ERGODIC THEOREMS FOR QUANTUM KRONECKER FLOWS

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Abstract. We discuss examples of ergodic theorems in quantum dynamics. Our focus is on infinite dimensional quantum dynamical systems arising from quantization of Kronecker type flows.

1. Quantum ergodic theorems

In this talk we present some results on the quantum and semiclassical ergodic theory for a family of quantum dynamical systems called quantum Kronecker flows. These quantum dynamical systems arise as quantization of finite and infinite dimensional classical Kronecker flows. They play an important role in quantum field theory, see e.g. [KL] where this point is emphasized. Also, special cases of these systems have a natural interpretation in the context of number theory.

For a background on quantum maps and quantum ergodic theory, see our contribution "Quantum maps" to this volume. Our presentation is pedagogical; we explain the ideas, formulate the results, and outline some of the proofs. Complete discussion may be found in the literature cited.

1.1. Quantum ergodicity. We introduce some notation. Let $\mathcal{A}$ be a ($\mathbb{C}^*$- or von Neumann) algebra of operators acting on a Hilbert space $\mathcal{H}$, and let ρ_t be a one-parameter group of $*$-automorphisms of $\mathcal{A}$. We will assume that

$$\rho_t(a) = e^{itH} a e^{-itH}, \quad a \in \mathcal{A},$$

where H is a self-adjoint, possibly unbounded operator on $\mathcal{H}$. Furthermore, we assume that there is a ρ_t-invariant state τ on $\mathcal{A}$. The main object of ergodic theory is the time average of an observable,

$$< a > := \lim_{T\to\infty} \frac{1}{T}\int_0^T \rho_t(a)\,dt\ , \tag{1.1.1}$$

where, the limit in (1.1.1) is taken with respect to a suitable (usually τ dependent) topology. Under suitable assumptions, this limit exists and is ρ_t-invariant. The dynamical system $(\mathcal{A}, \rho_t, \tau)$ is then called ergodic if the following statement holds (ergodic theorem):

$$< a > = \tau(a) I\ . \tag{1.1.2}$$

Here is an example.

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Kronecker flow. Consider the following dynamics on a quantum torus, namely the quantum Kronecker flow. Recall that the algebra $\mathcal{A}$ of observables on a quantum torus is defined as the universal ($\mathbb{C}^*$- or von Neumann) algebra generated by two unitary generators U, V satisfying the relation

$$UV = e^{i\lambda} VU \ .$$

Here $\lambda = 2\pi h$, where h is Planck's constant. One can think of the elements of $\mathcal{A}$ as series of the form $a = \sum a_{n,m} U^n V^m$. A natural trace on $\mathcal{A}$ is simply given by $\tau(a) = a_{0,0}$. The quantum Kronecker flow is defined on generators by:

$$\rho_t(U) := e^{2\pi i \omega_1 t} U, \quad \rho_t(V) := e^{2\pi i \omega_2 t} V,$$

in analogy with the classical situation. It extends to a one-parameter group of automorphisms of $\mathcal{A}$. If we assume that ω_1, ω_2 are linearly independent over $\mathbb{Z}$ then (1.1.2) holds with the limit taken in the operator norm sense, see [KLMR].

1.2. Semiclassical ergodicity. In many important quantum mechanical problems the scenario of the previous section does not apply. Let us assume that the spectrum of H is purely discrete:

$$H\phi_n = \lambda_n \phi_n \ .$$

Now, if $\mathcal{A}$ contains compact operators, then we cannot expect an ergodic theorem of the type explained in the previous section. Indeed, for a continuous function f vanishing at infinity,

$$< f(H) > = f(H).$$

To account for that one can reasonably expect the following variant of the ergodic theorem:

$$< a > = \tau(a) I + C_a \ , \tag{1.2.1}$$

where C_a is compact. If this is the case, then we have

$$\begin{aligned} (\phi_n, a\phi_n) &= (\phi_n, < a > \phi_n) \\ &= \tau(a) + (\phi_n, C_a \phi_n) \\ &\to \tau(a), \ \text{as } n \to \infty \ . \end{aligned} \tag{1.2.2}$$

