODLRO.*

This is equivalent to the demand that

$$f_{\mathcal{O}}(T) < f_{\mathcal{N}}(B, T) + \frac{1}{2}B^2 \ for \ T < T_c, \ B \neq 0,$$

and implies, by equns. (3.4)-(3.6), that

$$f_{\mathcal{O}}(T) \leq \tilde{f}_{\mathcal{N}}(0, T) \ for \ T < T_c. \tag{3.7}$$

Our final assumption concerning the system $\tilde{\Sigma}$ is the following, whose basis we shall discuss below.

(III.5) *(a) The strict inequality prevails in the condition (3.7), i.e.,*

$$f_{\mathcal{O}}(T) < \tilde{f}_{\mathcal{N}}(0,T) \ for \ T < T_c. \tag{3.8}$$

(b) The potential $\tilde{f}_{\mathcal{N}}(H_{ex},T)$ is continuous in both its arguments.

Here, (III.5a) is based on the idea that the radical difference in structure between the ODLRO and normal phases at $T < T_c, \ H_{ex} = 0$ should correspond to a difference between their Gibbs potentials, as in phase transitions of the first kind[45,46]. (III.5b), on the other hand, represents the assumption that there are no sub-phase transitions within the normal phase for $T < T_c$.

It follows immediately from assumptions (III.5) that, for $T < T_c$ and $|H_{ex}|$ sufficiently small,

$$f_{\mathcal{O}}(0,T) < \tilde{f}_{\mathcal{N}}(H_{ex},T),$$

and hence, by equn. (3.5),

$$f_{\mathcal{O}}(0,T) < f_{\mathcal{N}}(B,T) + \frac{1}{2}B^2 - H_{ex}.B \ \forall T < T_c, \ B \neq 0. \tag{3.9}$$

This signifies that the equilibrium state enjoys the property of ODLRO and is induction-free. In other words, we have the Meissner effect. We summarise this result as follows.

Proposition 3.3. *Under the above assumptions, and for $T < T_c$ and $|H_{ex}|$ less than some critical value, $H_c(T)$, the equilibrium state of the system $\tilde{\Sigma}$ has the property of ODLRO and carries no magnetic induction. Thus, it exhibits the Meissner effect.*

Comments. (1) This result stems from the stability ODLRO against applied magnetic fields, and corresponds to a rigidity of the macroscopic wave function similar to that envisaged by London[1] for the microstate of the system.

(2) In general, when B and H_{ex} might not be uniform, the quantum statistics of Σ still yields a constitutive equation for the position-dependent current density, $J(x)$, of the form

$$J(x) \equiv \langle j(x) \rangle_A = F(A,x), \tag{3.10}$$

where the functional F satisfies the gauge invariance condition

$$F(A,x) = F(A + \nabla\chi; x). \tag{3.11}$$

(3) According to the London theory[2], the essential difference between a Meissner effect and a normal diamagnetism may be simply expressed in terms of the linearised form, F_{lin}, of F as follows. For the Meissner effect, F_{lin} is proportional to the transversely gauged vector potential A_{tr} in the large scale limit; while for normal magnetism it is proportional to $\nabla^2 A_{tr}$ $(= -curl B)$ in this limit.

(4) The relation between the critical field, $H_c(T)$, and the thermodynamic potentials of the model takes a particularly simple form in the case where the magnetic susceptibility is negligible in the normal phase, e.g. in type-I metallic superconductors. For, in this case, $f_{\mathcal{N}}(B,T)$ may be equated with $f_{\mathcal{N}}(0,T) \equiv f_{\mathcal{N}}(T)$. Hence, the condition (3.9) for the Meissner effect reduces to the form

$$f_{\mathcal{O}}(T) < f_{\mathcal{N}}(T) - \frac{1}{2} H_{ex}^2,$$

and, consequently, the critical field, H_c is given by the standard formula[2]

$$H_c(T) = (f_{\mathcal{N}}(T) - f_{\mathcal{O}}(T))^{1/2}. \tag{3.12}$$

We shall conclude this Section with the proofs of Props. 3.1 and 3.2.

