THERMODYNAMIC CHAOS AND THE STRUCTURE OF SHORT-RANGE SPIN GLASSES

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Abstract

This paper presents an approach, recently introduced by the authors and based on the notion of "metastates", to the chaotic size dependence expected in systems with many competing pure states, and applies it to the Edwards-Anderson (EA) spin glass model. We begin by reviewing the standard picture of the EA model based on the Sherrington-Kirkpatrick (SK) model and why that standard SK picture is untenable. We then introduce metastates, which are the analogues of the invariant probability measures describing chaotic dynamical systems and discuss how they should appear in several models simpler than the EA spin glass. Finally, we consider possibilities for the nature of the EA metastate, including one which is a nonstandard SK picture, and speculate on their prospects. An appendix contains proofs used in our construction of metastates and in the earlier construction by Aizenman and Wehr.

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1 Introduction

The thermodynamic, or infinite volume, limit is a convenient mathematical device for analyzing the thermal equilibrium properties of systems with a finite, but very large, number of microscopic components. The utility of this device has been particularly apparent in its contribution to our conceptual understanding of phase transitions in relatively simple models, such as the homogeneous Ising ferromagnet (or equivalent lattice gas). In that context, it led both to Onsager's exact calculation of the two-dimensional free energy [O44] and to the Lee-Yang mechanism (via partition function zeros) for obtaining macroscopically distinct equations of state for different phases with a single microscopic Hamiltonian [YL52], [LY52].

Behind these successes lie the related facts that for such simple models, (a) one knows how to choose finite volume boundary conditions (b.c.'s) so that the infinite volume limit is a pure phase and (b) for b.c.'s not specially chosen to yield a pure phase, one knows the resulting limiting mixture of pure phases. For more complex models with an unknown relation between b.c.'s and pure phases, the nature, meaning, and even utility of the infinite volume limit (for local state observables, such as microscopic correlations) may be more problematic. In particular, it was unclear how to deal constructively with chaotic size dependence (CSD) [NS92], the phenomenon that (with not specially chosen b.c.'s) there may be many distinct limiting states along different subsequences of volumes.

In this paper, we report on a new approach to the thermodynamic limit [NS96c] which harnesses CSD by means of "metastates", ensembles of (i.e., probability measures on) the possible limiting (mixed) states. As in [NS96c], we present this approach primarily in the context of the Edwards-Anderson (EA) spin glass model [EA75], although our arguments apply to a very wide class of disordered systems. In fact, the metastate approach should be applicable also to nondisordered systems with a complex structure of pure phases.

In addition to this introductory section, numbered 1, this paper consists of seven more sections and an appendix. In Section 2, we introduce the EA model, the number $\mathcal{N}$ of pure phases for it, and the main competing predictions from the physics literature that when $\mathcal{N} \neq 1$, it must be two according to one view and infinity according to another. The prediction that $\mathcal{N} = \infty$, based on the Sherrington-Kirkpatrick (SK) spin glass model [SK75], is part of what we call an SK picture (of the EA model). In Section 3, we present the "standard" SK picture (which does not utilize the metastate approach) as in the physics literature but made more precise, and then explain why it breaks down [NS96b]. This breakdown leads us, in Section 4, to our metastate approach. In Section 5, we digress to discuss several examples of metastates in systems simpler than the EA model and then in Section 6, we present several possibilities concerning the nature of the metastate in the EA model itself. One of these is a new possibility intermediate between an SK picture and the prediction that $\mathcal{N} = 2$. To explore another of the possibilities, that a nonstandard SK picture, within the framework of the metastate approach, might be valid for the EA model, we begin by discussing, in Section 7, the use of replicas and their overlaps in the metastate approach. In Section 8, we discuss the nonstandard SK picture and its prospects. Finally, the Appendix concerns various technical arguments needed for the constructions of metastates in Section 4.

2 The Edwards Anderson Model

An Edwards-Anderson (EA) spin glass model [EA75] is a disordered nearest neighbor Ising model on $\mathbb{Z}^d$ whose couplings $J = (J_e : e \in \mathbb{E}^d)$ are i.i.d. random variables (on some probability space $(\Omega, \mathcal{F}, \nu)$) with a common symmetric distribution μ (i.e., J_e and $-J_e$ are equidistributed). Here $\mathbb{E}^d$ denotes the set of nearest neighbor bonds $\{x, y\}$, i.e., pairs of sites of $\mathbb{Z}^d$, whose Euclidean distance $||x-y||$ equals 1. The most common examples are the Gaussian (where μ is a mean zero normal distribution) and the $\pm\tilde{J}$ (where $\mu = \frac{1}{2}\delta_{\tilde{J}} + \frac{1}{2}\delta_{-\tilde{J}}$) models. Unless otherwise noted, we place no restrictions on μ beyond symmetry.

Let Λ be a finite subset of $\mathbb{Z}^d$, for example, $\Lambda = \Lambda_L = \{-L, -L+1, \cdots, L\}^d$. The EA model on Λ at inverse temperature $\beta \geq 0$ (with some b.c., temporarily denoted by *) is a family $\{S_x : x \in \Lambda\}$ of random variables (or spins) taking values $+1$ or -1, whose joint distribution is the finite volume Gibbs measure $P^*_{\Lambda,\beta}$ given by

$$P^*_{\Lambda,\beta}(\{s\}) = Z^{*-1}_{\Lambda,\beta} \exp\{-\beta H^*_\Lambda(s)\}. \tag{2.1}$$

Here $s \in \mathcal{S}_\Lambda = \{-1, +1\}^\Lambda$, $H^*_\Lambda(s)$ is the finite volume Hamiltonian, and the partition function, $Z^*_{\Lambda,\beta}$, is such that $P^*_{\Lambda,\beta}(\mathcal{S}_\Lambda) = 1$.

The free b.c. Hamiltonian (with $* = f$) is

$$H^f_\Lambda(s) = \sum_{\substack{x,y\in\Lambda \\ \{x,y\}\in\mathbb{E}^d}} -J_{\{x,y\}} s_x s_y. \tag{2.2}$$

One type of b.c. (called a fixed b.c.) is the specification of a spin configuration, $\bar{s} \in \{-1, +1\}^{\partial\Lambda}$, on the boundary of Λ,

$$\partial\Lambda = \{y \in \mathbb{Z}^d \setminus \Lambda : \{x, y\} \in \mathbb{E}^d \text{ for some } x \in \Lambda\}, \tag{2.3}$$

and the replacement of (2.2) by

$$H^{\bar{s}}_\Lambda(s) = H^f_\Lambda(s) - \sum_{\substack{x\in\Lambda, y\in\partial\Lambda \\ \{x,y\}\in\mathbb{E}^d}} J_{\{x,y\}} s_x \bar{s}_y. \tag{2.4}$$

A more general type of b.c. is specified by a probability measure $\bar{\rho}$ on $\mathcal{S}_{\partial\Lambda}$. The corresponding Gibbs distribution on $\mathcal{S}_\Lambda$, denoted by $P^{\bar{\rho}}_{\Lambda,\beta}$, is the mixture of fixed b.c. Gibbs distributions,

$$P^{\bar{\rho}}_{\Lambda,\beta} = \sum_{\bar{s}\in\mathcal{S}_{\partial\Lambda}} \bar{\rho}(\{\bar{s}\}) P^{\bar{s}}_{\Lambda,\beta}. \tag{2.5}$$

Later on, we will introduce periodic b.c.'s when Λ is the cube Λ_L. Meanwhile, we recall that an infinite volume Gibbs state (at inverse temperature β) for given $\{J_e : e \in \mathbb{E}^d\}$ is any measure P_β on $\mathcal{S} = \{-1,+1\}^{\mathbb{Z}^d}$ such that there is some sequence of b.c.'s (on the cubes Λ_L), i.e., $\bar\rho_L$ on $\mathcal{S}_{\partial\Lambda_L}$, such that P_β is the limit (in the sense of convergence of finite dimensional distributions) of $P^{\bar\rho_L}_{\Lambda_L,\beta}$ as $L \to \infty$. We remark that there is no loss of generality resulting from the restriction to cubes or to this class of b.c.'s. The intrinsic characterization of P_β is that it satisfies the Dobrushin-Lanford-Ruelle (DLR) equations (see, e.g., [Ge88] for a more complete discussion and for historical references), i.e., for every finite Λ, the conditional distribution of P_β, conditioned on the σ-field generated by $\{s_x : x \in \mathbb{Z}^d \setminus \Lambda\}$, is $P^{\bar s}_{\Lambda,\beta}$ where $\bar s \in \mathcal{S}_{\partial\Lambda}$ is given by the (conditioned) values of s_x for $x \in \partial\Lambda$.

For a given ω and hence a given coupling configuration $J = J(\omega)$, we may consider the set $\mathcal{G} = \mathcal{G}(J(\omega),\beta)$ of all infinite volume Gibbs states for $J(\omega)$ at inverse temperature β. In analyzing $\mathcal{G}$ it is natural to consider the set of extremal (or pure) Gibbs states,

$$\text{ex } \mathcal{G} = \mathcal{G} \setminus \{\alpha P_1 + (1-\alpha)P_2 : \ \alpha \in (0,1); \ P_1, P_2 \in \mathcal{G}; \ P_1 \neq P_2\}. \tag{2.6}$$

We define $N = N(J(\omega),\beta)$ to be the cardinality, $|\text{ex } \mathcal{G}|$, of ex $\mathcal{G}$. N can (a priori) take any of the values $1, 2, \ldots,$ or ∞ (we do not distinguish here between countably and uncountably infinite). The next proposition shows that N is "self-averaged", i.e., it is the same value for almost all ω's. The "hard" part of the proof is to show measurability; self-averaging then follows easily from invariance under spatial translations. In the next section of the paper, the same translation invariance argument will rule out the standard SK picture.

Proposition 2.1 *$N(J(\omega),\beta)$ is measurable and a.s. equals a constant, $\mathcal{N}(d,\mu,\beta)$.*

Proof. We will use the fact from the general theory of Gibbs distributions (see Corollary 7.28 of [Ge88]) that $N \geq k$ if and only if $\mathcal{G}$ contains at least k linearly independent measures $P_1, \ldots, P_k$. But this is so if and only if for some finite $\Lambda \subset \mathbb{Z}^d$, there is linear independence of the ($2^{|\Lambda|}$-dimensional) vectors $\vec p_\Lambda(P_1), \ldots, \vec p_\Lambda(P_k)$, defined as

$$\vec p_\Lambda(P) = (P(s = t \text{ on } \Lambda) : t \in \mathcal{S}_\Lambda)\,. \tag{2.7}$$

Thus $N \geq k$ if and only if for some finite Λ, the closed convex set,

$$G_\Lambda \equiv \{\vec p_\Lambda(P) : P \in \mathcal{G}\}\,, \tag{2.8}$$

is at least k-dimensional.

For L sufficiently large (so that Λ is contained in the cube Λ_L), let us consider the finite volume approximation to G_Λ, obtained by replacing $\mathcal{G}$ in (2.8) by the finite volume Gibbs measures $P^{\bar\rho}_{\Lambda_L,\beta}$:

$$G^L_\Lambda \equiv \left\{\vec p_\Lambda(P^{\bar\rho}_{\Lambda_L,\beta}) : \bar\rho \text{ is a probability measure on } \mathcal{S}_{\partial\Lambda_L}\right\}. \tag{2.9}$$

Then $G_\Lambda^{L''} \subseteq G_\Lambda^{L'}$ for $L'' > L'$ and $G_\Lambda = \lim_{L\to\infty} G_\Lambda^L$. Let $\mathcal{R}_{\Lambda,k}$ denote the countable set of linearly independent k-tuples $(\vec{p}_1, \ldots \vec{p}_k)$ of vectors in $\mathbb{R}^{2^{|\Lambda|}}$ with rational coordinates. Then G_Λ is at least k-dimensional if and only if there exists $(\vec{p}_1, \ldots \vec{p}_k) \in \mathcal{R}_{\Lambda,k}$ such that for each j, $\vec{p}_j \in G_\Lambda^L$ for all (large) L.

Next we note that since $P_{\Lambda_L,\beta}^{\bar{\rho}}$ is just the convex combination (corresponding to $\bar{\rho}$) of the fixed b.c. $P_{\Lambda_L,\beta}^{\bar{s}}$'s,

$$G_\Lambda^L = \text{ convex hull } \left(\vec{p}_\Lambda(P_{\Lambda_L,\beta}^{\bar{s}}) : \bar{s} \in \mathcal{S}_{\partial\Lambda_L}\right). \tag{2.10}$$

From the definition of $P_{\Lambda_L,\beta}^{\bar{s}}$ we clearly have that the countably many points $\vec{p}_\Lambda(P_{\Lambda_L,\beta}^{\bar{s}})$, as Λ, L and $\bar{s}$ vary, are all measurable functions of J. Thus

$$\begin{aligned} \{N \geq k\} \quad = \quad & \{\exists\Lambda,\ \exists(\vec{p}_1, \ldots \vec{p}_k) \in \mathcal{R}_{\Lambda,k}, \\ & \forall j,\ \forall \text{ large } L, \\ & \vec{p}_j \in \text{ convex hull } \left(\vec{p}_\Lambda(P_{\Lambda_L,\beta}^{\bar{s}}) : \bar{s} \in \mathcal{S}_{\partial\Lambda_L}\right)\} \end{aligned} \tag{2.11}$$

is measurable.

Finally, to complete the proof, note that the joint distribution $\bar{\nu}$ of $J = (J_e : e \in \mathbb{E}^d)$ is an i.i.d. product measure which is invariant and ergodic with respect to translations on $\mathbb{Z}^d$. Since N is a measurable function of J which is clearly translation invariant, it follows that N is a.s. constant.

□

For a given J and β, $N \geq 2$ if and only if there is non-uniqueness of infinite volume Gibbs distributions. Thus one may define a critical inverse temperature in terms of the value of N. There are various ways to do this, one of which is

$$\begin{aligned} \beta_c^* = \inf\{\beta \geq 0 : \mathcal{N} \geq 2\} = \inf\{\beta \geq 0 :\ & \text{a.s, there is} \\ & \text{nonuniqueness of the infinite} \\ & \text{volume Gibbs states for } \beta\}. \end{aligned} \tag{2.12}$$

If there is a.s. uniqueness of the infinite volume Gibbs state for all finite β, then we set $\beta_c^* = \infty$. Of course, this critical value is a function, $\beta_c^*(d, \mu)$ of the spatial dimension and the common distribution of the couplings. It is not hard to see that for any μ, $\beta_c^* = \infty$ for d=1. The critical value is also infinite for $d > 1$ if μ has too large of an atom at zero. It is also known that for any d and μ, β_c^* is strictly positive. (For more details and for a discussion of some other definitions of critical values, see Chapters 3 and 4 of [N97].)

The general belief in the physics literature, based primarily on numerical studies, is that there is some critical dimension d_c (probably with $d_c \leq 3$ [FS90]) such that for $d \geq d_c$, the critical β is finite (at least for the standard examples of μ). It should be noted that in this literature, it is not usually clear which of the possible definitions of the critical value is under consideration. It is also believed that, like in the random field Ising model [AW90], the critical β is infinite for $d = 2$. Thus we have the following open problems.

Open Problem 2.2 *Determine whether for all* μ, $\beta_c^*(2,\mu) = \infty$.

Open Problem 2.3 *Determine whether for some* d *and some* μ, $\beta_c^*(d,\mu) < \infty$.

Although it seems to be generally believed that (at least for some μ's), $\mathcal{N}(d,\mu,\beta) \geq 2$ for $d \geq$ some d_c and large β, there is a lively controversy in the physics literature on the nature of the pure Gibbs states and, in particular, on the value of $\mathcal{N}$ (when $\mathcal{N} \geq 2$).