In physics, H is the Hamiltonian of a system, and the limit $\lambda_n \to \infty$ is the semiclassical limit. For this reason, we refer to the above type of ergodic behavior as *semiclassical ergodicity.*

It is very difficult to establish (1.2.1) in examples of quantum dynamical systems. A weaker version of (1.2.1) has been studied extensively in the literature, see [Z] and references therein. The starting point is equation (1.2.2),

$$\tau(a) = \lim_{n\to\infty} (\phi_n, a\phi_n).$$

Consider a possibility where the above limit exists for almost all subsequences only. More precisely, assume there is a subset $S \subset \mathbb{N}$ of density 1 such that

$$\tau(a) = \lim_{n \in S,\ n \to \infty} (\phi_n, a\phi_n).$$

Alternatively, by a well known lemma in ergodic theory, the above equation can be rephrased as

$$\tau(a) = \lim_{E \to \infty} \tau_E(a) := \lim_{E \to \infty} \frac{1}{\#\{\lambda_n \le E\}} \sum_{\lambda_n \le E} (\phi_n, a\phi_n). \tag{1.2.3}$$

The ergodic theorem which holds within this framework reads:

$$< a >= \tau(a)I + C_a, \quad \text{and} \quad \lim_{E \to \infty} \tau_E(C_a^* C_a) = 0 \tag{1.2.4}$$

More precisely, the following theorem due to Zelditch [Z] gives sufficient conditions for (1.2.4) to hold.

Theorem 1.2.1. *Let π_τ the GNS representation of $\mathcal{A}$ associate with τ. Assume that (i) the algebra $\tau(\mathcal{A})$ is commutative, and (ii) $\tau(\rho_t)$ is ergodic (in the classical sense). Then the system $(\mathcal{A}, \rho_t, \tau)$ is semiclassically ergodic i.e. (1.2.4) holds.*

Here is an example of this scenario.

Geodesic flow. Consider the following quantum version of the geodesic flow on a Riemann surface M with constant negative curvature. The geodesic flow on such a manifold is known to be ergodic (in fact, mixing). Consider the Hilbert space $\mathcal{H} := L^2(M, d\mu)$ of functions on M square integrable with respect to the measure μ defined by the metric. Take the corresponding Laplace operator Δ and set $H := \sqrt{-\Delta}$. Let $\mathcal{A}$ be a completion of the algebra of order 0 pseudodifferential operators on M, and define the trace τ on $\mathcal{A}$ by

$$\tau(a) := \int_{ST^*M} \sigma(a)\, d\nu.$$

Here ST^*M is the unit sphere in the cotangent bundle of M, $\sigma(a)$ is the principal symbol of $a \in \mathcal{A}$, and ν is the normalized Liouville measure on ST^*M. The results of [C], [S] state that (1.2.4) holds in this case. In fact, as remarked in [Z], the algebra $\mathcal{A}$ becomes the commutative algebra of functions on ST^*M in the GNS representation with respect to τ, and ρ_t becomes the geodesic flow on M. One can then apply theorem 1.2.1 to yield the results of [C] and [S].

Remark. It was conjectured in [RS] that the remainder term C_a in (1.2.4) is compact so that the limit in (1.2.2) exists for every subsequence. If true, this would imply that no "scars" exist for such systems.

2. Quantum Kronecker flow

Another example of semiclassical ergodicity is a variant of quantized Kronecker flow discussed below. It is a quantization of the classical Kronecker flow different from the one consider in Section 1.1. Interesting technical aspects appear when the number of degrees of freedom is infinite. It turns out that the proof of ergodicity in this case requires an analysis of a generalized partition problem of number theory. To explain this point we take a historical approach and first discuss the Hardy-Ramanujan solution of the partition problem. The main question in that theory is to obtain an asymptotic expansion of the Laplace transform of certain functions. This is typically done by using the method of stationary phase.