Proof of Proposition 3.1. (1) Assuming that Φ_1, Φ_2 both satisfy the ODLRO condition (3.2), it follows from that equation that

$$\lim_{|y|\to\infty} \left[\Phi_1(x_1,x_2)\Phi_1^\star(x_1'+y, x_2'+y) - \Phi_2(x_1,x_2)\Phi_2^\star(x_1'+y, x_2'+y)\right] = 0. \tag{3.13}$$

Since this is valid for all $x_1, x_2, x_1', x_2' \in X$, we may replace x_1, x_2 here by arbitrary points x_1'', x_2'', thereby obtaining

$$\lim_{|y|\to\infty} \left[\Phi_1(x_1'',x_2'')\Phi_1^\star(x_1'+y, x_2'+y) - \Phi_2(x_1'',x_2'')\Phi_2^\star(x_1'+y, x_2'+y)\right] = 0. \tag{3.14}$$

On multiplying (3.13) by $\Phi_2(x_1'',x_2'')$ and (3.14) by $\Phi_2(x_1,x_2)$, and then taking the difference, we see that

$$\lim_{|y|\to\infty} \Phi_1^\star(x_1'+y, x_2'+y)\left[\Phi_1(x_1,x_2)\Phi_2(x_1'',x_2'') - \Phi_1(x_1'',x_2'')\Phi_2(x_1,x_2)\right] = 0.$$

Since, by the definition of ODLRO, Φ does not tend to zero at infinity, it follows that the quantity in the square brackets of this last equation vanishes. Therefore, as Φ_1 and Φ_2 are non-zero, by the same stipulation,

$$\Phi_2(x_1,x_2) = c\Phi_1(x_1,x_2) \ \forall x_1, x_2 \in X,$$

where c is a complex-valued constant. Consequently, as Φ_1, Φ_2 both satisfy (3.2), it follows immediately that c is just a constant phase factor $\exp(i\alpha)$, as required.

(2) Assuming now that the state is invariant under time-reversals and that $A = 0$, it follows from equns. (2.2) and (2.7) that, if Φ satisfies the ODLRO condition (3.2), then so too does $\Phi^\star$. Hence, by part (1) of this Proposition, $\Phi^\star = \exp(i\alpha)\Phi$, where α is a real-valued constant. In other words, the function $\Phi\exp(i\alpha/2)$, is real-valued, and this is the required result.

Proof of Proposition 3.2. Let $\Psi_a \equiv \gamma(a)\Psi$ be the transform of Ψ under the regauged space translation (2.8). Then, by equn. (3.1),

$$\Psi_a(x_1,x_2) \equiv (\gamma(a)\Psi)(x_1,x_2) = \Psi(x_1+a,x_2+a)\exp\big(\frac{-ie(B\times(x_1+x_2)).a}{2\hbar c}\big). \quad (3.15)$$

It follows immediately from this formula that the transformations $\{\gamma(a)|a\in X\}$ do not all intercommute and that, in fact,

$$\gamma(-a-b)[\gamma(a)\gamma(b)-\gamma(b)\gamma(a)] = 2i\sin\big(\frac{e(B\times a).b}{\hbar c}\big). \quad (3.16)$$

We now note that, by the regauged translational invariance of the equilibrium state,

$$\langle\Psi_a(x_1,x_2)\Psi_a^\star(x_1',x_2')\rangle_A \equiv \langle\Psi(x_1,x_2)\Psi^\star(x_1',x_2')\rangle_A. \quad (3.17)$$

Further, it follows from (3.15), (3.17) and the ODLRO condition (3.2) that

$$\lim_{y\to\infty}\big[\langle\Psi(x_1,x_2)\Psi^\star(x_1'+y,x_2'+y)\rangle_A - \Phi_a(x_1,x_2)\Phi_a^\star(x_1'+y,x_2'+y)\big] = 0, \quad (3.18)$$

where

$$\Phi_a \equiv \gamma(a)\Phi, \quad (3.19)$$

with γ defined as in (3.15). Since equations (3.2) and (3.17) imply that Φ_a, as well as Φ, serves as a macroscopic wave function for the equilibrium state, it follows from Prop. 3.1(1) that $\Phi_a \equiv \gamma(a)\Phi = c(a)\Phi$, where $c(a)$ is a complex number of unit modulus. As this is valid for all spatial displacements a, it follows that

$$\gamma(a)\gamma(b)\Phi = \gamma(b)\gamma(a)\Phi = c(a)c(b)\Phi \ \forall a,b\in X,$$

and therefore, by (3.16),

$$\sin\big(\frac{eB.(a\times b)}{\hbar c}\big)\Phi = 0 \ \forall a,b\in X.$$

Consequently, since $\Phi \neq 0$, by the assumption of ODLRO,

$$\sin\left(\frac{eB.(a \times b)}{\hbar c}\right) = 0 \ \forall a, b \in X,$$

which is equivalent to the required result.