One side of the controversy is fairly easy to describe. This side predicts, on the basis of nonrigorous scaling arguments [M84], [FH86], [BM87], [FH88], that $\mathcal{N} = 2$ and that the two pure states, P' and P'', are global flips of each other, i.e., that the mapping $s \to -s$ on $\mathcal{S}$ transforms P' into P'' and vice versa. (See [BF86] for different arguments that predict $\mathcal{N} = 2$ only for $d = 3$ and [En90] for a critique of the scaling argument prediction that $\mathcal{N} = 2$ for all d.) This would be analogous to the situation for the homogeneous $d = 2$ Ising ferromagnet [A80], [Hi81]. The local magnetizations $\langle S_x \rangle'$ (where $\langle \cdot \rangle'$ denotes expectation w.r.t. P') and $\langle S_x \rangle'' = -\langle S_x \rangle'$ would depend on both x and ω and would be nonvanishing (at least for some x's with a positive density in $\mathbb{Z}^d$). The mean magnetization should vanish; i.e., w.r.t. P',

$$\lim_{L\to\infty} |\Lambda_L|^{-1} \sum_{y\in\Lambda_L} S_y = \lim_{L\to\infty} |\Lambda_L|^{-1} \sum_{y\in\Lambda_L} \langle S_y \rangle' = 0. \tag{2.13}$$

However, the EA order parameter should be a strictly positive constant, $r_{EA} = r_{EA}(d,\mu,\beta)$, given by

$$r_{EA} = \lim_{L\to\infty} |\Lambda_L|^{-1} \sum_{y\in\Lambda_L} [\langle S_y \rangle']^2 = E\left([\langle S_x \rangle']^2\right), \tag{2.14}$$

where E denotes expectation with respect to ν (on the probability space of J).

One feature of the other side of the controversy is also easy to describe – i.e., the prediction that $\mathcal{N} = \infty$. Other features are more difficult and much of this paper will be devoted to presenting and analyzing different interpretations of how $\mathcal{N} = \infty$ could or should manifest itself. Before we begin that, we list an obvious open problem.

Open Problem 2.4 *Determine whether* $\mathcal{N}(d,\mu,\beta) = \infty$, *or at least whether* $\mathcal{N}(d,\mu,\beta) > 2$, *for some* d, μ *and* β.

The prediction that $\mathcal{N} = \infty$ is based firstly on the notion that a short-ranged spin glass like the EA model should behave qualitatively like an "infinite-ranged" or "mean-field" model and based secondly on the nonrigorous analysis by Parisi and others [P79], [P83], [HoJY83], [MPSTV84] of the Sherrington-Kirkpatrick (SK) model [SK75], the standard mean-field spin glass. This line of reasoning leads not only to the prediction that $\mathcal{N} = \infty$ (for appropriate d, μ and β) but also to many other predictions concerning the Gibbs states for large finite volume and for infinite volume.

These predictions, taken as a whole, constitute what we call an SK picture of the EA model. Unfortunately the predictions (other than that $\mathcal{N} = \infty$) have not been formulated in the literature with much precision so that before attacking the problem of whether the SK picture (of the EA model) is valid, it is first necessary to consider the question, "what is the SK picture?". One of our main objectives here, following [NS92], [NS96b], [NS96c], is to answer that question. To begin, we start the next section with a brief introduction to the SK model itself and to Parisi's analysis of it. For a comprehensive discussion, the reader is referred to [MPV87].

3 The Standard SK Picture

Unlike the EA model, the SK model has no a priori connection with the lattice $\mathbb{Z}^d$ and so it is conventional to replace subsets of $\mathbb{Z}^d$ by subsets of $\mathbb{N}$, the set of positive integers. The SK model of size n at inverse temperature β has the Gibbs measure on $\mathcal{S}_n = \{-1, +1\}^n$,

$$P_{n,\beta}(\{s\}) = Z_{n,\beta}^{-1} \exp\left(\beta \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} J_{ij}^n s_i s_j\right), \tag{3.1}$$

with the n-dependent couplings given by

$$J_{ij}^n = n^{-1/2} K_{ij} \quad \text{for } 1 \leq i < j < \infty, \tag{3.2}$$

where the K_{ij}'s are i.i.d. random variables (on some $(\Omega, \mathcal{F}, \nu)$) with common symmetric distribution μ.

This Gibbs measure may be regarded, in an obvious sense, as having a free b.c. In another sense, since there is no natural notion of a boundary at all in the SK model, it may be regarded as more analogous to taking periodic b.c.'s in nearest neighbor models, since those are generally thought of as minimizing the effects of the boundary. There seems to be no obvious analogue in the SK model to nearest neighbor model b.c.'s such as fixing $\bar{s}$ on $\partial\Lambda$.

The ferromagnetic infinite-ranged or mean-field model, known as the Curie-Weiss (CW) model, has a Gibbs distribution also of the form of (3.1) but where (3.2) is replaced by $J_{ij}^n = n^{-1}$. In both SK and CW models, the scaling factors, $n^{-1/2}$ and n^{-1}, are chosen so that the mean energy per site,

$$n^{-1} \left\langle -\sum_{i=1}^{n-1} \sum_{j=i}^{n} J_{ij}^n s_i s_j \right\rangle_{n,\beta}, \tag{3.3}$$

is bounded away from 0 and ∞; the difference in scaling exponents is due to the presence of sign cancellations in the SK model and their absence in the CW model. The CW model has a simple and easily derived exact solution – e.g., for the (infinite volume limit) free energy per site,

$$f(\beta) = \lim_{n\to\infty} -\frac{1}{n\beta} \ln Z_{n,\beta}. \tag{3.4}$$

This is not the case for the SK model.

In the CW model (as in the homogeneous (nearest-neighbor) ferromagnet on $\mathbb{Z}^d$), a natural object to study is the magnetization per site, i.e., the random variable (on $\mathcal{S}_n$ with probability measure $P_{n,\beta}$), $n^{-1}\sum_{i=1}^n s_i$. The derivation of the exact formula for the free energy also yields information on the $n \to \infty$ limit of the distribution of this random variable. There is a critical value β_c^{CW} such that for $\beta \leq \beta_c^{CW}$, the limiting distribution is simply δ_0, the point measure at the origin, while for $\beta > \beta_c^{CW}$, the limit is $\frac{1}{2}\delta_{M^*} + \frac{1}{2}\delta_{-M^*}$, where $M^* = M^*(\beta)$, the spontaneous magnetization, is strictly positive for $\beta > \beta_c^{CW}$. In the SK model, the magnetization per site is not a fruitful choice of variable since, due to sign cancellations, it should have a trivial limiting distribution, δ_0, for all β. Indeed, Parisi's analysis of the free energy for the SK model led him to study the distribution of a different random variable, the replica overlap.

The replica overlap is the random variable (on $\mathcal{S}_n \times \mathcal{S}_n = \{(s^1, s^2)\}$ with probability measure $P_{n,\beta} \times P_{n,\beta}$)

$$R_n = \frac{1}{n}\sum_{i=1}^n s_i^1 s_i^2. \tag{3.5}$$

The formal density of its distribution (on the interval $[-1,1]$) is denoted $\mathcal{P}_n(r)$. For β below a critical value β_c^{SK}, the $n \to \infty$ limit of $\mathcal{P}_n(r)$ is $\delta(r)$, the trivial point density at 0. But above β_c^{SK}, Parisi found highly nontrivial behavior: namely, that as $n \to \infty$, $\mathcal{P}_n$ approximates a sum of many delta functions, at locations and with weights which depend on ω, with the weights *not* tending to zero and with the dependence of the locations and weights on ω also *not* tending to zero as $n \to \infty$.

The usual explanation given for this behavior (see, e.g., [MPV87]) is that, as $n \to \infty$, the Gibbs measure $P_{n,\beta}$ has a decomposition into many pure states $P^\alpha = P^\alpha_\beta$,

$$P_{n,\beta} \approx \sum_\alpha W^\alpha P^\alpha, \tag{3.6}$$

with weights W^α depending on ω, with neither they nor their dependence on ω tending to zero. [Warning: The reader should not be alarmed if the meaning of a pure state in the SK context is not precisely clear; it is certainly not clear to the authors.]

If the replicas s^1 and s^2 were chosen from $P^\alpha \times P^\gamma$ rather than from $P_{n,\beta} \times P_{n,\beta}$, then, as $n \to \infty$, the overlap would have a point density

$$\delta(r - r_{\alpha\gamma}), \qquad r_{\alpha\gamma} \approx \frac{1}{n}\sum_{i=1}^n \langle S_i\rangle^\alpha \langle S_i\rangle^\gamma, \tag{3.7}$$

and so

$$\mathcal{P}_n(r) \approx \sum_\alpha \sum_\gamma W^\alpha W^\gamma \delta(r - r_{\alpha\gamma}). \tag{3.8}$$

Like the W^α's, the $r_{\alpha\gamma}$'s depend on ω with a dependence not tending to zero. This persistence in the dependence of various quantities on ω in the infinite volume limit, $n \to \infty$, is called non-self-averaging (NSA) and is one of the essential features of Parisi's analysis. (For a rigorous approach to NSA in the SK model, see [PS91].)

A second essential feature is that there are many α's (with non-negligible weights) appearing in (3.6) so that the two replicas have non-negligible probability of appearing in different (and "unrelated") pure states α and γ (with overlap value $r_{\alpha\gamma}$). This is called replica symmetry breaking (RSB). By unrelated, we mean that γ is neither α nor the "negative" of α, i.e., that P^γ is not P^α and is also not the global spin flip of P^α. When $\gamma = \alpha$, $r_{\alpha\alpha}$ should, for all α, be the Edwards-Anderson order parameter r_{EA} (the analogue of (2.14), but for the SK model), while for $\gamma = -\alpha$, $r_{\alpha\gamma} = -r_{EA}$. The value of r_{EA} (unlike other $r_{\alpha\gamma}$'s) should *not* depend on ω.

A third essential feature, closely related to NSA, is that the discrete nature of the Parisi order parameter distribution as a countable sum of delta functions is only for fixed ω. As ω varies, the $r_{\alpha\gamma}$'s vary in such a way that, as $n \to \infty$, $\bar{\mathcal{P}}_n(r)$, the average over the couplings (i.e., over the underlying probability measure ν on Ω) is continuous (except for two delta functions at $\pm r_{EA}$ since those locations do not depend on ω).

A fourth essential feature, closely related to, but going well beyond, the discreteness of the order parameter distribution, is ultrametricity of the $r_{\alpha\gamma}$'s (for fixed ω). Here one regards $d_{\alpha\gamma} \equiv r_{EA} - r_{\alpha\gamma}$ as defining a metric on the pure states. The ultrametric property of $d_{\alpha\gamma}$ is that among any three pure states α, γ and δ, the largest two of $d_{\alpha\gamma}$, $d_{\alpha\delta}$, $d_{\gamma\delta}$ are equal.

In formulating a precise interpretation of the SK picture of the EA model, it is clear that a primary role will be played by an EA model analogue of the approximate pure state decomposition (3.6). In our EA analogues of (3.6), we will replace $P_{n,\beta}$ on the LHS by $P_L = P^{per}_{\Lambda_L,\beta}$, the ($\omega$-dependent) Gibbs measure for the EA model on the cube Λ_L with periodic b.c.'s (although for many purposes, free b.c.'s or, indeed, any b.c. not depending on ω could equally well be chosen). The EA model has a great advantage over the SK model in that there is already a well defined meaning of pure states as extremal infinite volume Gibbs states. Thus in the EA model, the role of pure states will be played by ... the pure states. The real issue for EA analogues of (3.6) is interpreting the approximate equality.

The most straightforward interpretation of the SK picture, which we call the standard SK picture, is also the one that most closely matches the presentations in the physics literature. In this picture, (3.6) is replaced by the identity

$$P_{\mathcal{J}} = \sum_\alpha W^\alpha_{\mathcal{J}} P^\alpha_{\mathcal{J}}, \tag{3.9}$$

where $\mathcal{J} = (\mathcal{J}_e : e \in \mathbb{E}^d)$ represents a particular configuration of the random couplings $J = (J_e : e \in \mathbb{E}^d)$, $P_{\mathcal{J}}$ is an infinite volume Gibbs state for $\mathcal{J}$ (and some fixed β) obtained in some natural way from the finite volume periodic b.c. Gibbs states $P_L = P_{\mathcal{J},L}$ by letting $L \to \infty$, and the $P^\alpha_{\mathcal{J}}$'s are pure (i.e., extremal infinite volume) Gibbs states for that same $\mathcal{J}$. The identity (3.9) is to be valid for $\bar{\nu}$-a.e. $\mathcal{J}$, where $\bar{\nu}$ is the joint distribution of the

J_e's. The replica overlap in this picture is the random variable (on $\mathcal{S} \times \mathcal{S}$ with probability measure $P_{\mathcal{J}} \times P_{\mathcal{J}}$),

$$R_{\mathcal{J}} = \lim_{L \to \infty} |\Lambda_L|^{-1} \sum_{x \in \Lambda_L} s_x^1 s_x^2, \tag{3.10}$$

and (the formal density of) its distribution is the Parisi overlap distribution $\mathcal{P}_{\mathcal{J}}$. The limit over cubes in (3.10) is to exist, a.s. with respect to $P_{\mathcal{J}} \times P_{\mathcal{J}}$, for $\bar{\nu}$-a.e. $\mathcal{J}$ and the overlap distribution $\mathcal{P}_{\mathcal{J}}$ is to depend measurably on $\mathcal{J}$.

If all this were so, then it would follow from (3.9) and the tail σ-field triviality of pure Gibbs states (see Chapter 7 of [Ge88]), that

$$\mathcal{P}_{\mathcal{J}}(r) = \sum_{\alpha} \sum_{\gamma} W_{\mathcal{J}}^{\alpha} W_{\mathcal{J}}^{\gamma} \delta(r - r_{\alpha\gamma}^{\mathcal{J}}), \tag{3.11}$$

with

$$r_{\alpha\gamma}^{\mathcal{J}} = \lim_{L \to \infty} |\Lambda_L|^{-1} \sum_{x \in \Lambda_L} \langle S_x \rangle_{\mathcal{J}}^{\alpha} \langle S_x \rangle_{\mathcal{J}}^{\gamma}. \tag{3.12}$$

The standard SK picture thus predicts that the overlap distribution $\mathcal{P}_{\mathcal{J}}$, obtained from $P_{\mathcal{J}}$ in this way, has the following four essential features. (1) NSA $-$ $\mathcal{P}_{\mathcal{J}}$ does depend on $\mathcal{J}$; (2) nontrivial discreteness $-$ $\mathcal{P}_{\mathcal{J}}$ is a sum of (countably) infinitely many delta functions; (3) continuity of its $\mathcal{J}$-average $-$ $\bar{\mathcal{P}} \equiv \int \mathcal{P}_{\mathcal{J}} d\bar{\nu}(\mathcal{J})$ is continuous (except for two δ-functions at $\pm r_{EA}$, whose weights add up to less than one); (4) ultrametricity of the $r_{\alpha\gamma}^{\mathcal{J}}$'s.

We remark that although the discreteness of $\mathcal{P}_{\mathcal{J}}$ is essential to this SK picture, it has been pointed out to us by A. van Enter that this could be the case without discreteness of the pure state decomposition (3.9); indeed, such a situation occurs in some deterministic models considered in [EnHM92]. However, there is another feature of the SK picture, as usually presented in the physics literature, which suggests discreteness of the pure state decomposition. This is generally described in terms of there being free energy gaps of order unity between the low-lying states in any (large) volume, accompanied by an exponentially increasing density of states as the free energy increases from the bottom of the spectrum [MPV85, DT85, B92]. Throughout this paper, we will take a countable pure state decomposition in various SK pictures. However, the reader should bear in mind that, at least in principle, it might be possible to have a continuum pure state decomposition without violating those features of the SK pictures that concern only the overlaps.

Now that we have formulated the standard SK picture, we can ask whether this picture of the EA model can be valid (for some dimensions and some temperatures). This question has two parts. First, does there exist some natural construction which begins with the finite volume states, $P_{\mathcal{J},L}$, takes $L \to \infty$, and ends with an infinite volume state, $P_{\mathcal{J}}$, and its accompanying overlap distribution $\mathcal{P}_{\mathcal{J}}$? Second, can such a $\mathcal{P}_{\mathcal{J}}$ exhibit all the essential features of the SK picture? The answers to these two parts, given in [NS96b] are, respectively, yes and no, as we now explain. We will not formalize the answer to the first part as a proposition or theorem because it will be implicitly included as a part of the more comprehensive "metastate" approach given in the next section.