2.1. Method of stationary phase.

We begin by reviewing the method of stationary phase. The argument presented here, while very simple, clearly illustrates the main ideas of the method.

Let f be a function on $[-N,N]$ with a simple maximum at $x=0$ and $f''(0)<0$. We consider the problem of finding the asymptotic behavior of the integral $\int_{-N}^{N} e^{Ef(x)}\,dx$ as $E\to\infty$.

Proposition 2.1.1. *Under the above assumptions,*

$$\int_{-N}^{N} e^{Ef(x)}\,dx = \sqrt{\frac{2\pi}{-f''(0)E}}\, e^{Ef(0)}\,(1+o(1)),\ \ \textit{as } E\to\infty. \tag{2.1.1}$$

Proof. For large E, the main contribution to the integral comes from the vicinity of the critical point. To see it precisely, we write:

$$\int_{-N}^{N} e^{Ef(x)}\,dx = \int_{|x|<\delta} e^{Ef(x)}\,dx + \int_{\delta<|x|<N} e^{Ef(x)}\,dx =: I + II,$$

where $\delta=\delta(E)$ will be chosen so that

$$\delta\to 0,\ \text{ as } E\to\infty. \tag{2.1.2}$$

For $|x|<\delta$, we expand $f(x)$ in a Taylor series around 0,

$$f(x) = f(0) - \big(-\frac{1}{2}f''(0)\big)x^2\,(1+o(1)).$$

The term I can be then written as

$$\begin{aligned} I &= \int_{|x|<\delta} e^{Ef(0)} e^{-(-f''(0))Ex^2(1+o(1))/2}\,dx \\ &= \frac{e^{Ef(0)}}{\sqrt{E}} \int_{|x|<\delta\sqrt{E}} e^{-(-f''(0))x^2(1+o(1))/2}\,dx. \end{aligned} \tag{2.1.3}$$

We now choose $\delta = E^{-1/4}$. Then $\delta\sqrt{E}\to\infty$, as $E\to\infty$, and (2.1.2) is satisfied. We can then perform the gaussian integration in (2.1.3) to obtain

$$I = \sqrt{\frac{2\pi}{-f''(0)E}}\, e^{Ef(0)}\,(1+o(1)).$$

The second term can be estimated as follows:

$$\begin{aligned}|II| &\leq 2N \max_{\delta<|x|<N} e^{Ef(x)} \\ &= 2N \max_{\delta=|x|} e^{Ef(x)} \\ &= 2Ne^{Ef(0)}e^{-(-f''(0))E\delta^2(1+o(1))/2} \\ &= O\left(E^{1/2}\, e^{f''(0)\sqrt{E}/2}\right) I = o(1)\, I \ ,\end{aligned}$$

and the proof is complete. $\square$

More sophisticated versions of the above arguments can handle situations with more then one maximum or when the exponent depends nonlinearly on E. But the basic idea remains the same.

2.2. Partition problem.

Let $p(n)$ be the number of partitions of a natural number n. For example,

$$\begin{aligned}5 &= 1+1+1+1+1 = 1+1+1+2 = 1+1+3 = 1+4 \\ &= 1+2+2 = 2+3,\end{aligned}$$

and so $p(5) = 7$. A famous example is

$$p(200) = 3972999029388.$$

Since no compact formula for $p(n)$ is known, the question arises to find the asymptotic behavior of $p(n)$, as $n \to \infty$. We will sketch the basic steps of the proof of the following classical theorem due to Hardy and Ramanujan, see [P]. The method of stationary phase is used to obtained the asymptotics.