IV. Flux Quantisation.

We now consider now the situation where a cylindrical region $\Gamma(\subset X)$ is removed from the body of Σ, so that the system occupies the complementary region $X' = X - \Gamma$. We assume that, for $T < T_c$, the system exhibits ODLRO and the Meissner effect, and that the constitutive equation (3.10) remains unchanged, except possibly for inessential modifications due to the new boundary conditions. Thus, a sufficiently weak magnetic field in the tunnel Γ will be screened from the body of the system by currents in a 'surface layer', and magnetic flux will be trapped in Γ and that layer.

We shall derive the quantisation rule for this trapped flux, $\mathcal{F}$, by an adaptation of the argument of Ref. 36, from general properties of the macroscopic wave function Φ that stem from gauge covariance, symmetry and topology. This argument will be centred exclusively on the properties of the model in the asymptotic region, far from the tunnel, where the magnetic induction can be taken to be zero and where we shall assume that translational covariance prevails. Thus, although we shall not keep repeating the point, the properties we invoke here will all be asymptotic ones, that become exact only in a limit where the regions in question are infinitely far from the tunnel.

Suppose now that C is a closed loop in X', in the deep interior of the body, that encircles the tunnel Γ. Then it follows immediately from the above specifications and Stokes's theorem that

$$\int_C A.ds = \mathcal{F}. \tag{4.1}$$

Thus, as in the Aharonov-Bohm effect[47], the magnetic vector potential survives, with non-trivial circulation, even in the field-free region.

On the other hand, if C' is a loop in the asymptotic field-free region that does not encircle the tunnel, then, by Stokes's theorem,

$$\int_{C'} A.ds = 0. \tag{4.2}$$

We now use this formula to obtain an asymptotic covariance w.r.t. certain regauged space translations, similar to those given by equn. (2.8). To this end, we choose C' to be the parallelogram whose vertices are $x-a,\ x,\ x+h$ and $x-a+h$. Thus, denoting by $\int_y^z A.dl$ the integral of A along the straight line leading from y to z, we see from equn. (4.2) that

$$\Big(\int_{x-a}^{x}+\int_{x}^{x+h}+\int_{x+h}^{x+a+h}+\int_{x+a+h}^{x-a}\Big)A.dl=0.$$

Hence, defining

$$\theta_a(x)=\int_{x-a}^{x}A.dl, \tag{4.3}$$

we obtain the formula

$$\theta_a(x+h)-\theta_a(x)=\Big(\int_{x}^{x+h}-\int_{x-a}^{x-a+h}\Big)A.dl,$$

from which it follows, on applying the gradient operator w.r.t. h, at $h=0$, that

$$\nabla\theta_a(x)=A(x-a)-A(x). \tag{4.4}$$

We now invoke the covariance of the model w.r.t. the gauge transformation (2.3), and observe that, in view of equn. (4.4), this reduces to the following form when χ is chosen to be θ_a.

$$A(x)\rightarrow A(x-a);\ \psi(x)\rightarrow\psi(x)\exp\Big(\frac{ie\theta_a(x)}{\hbar c}\Big). \tag{4.5}$$

Thus, the dynamics is covariant w.r.t. this transformation in the asymptotic field-free region.

We also assume that it is covariant there w.r.t. space translations

$$A(x)\rightarrow A(x+a);\ \psi(x)\rightarrow\psi(x+a). \tag{4.6}$$

On combining this with the gauge transformation (4.5), we obtain the *regauged space translation* (cf. equn. (2.8))

$$A(x)\rightarrow A(x);\ \psi(x)\rightarrow\psi(x+a)\exp\Big(\frac{ie\theta_a(x+a)}{\hbar c}\Big). \tag{4.7}$$

Thus, the dynamics is (asymptotically) covariant w.r.t. these regauged translations. Correspondingly, assuming uniqueness of the equilibrium state for given (A,T), as in Sec.III (cf. assumption (III.1)), the expectation value of

$\Psi(x_1,x_2)\Psi^\star(x_1',x_2')$ becomes (asymptotically) invariant under this transformation. Hence, defining

$$\Phi_a(x_1,x_2)=\Phi(x_1+a,x_2+a)\exp\big(\frac{i(\theta_a(x_1+a)+\theta_a(x_2))}{\hbar c}\big), \qquad (4.8)$$

it follows from equns. (3.1), (4.7) and (4.8) that the ODLRO condition (3.2) remains valid when Φ is replaced is Φ_a. Consequently, by Prop. 3.1(1), $\Phi_a = \exp(i\eta(a))\Phi$, for some $a-$dependent phase angle, $\eta(a)$, i.e., by equns. (4.3) and (4.8),

$$\Phi(x_1+a,x_2+a)=e^{i\eta(a)}\Phi(x_1,x_2)\exp\big(\frac{-ie}{\hbar c}(\int_{x_1}^{x_1+a}+\int_{x_2}^{x_2+a})A.dl\big). \qquad (4.9)$$