We begin our answer to the first part of the question by noting that we cannot simply fix $\mathcal{J}$ and take an ordinary limit of the finite cube, periodic b.c. state $P_{\mathcal{J},L}$, as $L \to \infty$. Unlike, say, the $d = 2$ homogeneous Ising ferromagnet, where such a limit exists (and equals $\frac{1}{2}P^+ + \frac{1}{2}P^-$) by spin flip symmetry considerations (and the fact that P^+ and P^- are the only pure states [A80], [Hi81]), there is no guarantee for a spin glass that there is a well defined limit. By compactness arguments, one can easily obtain, for each $\mathcal{J}$, convergence along subsequences of L's. But these subsequences may be $\mathcal{J}$-dependent and there seems to be no natural way to patch together the limits for different $\mathcal{J}$'s to yield $P_{\mathcal{J}}$. This is more than a technical problem. As first discussed in [NS92] and as we will explain below, when there are many competing pure states (as in an SK picture), there should be chaotic size dependence (CSD) – i.e., the existence, for typical configurations of $\mathcal{J}$, of different limits along different ($\mathcal{J}$-dependent) subsequences.

In spite of the "problem" of CSD, a limit can be taken, not by fixing $\mathcal{J}$, but by considering the joint distribution $\bar{\nu}(\mathcal{J}) \times P_{\mathcal{J},L}((s_x : x \in \Lambda_L))$, of the J_e's and the S_x's. (We note that such joint distribution limits were considered, implicitly or explicitly, in [L77], [Co89], [GKN92] and [S95].) That is, by choosing a subsequence of L's (not depending on $\mathcal{J}$), one has convergence of all the finite dimensional distributions of the J_e's and S_x's to those of a probability measure on $\mathbb{R}^{\mathbb{E}^d} \times \mathcal{S}$, whose marginal distribution of $\mathcal{J}$ is $\bar{\nu}$ and whose conditional distribution of s, given $\mathcal{J}$, is some $P_{\mathcal{J}}$. All this follows from standard compactness arguments with $P_{\mathcal{J}}$ a probability measure on $\mathcal{S}$ defined for $\bar{\nu}$-a.e. $\mathcal{J}$ and depending measurably on $\mathcal{J}$. What doesn't follow from general compactness arguments is that (for $\bar{\nu}$-a.e. $\mathcal{J}$) $P_{\mathcal{J}}$ is an infinite volume Gibbs state for $\mathcal{J}$. This however can be shown either directly, or as a corollary of a more comprehensive result from [AW90] that is also presented below (see Theorem 4.2 and the remark following it).

This construction has certain translation invariance properties that are important, both technically and conceptually. Because of the periodic b.c.'s on the cube Λ_L, the couplings and spins are really defined on a (discrete) torus of size L, with a joint distribution invariant under torus translations. This implies that any (subsequence) limit joint distribution on $\mathbb{R}^{\mathbb{E}^d} \times \mathcal{S}$ is invariant under translations of $\mathbb{Z}^d$, which in turn implies that $P_{\mathcal{J}}$ is translation covariant; i.e., under the translation of $\mathcal{J}$ to $\mathcal{J}^a$, where $\mathcal{J}^a_{\{x,y\}} = \mathcal{J}_{\{x+a,y+a\}}$, $P_{\mathcal{J}}$ transforms so that

$$P_{\mathcal{J}^a}(S_{x_1} = s_1, \ldots, S_{x_m} = s_m) = P_{\mathcal{J}}(S_{x_1-a} = s_1, \ldots S_{x_m-a} = s_m). \tag{3.13}$$

The conceptual significance of translation covariance is that the mapping from $\mathcal{J}$ to $P_{\mathcal{J}}$, being a natural one, should not (and in this construction does not) depend on the choice of an origin. The technical significance is that it implies that the joint measure for $\mathcal{J}$ and the two replicas s^1 and s^2, $\bar{\nu}(\mathcal{J})P_{\mathcal{J}}(s^1)P_{\mathcal{J}}(s^2)$, on $\mathbb{R}^{\mathbb{E}^d} \times \mathcal{S} \times \mathcal{S}$ is translation invariant (under $(\mathcal{J}, S^1, S^2) \to (\mathcal{J}^a, S^{1a}, S^{2a})$) which implies, by the ergodic theorem, that $|\Lambda_L|^{-1} \sum_{x \in \Lambda_L} s^1_x s^2_x$ has an a.s. limit R (with respect to $\bar{\nu}(\mathcal{J})P_{\mathcal{J}}(s^1)P_{\mathcal{J}}(s^2)$) and thus the $R_{\mathcal{J}}$ of (3.10) exists a.s. (with respect to $P_{\mathcal{J}} \times P_{\mathcal{J}}$) for $\bar{\nu}$-a.e. $\mathcal{J}$, as desired. $\mathcal{P}_{\mathcal{J}}$, the distribution of $R_{\mathcal{J}}$, then exists for $\bar{\nu}$-a.e. $\mathcal{J}$ and depends measurably on $\mathcal{J}$. Indeed $\mathcal{P}_{\mathcal{J}}$ is simply the conditional distribution of the random variable R, given $\mathcal{J}$.

We have answered the first part of our question on the validity of the standard SK picture by showing that, yes, there does exist a natural $P_{\mathcal{J}}$ and $\mathcal{P}_{\mathcal{J}}$, that are related as required by that picture and that depend on $\mathcal{J}$ measurably. To begin our answer to the second part of the question, we see what the translation covariance of $P_{\mathcal{J}}$ implies about $\mathcal{P}_{\mathcal{J}}$. By translation covariance, $R_{\mathcal{J}^a}$ is equidistributed with the random variable (on $\mathcal{S}\times\mathcal{S}$ with measure $P_{\mathcal{J}} \times P_{\mathcal{J}}$)

$$R_{\mathcal{J}}^{-a} \equiv \lim_{L\to\infty} |\Lambda_L|^{-1} \sum_{x\in\Lambda_L} s^1_{x-a} s^2_{x-a} = R_{\mathcal{J}}. \tag{3.14}$$

Thus $\mathcal{P}_{\mathcal{J}^a} = \mathcal{P}_{\mathcal{J}}$ for $\bar{\nu}$-a.e. $\mathcal{J}$ and all $a \in Z^d$; i.e., $\mathcal{P}_{\mathcal{J}}$ is translation invariant. As in the case of the translation covariance of $P_{\mathcal{J}}$, the translation invariance of $\mathcal{P}_{\mathcal{J}}$ has the conceptual significance that a natural object like the Parisi order parameter distribution should not (and in this construction does not) depend on the choice of an origin. But it also has an important technical significance which, in the next proposition, explains why the answer to the second part of our question on the validity of the standard SK picture is, no, such a $\mathcal{P}_{\mathcal{J}}$ *cannot* exhibit all the essential features of the SK picture.

Proposition 3.1 ([NS96b]) *If $\mathcal{P}_{\mathcal{J}}$ is translation invariant, then it is self-averaged; i.e., it equals a fixed probability measure $\mathcal{P}$ on $[-1,1]$ for $\bar{\nu}$-a.e. $\mathcal{J}$. Thus it does not exhibit (1) non-self-averaging and consequently also does not exhibit at least one of (2) nontrivial discreteness or (3) continuity of its $\mathcal{J}$-average.*

Proof. Consider for any k, the moment $\int_{-1}^{1} q^k \mathcal{P}_{\mathcal{J}}(q)dq$. This is a measurable function of $\mathcal{J}$ defined for $\bar{\nu}$-a.e. $\mathcal{J}$ which is invariant under the translation $\mathcal{J} \to \mathcal{J}^a$ for every $a \in \mathbb{Z}^d$. By the spatial translation invariance and ergodicity of $\bar{\nu}$, this implies that this function is $\bar{\nu}$-a.s. a constant. These moments, for all k, determine $\mathcal{P}_{\mathcal{J}}$; thus $\mathcal{P}_{\mathcal{J}}$ itself is $\bar{\nu}$-a.s. a constant $\mathcal{P}$. Since $\mathcal{P} = \int \mathcal{P}_{\mathcal{J}} d\bar{\nu}(\mathcal{J})$, the rest of the proposition follows.

□

4 Chaotic Size Dependence and Metastates

Technically, the feature of the standard SK picture which led to its demise was the translation covariance of the infinite volume state $P_{\mathcal{J}}$. In pursuing other interpretations of the SK picture, and in analyzing disordered systems more generally, we will not give up translation covariance, but we will give up the idea that a disordered system, in the infinite volume limit, should necessarily be described by a single $P_{\mathcal{J}}$, i.e., by a function from coupling configurations to *single* infinite volume Gibbs states. Indeed, rather than finessing CSD (by constructing our $P_{\mathcal{J}}$), we will try to understand (or at least describe) CSD by analyzing the way in which $P_{\mathcal{J},L}$ samples from its various possible limits as $L \to \infty$. A major contribution of [NS96c] was the proposal that this sampling is naturally understood in terms of a "metastate", a probability measure $\kappa_{\mathcal{J}}$ on the infinite volume Gibbs states for the given $\mathcal{J}$. (We will give a more precise definition below.)

This proposal of [NS96c] was based on an analogy with chaotic deterministic dynamical systems, where the chaotic motion along a deterministic orbit is analyzed in terms of some appropriately selected probability measure, invariant under the dynamics. Time along the orbit is replaced, in our context, by L and the state space (or configuration space or phase space) of the dynamical system is replaced by the space of Gibbs states (for a fixed $\mathcal{J}$). We will delay until Section 8 any discussion of the issue of what, in disordered systems, replaces invariance of the probability measure under the dynamics. Rather we will now explain how the same metastate can be constructed by two different approaches, one based on the randomness of the $\mathcal{J}$'s and the other based on CSD for a fixed $\mathcal{J}$.

The approach based on $\mathcal{J}$-randomness is due to Aizenman and Wehr [AW90]. This approach is analogous to the construction of $P_{\mathcal{J}}$ described in the last section, except that instead of considering the random pair (J, S), distributed for finite L by $\bar{\nu}(\mathcal{J}) \times P_{\mathcal{J},L}$, and then taking the limit (along a subsequence) of finite dimensional distributions, one considers the random pair $(J, P_{J,L})$, defined on the underlying probability space $(\Omega, \mathcal{F}, \nu)$ of J, and takes the limit of its finite dimensional distributions. The finite dimensional distributions can be defined in a number of equivalent ways. We will consider (for $P_{J,L}$) the random (because of J) probabilities of cylinder sets. I.e., for each finite $\Lambda \subset \mathbb{Z}^d$ and $s \in \mathcal{S}_\Lambda = \{-1, +1\}^\Lambda$ (we denote by $\mathcal{A}$ the set of all such pairs (Λ, s)) and for L sufficiently large so that $\Lambda \subseteq \Lambda_L$, we define the random variable (on $(\Omega, \mathcal{F}, \nu)$)

$$Q^{(L)}_{(\Lambda,s)} = P_{J,L}\left(\{s' \in \mathcal{S}_{\Lambda_L} : s'_x = s_x \ \forall x \in \Lambda\}\right). \tag{4.1}$$

Let $\kappa^\dagger$ denote a probability measure on $\Omega^\dagger = \mathbb{R}^{\mathbb{E}^d} \times \mathbb{R}^{\mathcal{A}}$, with the product Borel σ-field $\mathcal{F}^\dagger$. We say that $(J, P_{J,L}) \to \kappa^\dagger$ as $L \to \infty$, if each of the finite dimensional distributions of $\left(J_e, Q^{(L)}_{(\Lambda,s)} : e \in \mathbb{E}^d,\ (\Lambda, s) \in \mathcal{A}\right)$ converges as $L \to \infty$ to the corresponding finite dimensional marginal distribution of $\kappa^\dagger$.

Open Problem 4.1 *Prove that $(J, P_{J,L}) \to \kappa^\dagger$ for some probability measure $\kappa^\dagger$ on $(\Omega^\dagger, \mathcal{F}^\dagger)$.*

Although convergence of $(J, P_{J,L})$ has not been proved, it can be shown [AW90] (see also Lemmas A.2, A.3 and A.4 of the Appendix) that there is sequential compactness and that every subsequence limit $\kappa^\dagger$ has a conditional distribution $\kappa_{\mathcal{J}}$ (of $q \in \Omega^1 \equiv \mathbb{R}^{\mathcal{A}}$, given $\mathcal{J} \in \Omega^0 \equiv \mathbb{R}^{\mathbb{E}^d}$) that, for $\bar{\nu}$-a.e. $\mathcal{J}$, is supported on infinite volume Gibbs distributions for that $\mathcal{J}$. Thus we have the following result.

Theorem 4.2 ([AW90]) *There exists a subsequence L_n of the L's such that $(J, P_{J,L_n}) \to \kappa^\dagger$. Here $\kappa^\dagger$ is a probability measure on $(\Omega^\dagger, \mathcal{F}^\dagger)$ whose marginal distribution $\kappa_{\mathcal{J}}$ (of q, given $\mathcal{J}$) satisfies: for $\bar{\nu}$-a.e. $\mathcal{J}$, $\kappa_{\mathcal{J}}(\{q : q \text{ is an infinite volume Gibbs state for } \mathcal{J}\}) = 1$.*

Proof. This follows directly from Lemmas A.2, A.3 and A.4 of the Appendix.

□

Remark 4.3 There is more than an analogy relating the construction of $\kappa_{\mathcal{J}}$ given in the last proposition to the construction of the Gibbs distribution $P_{\mathcal{J}}$ given earlier. By restricting the function f appearing in (A.8) of the Appendix to be linear in q, we see that, as $n \to \infty$,

$$E(g(J)\langle S_A\rangle_{J,L_n}) \to \int_{\Omega^0} g(\mathcal{J}) \left[\int_{\Omega^1} \langle S_A\rangle_q d\kappa_{\mathcal{J}}(q)\right] d\bar{\nu}(\mathcal{J}), \tag{4.2}$$

where g is a (continuous, bounded) function of finitely many couplings, A is a finite subset of $\mathbb{Z}^d$, $S_A = \prod_{x\in A} S_x$, $\langle\cdot\rangle_{J,L_n}$ denotes expectation with respect to P_{J,L_n}, and $\langle\cdot\rangle_q$ denotes the expectation with respect to q (for q a probability measure on $\mathcal{S}$). On the other hand, the construction of $P_{\mathcal{J}}$ yields

$$E(g(J)\langle S_A\rangle_{J,L_n}) \to \int_{\Omega^0} g(\mathcal{J})\langle S_A\rangle_{P_{\mathcal{J}}} d\bar{\nu}(\mathcal{J}). \tag{4.3}$$

Since these last two equations are valid for general g and A, we conclude (see [AW90]) that the state $P_{\mathcal{J}}$ is the mean of the metastate $\kappa_{\mathcal{J}}$, i.e.,

$$P_{\mathcal{J}} = \int_{\Omega^1} q \ d\kappa_{\mathcal{J}}(q). \tag{4.4}$$

Theorem 4.2 then implies that $P_{\mathcal{J}}$ is indeed a Gibbs distribution.

The second approach to constructing a metastate takes a fixed $\mathcal{J}$ and replaces $\mathcal{J}$-randomness, roughly speaking, by regarding L as random, i.e., by considering the empirical distribution of $P_{J,L}$ as L varies. We define the empirical distribution along a given subsequence L_n of the L's as follows. First, we make the convention that $Q^{(L)}_{(\Lambda,s)}$, defined by (4.1) for $\Lambda \subseteq \Lambda_L$, is zero when $\Lambda \not\subseteq \Lambda_L$, and then we define $\vec{Q}^{(L)}$ as the Ω^1-valued random variable (on the underlying probability space $(\Omega, \mathcal{F}, \nu)$ of J),

$$\vec{Q}^{(L)} = \left(Q^{(L)}_{(\Lambda,s)} : (\Lambda, s) \in \mathcal{A}\right). \tag{4.5}$$

The empirical distribution, along the subsequence (L_n), is the random discrete measure on Ω^1,

$$\kappa_J^K((L_n)) = \frac{1}{K}\sum_{k=1}^{K} \delta_{\vec{Q}^{(L_k)}}. \tag{4.6}$$

As in the first approach to metastates, the empirical distribution approach has an open problem about convergence (which will be discussed again in later sections of the paper) along with a partial solution based on taking subsequences. For probability measures κ'_m and κ' on Ω^1, we say $\kappa'_m \to \kappa'$ if each of the finite dimensional marginal distributions of κ'_m converges as $m \to \infty$ to the corresponding marginal distribution of κ'.