Theorem 2.2.1.

$$p(n) = \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}}\,(1+o(1)), \ \textit{as } n \to \infty.$$

Idea of proof. Let us first note that $p(n)$ is the number of solutions of the equation

$$n = 1n_1 + 2n_2 + 3n_3 + \ldots$$

in non-negative integers (n_i). This will be important later. We consider a ζ-type generating function,

$$F(s) := \sum_{n=0}^{\infty} p(n)e^{-sn}. \tag{2.2.1}$$

It has the following infinite product representation:

$$F(s) = \prod_{k=1}^{\infty} \left(1-e^{-ks}\right)^{-1}. \tag{2.2.2}$$

Using (2.2.2), we can rewrite (2.2.1) as

$$\sum_{n=0}^{\infty} p(n)e^{-sn} = e^{\phi(s)}, \tag{2.2.3}$$

where $\phi(s) := -\sum_{k=1}^{\infty} \log\left(1-e^{-ks}\right)$. Remarkably, the function $\phi(s)$ can be studied by elementary means. The relation between $p(n)$ and $\phi(s)$ is similar to the Laplace transformation. Consequently, the behavior of $p(n)$ for n large is related to the behavior of $\phi(s)$ for s small. Results relating these functions are known as tauberian theorems.

Formula (2.2.3) can be easily inverted as follows:

$$p(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\phi(\sigma+it)} e^{n(\sigma+it)}\, dt, \tag{2.2.4}$$

for any $\sigma > 0$. Now, the idea is to choose σ so that the integrand of (2.2.4) has a critical point at $t = 0$, and use the stationary phase method to get the asymptotic of $p(n)$ as $n \to \infty$. Expanding the exponent of (2.2.4) to the second order around $t = 0$ and assuming that the stationary phase method can be justified, we obtain

$$\begin{aligned} p(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\phi(\sigma_n)+n\sigma_n} e^{-\phi''(\sigma_n)t^2/2}\, dt \left(1 + o(1)\right) \\ &= \frac{e^{\phi(\sigma_n)+n\sigma_n}}{\sqrt{2\pi\phi''(\sigma_n)}} \left(1 + o(1)\right), \end{aligned}$$

where σ_n satisfies $\phi'(\sigma_n) + n = 0$. All we need to do now is to wok out the asymptotics of σ_n, $\phi(\sigma_n)$ and $\phi''(\sigma_n)$. As an example, we will find the asymptotic behavior of σ_n. To this end, we approximate the derivative of $\phi(s)$ for small s,

$$\begin{aligned} \phi'(s) &= -\sum_{k=1}^{\infty} \frac{ke^{-ks}}{1-e^{-ks}} \\ &= -\frac{1}{s^2} \sum_{k=1}^{\infty} \frac{kse^{-ks}}{1-e^{-ks}}\, s \\ &\approx -\frac{1}{s^2} \int_0^{\infty} \frac{xe^{-x}}{1-e^{-x}}\, dx \\ &= \frac{-\pi^2}{6s^2}\,. \end{aligned}$$

Consequently, $\sigma_n = \pi/\sqrt{6n}$, as $n \to \infty$. Similar arguments give the asymptotics of $\phi(\sigma_n)$ and $\phi''(\sigma_n)$.

□

2.3. Kronecker flow.

In this section we define a non-commutative variant of the infinite dimensional Kronecker flow. We then sketch the proof in [KL] of semiclassical ergodicity of that flow. The proof is based on the ideas of Theorem 1.2.1.

Definition 2.3.1. A countable ordered subset Ω of $\mathbb{R}$ is called a *Kronecker system* if it satisfies conditions below:

(1) Ω consists of positive numbers.
(2) Let ω_n, $n = 1, \dots,$ be the elements of Ω listed in increasing order. Then $\omega_n \to \infty$, as $n \to \infty$.
(3) The elements of Ω are algebraically independent over $\mathbb{Z}$.

To prove ergodicity, assumption 2 will be considerably strengthened later. Furthermore, define $\mathbb{N}[\Omega] := \{\sum n_i\omega_i,\ 0 \le n_i \in \mathbb{Z}\}$, the additive group of finite linear combinations of elements of Ω with non-negative integer coefficients. Note that, as a consequence of assumption 3, all numbers $\sum n_i\omega_i$ are different, so that $\mathbb{N}[\Omega]$ can be viewed as subset of $\mathbb{R}$. We set $\mathcal{A}$ to be the (reduced) $\mathbb{C}^*$-algebra of the semigroup $\mathbb{N}[\Omega]$. Recall that it is defined as the algebra generated by the (left) regular representation of $\mathbb{N}[\Omega]$ in $\mathcal{H} := l^2(\mathbb{N}[\Omega])$.