Now let $a_0(=0),\ a_1,\ a_2,..,a_n$ be a sequence of displacements, whose sum is zero, and let $C(x)$ be the closed contour formed from the lines joining $x+a_0+\ \ +a_{r-1}$ to $x+a_0+\ \ +a_r$, for $r=1$ to n. For each pair of points x_1 and x_2, we choose the a_r's so that the contours $C(x_1)$ and $C(x_2)$ lie in the asymptotic field-free region and encircle the tunnel, Γ. Thus, by equn. (4.1),

$$\int_{C(x_j)} A.dl=\mathcal{F},\ for\ j=1,2.$$

It follows now from these specifications and that the application of equn. (4.9) to the succession of spatial displacements by $a_1,..,a_n$ leads to the formula

$$\Phi(x_1,x_2)=\Phi(x_1,x_2)\exp\big(i(\frac{2e\mathcal{F}}{\hbar c}+\eta)\big),$$

where $\eta=\eta(a_1)+..+\eta(a_n)$. Hence

$$\exp(i(\frac{2e\mathcal{F}}{\hbar c}+\eta))=1. \qquad (4.10)$$

Furthermore[36], on replacing x_1,x_2 by points y_1,y_2, that are chosen so that the contours $C(y_1)$ and $C(y_2)$ do not encircle Γ, we are led by the same argument to the formula

$$\exp(i\eta)=1.$$

Consequently, by equn. (4.10), $\exp(2ir\mathcal{F}/\hbar c)=1$. This immediately implies the following result, which accords with the experiment of Deaver and Fairbank[5].

Proposition 4.1. *Under the above assumptions, the trapped flux, $\mathcal{F}$, is quantised in integral multiples of $nhc/2e$.*

V. Supercurrents and Superselection Rules.

The phenomenon of superconductivity itself is that of the persistence of currents induced in a ring, in the absence of any applied electric field. According to the London macroscopic theory[2], this phenomenon is intimately connected to the Meissner effect. Specifically, the supercurrents are the source of a magnetic field, which they themselves screen from the interior of the body. Thus, the currents and magnetic field stabilise one another. Further, it is clear that the supercurrent-carrying states must be *metastable*, rather than absolutely stable, since both the currents and the magnetic field carry positive energy.

Our objective now is to provide a quantum mechanical basis for the above phenomenological picture. We base our considerations on the quantum mechanics of a system occupying a multiply-connected region, e.g. with the topology of a torus. For simplicity, we employ the model of the Sec. IV and, as in Secs. II and III, we shall denote by $\tilde{\Sigma}$ the system comprising the matter, Σ, and the magnetic induction, B.

So again we consider the state of the system in which magnetic induction is trapped in the tunnel, Γ, and a 'surface region' of the matter. The supercurrents are then the surface currents that implement the Meissner effect by screening the induction out of the deep interior of the body. Our objective now is to investigate the stability properties of the trapped magnetic flux and the associated supercurrents, corresponding to the flux quantum number n ($\neq 0$), as given by the formula

$$\int_S B.dS = nhc/2e, \tag{5.1}$$

where S is a surface orthogonal to the generators of the tunnel Γ.

We start by noting that since, by equn. (2.4), the densities of charge, current and polarisation are fully gauge invariant, their expectation values in the equilibrium state of Σ corresponding to (A,T) are functionals of the induction B, as defined throughout the space X, and not merely of A. Thus, we have constitutive equations of the form

$$P(x)\equiv\langle\rho(x)\rangle_A = \mathcal{P}(B;x) \tag{5.2}$$

$$J(x)\equiv\langle j(x)\rangle_A = \mathcal{J}(B;x) \tag{5.3}$$

and

$$M(x)\equiv\langle m(x)\rangle_A = \mathcal{M}(B;x). \tag{5.4}$$

These equations apply, of course, only to points x in the matter. We extend the definitions of the functionals $\mathcal{P}$, $\mathcal{J}$ and $\mathcal{M}$ to the whole space X by the specification that they vanish in the tunnel, Γ. Thus, the magnetic field $H(x) = B(x) - \mathcal{M}(B; x)$ and induction B satisfy the Maxwell equations

$$curl H(x) = c^{-1}\mathcal{J}(B; x);\ and\ div B = 0,$$

i.e.,

$$curl B(x) = curl \mathcal{M}(B; x) + \mathcal{J}(B; x);\ and\ div B(x) = 0,\ \forall x \in X. \qquad (5.5)$$

We now make the following assumptions concerning the solution of these equations for B.