Open Problem 4.4 *Determine whether for ν-a.e. ω, $\kappa^K_{J(\omega)}((L_n \equiv n)) \to \kappa'_\omega$, as $K \to \infty$, for some probability measure κ'_ω on Ω^1.*

The partial solution to this problem is based on Theorem 4.2. If (L_n) is a subsequence such that $(J, P_{J,L_n}) \to \kappa^\dagger$, it can be shown (see Proposition A.6 of the Appendix) that by taking a subsequence n_k of the n's and a subsequence K_m of the K's, that for ν-a.e. ω, $\kappa^{K_m}_{J(\omega)}((L_{n_k}))$ converges as $m \to \infty$ to $\kappa_{J(\omega)}$, where $\kappa_{\mathcal{J}}$ is the conditional distribution of $\kappa^\dagger$ (of q, given $\mathcal{J}$). Combining this with Theorem 4.2, we have the following result, announced in [NS96c].

Theorem 4.5 ([NS96c]) *There is a sub-subsequence L_{n_k} of the L_n's of Theorem 4.2 such that along some subsequence K_m of the K's,*

$$\text{for } \nu\text{-a.e. } \omega, \quad \kappa^{K_m}_{J(\omega)}((L_{n_k})) \to \kappa_{J(\omega)} \quad \text{as } m \to \infty, \tag{4.7}$$

where $\kappa_{\mathcal{J}}$ is the same metastate as in Theorem 4.2.

Proof. This result follows from Proposition A.6 of the Appendix.

□

5 Metastates in Simpler Models

Before going on, in the next section of the paper, to analyze various possibilities for the nature of the metastate in the EA model, we digress in this section to discuss the nature of metastates in some much simpler models, where we do have some knowledge about the structure of the pure phases. The purpose of this digression is not to increase our understanding of these models, but rather to shed some light on the general metastate approach. Consequently, in this section we will primarily give heuristic arguments and not focus on results that have been or can currently be proved rigorously. Indeed, even though the models we will discuss are much simpler than the EA model, in some of the cases, it appears that obtaining a rigorous proof of the conjectured nature of the metastate would be highly nontrivial.

All the models we will consider in this section are nearest neighbor ones on the usual lattice $\mathbb{Z}^d$. These are: a) homogeneous Ising ferromagnet, b) random field Ising model (RFIM), c) highly disordered model of spin glass ground states (see [NS94], [NS96a]) and d) homogeneous XY (i.e., two-component rotator) model. In cases a) and d) where the Hamiltonians are not themselves disordered, we will introduce disorder by means of random b.c.'s. It should be noted that there already exist some results, which implicitly or explicitly concern metastates, in certain mean-field models, including in particular the random field Curie-Weiss model [AmPZ92], [Ku96]. Since the mechanisms which control the nature of the metastate for random field models seem to be essentially the same for nearest-neighbor as for mean-field models, we will return to these mean-field results below.

Homogeneous Ising Ferromagnet. This is the standard Ising model in which the all the nearest neighbor couplings have a constant positive value, $\check{J}$. At high temperature (at all nonzero temperatures, if $d = 1$), there is a unique infinite volume Gibbs state P and so any sequence of finite volume states (with Λ_n tending to all of $\mathbb{Z}^d$) converges (in the sense of finite dimensional distributions) to P. In particular, this is so for the periodic b.c. states P_L^{per} on cubes. Thus there is no CSD and the metastate is simply the point measure supported on the single (necessarily pure) state P.

Let us now consider the $d = 2$ model at low temperature (i.e., below the critical temperature). Here, it has been proved [A80], [Hi81] that there are exactly two pure infinite volume Gibbs states, P^+ and P^-; these are the ones obtained with plus and minus b.c.'s and thus are related by a global spin flip. It follows that, since periodic b.c's do not break spin flip symmetry, P_L^{per} must converge to $P = (P^+ + P^-)/2$. Thus, with periodic b.c.'s, there is again no CSD and the metastate is the point measure supported on the single state P. The difference with the high temperature metastate is that P is not pure but rather a mixture of the two pure states, P^+ and P^-. Free b.c.'s would behave the same as periodic.

The situation becomes a bit more interesting if we introduce disorder by taking random b.c.'s. I.e., we consider the finite volume Gibbs states on cubes, $P_L^{\bar{s}(\omega)}$, where for each L the b.c. spins at different sites on the boundary are i.i.d. symmetric random variables and where we also assume independence among the b.c.'s for different L's. Heuristically, we expect that (with probability close to 1) $P_L^{\bar{s}(\omega)}$ is approximated by $W_{\omega,L}^+ P^+ + W_{\omega,L}^- P^-$ with

$$W_{\omega,L}^{\pm} = e^{-\beta G_L^{\pm}(\omega)} / [e^{-\beta G_L^{+}(\omega)} + e^{-\beta G_L^{-}(\omega)}], \tag{5.1}$$

where

$$G_L^{\pm}(\omega) = \pm \sum_{x \in \partial\Lambda_L} (-\check{J}) \bar{s}_x(\omega). \tag{5.2}$$

$G_L^{\pm}$ represents the interaction energy between the b.c. spins and either the constant (identically $+1$ or identically -1) spin configuration on Λ_L or equivalently the constant configuration on a single interior boundary layer of spins at the sites $y \in \Lambda_L$ which are nearest neighbors of $\partial\Lambda_L$. The asymptotic behavior of $(W_{\omega,L}^+, W_{\omega,L}^-)$ is easy to see. By the central limit theorem, $G_L^{\pm}/\sqrt{L}$ converges in distribution to a normal random variable, and furthermore the G_L's are independent random variables for different L's. For almost every fixed ω, $(W_{\omega,L}^+, W_{\omega,L}^-)$ depends (extremely) chaotically on L, taking values approaching $(1, 0)$ for about half the L's and approaching $(0, 1)$ for the rest; thus the empirical distribution of $(W_{\omega,L}^+, W_{\omega,L}^-)$ for $L = 1, 2, \ldots, K$ converges to $[\delta_{(1,0)} + \delta_{(0,1)}]/2$. Hence we expect that the metastate for random b.c.'s is the corresponding $[\delta_{P^+} + \delta_{P^-}]/2$ (and that there is an affirmative answer to the analogue of Open Problem 4.4).

Should there be any differences between $d \geq 3$ and $d = 2$? The short answer is: *yes and no*. A slightly longer answer is: *yes*, there are of course differences, since for $d \geq 3$ and sufficiently low temperature, the number of pure states is infinite. But nevertheless, we expect *no* difference between $d = 2$ and $d \geq 3$ in the nature of the metastates for periodic (or free) and random b.c.'s. This is due to the nature of the additional pure states, which are non-translation-invariant interface states obtained by taking an infinite volume limit with Dobrushin b.c.'s (e.g., plus above or at the equator and minus below) chosen so as to force an interface (between configurations looking like the plus and minus states far from the interface) at some plane perpendicular to some coordinate direction (e.g., the equator).

For periodic or free b.c.'s, the finite volume states converge to a translation invariant (and spin flip symmetric) Gibbs state which (at least for very low temperature) must be $P = (P^+ + P^-)/2$ and so, as for $d = 2$, the metastate is just δ_P. Physically, one would say that the weights of the pure states, other than P^+ and P^-, in P_L^{per} are suppressed because these other states are much less consistent with periodic b.c.'s. For random b.c.'s, the ratios of the weight of P^+ to that of P^- should be, as for $d = 2$, of order $e^{-G_L(\omega)}$ with $G_L/L^{(d-1)/2}$ random and of order 1, but the weights of the other pure states should be suppressed by extra factors of order $\exp(-L^{d-1})$ due to the presence of the interface. Thus we still expect the random b.c. metastate to be $(\delta_{P^+} + \delta_{P^-})/2$. (We note that essentially the same reasoning was used by van Enter [En90] to argue the absence for $d = 2$ of interface ground states at zero temperature with random b.c.'s.)

Random Field Ising Model. This model has homogeneous ferromagnetic nearest neighbor couplings along with independent random magnetic fields at each site. The (formal) Hamiltonian is

$$-\check{J} \sum_{e=\{x,y\} \in \mathbb{E}^d} s_x s_y - \sum_{x \in \mathbb{Z}^d} h_x(\omega) s_x, \tag{5.3}$$

where the h_x's are i.i.d. symmetric random variables on some probability space. For $d = 2$ and any β, there is a.s. a unique infinite volume Gibbs state [AW90], while for $d = 3$ and sufficiently large $\check{J}$ and β (with some assumptions on the common distribution of the h_x's), the plus and minus b.c. infinite volume Gibbs states P_ω^+ and P_ω^- are a.s. distinct [BrK88] (see also [I85]). Note that because of the random field, these two states do *not* map into each other under a global spin flip and thus, for fixed ω, $P_{\omega,L}^{per}$ should *not* converge to $(P_\omega^+ + P_\omega^-)/2$. Here we expect $P_{\omega,L}^{per}$ to be approximated by $W_{\omega,L}^+ P_\omega^+ + W_{\omega,L}^- P_\omega^-$ with $W_{\omega,L}^\pm$ given as in (5.1), but with (5.2) replaced by

$$G_L^\pm(\omega) = \pm \sum_{x \in \Lambda_L} -h_x(\omega). \tag{5.4}$$

For the RFIM, $G_L^\pm/L^{d/2}$ converges in distribution to a normal random variable, but, unlike for the homogeneous ferromagnet with random b.c.'s, the G_L's are not independent for different L's. As in the homogeneous case, but without the need for random b.c.'s, we expect CSD to occur and the metastate to be $(\delta_{P_\omega^+} + \delta_{P_\omega^-})/2$. Indeed, precisely this behavior was proved for the Curie-Weiss model in a random field, where the necessary approximations can be controlled rigorously [AmPZ92], [Ku96]. But it was also proved in [Ku96] for the Curie-Weiss random field model, that if a (sparse) subsequence of volumes is not chosen, then a.s. the empirical distribution does *not* converge to the metastate. Heuristically, one expects essentially the same phenomenon to occur for the short range RFIM. The basic mechanism is that the fraction of L's between 1 and K for which $G_L^+(\omega) > 0$ does not converge a.s. to $1/2$ as $K \to \infty$, but rather converges in distribution to a random variable with a continuous distribution between 0 and 1 (related to the arcsine law of Brownian motion). The reader is referred to [Ku96] for more precise statements and for proofs, in the Curie-Weiss context.

Nevertheless, the empirical distribution of $(W_{\omega,L}^+, W_{\omega,L}^-)$, along a (deterministic) subsequence $L = L_1, \ldots, L_K$ with $L_{j+1} - L_j$ growing rapidly enough, does converge a.s. to $[\delta_{(1,0)} + \delta_{(0,1)}]/2$. The subsequence should be chosen so that for $j, j' \to \infty$ with $j'/j \leq \eta < 1$, $(L_{j'})^d/(L_j)^d \to 0$ in order that $G_{L_j}^+/(L_j)^{d/2}$ be asymptotically independent of $G_{L_{j'}}^+/(L_{j'})^{d/2}$. This will guarantee that the fraction of j's between 1 and K for which $G_{L_j}^+(\omega) > 0$ *does* converge a.s. to $1/2$ as $K \to \infty$. For example, constructing the subsequence by taking $L_{j+1} = cL_j$ with a fixed $c > 1$ is much more than sufficient for this purpose.

Highly Disordered Spin Glass. This is a nearest neighbor Ising spin glass model (in which the couplings depend [nonlinearly] on the volume) that was introduced and analyzed in [NS94], [NS96a] (see also [CiMB94]). At zero temperature, the model has a quite interesting dimension dependent ground state structure, while at strictly positive temperature, the model makes little or no thermodynamic sense. Before reviewing the model and its ground state structure, we briefly discuss the construction of metastates at zero temperature, for models where the couplings do not depend on the volume.

The ground states in a finite volume Λ for b.c. * are the spin configurations s in $\mathcal{S}_\Lambda$ that minimize the finite volume Hamiltonian H_Λ^*. The natural replacement, at zero temperature, for the Gibbs distribution $P_{\Lambda,\beta}^*$ is $P_{\Lambda,\infty}^* = \lim_{\beta\to\infty} P_{\Lambda,\beta}^*$, the probability measure which assigns equal probability to all the ground states and zero probability to all other spin configurations. Using $P_{\Lambda,\infty}^*$, one can construct metastates $\kappa_{\mathcal{J}}$ by the zero temperature analogues of the lemmas and propositions of the Appendix. The DLR equations for the Γ's in the support of $\kappa_{\mathcal{J}}$ are replaced by the property that Γ is supported on infinite volume ground state configurations, i.e., on spin configurations on $\mathbb{Z}^d$ such that the flip of any finite set of spins yields a non-negative change in the energy.

It is worth noting that in the EA model, where spin flip symmetry is broken only by (certain) b.c.'s, if the common distribution μ of the individual couplings is continuous, then H_Λ^* with a b.c. such as periodic or free (resp., such as plus or random) a.s. has exactly two ground states, related by a spin flip (resp., one ground state). Thus the Γ's in the support of the metastate would all be of the form $(\delta_s + \delta_{-s})/2$ (resp., δ_s) for an infinite volume ground state configuration s.

The highly disordered spin glass model of [NS94] is one defined on the sequence of cubes Λ_L, where the nearest neighbor Hamiltonians, $H^*_{\Lambda_L}$ have couplings $J_e^{(L)}$ depending nonlinearly on L as follows.

$$J_e^{(L)} = \alpha_e \exp(-\lambda^{(L)} K_e), \tag{5.5}$$

where the K_e's are i.i.d. continuous random variables, the α_e's are i.i.d., independent of the K_e's, with $\nu(\alpha_e = +1) = \nu(\alpha_e = -1) = 1/2$, and the $\lambda^{(L)}$'s are positive constants. The K_e's are not restricted as to sign and in fact can have any continuous common distribution $\tilde{\mu}$, but it is convenient to take $\tilde{\mu}$ to be the uniform distribution on $(0, 1)$. The $\lambda^{(L)}$'s are chosen so that each of the $J_e^{(L)}$'s appearing in $H^{\bar{s}^{(L)}}_{\Lambda_L}$ has a magnitude "on its own scale". More precisely, let m_L denote the number of edges $e = \{x, y\} \in \mathbb{E}^d$ with x or y (or both) in Λ_L and arrange the m_L random variables $|J_e^{(L)}|$, for these e's, in rank order,

$$|J_{(1)}^{(L)}| \geq |J_{(2)}^{(L)}| \geq \ldots \geq |J_{(m_L)}^{(L)}|; \tag{5.6}$$

$\lambda^{(L)}$ is chosen so that a.s., for all large L,

$$|J_{(j)}^{(L)}| \geq 2|J_{(j+1)}^{(L)}| \quad \text{for } 1 \leq j < m_L. \tag{5.7}$$

This immediately implies that a.s., for all large L,

$$|J_{(j)}^{(L)}| > \sum_{k=j+1}^{m_L} |J_{(k)}^{(L)}| \quad \text{for } 1 \leq j < m_L. \tag{5.8}$$

For L large enough so that (5.8) is valid, the finite volume ground states can be determined in terms of certain finite volume tree graphs depending only on the K_e's and on the type of b.c. We will restrict attention to fixed and free b.c.'s. For fixed b.c.'s, we construct a random graph $F^{(L)}$ which is a subset of the nearest neighbor graph on the sites of $\Lambda_L \cup \partial\Lambda_L$. The set of edges in $F^{(L)}$ is defined inductively as follows. (The vertices in $F^{(L)}$ are all sites in $\Lambda_L \cup \partial\Lambda_L$ touching these edges). Begin with the edge (among those with at least one endpoint in Λ_L) with the smallest value of K_e (i.e., the one corresponding to $J_{(1)}^{(L)}$), then sequentially add edges with the smallest values of K_e among the remaining edges, except not any edge that would either create a closed loop or would cause some site in Λ_L to be connected to more than one site in $\partial\Lambda_L$. Stop when every site in Λ_L is connected to exactly one site in $\partial\Lambda_L$. $F^{(L)}$ is a forest (i.e., a union of site-disjoint trees) such that each site $x \in \Lambda_L$ is connected to a unique site $w_L(x)$ in $\partial\Lambda_L$ by a unique path $r_L(x)$ in $F^{(L)}$. The fixed b.c. ground states are given as follows. Define

$$\eta_L(x) = \prod_{e \in r_L(x)} \alpha_e. \tag{5.9}$$

Then the ground state $s^{(L)}$ for $H^{\bar{s}^{(L)}}_{\Lambda_L}$ is given by

$$s_x^{(L)} = \eta_L(x) \bar{s}_{w_L(x)}^{(L)}, \ \text{for } x \in \Lambda_L. \tag{5.10}$$

For free b.c.'s, $F^{(L)}$ is replaced by $F^{(L)f}$, a subgraph of the nearest neighbor graph on Λ_L in which the edges with smallest K_e are sequentially added, with no loop creation allowed, until the graph becomes a single tree touching every site in Λ_L. The pair of free b.c. ground states, $\pm s^{(L)f}$, is determined by the requirement that

$$s_x^{(L)f} s_{x'}^{(L)f} = \eta_L(x,x') = \prod_{e \in r_L(x,x')} \alpha_e, \tag{5.11}$$

where, for $x, x' \in \Lambda_L$, $r_L(x,x')$ is the unique path in $F^{(L)}$ connecting x and x'.