It turns out that the $\mathbb{C}^*$-algebra $\mathcal{A}$ can be identified with an infinite tensor product of standard Toeplitz $\mathbb{C}^*$-algebras $\mathfrak{T}_\omega$,

$$\mathcal{A} = \bigotimes_{\omega\in\Omega} \mathfrak{T}_\omega \ ,$$

where $\mathfrak{T}_\omega$ is a copy of the (reduced) $\mathbb{C}^*$-algebra of the semigroup $\mathbb{N}$.

For $\eta \in \mathbb{N}[\Omega]$, let $e(\eta)$ denote the canonical basis element in $l^2(\mathbb{N}[\Omega])$. Let H be the unbounded, self-adjoint operator in $\mathcal{H}$ defined by

$$He(\eta) = \eta e(\eta),$$

and let $\rho_t(a) := e^{itH}ae^{-itH},\ a \in \mathcal{A}$. By $N(E)$ we denote the counting function for the eigenvalues of H i.e.

$$N(E) := \#\{\eta \in \mathbb{N}[\Omega], \eta \le E\}.$$

The dynamical system $(\mathcal{A}, \rho_t)$ is the quantum Kronecker flow which we will study in this section. The ρ_t-invariant state τ will be defined later in (1.2.3). The relation of $(\mathcal{A}, \rho_t)$ to the classical Kronecker flow is best described in the language of Toeplitz operators to be discussed next.

Let X denote the infinite cartesian product of unit circles, $X = \prod_{\omega\in\Omega} S^1$, equipped with the Tikhonov topology. Let μ be the normalized Haar measure on X. The classical Kronecker flow α_t on X is given by

$$\alpha_t\Big(\prod_{\omega\in\Omega} e^{ix_\omega}\Big) = \prod_{\omega\in\Omega} e^{ix_\omega + it\omega} \ .$$

As a consequence of our assumptions on Ω, this flow is ergodic. In quantum theory, one associates with a function $f \in C(X)$ an operator. This can be done in the following way. Let $L^2_+(X,\mu) \subset L^2(X,\mu)$ be the subspace of $L^2(X,\mu)$ consisting of functions whose Fourier transformations involve non-negative frequencies only, and let P be the orthogonal projection onto $L^2_+(X,\mu)$. Notice that the Fourier transformation gives a natural isomorphism $L^2_+(X,\mu) \simeq l^2(\mathbb{N}[\Omega]) = \mathcal{H}$. Every $f \in C(X)$ defines a *Toeplitz operator* $T(f)$ on $L^2_+(X,\mu) \simeq \mathcal{H}$ by

$$T(f) = PM(f)P,$$

where $M(f)$ is the operator of multiplication by f on $L^2(X,\mu)$.

It is not difficult to see that $\mathcal{A}$ coincides with the C^*-algebra generated by the Toeplitz operators. Moreover, we have

$$\rho_t(T(f)) = T(\alpha_t(f)) \ ,$$

which justifies the view that the dynamical system $(\mathcal{A}, \rho_t)$ is a non-commutative analog of the "classical" Kronecker flow.

We set

$$\tau_E(x) := \frac{1}{N(E)} \sum_{\mathbb{N}[\Omega] \ni \eta \leq E} (e(\eta), ae(\eta)),$$

and study $\tau(a) := \lim_{E\to\infty} \tau_E$, the semiclassical limit. Simple calculation shows that for every $f \in C(X)$,

$$\tau_E(T(f)) = \int f(x)\, d\mu(x).$$

To proceed further, we need to know the structure of $\mathcal{A}$. Let $\mathcal{I}$ be the commutator ideal of $\mathcal{A}$. The structure of the standard Toeplitz algebra implies that the quotient $\mathcal{A}/\mathcal{I}$ is isomorphic to $C(X)$. The quotient map $\pi : \mathcal{A} \to C(X)$ is called the symbol map, and $\pi(T(f)) = f$. The crucial observation here is that if ω_n grow sufficiently fast then τ is zero on the commutator ideal $\mathcal{I}$. To see that we analyze $\tau_E(a)$ for $a \in \mathcal{I}$.