(V.1) *There is a unique solution, B_n, of the equations (5.5) for B, subject to the constraint (5.1) and the standard conditions on the boundary of Γ and infinitely far from this cylinder, where B vanishes, by Meissner effect.*

(V.2) *For $B = B_n$, there is a unique equilibrium state of Σ, as given by the KMS equilibrium conditions (2.9), corresponding to the vector potential A_n chosen to represent this induction.*

It follows now from our specifications that both J and B are non-zero in a surface region of the matter when $B = B_n$, for $n \neq 0$, and Σ is in the corresponding equilibrium state. Thus, the question of the persistence of the supercurrent and trapped magnetic field now reduces to that of the stability of the state of $\tilde{\Sigma}$ defined by these conditions. Moreover, any such stability must, at most, be a metastability, since the advent of the current and trapped flux ensure that the free energy of this state exceeds that of true thermal equilibrium. To be precise, a characterisation of the highest grade of metastability corresponds to thermodynamic stability against strictly localised perturbations[14], as represented by the condition that such perturbations cannot lead to a decrease in the free energy of the system. By contrast, true equilbrium corresponds to a global thermodynamic stability.

Thus, we reduce the problem of the persistence of supercurrents to that of the thermodynamic stability of the above-described state of $\tilde{\Sigma}$ against local perturbations. To formulate this problem, we define the local perturbations of the induction, B_n, to comprise the set, $\mathcal{B}$, of divergence-free vector fields, b, such that

(b.1) $b(x) = 0$ unless x lies inside some bounded region, D_b, and (b.2) the field b does not change the value of the flux, as given by the l.h.s. of equn. (5.1).

In fact, the second condition is redundant if the dimensionality of X exceeds 2, since then the divergence-free property of b implies that $\int_S b.dS$, the flux due to b, is independent of the particular choice of S; and therefore, by choosing this surface to lie in the region where b vanishes, we see that this integral is zero. On the other hand, if X is two-dimensional and D_b is just a bounded region of that space, then (b) is equivalent to the condition that

(b.2)′ $\int_{D_b} b.dS = 0$.

Further, as we shall prove in the Appendix, the conditions (b.1,2) imply that b may be represented by a vector potential a that vanishes outside some bounded region.

We formulate the increment $\phi_n(b)$ in the free energy of Σ, corresponding to the perturbation of B_n by b, by the following procedure. We first note that it follows from our above specifications and equns. (2.4) and (2.5) that the interaction Hamiltonian of the system due to this perturbation is the localised observable

$$K = \int_{X'} (-c^{-1}j.a + \frac{1}{2}c^{-2}\rho a^2 - m.b)dx, \tag{5.6}$$

where j, ρ and m are as defined for $A = A_n$. The corresponding interaction Hamiltonian for the perturbative field λb, with λ real, is therefore

$$K(\lambda) = \int_{X'} (-\lambda c^{-1}j.a + \frac{1}{2}\lambda^2 c^{-2}\rho a^2 - \lambda m.b)dx. \tag{5.7}$$

We now invoke the fact that, by classical thermodynamics, the increment in free energy of a system, due to an infinitesimal change ΔH in its Hamiltonian, is given by the equilibrium expectation value of ΔH. In the present situation, this implies that the change in free energy of Σ due to an infinitesimal increment increment $d\lambda$ in the parameter λ of equn. (5.7), is

$$\langle K'(\lambda)\rangle_{A_n+\lambda a} d\lambda, \tag{5.8}$$

where

$$K'(\lambda) = \int (-c^{-1}j.a + \lambda c^{-2}\rho a^2 - m.b)dx. \tag{5.9}$$

Thus, by (5.8), the incremental free energy, due to the perturbation b of the magnetic induction B_n, is

$$\phi_n(b) = \int_0^1 \langle K'(\lambda)\rangle_{A_n+\lambda a} d\lambda. \tag{5.10}$$

The corresponding increment in the free energy of $\tilde{\Sigma}$ is therefore

$$\tilde{\phi}_n(b) = \phi_n(b) + \int_X (b.B_n + \frac{1}{2}b^2)dx, \tag{5.11}$$

the integral being the contribution due to the magnetic field energy.