The set of all (subsequence) limits s of fixed b.c. ground states $s^{(L)}$, for all possible choices of b.c. $\bar{s}^{(L)}$, is completely characterized in terms of F_∞ (the limit of $F^{(L)}$ or of $F^{(L)f}$ as $L \to \infty$), which is a forest, spanning all of $\mathbb{Z}^d$, each of whose trees is infinite. Namely, for each tree in F_∞, the relative signs of the spins in s on that tree are determined by the analogue of (5.11), with the absolute sign of a single spin in each tree undetermined. The number of such infinite volume configurations is thus $2^{\mathcal{M}}$, where $\mathcal{M}$ is the (a.s. constant) number of trees in the forest F_∞. This forest is a natural object in the context of invasion percolation (see [NS94], [NS96a] for further discussion and for historical references) and in that context, it was proved rigorously that $\mathcal{M} = 1$ for $d = 2$ [ChCN85]. It has been conjectured [NS94], [NS96a] but not proved that $\mathcal{M} = 1$ for $d < 8$ and that $\mathcal{M} = \infty$ for $d > 8$.

When $\mathcal{M} = 1$, the metastate structure is like that for the homogeneous Ising ferromagnet. For free b.c.'s, there will be no CSD and the metastate will be the point measure supported on $\Gamma = (\delta_s + \delta_{-s})/2$ where $\pm s$ is the pair of infinite volume ground states. For plus b.c.'s, the behavior will be like that for the random b.c. homogeneous ferromagnet, i.e., the metastate will be $(\delta_{\delta_s} + \delta_{\delta_{-s}})/2$. Note in this regard that for plus b.c.'s, the ground state spins, $s_x^{(L)}$ at a fixed x as L varies are i.i.d. symmetric random variables (in their dependence on the α_e's for fixed K_e's), because each time L is increased, at least one new α_e appears in (5.9).

When $\mathcal{M} = \infty$, similar reasoning shows that with plus b.c.'s, the signs of $s_{x_1}^{(L)}, \dots, s_{x_j}^{(L)}$ for fixed $x_1, \dots, x_j$ in distinct trees are i.i.d. symmetric random variables for fixed L and independent as L varies. Here, the metastate corresponds to the measure supported on the infinite volume configurations s corresponding to independent (fair) coin tosses that determine the overall sign of each of the trees in F_∞.

When $\mathcal{M} = \infty$, but free b.c.'s are used, the situation is trickier. For each L, the edges of $F^{(L)f}$ consist of those edges of F_∞ within Λ_L plus enough extra edges (with the smallest posible K_e values) to form a *single* tree. These edges are known to run off to infinity as $L \to \infty$; this gives asymptotic independence (between L and L') of the relative signs of the trees touching fixed locations, as determined by the signs of these extra edges, providing L' is sufficiently larger than L. Thus the metastate will be supported on Γ's of the form $\Gamma = (\delta_s + \delta_{-s})/2$, with the metastate measure on the pairs $\{s, -s\}$ corresponding to independent coin tosses to determine the relative signs between different trees. However, since the choices for L and for $L+1$ are (unlike in the plus b.c. case) not independent, it is a priori possible that a phenomenon like that found in [Ku96] for the random field Curie-Weiss model could occur; i.e., it could be that the empirical distribution over $L = 1, 2, \ldots, K$ would not converge a.s. to the metastate. We believe however, based on heuristic arguments, that this is not so, although we have not worked out the details of a complete proof. We conclude our discussion of this model by noting that the situation for periodic b.c.'s (when $\mathcal{M} = \infty$) seems more difficult to analyze than for free b.c.'s.

Homogeneous XY Model. In this model, the spin variables $s_x = (s_x^1, s_x^2)$ take values in the unit circle and the (formal) Hamiltonian is

$$-\check{J} \sum_{\{x,y\} \in \mathbb{Z}^d} (s_x^1 s_y^1 + s_x^2 s_y^2). \tag{5.12}$$

We take $d = 3$, since in that dimension, at low temperature, there exist distinct infinite volume Gibbs states P^θ obtained by taking fixed constant b.c.'s, in which $\bar{s}$ is chosen by setting $\bar{s}_x$ to be $(\cos\theta, \sin\theta)$ for every x in $\partial\Lambda_L$ [FSS76]. We will assume, for the purposes of this discussion, that the set of all pure infinite volume Gibbs states is exactly $\{P^\theta : 0 \leq \theta < 2\pi\}$.

If one takes free or periodic b.c.'s, which do not break the spin rotation symmetry, then there would be no CSD and the finite volume Gibbs states would converge to $P = \int_0^{2\pi} P^\theta d\theta$. The metastate is then δ_P, analogously to the homogeneous Ising model (where $P = (P^+ + P^-)/2$).

On the other hand, the situation is less clear if we take random b.c.'s. I.e., suppose for each L, the b.c. $\bar{s}(\omega)$ has i.i.d. $\bar{s}_x$'s for $x \in \partial\Lambda_L$ (each with a uniform distribution on the unit circle) and we also take the b.c.'s for different L's to be independent. Now, for each fixed ω, the rotation symmetry is broken, although the ensemble of b.c.'s is of course (statistically) rotation invariant. There are (at least) two natural a priori guesses for the metastate in this case: 1. The "competition" among pure states for the random b.c.'s is so balanced that we end up with the same metastate as for free or periodic b.c.'s, i.e., δ_P with P the uniform mixture of the pure states P^θ over θ, as above. 2. The competition is so unbalanced that for large L, essentially one $\theta = \theta_L(\omega)$ wins out (decisively) with that θ_L distributed uniformly (as a function of ω for fixed L) and depending chaotically on L (for fixed ω); the metastate will then be supported on Γ's each of which is one of the pure states P^θ, with a uniform distribution of the θ's. In fact, we conjecture that what actually happens is case 2, based on the following heuristic arguments modelled after the ones we used above for the homogeneous Ising ferromagnet.

We expect that $P_L^{\bar{s}(\omega)}$ is approximated by $\int_0^{2\pi} W_{\omega,L}^\theta P^\theta d\theta$ with

$$W_{\omega,L}^\theta = e^{-\beta G_L^\theta(\omega)} / \int_0^{2\pi} e^{-\beta G_L^\theta(\omega)} d\theta, \tag{5.13}$$

where

$$G_L^\theta(\omega) = \sum_{x\in\partial\Lambda_L} (-\check{J})(\bar{s}_x^1(\omega)\cos\theta + \bar{s}_x^2(\omega)\sin\theta). \tag{5.14}$$

The metastate is supported on states Γ of the form $\int_0^{2\pi} P^\theta \rho(d\theta)$, where ρ is a probability measure on $[0, 2\pi)$, with some distribution of ρ's. With our ansatz for approximating $P_L^{\bar{s}}(\omega)$, this distribution would be the limiting distribution of the sequence of random measures $\rho_L(d\theta) = W_{\omega,L}^\theta d\theta$.

What is this limit in distribution? By using the central limit theorem for the random vectors, $\sum_{x\in\partial\Lambda_L}(\bar{s}_x^1, \bar{s}_x^2)$, one can prove that $W_{\omega,L}^\theta d\theta$ does converge in distribution to the random point measure $\delta_{\theta'}$ with θ' uniformly distributed on $[0, 2\pi)$. Although this supports our conjecture that case 2 is the correct guess, it does not rigorously prove the conjecture, since the connection between $P_L^{\bar{s}}$ and $W_{\omega,L}^\theta$ is an ansatz. Obtaining a complete proof is an interesting open problem.

6 Metastates in the Edwards-Anderson Model

We proceed, as in [NS96c], by giving a partial classification of the possible types of metastates $\kappa_{\mathcal{J}}$ which could occur in the EA model. The simplest of these, and one which of course does occur, at least for small β or for $d = 1$ (as in the homogeneous ferromagnet), is possibility 1) that $\kappa_{\mathcal{J}}$ is (for $\bar{\nu}$-a.e. $\mathcal{J}$) supported on a single pure Gibbs state $P = P_{\mathcal{J}}$. This is the case, for example, if $\mathcal{N} = \mathcal{N}(d, \mu, \beta) = 1$ (e.g., for $\beta < \beta_c^*(d, \mu)$), since then $P_{\mathcal{J},L} \to P_{\mathcal{J}}$ as $L \to \infty$. It is important to note however, that this possibility could occur (at least, in principle) even if $\mathcal{N} \neq 1$; indeed, just such a situation of “weak uniqueness” (see [COE87] and [EnF85]) happens in very long range spin glasses at high temperature [FZ87], [GNS93].

Another simple possibility 2) is that $\kappa = \kappa_{\mathcal{J}}$ is supported on a single Gibbs state, which is a mixture of two distinct pure states:

$$\kappa = \delta_P, \quad P = \frac{1}{2}P' + \frac{1}{2}P'', \tag{6.1}$$

where $P' = P'_{\mathcal{J}}$ and $P'' = P''_{\mathcal{J}}$ are pure states that are global flips of each other. As in the $d = 2$ homogeneous Ising ferromagnet, this would be the case according to the Fisher-Huse (FH) scaling picture discussed earlier, which predicts that (when $\mathcal{N}(d, \mu, \beta) \neq 1$) such a $P'_{\mathcal{J}}$ and $P''_{\mathcal{J}}$ are the only pure states so that (by obvious spin flip symmetry arguments) $P_{\mathcal{J},L} \to \frac{1}{2}P'_{\mathcal{J}} + \frac{1}{2}P''_{\mathcal{J}}$ as $L \to \infty$. (We deal here [and in the rest of the paper, unless otherwise noted] with $\kappa_{\mathcal{J}}$ constructed, as in Sect. 4, with periodic b.c.'s. We note that the same analysis should apply to other b.c.'s that are spin-flip symmetric.) Of course, when $P_{\mathcal{J},L}$ converges to some $P_{\mathcal{J}}$, there is no CSD and no need for a metastate description. When the metastate is supported on a single $P_{\mathcal{J}}$, we will say that it is not dispersed. Again, it is important to note that possibility 2 can occur even if $\mathcal{N} > 2$, like in the $d = 3$

homogeneous Ising ferromagnet at low temperature (below the roughening temperature). That would be an example of a "weak scaling" picture.

A trivial (i.e., over only two states) sort of dispersal, analogous to what happens in the homogeneous ferromagnet with random b.c.'s, should occur in the scaling picture if one replaces periodic boundary conditions by b.c.'s which break the spin-flip symmetry (but still do not depend on J) such as random b.c.'s. or more simply plus b.c's. Here, one expects the finite volume state to exhibit CSD, approximating P' for (roughly) half of the L's and P'' for the other half so that the metastate (for such a b.c.) should be $\frac{1}{2}\delta_{P'} + \frac{1}{2}\delta_{P''}$, which is quite different from (6.1). The phenomenon observed by Külske (in the random field Curie-Weiss model) [Ku96], that subsequences of volumes are essential in order to have a.s. convergence of the empirical distribution of the metastate, should not occur in the context of this possibility, but conceivably could in some other possibilities we discuss below.

Other sorts of dispersal could happen if there are more then two pure states, as in an SK picture. Roughly speaking, according to Theorem 4.5, a dispersed metastate describes the way in which, for large L,

$$P_{\mathcal{J},L} \approx \sum_{\alpha} W^{\alpha}_{\mathcal{J},L} P^{\alpha}_{\mathcal{J}}, \tag{6.2}$$

when there is CSD and $P_{\mathcal{J},L}$ does *not* converge to a single $P_{\mathcal{J}}$.

In possibilities 1 and 2 that we discussed above, most of the weight (represented by the $W^{\alpha}_{\mathcal{J},L}$'s) is concentrated on one or two pure states as $L \to \infty$. Two other possibilities (numbers 5 and 6, the latter of which includes the nonstandard SK picture) that we will soon discuss also have the property that most weight in a given volume is concentrated on a few states, even though there are (uncountably) many pure states. But first, we briefly mention two possibilities (very different from either FH or SK pictures) where the weights are shared more equitably. In both of these possibilities $\kappa_{\mathcal{J}}$ is supported on infinite volume Gibbs states Γ whose decomposition into pure states is continuous. Possibility 3) is that there is no dispersal and $\kappa_{\mathcal{J}}$ is supported on a single such Γ, while possibility 4) is that there is dispersal and $\kappa_{\mathcal{J}}$ is supported on multiple such Γ's. As there seems to be no particular reason to suppose that either of these possibilities occurs in the EA model, we proceed to the next possibility, which we find rather intriguing.

Possibility 5) is one in which there is CSD and dispersal among (uncountably) many Γ's, but each of these Γ's decomposes into just two pure states related by a global flip, like in possibility 2. This possibility, where $\mathcal{N} = \infty$ but for each large L one "only sees two pure states" (in the sense that $P_{\mathcal{J},L}$ is approximately a mixture of just two pure states) is intermediate between the scaling picture where $\mathcal{N} = 2$ (or more generally, where possibility 2 occurs even if $\mathcal{N} > 2$) and the next possibility (which leads to the nonstandard SK picture) where $\mathcal{N} = \infty$ and for large L one "sees many pure states". In the context of possibility 5, it would appear that there is a chance for a phenomenon like the one seen in [Ku96] for the random field Curie-Weiss model to occur. This would mean, roughly speaking, that the pair of pure states seen for a given large L would tend to be the same pair of states seen for $L' > L$ until L'/L became large. Before leaving possibility 5, we note that its analogue for the situation where periodic b.c.'s are replaced by plus or by random b.c.'s would be one with a metastate supported on Γ's that are single pure states chosen from the continuum of all pure states. This would be analogous to the behavior conjectured in the last section for the homogeneous XY model with random b.c.'s.

Possibility 6) has, like possibility 5, nontrivial dispersal over many Γ's, but unlike possibility 5, its Γ's have a nontrivial decomposition into pure states – namely

$$\Gamma = \sum_{\alpha} W_{\Gamma}^{\alpha} P_{\mathcal{J}}^{\alpha} \; ; \tag{6.3}$$

this decomposition should be discrete (i.e., a countable sum, as indicated) but with many (in particular, more than two) nonzero weights W_{Γ}^{α}. As we shall see, the nonstandard SK picture will require that the dispersal of $\kappa_{\mathcal{J}}$ over Γ's of the form (6.3) is a dispersal not only over the weights W_{Γ}^{α} but also over the selection of the countably many pure states appearing in the discrete sum (from an uncountable family of all pure states for the given $\mathcal{J}$). To explain why that is so, we first need to discuss replicas and their overlaps within the metastate framework. (We remark, as previously discussed in the context of the standard SK picture, that, strictly speaking, it is the overlaps which must be discrete rather than the pure state decomposition.)

7 Replicas and Overlaps

Replicas and overlaps are a way of probing the meaning of the approximate equation (6.2), our EA model replacement for the SK approximate equation (3.6). It is therefore natural to take replicas for fixed L and only afterward let $L \to \infty$. As emphasized by Guerra [Gu95], this order of operations could yield a different result than that obtained by first letting $L \to \infty$ and then taking replicas, as was done in the standard SK picture. In other words, rather than letting $L \to \infty$ first to obtain a single infinite volume state $P_{\mathcal{J}}$ and then defining replicas $s^1, s^2, s^3, \ldots$ by using the product measure $P_{\mathcal{J}}(s^1) \times P_{\mathcal{J}}(s^2) \times \ldots$, we will first take the product measure for finite volume, $P_{\mathcal{J},L}(s^1) \times P_{\mathcal{J},L}(s^2) \times \ldots$, and then let $L \to \infty$.