Again, the structure of the standard Toeplitz algebra implies that $\mathcal{I}$ is generated by the operators of the form

$$a = a_{\omega_1} \otimes a_{\omega_2} \otimes \ldots \otimes a_{\omega_N} \otimes I \otimes I \ldots \ ,$$

where at least one of the operators $a_{\omega_1} \ldots a_{\omega_N}$, say a_{ω_k}, is compact. By a density argument, it is no loss of generality to assume that a_{ω_k} is a finite rank operator whose range is spanned by finitely many elements of the canonical basis. Then

$$\tau_E(a) \leq \frac{\|a\|}{N(E)} \#\{n_\omega \geq 0 \ , \ n_{\omega_k} \leq M : \sum n_\omega \omega \leq E\}.$$

But $\#\{n_\omega \geq 0 \ , \ n_{\omega_k} \leq M : \sum n_\omega \omega \leq E\} = N(E) - N(E - \omega_k(M+1))$ by a simple counting argument. The *main difficulty* is to show that, under additional assumptions,

$$\frac{N(E) - N(E - \omega_k(M+1))}{N(E)} \longrightarrow 0, \ \text{as } E \to 0. \tag{2.3.1}$$

Assume that (2.3.1) is true. Then

$$\tau(a) = \lim_{E\to\infty} \tau_E(a) = \int \pi(a)(x)\, d\mu(x),$$

for all $a \in \mathcal{A}$. In other words, the state τ is trivial on the commutator ideal, and it coincides with the Lebesgue integral on the abelian quotient. Moreover, let π_τ be the GNS representation of $\mathcal{A}$ with respect to the state τ. Then,

$$\pi_\tau(\mathcal{A}) \simeq C(X),$$

and for every $a \in \mathcal{A}$ we have

$$\pi_\tau(\rho_t(a)) = \alpha_t(\pi_\tau(a)).$$

Theorem 1.2.1 implies now the semiclassical ergodicity.

Remark 2.3. The ratio $\frac{N(E)-N(E-\omega_k(M+1))}{N(E)}$ does not always go to zero as $E \to \infty$. For example if $\Omega = \{\log p,\ p \,\text{prime}\}$, see [BC], then it is easy to see that $N(E) \sim e^E$ and (2.3.1) is not true.

Notice that (2.3.1) is equivalent to:

$$N(E+1) = N(E)(1+o(1)) \text{ as } E \to \infty\ . \tag{2.3.2}$$

Also let us remark that $N(E)$ is the number of solutions of

$$\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 + \ldots \le E$$

in nonnegative integers (n_i). This is strikingly similar to the formula for the number of partitions $p(n)$. It also suggests to use the Hardy-Ramanujan argument to obtain the asymptotic behavior of $N(E)$, and from there deduce (2.3.2).

In order to apply the method of Theorem 2.2.1 we will need additional assumptions on the set Ω. Specifically, assume that for every $s > 0$,

$$\theta(s) := \sum_{n=1}^{\infty} e^{-s\omega_n} < \infty.$$

This implies that the following ζ-type function (see (2.2.2))

$$\zeta_\Omega(s) := \prod_{n=1}^{\infty} (1 - e^{-s\omega_n})^{-1}$$

converges for all $s > 0$. Expanding each term of $\zeta_\Omega(s)$ in a power series and multiplying out the terms, we can express $\zeta_\Omega(s)$ as the following Lebesgue-Stieltjes integral:

$$\zeta_\Omega(s) = 1 + \int_0^\infty e^{-sx}\, dN(x),$$

where, as above, $N(x)$ is the counting function for the eigenvalues of H. Equivalently, we can write this formula as