We now recall that[14], for fixed A, the KMS condition (2.9) ensures that the free energy of Σ cannot be reduced by any localised modification of its equilibrium state. Hence, by equn. (5.11), the condition for metastability of the state of $\tilde{\Sigma}$, in which $B = B_n$ and Σ satisfies the corresponding KMS condition, is that $\tilde{\phi}_n$ is minimised at $b = 0$. In fact, it is a simple matter to infer from our constructions that $\tilde{\phi}_n$ is stationary at $b = 0$. For, by equns. (5.2)-(5.4) and (5.9)-(5.11), together with the stipulation that $\mathcal{J}$ and $\mathcal{M}$ vanish in the tunnel,

$$\frac{\partial}{\partial\lambda}\phi_n(\lambda b)_{|\lambda=0} = -\int_X \big(c^{-1}\mathcal{J}(B_n;x).a(x) + \mathcal{M}(B_n;x).b(x)\big)dx.$$

Consequently, by equn. (5.11),

$$\frac{\partial}{\partial\lambda}\tilde{\phi}_n(\lambda b)_{|\lambda=0} = \int_X \big(B_n(x).b(x) - \mathcal{M}(B_n,x).b(x) - c^{-1}\mathcal{J}(B_n,x).a(x)\big)dx.$$

Since $b = curla$ and $\int_X a.curl\mathcal{J} \equiv \int_X \mathcal{J}.curla$, it follows from equn. (5.5) that the r.h.s. of this last equation vanishes. Hence, we have the following result.

Proposition 5.1. *Under the above assumptions, the incremental free energy function $\tilde{\phi}_n$ is stationary at $b = 0$, i.e.*

$$\frac{\partial}{\partial\lambda}\tilde{\phi}_n(\lambda b)_{|\lambda=0} = 0 \ \forall b \in \mathcal{B}.$$

Comment. The problem of the metastability of the current-carrying state $\tilde{\rho}_n$ now reduces to the question of whether the stationary point $b = 0$ corresponds to a minimum of $\tilde{\phi}_n$. In fact[14], this state will be metastable if and only if the following assumption is valid.

(V.3) *The stationary point of $\tilde{\phi}_n$, at $b = 0$, corresponds to either*

(a) an absolute minimum, in which case the lifetime of the above- described current-carrying state is infinite; or

(b) a local minimum, surrounded by maxima in such a way that the activation energy required for escape is sufficently large by comparison with $k_{Boltzmann}T$ to ensure that the state has an enormously long lifetime.

Thus, the basic problem concerning the metastability of supercurrents, i.e. of *superconductivity*, is that of determining which of these alternatives prevails. We remark here that a similar problem was addressed in Ref. 14, Sec.5, where we proved that, in the absence of a magnetic field, persistent translationally invariant states satisfying the KMS conditions could not carry a current. In the present different context, the corresponding result would be be that $\tilde{\phi}_n$ could not support an absolute minimum at $b = 0$, which would mean that only the second alternative of (V.3) could be viable.

We may gain another perspective of this picture by relating the metastability of the superconductive phase to a *superselection rule*, forbidding transitions between states with different flux quantum numbers. To this end, we note that, in the asymptotic field-free region, A_n is the gradient of a scalar potential, χ_n. Further, by equn. (4.1) and Prop. 4.1, this potential is many-valued, its change over a contour passing once round the tunnel being

$$[\chi_n] = nhc/2e. \tag{5.12}$$

Furthermore, since a reversal of the magnetic field in the tunnel corresponds to a reversal of the quantum number n, we see that

$$\chi_{-n} = -\chi_n. \tag{5.13}$$

We now observe that, by the quantisation rule (5.12), $\exp(2ie\chi_n/\hbar c)$ is a single-valued function of position, even though, for n odd, $\exp(ie\chi_n/\hbar c)$ is not. Thus, by gauge covariance of the second kind, the dynamics of the model is covariant w.r.t. the transformation

$$A \rightarrow A + 2\nabla\chi_n;\ \ \psi \rightarrow \psi \exp(2ie\chi_n/\hbar c). \tag{5.14}$$

Hence, by equns. (3.1) and (3.2), the macroscopic wave-function, Φ_A, corresponding to the vector potential A, satisfies the condition

$$\Phi_{A+2\nabla\chi_n}(x_1, x_2) = \Phi_A(x_1, x_2)\exp\bigl(\frac{2ie}{\hbar c}(\chi_n(x_1 + \chi_n(x_2))\bigr).$$

Consequently, choosing $A = -\nabla\chi_n \equiv -A_n$, and denoting Φ_{A_n} by Φ_n, it follows from equn. (5.13) that

$$\Phi_{-n}(x_1, x_2) = \Phi_n(x_1, x_2)\exp\bigl(\frac{2ie}{\hbar c}(\chi_n(x_1 + \chi_n(x_2))\bigr). \tag{5.15}$$

On the other hand, the assumption of covariance w.r.t. time- reversals implies, by equns. (2.2), (2.7), (3.1) and (3.2), that

$$\Phi_n^\star = \Phi_{-n}. \tag{5.16}$$

On combining these last two equations, we obtain the following result.