We have already encountered three (related) ways to consider the limit as $L \to \infty$ of $P_{\mathcal{J},L}$ (without replicas). These are: (a) to obtain $P_{\mathcal{J}}$ via the limit of the joint distribution $(\bar{\nu}(\mathcal{J}) \times P_{\mathcal{J},L}(s))$ of J and S (see the remark following Theorem 4.2), (b) to obtain $\kappa_{\mathcal{J}}$ via the limit of the joint distribution of J and $P_{J,L}$ (see Theorem 4.2) and (c) to obtain $\kappa_{\mathcal{J}}$ via the limit for fixed $\mathcal{J}$ of the empirical distribution of $P_{\mathcal{J},L}$ (see Theorem 4.5). We want to see what happens to these three types of limits when replicas are taken first, i.e., when $P_{\mathcal{J},L}$ is replaced by the infinite product measure (on $\mathcal{S}_{\Lambda_L} \times \mathcal{S}_{\Lambda_L} \times \ldots$),

$$P_{\mathcal{J},L}^{\infty}(s^1, s^2, \ldots) = P_{\mathcal{J},L}(s^1) \times P_{\mathcal{J},L}(s^2) \times \ldots, \tag{7.1}$$

which is the finite volume Gibbs state for (arbitrarily many) replicas. The next proposition (the first part of which is explicit in [NS96c] with the rest implicit) gives the answer, which is that $\kappa_{\mathcal{J}}(\Gamma)$ is replaced by $\kappa_{\mathcal{J}}^{\infty}$, the probability measure on $\Omega^1 \times \Omega^1 \times \ldots$, supported on elements of the form $\Gamma \times \Gamma \times \ldots$, with Γ distributed by $\kappa_{\mathcal{J}}$. Convergence of measures or random variables in the proposition denotes, as usual, (weak) convergence of finite dimensional distributions.

Proposition 7.1 ([NS96c]) *Let L_n, n_k and K_m be as in Theorems 4.2 and 4.5. Then*

$$\bar{\nu}(\mathcal{J}) \times P^{\infty}_{\mathcal{J},L_n} \to \bar{\nu}(\mathcal{J}) \times P^{\infty}_{\mathcal{J}} \quad \text{as } n \to \infty, \tag{7.2}$$

where $P^{\infty}_{\mathcal{J}}$ is the measure on $\mathcal{S}^{\infty}$,

$$P^{\infty}_{\mathcal{J}}(s^1, s^2, \ldots) = \int \left[\Gamma(s^1) \times \Gamma(s^2) \times \ldots\right] d\kappa_{\mathcal{J}}(\Gamma); \tag{7.3}$$

$$(J, P^{\infty}_{J,L_n}) \to \bar{\nu}(\mathcal{J}) \times \kappa^{\infty}_{\mathcal{J}} \quad \text{as } n \to \infty; \tag{7.4}$$

and finally, defining the empirical distribution for replicas to be the measure on $\Omega^1 \times \Omega^1 \times \ldots$,

$$\kappa_J^{K,\infty}\left((L_n)\right) = \frac{1}{K}\sum_{k=1}^{K} \delta_{(\vec{Q}^{(L_k)},\vec{Q}^{(L_k)},\ldots)}, \tag{7.5}$$

we have

$$\text{for } \nu\text{-a.e. } \omega, \quad \kappa_{J(\omega)}^{K_m,\infty}\left((L_{n_k})\right) \to \kappa^{\infty}_{J(\omega)} \quad \text{as } m \to \infty. \tag{7.6}$$

Proof. We sketch the proofs in the reverse order of the proposition. The last claim (7.6) easily reduces to the following fact, whose proof we leave to the reader: if $\vec{Q}_l$ is a sequence of random vectors whose sequence of distributions (on $\mathbb{R}^M$, for some M) converges as $l \to \infty$ to some ρ, then the sequence of distributions of $(\vec{Q}_l, \vec{Q}_l, \ldots, \vec{Q}_l)$ (on $(\mathbb{R}^M)^k$)) converges to $\rho \times \rho \times \ldots \times \rho$. Similarily, the middle claim (7.4) reduces to: if $(\vec{J}_l, \vec{Q}_l)$ converges in distribution to some $\rho^{\dagger}$ whose marginal distribution (of $\vec{q}$, given $\vec{\mathcal{J}}$) is $\rho_{\vec{\mathcal{J}}}$, then $(\vec{J}_l, \vec{Q}_l, \ldots, \vec{Q}_l)$ converges to a measure whose marginal distribution (of $(\vec{q}, \ldots, \vec{q})$, given $\vec{\mathcal{J}}$) is $\rho_{\vec{\mathcal{J}}} \times \ldots \times \rho_{\vec{\mathcal{J}}}$. The first claim (7.2) follows from (7.4) by the replica version of the remark following Theorem 4.2. This is essentially the same as arguing directly from Theorem 4.2 by using "metacorrelations", as in [NS96c]. That argument makes clear that replicas are already implicit in the very notion of metastates, since the natural generalization of (4.2) beyond functions linear in q replaces $\langle S_A \rangle_{J,L_n}$ by the metacorrelation

$$\langle S_{A_1} \rangle_{J,L_n} \cdots \langle S_{A_m} \rangle_{J,L_n} = \langle S^1_{A_1} \cdots S^m_{A_m} \rangle^{\infty}_{J,L_n}, \tag{7.7}$$

where the latter bracket denotes the expectation with respect to P^{∞}_{J,L_n}.

□

Remark 7.2 There are two conceptually important points hidden in the last proposition. One, which is the essential content of (7.6), is that replicas must be described in the infinite volume limit by $\Gamma \times \Gamma \times \ldots$ with Γ distributed by $\kappa_{\mathcal{J}}$ because replicas in finite volume are taken from the *same* L and $\kappa_{\mathcal{J}}$ (according to Theorem 4.5) describes the sampling of states as L varies. The second, emphasized by Guerra [Gu95], is that in the infinite volume replica measure $P^{\infty}_{\mathcal{J}}$, the replicas need not be independent, even though they of course are independent (by construction) in the finite volume measure $P^{\infty}_{\mathcal{J},L}$. Indeed, from the form of (7.3), we see that they will only be independent if $\kappa_{\mathcal{J}}$ is non-dispersed. In [NS96c], this lack of independence when the metastate is dispersed is called replica non-independence (RNI). It is equivalent to the non-interchangeability of the operations of taking replicas and letting $L \to \infty$, since done in the other order (as in the standard SK picture) the measure obtained is

$$P_{\mathcal{J}}(s^1) \times P_{\mathcal{J}}(s^2) \times \ldots = \int \Gamma_1(s^1) d\kappa_{\mathcal{J}}(\Gamma_1) \times \int \Gamma_2(s^2) d\kappa_{\mathcal{J}}(\Gamma_2) \times \ldots, \tag{7.8}$$

which does not agree with $P^{\infty}_{\mathcal{J}}$ unless $\kappa_{\mathcal{J}}$ is non-dispersed.

Now that we have seen that replicas are described by $\kappa^{\infty}_{\mathcal{J}}$, it is fairly clear what the appropriate replacements are for the overlap $R_{\mathcal{J}}$ and overlap distribution $\mathcal{P}_{\mathcal{J}}$ as defined (in the standard SK picture) by using the two-replica product measure $P_{\mathcal{J}} \times P_{\mathcal{J}}$. Namely, on $\mathcal{S} \times \mathcal{S}$ with product measure $\Gamma \times \Gamma$, we define the random variable

$$R_{\Gamma} = \lim_{L \to \infty} |\Lambda_L|^{-1} \sum_{x \in \Lambda_L} s^1_x s^2_x, \tag{7.9}$$

and define $\mathcal{P}_{\Gamma}$ to be (the formal density of) its distribution.

The nature of $\mathcal{P}_{\Gamma}$ depends of course on the nature of Γ, and more particularly on its pure state decomposition. Thus if Γ is a pure state (and the limit in (7.9) exists), R_{Γ} will be the constant r_{Γ} in which $s^1_x s^2_x$ in (7.9) is replaced by the expectation $\langle s_x \rangle_{\Gamma}{}^2$; if Γ is pure and flip invariant then $\langle s_x \rangle_{\Gamma} \equiv 0$ and $\mathcal{P}_{\Gamma} = \delta(r)$. If Γ has the decomposition $\Gamma = \frac{1}{2}P' + \frac{1}{2}P''$ with P' and P'' related by a global flip (as in possibilities 2 and 5 discussed in the previous section), then $\mathcal{P}_{\Gamma} = \frac{1}{2}\delta(r - r_0) + \frac{1}{2}\delta(r + r_0)$, where $r_0 = r_{P'} = r_{P''}$. If Γ is as in (6.3) of possibility 6, then $\mathcal{P}_{\Gamma}$ has the SK type form (see (3.8)),

$$\mathcal{P}_{\Gamma} = \sum_{\alpha} \sum_{\gamma} W^{\alpha}_{\Gamma} W^{\gamma}_{\Gamma} \delta(r - r_{\alpha\gamma}). \tag{7.10}$$

It is important to note that our analysis of the relation between the state Γ and the overlap distribution $\mathcal{P}_{\Gamma}$ is based on the definition (7.9) for the overlap. The limit in (7.9) is taken within an already infinite system, corresponding physically to taking the overlap in a box which is large but much smaller than the overall system size. Were one to take the overlap box and system size the same, one might, in principle, obtain a different limiting distribution. For example, Huse and Fisher [HuF87] noted that in the $d = 2$

homogeneous Ising ferromagnet, such a limit (with antiperiodic b.c.'s) yields a *continuous* overlap distribution (rather than $\frac{1}{2}\delta(r-r_0)+\frac{1}{2}\delta(r+r_0)$) despite the presence of only two pure states.

In the metastate approach, even with $\mathcal{J}$ fixed, Γ should itself be regarded as random with the distribution $\kappa_{\mathcal{J}}(\Gamma)$. Thus $\mathcal{P}_\Gamma$ should be regarded as random with the distribution induced on it by $\kappa_{\mathcal{J}}$. In possibilities 1 and 2 (and 3), $\kappa_{\mathcal{J}}$ is non-dispersed, so $\mathcal{P}_\Gamma$ is non-random. In possibility 5, $\kappa_{\mathcal{J}}$ is dispersed, but if we postulate (as in the SK pictures) that $r_{\alpha\alpha}$ (see (3.7)) is the same ($= r_{EA}$) for all pure states $P^\alpha_{\mathcal{J}}$, then $\mathcal{P}_\Gamma$ is again non-random. In possibility 6, $\kappa_{\mathcal{J}}$ is dispersed and $\mathcal{P}_\Gamma$ should really be random, i.e., should depend on Γ since the W^α_Γ's and the selection of the $r_{\alpha\gamma}$'s that have nonzero weights should depend on Γ.

8 The Nonstandard SK Picture

The dependence of $\mathcal{P}_\Gamma$ on Γ (for fixed $\mathcal{J}$), just discussed in the previous section of the paper, is crucial for the nonstandard SK picture because there can be no dependence on $\mathcal{J}$. More precisely, if one considers the distribution of the random P_Γ (that it inherits from $\kappa_{\mathcal{J}}$) as a function of $\mathcal{J}$, there is a translation invariance/ergodicity argument, essentially the same as in Proposition 3.1 and the discussion preceding it, which shows that that distribution is self-averaged. Such a self-averaging property killed the standard SK picture. It does not kill the nonstandard picture (at least, not on its own) because there *dependence on $\mathcal{J}$ is replaced by dependence on Γ for fixed $\mathcal{J}$*. In particular $\bar{\mathcal{P}} = \int \mathcal{P}_{\mathcal{J}} d\bar{\nu}(\mathcal{J})$ is now replaced by

$$\tilde{\mathcal{P}} = \int \mathcal{P}_\Gamma d\kappa_{\mathcal{J}}(\Gamma). \tag{8.1}$$

The nonstandard SK picture thus includes the following requirements within the context of possibility 6: (1) $\mathcal{P}_\Gamma$ does depend on Γ as Γ varies according to $\kappa_{\mathcal{J}}$; (2) $\mathcal{P}_\Gamma$ is (for $\kappa_{\mathcal{J}}$-a.e. Γ) a sum of (countably) infinitely many delta functions; (3) $\tilde{\mathcal{P}}$ is continuous (except for two δ-functions at $\pm r_{EA}$, whose weights add up to less than one); and (4) ultrametricity of $\{r_{\alpha\gamma} : W^\alpha_\Gamma W^\gamma_\Gamma \neq 0\}$ (for $\kappa_{\mathcal{J}}$-a.e. Γ).

Before going on to discuss the prospects for the validity of the nonstandard SK picture, we point out that replica symmetry breaking (RSB), which plays an important role in the usual analyses of the SK model, has a very natural interpretation within the metastate framework. Replica symmetry just refers to permutation invariance among the replica labels. To be broken, there must be something which distinguishes between the replicas, such as nontrivial $q^{\alpha\gamma}$'s with α and γ coming from different replicas, or more generally different pure states coming from different replicas. Thus we consider RSB to occur whenever the Γ's in the support of the metastate are not pure states so that when $\Gamma \times \Gamma \times \ldots$ is decomposed into pure states, different α's and γ's will appear for different replicas. In the physics literature, it is conventional to regard situations such as possibility

2, where the replicas appear in pure states which only differ by a global flip, as a trivial example of RSB. Thus we would describe possibility 2 as having no dispersal of the metastate and only trivial RSB while possibility 5 has dispersal of the metastate (along with CSD and RNI) but still only trivial RSB. The nonstandard SK picture has both dispersal of the metastate and nontrivial RSB.

Should there be an affirmative answer to Open Problem 2.4, i.e., should it be shown that the number of pure states (sometimes) exceeds 2, it would remain to determine whether the many states are "seen" in $P_{\mathcal{J},L}$ for large L and if so, how. The nonstandard SK picture is only one of a number of ways in which the many states might be seen, but it appears to be the only way in which the essential features (suitably reinterpreted) of Parisi's mean-field analysis would be present. It is an open problem to answer the question: is the nonstandard SK picture valid for any d, μ and β?

Open Problem 8.1 *Determine whether for some d, μ and β, the metastate $\kappa_{\mathcal{J}}$ is (for $\bar{\nu}$-a.e. $\mathcal{J}$) supported on (a) many Γ's, (b) Γ's which decompose into more than two pure states, (c) Γ's which satisfy the other requirements of the nonstandard SK picture.*

What are the prospects for the nonstandard SK picture to be valid? In discussing this question, we will focus on a certain property of the metastate that already appears in [AW90]; namely, covariance with respect to changes of finitely many couplings. Let us assume, for simplicity, that the common distribution μ of the individual couplings is continuous with a strictly positive density on all of $(-\infty, \infty)$, as in the usual Gaussian EA model. Consider changing $\mathcal{J}$ to $\mathcal{J} + \Delta\mathcal{J}$ where $\Delta\mathcal{J}$ is nonzero on some finite collection of edges of $\mathbb{E}^d$ and further consider the corresponding map $\phi_{\Delta\mathcal{J}}$ from infinite volume Gibbs states Γ for $\mathcal{J}$ to those, $\Gamma' = \phi_{\Delta\mathcal{J}}(\Gamma)$, for $\mathcal{J} + \Delta\mathcal{J}$ (all at a given β) defined by

$$\langle s_A \rangle_{\Gamma'} = \langle s_A e^{-\beta \Delta H} \rangle_{\Gamma} / \langle e{-}\beta \Delta H \rangle_{\Gamma}, \tag{8.2}$$

for all finite $A \subset \mathbb{Z}^d$, where ΔH is the (finite) sum

$$\Delta H = - \sum_{\{x,y\} \in \mathbb{E}^d} \Delta J_{\{x,y\}} s_x s_y. \tag{8.3}$$

The covariance property that a metastate $\kappa_{\mathcal{J}}$ satisfies is that under all such finite changes $\Delta\mathcal{J}$, the measure $\kappa_{\mathcal{J}+\Delta\mathcal{J}}$ on states for $\mathcal{J} + \Delta\mathcal{J}$ is the one induced by $\kappa_{\mathcal{J}}$ under the mapping $\Gamma \to \phi_{\Delta\mathcal{J}}(\Gamma)$. That is, changing from $\kappa_{\mathcal{J}}$ to $\kappa_{\mathcal{J}+\Delta\mathcal{J}}$ corresponds to simply a change of variables from Γ to $\Gamma' = \phi_{\Delta\mathcal{J}}(\Gamma)$.