$$\frac{e^{\phi(s)}}{s} = \int_0^\infty e^{-sx}(N(x)+1)\,dx, \tag{2.3.3}$$

where $\phi(s) := -\sum_{n=1}^{\infty}\log(1-e^{-s\omega_n})$, see (2.2.3). As before, the right hand side of (2.3.3) is essentially the Laplace transform of $N(x)$, while the left hand side is controllable and depends on the behavior of ω_n. The standard lore says that the behavior of $N(x)$, as $x \to \infty$, can be extracted from the behavior of its Laplace transform as $s \to 0$.

Informally, one proceeds as follows. Taking the inverse Laplace transform of both sides of (2.3.3) yields:

$$N(x)+1 = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{\phi(\sigma+it)+x(\sigma+it)}}{\sigma+it}\,dt\ ,$$

where $\sigma > 0$ is arbitrary. Now choose $\sigma = \sigma_x$ such that the function

$$t \mapsto \phi(\sigma+it) + x(\sigma+it)$$

has a critical point at $t = 0$. This leads to the following condition on σ_x:

$$\phi'(\sigma_x) + x = 0.$$

At this point we can use the method of stationary phase to get the asymptotic of $N(x)$ as $x \to \infty$:

$$\begin{aligned} N(x) &= \frac{1}{2\pi\sigma_x} e^{\phi(\sigma_x)+x\sigma_x} \int_{\mathbb{R}} e^{-\phi''(\sigma_x)t^2/2}\,dt\,(1+o(1)) \\ &= \frac{e^{\phi(\sigma_x)+x\sigma_x}}{\sqrt{2\pi\sigma_x^2\phi''(\sigma_x)}}(1+o(1))\ . \end{aligned} \tag{2.3.4}$$

The last step is to establish (2.3.2) by analyzing (2.3.4).

It is unlikely that the above strategy will work in such a generality. The main obstacle is the difficulty to control the other critical points of $\phi(\sigma+it)+x(\sigma+it)$. Such a control is possible, however, under the assumption of polynomial growth of ω_n. The following theorem was proved in [KL].

Theorem 2.3. *Let $\omega_n = An^\alpha(1+\mu_n)$, where $A > 0$ and $\alpha \geq 1$ are constant, and where $\mu_n = o(1)$, as $n \to \infty$. Then (2.3.1) is true and consequently for every $a \in \mathcal{A}$,*

$$w - \lim_{T\to\infty}\frac{1}{T}\int_0^T \rho_t(a)\,dt = \tau(a)I + C_a,$$

where C_a is in the weak closure of $\mathcal{A}$, and

$$\lim_{E\to\infty}\tau_E(C_a^* C_a) = 0\ .$$

References

[BC] J. P. Bost, A. Connes, *Hecke Algebras, Type III Factors and Phase Transitions with Spontaneous Symmetry Breaking in Number Theory*, preprint (1995).

[C] Y. Colin de Verdiere, *Ergodicite et fonctions propres du Laplacien*, Comm. Math. Phys. **102** (1985), 497–502.

[KL] S. Klimek, A. Lesniewski, *Quantized Kronecker flows and almost periodic quantum field theory.*

[KLMR] S. Klimek, A. Lesniewski, N. Maitra, R. Rubin, *Ergodic properties of quantized toral automorphisms*, J. Math. Phys. **to appear**.

[P] A. Postnikow, *Introduction to analytic number theory*, Amer. Math. Soc., Providence, RI, 1988.

[RS] Z. Rudnick, P. Sarnak, *The behavior of eigenstates of arithmetic hyperbolic manifolds*, Comm. Math. Phys. **161** (1994), 195–213.

[S] A. Schnirelman, *Ergodic properties of the eigenfunctions*, Usp. Math. Nauk **29** (1974), 181–182.

[Z] S. Zelditch, *Quantum ergodicity of $\mathbb{C}^*$ dynamical systems*, Comm. Math. Phys. **177** (1996), 507–528.

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