Proposition 5.2. *Assuming covariance w.r.t. gauge transformations and time reversals, the phase of the macroscopic wave-function Φ_n in the asymptotic field-free region is given by*

$$arg\big(\Phi_n(x_1,x_2)\big) = \frac{2e}{\hbar c}\big(\chi_n(x_1)+\chi_n(x_2)\big). \tag{5.17}$$

Comment. In view of equn. (5.12), this result implies that the equilibrium states corresponding to the different flux quantum numbers n are *globally, i.e. macroscopically,* different from one another, and thus that there is a *superselection rule*[48,49] that forbids transitions between them via the agency of localised operations. Consequently, the allowable states of the model fall into *superselection sectors*, indicated by the quantum number n, and the metastability property of (V.3) pertains to each of these sectors.

VI. The Josephson Effect

Suppose now that two superconductors, whose potential difference is ΔV, are separated by an insulating film. The Josephson effect[6] is that, in this situation, a current of frequency $2e\Delta V/h$ flows across the insulating barrier. Our aim here is to derive this frequency from simple general properties of the macroscopic wave-functions of the superconductors.

Our model, Σ, now consists of the two superconductors, $\Sigma_1,\ \Sigma_2$, and an insulating film, $\mathcal{I}$, which occupy disjoint spatial regions $X_1,\ X_2$ and W, respectively, whose union, X, is either a Euclidean space or a lattice, as in previous Sections. We assume that W is sandwiched between X_1 and X_2, and that its boundaries with these regions are parallel planes. We note here that the generalisation of the model to the form where $X_1,\ X_2$ and W have different crystallographic structures presents no serious problems.

We describe the electronic part of the *whole system* Σ, as before, in terms of the wave operator ψ and the pair field Ψ. Again, we assume that the system exhibits ODLRO at temperatures below a certain critical point, and we denote the macroscopic wave-function by Φ. We define Φ_1 and Φ_2, to be the restrictions of Φ the regions X_1 and X_2, respectively, i.e.,

$$\Phi_j(x,x') = \Phi(x,x') \ for \ x,x' \in X_j, \ j = 1,2.$$

We now introduce an assumption to the effect that

(a) at large distances form their boundaries with $\mathcal{I}$, the properties of the subsystems Σ_1 and Σ_2 reduce asymptotically to the form they would have when isolated; and

(b) in this latter sitation, each of them satisfies the conditions of Prop. 3.1.

Thus, in view of the second part of that Proposition, the assumption is the following.

(VI.1) *The asymptotic forms of* Φ_1 *and* Φ_2 *are given by*

$$\Phi_j(x,x') = \Phi_j^{(0)}(x,x')\exp(i\eta_j) \ for \ j = 1,2, \tag{6.1}$$

where $\Phi_1^{(0)}$, $\Phi_2^{(0)}$, η_1 *and* η_2 *are real, and the latter two quantities are independent of the positions* x *and* x'.

Comments. (1) The angles η_1 and η_2 are not neccessarily equal here, since the translational invariance condition of Prop. 3.1(2) is no longer applicable. We define

$$\eta = \eta_1 - \eta_2. \tag{6.2}$$

(2) Since equilibrium cannot be maintained when the superconductors are held at different potentials, the angles η, η_1 and η_2 might be time-dependent.

In order to consider the possible significance of any time- dependence of these phases, we first note that, in the case of a *single* superconductor, as represented by the model of Sections II and III, it follows from equn. (2.3) that the system is covariant w.r.t. the gauge transformation $\phi \to \phi + V$, $\psi \to \phi\exp(-ieVt/\hbar)$, $A \to A$, with V constant. Hence, putting $\phi = 0$, we see that the raising of the electric potential by V serves to change the time translates, ψ_t, Ψ_t of ψ, Ψ by the factors $\exp(-ieVt/\hbar)$, $\exp(-2ieVt/\hbar)$, respectively. On combining this observation with the above assumption that, in the compound system, Σ, the local properties of the deep interiors of Σ_1 and Σ_2 reduce asymptotically to the form they have when these subsystems are uncoupled, we are led to assume the following.