Although this property is not particularly surprising, given the nature of the construction of the metastate, we believe that it is important, both conceptually and for its possible relevance to such issues as whether the nonstandard SK picture is valid. First we discuss a general conceptual point.

Previously, when motivating the construction of metastates by analogy with dynamical systems, we did not give any analogue for the dynamical system probability measure having the property of being invariant under the dynamics. Now, we propose that the Aizenman-Wehr covariance property under coupling changes is the appropriate analogue. Our reasoning is as follows. Suppose we consider free b.c.'s. Changing from a cube of size L to one of size $L+1$ corresponds to taking a certain layer of couplings and changing them from zero to nonzero values. Having already made an analogy between L and the time t for the dynamical system, it seems appropriate to extend it to one between dynamical invariance ($t \to t+1$) and coupling covariance ($\mathcal{J} \to \mathcal{J} + \Delta\mathcal{J}$). The analogy is even clearer if we consider increasing volumes not by a whole layer at a time but by a single site at a time.

To explore the possible implications of this covariance property of the metastate to the nonstandard SK picture, we will focus on the distribution of the overlap distribution P_Γ as Γ varies according to the metastate, when $\mathcal{J} \to \mathcal{J} + \Delta\mathcal{J}$. First, we remark that it is not hard to see that for any pure state $P^\alpha_\mathcal{J}$ for $\mathcal{J}$, $\phi_{\Delta\mathcal{J}}(P^\alpha_\mathcal{J}) \equiv P^\alpha_{\mathcal{J}+\Delta\mathcal{J}}$ is a pure state for $\mathcal{J} + \Delta\mathcal{J}$. Next we note that if Γ is a countable mixture of pure states, then it transforms to Γ' given by

$$\phi_{\Delta\mathcal{J}}(\sum_\alpha W^\alpha_\Gamma P^\alpha_\mathcal{J}) = \sum_\alpha W^\alpha_{\Gamma'} P^\alpha_{\mathcal{J}+\Delta\mathcal{J}}, \tag{8.4}$$

where

$$W^\alpha_{\Gamma'} = W^\alpha_\Gamma \langle e^{-\beta\Delta H}\rangle^\alpha_\mathcal{J} / \sum_\alpha W^\alpha_\Gamma \langle e^{-\beta\Delta H}\rangle^\alpha_\mathcal{J}. \tag{8.5}$$

Furthermore, since $P^\alpha_\mathcal{J}$ and $P^\alpha_{\mathcal{J}+\Delta\mathcal{J}}$ are clearly equivalent measures (i.e., are supported on the same sets of spin configurations) it follows that the $r_{\alpha,\beta}$'s of Γ' are the same as those of Γ.

In possibilities 2 or 5 of Sec. 6, $\langle e^{-\beta\Delta H}\rangle^\alpha_\mathcal{J}$ is the same for each of the two α's and so the two weights for Γ' are both 1/2 as they were for Γ. But in the nonstandard SK picture, there seems every reason to expect nontrivial dependence of $\langle e^{-\beta\Delta H}\rangle^\alpha_\mathcal{J}$ on the many α's appearing for each Γ. Thus, under changes of finitely many couplings, each P_Γ would be changed to a $P_{\Gamma'}$ with the same $r_{\alpha,\beta}$'s but *different* weights. Nevertheless, by the translation invariance/ergodicity argument mentioned earlier in this section, the *distribution* of the P_Γ's (as Γ varies over the metastate) in fact does not depend on $\mathcal{J}$ and hence is unchanged by $\mathcal{J} \to \mathcal{J} + \Delta\mathcal{J}$.

Thus the Aizenman-Wehr covariance property under changes of couplings places a large number of constraints on the distribution of the P_Γ's that can arise in the nonstandard SK picture. We wonder whether all these constraints (which do *not* arise either in possibility 2 or in possibility 5) can actually be satisfied. Clearly, more study of this issue is needed.

We close by briefly discussing zero temperature metastates for the EA model. Let us assume that the common distribution μ of the individual couplings is continuous. As mentioned in the discussion of Sec. 5 on the highly disordered spin glass model, the metastate at zero temperature (constructed say with periodic b.c.'s) must be concentrated on Γ's of the form $\delta_s + \delta_{-s}$, where s is an infinite volume ground state. Thus at zero temperature, the nonstandard SK picture degenerates into the picture of our possibility 5. But this many state picture still differs from the scaling picture since the metastate would be dispersed over many choices of $\pm s$ and not on a single pair of ground states. It would be of some interest to determine which of these pictures occurs. The answer is not known even for $d = 2$.

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A Appendix: Disordered Systems and Metastates

An infinite volume Gibbs state for couplings $J = (J_e : e \in \mathbb{E}^d)$ (and inverse temperature β, which will henceforth be fixed) is a probability measure on $\mathcal{S} = \{-1, +1\}^{\mathbb{Z}^d}$ (with the usual σ-field generated by cylinder sets) that satisfies extra requirements (the DLR equations). Let us denote by $\mathcal{M}(\mathcal{S})$ the set of all probability measures on $\mathcal{S}$. In this appendix, we give some technical lemmas and propositions about $\mathcal{M}(\mathcal{S})$-valued random variables Q and their distributions κ (on $\mathcal{M}(\mathcal{S})$) and, for disordered systems, about the joint distributions $\kappa^\dagger$ of (J, Q). Our primary interest is in $\kappa^\dagger$'s which are supported on configurations $(\mathcal{J}, q)$ such that q is an infinite volume Gibbs state for $\mathcal{J}$. Then the conditional distribution of Q given J will be a metastate $\kappa_{\mathcal{J}}$ – i.e., a probability measure on $\mathcal{M}(\mathcal{S})$ supported on the infinite volume Gibbs states for $\mathcal{J}$. We will begin however with more general Q's, κ's and $\kappa^\dagger$'s. It should be noted that the material presented here (other than Lemma A.5 and Proposition A.6) is basically contained, implicitly or explicitly, in the appendix of [AW90].

It is convenient to regard a probability measure on $\mathcal{S}$ as a point q in the product space $\Omega^1 \equiv \mathbb{R}^{\mathcal{A}}$, where

$$\mathcal{A} = \bigcup_{\Lambda \subset \mathbb{Z}^d \ finite} \left\{(\Lambda, s) : s \in \mathcal{S}_\Lambda = \{-1, +1\}^\Lambda\right\}, \tag{A.1}$$

with the coordinate $q_{(\Lambda,s)}$ denoting the probability of the cylinder set $\{s' \in \mathcal{S} : s'_i = s_i \ \forall i \in \Lambda\}$. It is a standard fact that a point $q \in \Omega^1$ corresponds in this way to a probability measure if and only if it satisfies the three properties of positivity, normalization and consistency; i.e., for all finite subsets Λ, Λ' of $\mathbb{Z}^d$ with $\Lambda \subset \Lambda'$:

$$\forall s \in \mathcal{S}_\Lambda, \quad q_{(\Lambda,s)} \geq 0, \tag{A.2}$$

$$\sum_{s \in \mathcal{S}_\Lambda} q_{(\Lambda,s)} = 1, \tag{A.3}$$

$$\forall s \in \mathcal{S}_\Lambda, \quad q_{(\Lambda,s)} = \sum_{s' \in \mathcal{S}_{\Lambda'}:\ s'=s \text{ on } \Lambda} q_{(\Lambda',s')}. \tag{A.4}$$

We will identify such a q with the (unique) corresponding probability measure on $\mathcal{S}$, and the set of all q's in Ω^1 satisfying (A.2)-(A.4) for all finite $\Lambda' \supset \Lambda$ will be identified with $\mathcal{M}(\mathcal{S})$. Clearly $\mathcal{M}(\mathcal{S})$ (now regarded as a subset of Ω^1) is measurable, i.e., it belongs to $\mathcal{F}^1$, the product σ-field over $\mathcal{A}$ of the Borel σ-field $\mathcal{B}$ on each $\mathbb{R}$.

Our first lemma uses elementary compactness arguments to construct, via an infinite volume limit, a joint distribution $\kappa^\dagger$ (for some random (J, Q)) on the product space

$$(\Omega^\dagger, \mathcal{F}^\dagger) = (\Omega^0, \mathcal{F}^0) \times (\Omega^1, \mathcal{F}^1), \tag{A.5}$$

where $(\Omega^0, \mathcal{F}^0)$ is the product over $\mathbb{E}^d$ of $(\mathbb{R}, \mathcal{B})$, such that $\kappa^\dagger$ is supported on $\{(\mathcal{J}, q) : q \in \mathcal{M}(\mathcal{S})\}$. For a sequence of subsets B_n of $\mathbb{E}^d$ we write $B_n \to \mathbb{E}^d$ if for every finite $B \subset \mathbb{E}^d$, $B_n \supset B$ for all large n; for subsets C_n of $\mathbb{Z}^d$ we similarly define $C_n \to \mathbb{Z}^d$.

Lemma A.1 *Let $B_n \subset \mathbb{E}^d$ and $C_n \subset \mathbb{Z}^d$ be finite and such that $B_n \to \mathbb{E}^d$ and $C_n \to \mathbb{Z}^d$. Suppose that for each n there are families of real valued random variables (on some $(\Omega, \mathcal{F}, \nu)$), $J^n = (J^n_e : e \in B_n)$ and $Q^n = (Q^n_{(\Lambda,s)} : \Lambda \subseteq C_n,\ s \in \mathcal{S}_\Lambda)$ such that*

$$\forall \text{ finite } \Lambda' \supset \Lambda,\ \forall \text{ large } n,\ q = Q^n \text{ satisfies (A.2)-(A.4)} \tag{A.6}$$

and

$$\forall e, \text{ the family of distributions of } J^n_e \text{ (as } n \text{ varies) is tight.} \tag{A.7}$$

Then the finite dimensional distributions of (J^n, Q^n) converge along a subsequence to a probability measure $\kappa^\dagger$ on $(\Omega^\dagger, \mathcal{F}^\dagger)$ with $\kappa^\dagger(\Omega^0 \times \mathcal{M}(\mathcal{S})) = 1$; i.e., there exists a subsequence n_k of the n's such that for any m, $m' \in \mathbb{N}$, any $e_1, \ldots, e_m \in \mathbb{E}^d$, any $(\Lambda^1, s_1), \ldots, (\Lambda^{m'}, s_{m'}) \in \mathcal{A}$ and any bounded continuous real valued function f on $\mathbb{R}^{m+m'}$

$$\begin{aligned} &E\left(f\left(J^{n_k}_{e_1}, \ldots, J^{n_k}_{e_m}, Q^{n_k}_{(\Lambda^1,s_1)}, \ldots, Q^{n_k}_{(\Lambda^{m'},s_{m'})}\right)\right) \\ &\quad \to \int_{\Omega^\dagger} f\left(\mathcal{J}_{e_1}, \ldots, \mathcal{J}_{e_m}, q_{(\Lambda^1,s_1)}, \ldots, q_{(\Lambda^{m'},s_{m'})}\right) d\kappa^\dagger((\mathcal{J}, q)). \end{aligned} \tag{A.8}$$

Here, E denotes expectation with respect to ν.

Proof. For each L, and all sufficiently large n, let $\kappa^\dagger_{L,n}$ denote the joint distribution of the finitely many variables $J^n_{\{x,y\}}$, with x and y in the cube Λ_L, and $Q^n_{(\Lambda,s)}$ with $\Lambda \subseteq \Lambda_L$.

The assumption (A.6) implies that the one-dimensional marginal distribution of each $Q^n_{(\Lambda,s)}$ is supported on $[0,1]$ and so the family of marginal distributions, as n varies for a fixed (Λ, s), is tight; on the other hand, for a fixed e, the tightness of the J^n_e marginal distributions was assumed. Thus as n varies for fixed L, the $\kappa^\dagger_{L,n}$'s are tight. It follows that we may choose a subsequence of the n's for each L (with the one for $L'' > L'$ a sub-subsequence of the one for L') along which $\kappa^\dagger_{L,n}$ converges to some probability measure $\kappa^\dagger_L$. By diagonalization, there is a single subsequence n_k so that for every L, $\kappa^\dagger_{L,n_k} \to \kappa^\dagger_L$ as $k \to \infty$.

Now for $L'' > L'$, the marginal distribution of $\kappa^\dagger_{L'',n_k}$ for the variables coming from $\Lambda_{L'}$ (i.e., $J^{n_k}_{\{x,y\}}$ and $Q^{n_k}_{(\Lambda,s)}$ with $x, y \in \Lambda_{L'}$ and $\Lambda \subseteq \Lambda_{L'}$) is just $\kappa^\dagger_{L',n_k}$ and the same is true for $\kappa^\dagger_{L''}$ and $\kappa^\dagger_{L'}$. Thus the set of $\kappa^\dagger_L$'s is a consistent family of finite dimensional distributions corresponding to some $\kappa^\dagger$ on $(\Omega^\dagger, \mathcal{F}^\dagger)$. This proves (A.8).

It remains only to show that $\kappa^\dagger(\Omega^0 \times \mathcal{M}(\mathcal{S})) = 1$. But for any fixed finite $\Lambda' \supset \Lambda$, the set of q in Ω^1 satisfying (A.2)-(A.4) is a finite cylinder set corresponding to a *closed* subset $F_{\Lambda',\Lambda}$ of the finitely many variables appearing in (A.2)-(A.4). By assumption (A.6), this subset has $\kappa^\dagger_{L,n}$-probability one for all large L and n, and so, by the already proved convergence in distribution, the corresponding cylinder set has $\kappa^\dagger$-probability one. Since this is true for every finite $\Lambda' \supset \Lambda$, it follows that $\Omega^0 \times \mathcal{M}(\mathcal{S})$ has $\kappa^\dagger$-probability one, as desired.

□

In Section 4, we use the following special corollary of Lemma A.1, where $J^n_e = J_e$, with no dependence on n, and where Q^n is a functional of these fixed couplings. There Q^n is given by a Gibbs distribution on C_n for the fixed couplings.

Lemma A.2 *Let B_n and C_n be as in Lemma A.1 and let $(J_e : e \in \mathbb{E}^d)$ be a fixed family of random variables (on some $(\Omega, \mathcal{F}, \nu)$). Suppose for each n and each $\mathcal{J} \in \mathbb{R}^{B_n}$, $P^n_{\mathcal{J}}$ is a probability measure on $\mathcal{S}_{C_n}$ which depends measurably on $\mathcal{J}$, i.e., for each $s' \in \mathcal{S}_{C_n}$, $P^n_{\mathcal{J}}(\{s'\})$, as a function of $\mathcal{J} \in \mathbb{R}^{B_n}$, is Borel measurable. For $\Lambda \subseteq C_n$ and $s \in \mathcal{S}_\Lambda$, define random variables,*

$$Q^n_{(\Lambda,s)} = P^n_{(J_e:e\in B_n)}(\{t \in \mathcal{S}_{C_n} : t = s \text{ on } \Lambda\}); \tag{A.9}$$

then along some subsequence n_k of the n's,

$$(J, Q^{n_k}) = (J_e, Q^{n_k}_{(\Lambda,s)} : e \in \mathbb{E}^d,\ \Lambda \subseteq C_{n_k}\ , s \in \mathcal{S}_\Lambda) \to \kappa^\dagger \tag{A.10}$$

in the sense of (A.8), where $\kappa^\dagger$ is some probability measure on $\Omega^0 \times \mathcal{M}(\mathcal{S})$ (i.e., $\kappa^\dagger$ is a probability measure on $(\Omega^\dagger, \mathcal{F}^\dagger)$ with $\kappa^\dagger(\Omega^0 \times \mathcal{P}(\mathcal{S})) = 1$).

Proof. This is an immediate corollary of Lemma A.1 by defining, for each n and each $e \in B_n$, $J^n_e = J_e$.