(VI.2) *For $x_j, x'_j \in X_j$ $(j = 1, 2)$, the asymptotic form of the time-dependent expectation value of $\Psi(x_1, x'_1)\Psi^\star(x_2, x'_2)$ carries a time-dependence, given by the factor* $\exp(2ie\Delta Vt/\hbar)$, *where*

$$\Delta V = V_1 - V_2. \tag{6.3}$$

The following Proposition is now a simple consequence of assumptions (VI.1) and (VI.2), Prop. 3.2 and equns. (3.2) and 6.2.

Proposition 6.1. *Under the assumptions of (VI.1) and (VI.2), the time-dependence of phase difference η is given by*

$$\eta = \eta_0 + \frac{2e\Delta V}{\hbar} t, \tag{6.4}$$

where η_0 is a constant.

This result tells us that, in the case of the present compound system, the phases of Φ in the deep interiors of the regions X_1 and X_2 differ by the time-dependent angle η, whereas, by contrast, Prop. 3.1(2) implies that, in the equilibrium states of the homogeneous system of Secs. II and III, the phase angle of the macroscopic wave function is a constant. This suggests that, in the present non-equilibrium situation, η represents the thermodynamic driving force of the current across the insulating film, as proposed originally by Josephson[6]. Thus, we make the following further assumption.

(VI.3) *The phase difference η generates a current across the insulating film, whose (single-valued) density is of the form*

$$J = F(\eta). \tag{6.5}$$

Further, since equn. (6.1) implies the invariance of Φ_j under the transformation $\eta_j \rightarrow \eta_j + 2\pi$, the function F satisfies the periodicity condition

$$F(\eta) \equiv F(\eta + 2\pi). \tag{6.6}$$

The following Proposition, which corresponds to the Josephson effect, is now an immediate consequence of equations (6.4)-(6.6).

Proposition 6.2. *Under the above assumptions, the frequency of the current due to the potential difference ΔV is $2e\Delta V/h$.*

Comment. The derivation of this result from ODLRO and gauge covariance may be contrasted with both the original one of Josephson[6], based on the Ginzburg-Landau model, and also with the subsequent one obtained by Rieckers[50] based on a fully microscopic treatment of a mean-field-theoretic model, which, however, violated gauge covariance of the second kind.

VII. Concluding Remarks

The theory presented here provides a general derivation of the electrodynamics of superconductors from assumptions of off-diagonal long range order, gauge covariance and thermodynamic stability. In our view, the outstanding general problems are the following.

(1) The substantiation of our assumptions for concrete models.

(2) The extension of our treatment to type-II superconductors.

(3) The precise characterisation of the metastability of supercurrents, within the terms of Sec. V. This amounts to the determination of which, if either, of the two alternatives proposed in assumption (V.3) prevails.

(4) The reformulation of the whole theory on a fully quantum basis, i.e., with the electromagnetic field as well as the matter treated quantum mechanically. An indication of how this might be achieved has been given in Ref. 27, Sec. 4.

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Appendix

Our aim here is to show that, under the conditions (b.1,2) of Sec. V, the perturbative induction b may be represented by a vector potential, a, that vanishes in the region $D_b^{(c)}$, exterior to the bounded domain D_b.

Suppose, then, that a', is a vector potential that represents b, i.e. that satisfies the equation $b = curla'$, and that x_0 is some chosen reference point in $D_b^{(c)}$. Then, for $x \in D_b^{(c)}$, we construct a curve, C, leading from x_0 to x in $D_b^{(c)}$, and define

$$\chi(x) = \int_C a'.dl. \qquad (A.1)$$

To show that this function is independent of the choice of C, we note that, if C' is another curve joining x_0 to x in $D_b^{(c)}$, then

$$(\int_C - \int_{C'})a.dl = \int_S b.dS, \tag{A.2}$$

where S is a closed surface bounded by C and C'. In the case where the dimensionality of X is greater than 2, we may choose S so that it does not intersect D_b, thereby ensuring that the r.h.s. of (A.2) vanishes. In the two- dimensional case, on the other hand, there is the possibility that the closed curve formed by C and $-C'$ loops round D_b, in which case the r.h.s. of (A.2) is $\int_{D_b} b.ds$, which vanishes, by (b.2)$'$. Hence, in all cases, the function χ, defined by equn. (A.2), does not depend on the choice of the curve C. It is therefore a simple consequence of that formula that, outside the region D_b,

$$A = \nabla\chi.$$

Hence, on extending χ by continuity to a differentiable function on the whole of X and defining

$$a = a' - \nabla\chi,$$

we see that $curla = b$ and that a vanishes outside the bounded region D_b. Thus, the potential a meets all our requirements.

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