□

When B_n contains all $e = \{x, y\} \in \mathbb{E}^d$ with $x, y \in \Lambda_n$ and $P^n_{\mathcal{J}}$ is some Gibbs distribution on $\mathcal{S}_{C_n}$ (for the couplings $\mathcal{J}_e$ and the given β), then the probabilities $q^n_{(\Lambda,s)} \equiv P^n_{\mathcal{J}}(t = s \text{ on } \Lambda)$ satisfy the following finite volume version of the DLR equations: For $\Lambda, \Lambda' \subseteq C_n$ with $(\Lambda \cup \partial\Lambda) \subseteq \Lambda'$,

$$\forall s' \in \mathcal{S}_{\Lambda'}, \quad q^n_{(\Lambda',s')} = q^n_{(\Lambda'\setminus\Lambda, s'|_{\Lambda'\setminus\Lambda})}\ P^{s'|_{\partial\Lambda}}_{\Lambda,\beta}(\{s'|_\Lambda\}), \tag{A.11}$$

where $s'|_A$ denotes the restriction of s' to $A \subseteq \Lambda'$ and $P^{\bar{s}}_{\Lambda,\beta}$ denotes the usual finite volume Gibbs distribution on Λ for the couplings $\mathcal{J}_e$, with b.c. $\bar{s}$ on $\partial\Lambda$.

Lemma A.3 *Suppose that the hypotheses of Lemma A.2 are valid and, in addition, the probabilities $q^n_{(\Lambda,s)} \equiv P^n_{\mathcal{J}}(t = s$ on $\Lambda)$, satisfy (A.11) for all finite $\Lambda' \supseteq \Lambda \cup \partial\Lambda$ and all large n. Then the $\kappa^\dagger$ of Lemma A.2 has*

$$\kappa^\dagger\left(\{(\mathcal{J}, q) : q \text{ is an infinite volume Gibbs distribution for } \mathcal{J}\}\right) = 1. \tag{A.12}$$

Proof. For a fixed Λ', Λ and s', the condition (A.11) is of the form $h = 0$ where h is a continuous function of $q^n_{(\Lambda',s')}$, $q^n_{(\Lambda'',s'')}$ (with $\Lambda'' = \Lambda' \setminus \Lambda$ and $s'' = s'|_{\Lambda'\setminus\Lambda}$) and $J_{e_1}, \ldots, J_{e_m}$ for certain fixed edges $e_1, \ldots, e_m$. Thus the set of these finitely many variables satisfying (A.11) is a closed subset $F_{\Lambda',\Lambda,s'}$. It follows from convergence in distribution (as in the proof of Lemma A.1) that the corresponding cylinder set has $\kappa^\dagger$-probability one. Thus the set H of $(\mathcal{J}, q)$ such that $q \in \mathcal{M}(\mathcal{S})$ and (A.11) (with q^n replaced by q) is valid for all finite $\Lambda' \supseteq \Lambda \cup \partial\Lambda$ has $\kappa^\dagger(H) = 1$.

We consider a pair $(\mathcal{J}, P) \in H$. For a finite Λ and for L large enough so that the cube Λ_L contains $\Lambda \cup \partial\Lambda$, we denote by $\mathcal{F}_{\Lambda_L\setminus\Lambda}$ the σ-field generated by $\{s_x : x \in \Lambda_L \setminus \Lambda\}$. The conditions (A.11) imply that the conditional probability of P with respect to $\mathcal{F}_{\Lambda_L\setminus\Lambda}$ satisfies, for any $t \in \mathcal{S}_\Lambda$,

$$P\left(\{s \in \mathcal{S} : s = t \text{ on } \Lambda\} | \mathcal{F}_{\Lambda_L\setminus\Lambda}\right) = P^{\bar{s}}_{\Lambda,\beta}(t), \tag{A.13}$$

where $\bar{s} \in \mathcal{S}_{\partial\Lambda}$ is given by the (conditioned) values of s_x for $x \in \partial\Lambda$. By the martingale convergence theorem, we may let $L \to \infty$ to get

$$P\left(\{s \in \mathcal{S} : s = t \text{ on } \Lambda\} | \mathcal{F}_{\mathbb{Z}^d\setminus\Lambda}\right) = P^{\bar{s}}_{\Lambda,\beta}(t). \tag{A.14}$$

But the set of these equations for arbitrary finite Λ and $t \in \mathcal{S}_\Lambda$ is just the set of DLR equations which characterize P as an infinite volume Gibbs distribution for the couplings $(\mathcal{J}_e : e \in \mathbb{E}^d)$. Since the DLR equations also imply the validity of all the conditions (A.11), we see that the set of $(\mathcal{J}, q)$ appearing in (A.12) is identical to H. Since $H \in \mathcal{F}^\dagger$ with $\kappa^\dagger(H) = 1$, the proof is complete.

□

For any probability measure $\kappa^\dagger$ on $(\Omega^\dagger, \mathcal{F}^\dagger) = (\Omega^0, \mathcal{F}^0) \times (\Omega^1, \mathcal{F}^1)$, one can define the marginal distribution (of $\mathcal{J}$), $\bar{\nu}$, on $(\Omega^0, \mathcal{F}^0)$ and the conditional distribution (of q, given $\mathcal{J}$), $\kappa_{\mathcal{J}}$, which for $\bar{\nu}$-a.e. $\mathcal{J}$ is a probability measure on $(\Omega^1, \mathcal{F}^1)$. For any $F \in \mathcal{F}^\dagger$,

$$\kappa^\dagger(F) = \bar{E}[\kappa_{\mathcal{J}}(F_{\mathcal{J}})], \tag{A.15}$$

where $\bar{E}$ denotes expectation with respect to $\bar{\nu}$ and

$$F_{\mathcal{J}} = \{q \in \Omega^1 : (\mathcal{J}, q) \in F\}. \tag{A.16}$$

As a corollary of Lemma A.3, we have the following.

Lemma A.4 *Let $\kappa_{\mathcal{J}}$ be the conditional distribution of q, given $\mathcal{J}$, for the $\kappa^\dagger$ of Lemma A.2. Under the hypotheses of Lemma A.3, for $\bar{\nu}$-a.e. $\mathcal{J}$,*
$\kappa_{\mathcal{J}}\left(\{q : q \text{ is an infinite volume Gibbs distribution for } \mathcal{J}\}\right) = 1$.

Proof. This is an immediate consequence of (A.15) by taking F to be the event appearing in (A.12).

□

The rest of the appendix is concerned, in the context of Lemma A.2, with the convergence of empirical distributions to $\kappa_{\mathcal{J}}$. For each $(\Lambda, s) \in \mathcal{A}$, $Q^n_{(\Lambda,s)}$ is a random variable, defined for $\{n : \Lambda \subseteq C_n\}$ and thus for all large n; we make the convention that $Q^n_{(\Lambda,s)} = 0$ for those finitely many n's for which $\Lambda \not\subseteq C_n$. We consider, for each L, the finite dimensional random vector

$$\vec{Q}^n_L = \left(Q^n_{(\Lambda,s)} : \Lambda \subseteq \Lambda_L,\ s \in \mathcal{S}_\Lambda\right), \tag{A.17}$$

and the empirical distributions, along some subsequence n'_k,

$$\kappa^{L,K} = \frac{1}{K} \sum_{k=1}^{K} \delta_{\vec{Q}^{n'_k}_L}. \tag{A.18}$$

These are (random) probability measures on $\mathbb{R}^{m_L}$, for some finite m_L.

In the context of Lemma A.2, $\kappa^{L,K}$ is random because it depends (measurably) on J. To denote this dependence, we will sometimes write $\kappa^{L,K}_J$. We are interested in whether and in what sense $\kappa^{L,K}$ converges as $K \to \infty$ to some limit (random) probability measure on $\mathbb{R}^{m_L}$. We note that there is a natural candidate for the limit measure $-$ namely, take the conditional distribution $\kappa_{\mathcal{J}}$, of q, given $\mathcal{J}$, for the joint distribution $\kappa^\dagger$ and consider its marginal distribution $\kappa^L_{\mathcal{J}}$ for $\left(q_{(\Lambda,s)} : \Lambda \subseteq \Lambda_L,\ s \in \mathcal{S}_\Lambda\right)$. The next lemma is a technical result which leads to the subsequent proposition giving a.s. convergence to this limit along a subsequence.

Lemma A.5 *Assume the hypotheses of Lemma A.2. For any fixed L and fixed $\vec{r} \in \mathbb{R}^{m_L}$, there exists a sub-subsequence n'_k of the subsequence n_k of Lemma A.2, such that (with $\kappa^{L,K}$ given by (A.17)-(A.18)), the sequence of random variables,*

$$\Phi^K_{L,\vec{r}} = \int\limits_{\mathbb{R}^{m_L}} e^{i\ (\vec{r},\vec{q}_L)} d\kappa^{L,K}(\vec{q}_L) = \frac{1}{K}\ \sum_{k=1}^{K} e^{i\ (\vec{r},\vec{Q}_L^{n'_k})}, \tag{A.19}$$

(where $(\vec{r},\vec{q}_L)$ denotes the standard inner product on $\mathbb{R}^{m_L}$) converges in $L^2(\Omega,\mathcal{F},\nu)$, as $K\to\infty$, to some random variable $\Phi_{L,\vec{r}}$.

Proof. Let $Y_n = e^{i\ (\vec{r},\vec{Q}_L^n)}$. We must show that n'_k can be chosen so that

$$\begin{aligned} E\left(\Phi^{K'}_{L,\vec{r}} - \Phi^{K}_{L,\vec{r}}\right)^2 \ &= \ \frac{1}{K'^2}\sum_{k,j=1}^{K'} E\left(Y_{n'_k}Y_{n'_j}\right) + \frac{1}{K^2}\sum_{k,j=1}^{K} E\left(Y_{n'_k}Y_{n'_j}\right) \\ &\quad -\frac{2}{K'K}\sum_{k=1}^{K'}\sum_{j=1}^{K} E\left(Y_{n'_k}Y_{n'_j}\right) \to 0 \end{aligned} \tag{A.20}$$

as $K,K'\to\infty$. Note that the total contribution of the diagonal terms (where $k=j$) in the last expression is negligible as $K,K'\to\infty$, as is the contribution from terms with k (or j) fixed. It follows that it suffices to choose n'_k so that

$$\lim_{\substack{k,j\to\infty \\ k\neq j}} E\left(Y_{n'_k}Y_{n'_j}\right) \text{ exists.} \tag{A.21}$$

We will do this by a compactness argument.

We will use the facts that $C_{mn} \equiv E(Y_mY_n)$ is bounded in modulus by 1 and that $C_{mn} = C_{nm}$. We begin by choosing, for each l, a subsequence of the n's (with the one for $l'' > l'$ a subsequence of the one for l') along which C_{ln} converges to some α_l. By diagonalization, there is a fixed subsequence $\tilde{n}_k$ so that, for every l, $C_{l\tilde{n}_k} \to \alpha_l$ as $k\to\infty$. Since the α_l's are also bounded in modulus by one, we may next pick a sub-subsequence n^*_k of $\tilde{n}_k$ so that $\alpha_{n^*_k}\to\alpha$. We now have:

$$\forall k,\quad C_{n^*_k,n^*_{k'}} \to \alpha_{n^*_k} \quad \text{as } k'\to\infty, \quad \text{and} \quad \alpha_{n^*_k}\to\alpha \quad \text{as } k\to\infty. \tag{A.22}$$

Finally, we define n'_k inductively by taking $n'_1 = n^*_1$ ($= n^*_{k_1}$ with $k_1 = 1$) and, for $j\geq 1$, taking $n'_{j+1} = n^*_{k_{j+1}}$ with k_{j+1} the smallest $k' > k_j$ such that

$$\left|C_{n'_j n^*_k} - \alpha_{n'_j}\right| < \frac{1}{2^j} \quad \text{ for } k\geq k'. \tag{A.23}$$

It follows that

$$\lim_{j\to\infty}\sup_{k>j}\left|C_{n'_jn'_k} - \alpha\right| \leq \lim_{j\to\infty}\left(\frac{1}{2^j} + \left|\alpha_{n'_j} - \alpha\right|\right) = 0. \tag{A.24}$$

But this, together with $C_{mn} = C_{nm}$, implies (A.21).

□

Proposition A.6 *Assume the hypotheses of Lemma A.2. Then there exists a sub-subsequence n'_k of the subsequence n_k of Lemma A.2 and some subsequence K_m of the K's, such that, almost surely, the empirical distributions κ^{L,K_m} (given by (A.17)-(A.18)) converge (weakly) as $m \to \infty$ to the finite dimensional marginal distributions κ^L_J (of the conditional distribution $\kappa_{\mathcal{J}}$ of the $\kappa^\dagger$ of Lemma A.2, with $\mathcal{J} = J$). I.e., for ν-a.e. ω, $\forall L$, $\forall$ bounded continuous functions f on $\mathbb{R}^{m_L}$,*

$$\lim_{m\to\infty} \int\limits_{\mathbb{R}^{m_L}} f(\vec{q}_L) d\kappa^{L,K_m}_{J(\omega)}(\vec{q}_L) = \int_{\Omega^1} f\left((q_{(\Lambda,s)} : \Lambda \subseteq \Lambda_L,\ s \in \mathcal{S}_\Lambda)\right) d\kappa_{J(\omega)}(q). \tag{A.25}$$

Proof. By Lemma A.5 and a diagonalization argument, we can replace the n'_k of Lemma A.5 by a single further sub-subsequence (which we will still call n'_k) so that $\Phi^K_{L,\vec{r}}$ converges in L^2 to some $\Phi_{L,\vec{r}}$, as $K \to \infty$, for every L and every $\vec{r} \in \mathbb{R}^{m_L}$ with rational coordinates. But L^2 convergence implies convergence in probability, which implies a.s. convergence along a subsequence of K's. Thus, by a further diagonalization, we have for a single subsequence K_m of the K's that $\Phi^{K_m}_{L,\vec{r}}$ converges a.s. to $\Phi_{L,\vec{r}}$, as $m \to \infty$, for all L and rational $\vec{r}$.

Now for each ω, $\Phi^K_{L,\vec{r}}$, as a function of $\vec{r}$, is the characteristic function of a probability measure $\kappa^{L,K}_{J(\omega)}$ supported on the compact set $[0,1]^{m_L}$ (since each $Q^n_{(\Lambda,s)}$ takes values in $[0,1]$). The tightness of this family of measures (as K varies) along with convergence (for $K = K_m$) of the characteristic functions for a dense set of $\vec{r}$'s implies that for ν-a.e. ω,

$$\forall L,\ \forall \vec{r} \in \mathbb{R}^{m_L}, \quad \Phi^{K_m}_{L,\vec{r}} \to \Phi_{L,\vec{r}} = \int\limits_{\mathbb{R}^{m_L}} e^{i\ (\vec{r},\vec{q}_L)} d\kappa^L(\vec{q}_L) \quad \text{as } m \to \infty, \tag{A.26}$$

where κ^L is some probability measure on $\mathbb{R}^{m_L}$, depending on ω.

To complete the proof of the proposition, it only remains to identify κ^L with the marginal distribution of $\kappa_{J(\omega)}$. Because of the relation between $\kappa_{\mathcal{J}}$ and $\kappa^\dagger$, to do this, it suffices to show that for every bounded continuous function $g(\mathcal{J})$, depending on only finitely many $\mathcal{J}_e$'s,

$$E\left(g(J)\Phi_{L,\vec{r}}\right) = \int g(\mathcal{J}) e^{i\ (\vec{r},\vec{q}_L)} d\kappa^\dagger((\mathcal{J},q)). \tag{A.27}$$

But by (A.19) and Lemma A.2

$$\begin{aligned} E\left(g(J)\Phi_{L,\vec{r}}\right) = \lim_{m\to\infty} E\left(g(J)\Phi^{K_m}_{L,\vec{r}}\right) &= \lim_{m\to\infty} \frac{1}{K_m} \sum_{k=1}^{K_m} E\left(g(J) e^{i\ (\vec{r},\vec{Q}^{n'_k}_L)}\right) \\ &= \int_{\Omega^\dagger} g(\mathcal{J}) e^{i\ (\vec{r},\vec{q}_L)} d\kappa^\dagger((\mathcal{J},q)), \end{aligned} \tag{A.28}$$

as desired.

□

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