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# Determinant representation for dynamical correlation functions of the Quantum nonlinear Schrödinger equation.

Short title : Correlation functions of the Quantum nonlinear Schrödinger equation.

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Abstract

Painlevé analysis of correlation functions of the impenetrable Bose gas by M. Jimbo, T. Miwa, Y. Môri and M. Sato [1] was based on the determinant representation of these correlation functions obtained by A. Lenard [2]. The impenetrable Bose gas is the free fermionic case of the quantum nonlinear Schrödinger equation. In this paper we generalize the Lenard determinant representation for $\langle\psi(0,0)\psi^{\dagger}(x,t)\rangle$ to the non-free fermionic case. We also include time and temperature dependence. In forthcoming publications we shall perform the JMMS analysis of this correlation function. This will give us a completely integrable equation and asymptotic for the quantum correlation function of interacting fermions.

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## 1 Introduction

We consider exactly solvable models of statistical mechanics in one space and one time dimension. The Quantum Inverse Scattering Method and Algebraic Bethe Ansatz are effective methods for a description of the spectrum of these models. Our aim is the evaluation of correlation functions of exactly solvable models. Our approach is based on the determinant representation for correlation functions. It consists of a few steps: first the correlation function is represented as a determinant of a Fredholm integral operator, second — the Fredholm integral operator is described by a classical completely integrable equation, third — the classical completely integrable equation is solved by means of the Riemann-Hilbert problem. This permits us to evaluate the long distance and large time asymptotics of the correlation function. The method is described in [3].

The most interesting correlation functions are time dependent correlation functions. The determinant representation for time dependent correlation functions was known only for the impenetrable Bose gas (the spectrum of the Hamiltonian of this model is equivalent to free fermions). In this paper we have found the determinant representation for the time dependent correlation function of local fields of the penetrable Bose gas. The main idea for the construction of the determinant representation is the following. We introduce auxiliary Bose fields (acting in the canonical Fock space) in order to remove the two body scattering matrix and to reduce the model to the free fermionic case. We want to emphasize that all dual fields, which we introduce commute (belong to the same Abelian sub-algebra). Therefore we do not have any ordering problem. This will also permit us to perform nonperturbative calculations, which are necessary for the derivation of the integrable equation for the correlation function.

First we shall discuss our model.

Quantum nonlinear Schrödinger equation (equivalent to Bose gas with delta-function interaction) can be described by the canonical Bose fields $\psi(x)$ and $\psi^\dagger(x)$ with the commutation relations:

$$[\psi(x), \psi^\dagger(y)] = \delta(x-y), \quad [\psi(x), \psi(y)] = [\psi^\dagger(x), \psi^\dagger(y)] = 0, \tag{1.1}$$

acting in the Fock space. Fock vacuum $|0\rangle$ and dual vector $\langle 0|$ are important. They are defined by the relations

$$\psi(x)|0\rangle = 0, \qquad \langle 0|\psi^\dagger(x) = 0, \qquad \langle 0 \mid 0\rangle = 1. \tag{1.2}$$

The Hamiltonian of the model is

$$H = \int dx \left( \partial_x \psi^\dagger(x) \partial_x \psi(x) + c\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x) - h\psi^\dagger(x)\psi(x) \right), \tag{1.3}$$

Here $c$ is the coupling constant and $h > 0$ is the chemical potential. We shall consider the repulsive case $0 < c \leq \infty$.

The spectrum of the model was first described by E. H. Lieb and W. Liniger [4], [5]. The Lax representation for the corresponding classical equation of motion

$$i\frac{\partial}{\partial t}\psi = [\psi, H] = -\frac{\partial^2}{\partial x^2}\psi + 2c\psi^\dagger\psi\psi - h\psi, \tag{1.4}$$

was found by V. E. Zakharov and A. B. Shabat [6]. The Quantum Inverse Scattering Method for the model was formulated by L. D. Faddeev and E. K. Sklyanin [7].

In this paper we shall follow the notations of [3]. First the model is considered in a finite periodic box of length $L$. Later the thermodynamic limit is considered when the length of the box $L$ and the number of particles in the ground state go to infinity, with the ratio $N/L$ held fixed.

The Quantum nonlinear Schrödinger equation is equivalent to the Bose gas with delta-function interaction. In the sector with $N$ particles the Hamiltonian of Bose gas is given by

$$\mathcal{H}_N = -\sum_{j=1}^{N}\frac{\partial^2}{\partial z_j^2} + 2c\sum_{1\leq j<k\leq N}\delta(z_k - z_j) - Nh. \tag{1.5}$$

Now a few words about the organization of the paper.

In Section 2 we shall review the Algebraic Bethe Ansatz and collect all the known facts necessary for further calculations. In Section 3 we shall calculate the form factor of the local field in finite volume. In Section 4 we shall present the idea of summation with respect to all intermediate states. In Section 5 we introduce an auxiliary Bosonic Fock space and auxiliary Bose fields. This helps us to represent the correlation function as a determinant in the finite volume. In Section 6 we consider the thermodynamic limit of the determinant representation for correlation function. Length of the periodic box $L$ and number of particles in the ground state go to infinity but their ratio remains fixed. This leads us to the main result of the paper (see formulæ (6.24)–(6.27)). The correlation function of local fields in the infinite volume is represented as a determinant of a Fredholm integral operator. For evaluation of the thermodynamic limit it is necessary to sum up singular expressions. Appendix A is devoted to these summations. In Appendix B we present realization of quantum dual fields as linear combinations of the canonical Bose fields. Appendix C shows how to reduce the number of dual fields. Appendix D contains determinant representation for temperature correlation function.

In forthcoming publications we shall use the determinant representation for the derivation of completely integrable equation for correlation functions. Later we shall solve this equation by means of the Riemann-Hilbert problem and evaluate the long-distance asymptotic.

## 2 Algebraic Bethe Ansatz

Let us review some main features of the Algebraic Bethe Ansatz, which we shall use later. We consider the quantum nonlinear Schrödinger model. The starting point and central object of the Quantum Inverse Scattering Method is the R-matrix, which is a solution of the Yang-Baxter equation. For the case of the quantum nonlinear Schrödinger equation, it is of the form :

$$R(\lambda,\mu)=\begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix}, \tag{2.1}$$

where

$$g(\lambda,\mu)=\frac{ic}{\lambda-\mu}, \quad f(\lambda,\mu)=\frac{\lambda-\mu+ic}{\lambda-\mu}. \tag{2.2}$$

Later we shall also use functions

$$h(\lambda,\mu)=\frac{\lambda-\mu+ic}{ic}, \quad t(\lambda,\mu)=\frac{(ic)^2}{(\lambda-\mu)(\lambda-\mu+ic)}=\frac{g(\lambda,\mu)}{h(\lambda,\mu)}. \tag{2.3}$$

Another important object is the monodromy matrix

$$T(\lambda)=\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{2.4}$$

The operators $A,\ B,\ C,\ D$ are acting in the Fock space where the operator $\psi(x)$ was defined. Their commutation relations are given by

$$R(\lambda,\mu)\left(T(\lambda)\otimes T(\mu)\right)=\left(T(\mu)\otimes T(\lambda)\right)R(\lambda,\mu). \tag{2.5}$$

These relations are written out explicitly in Section VII.1 of [3].

The hermiticity properties of $T(\lambda)$ are

$$\sigma_x T^*(\bar\lambda)\sigma_x = T(\lambda), \tag{2.6}$$

so that $B^\dagger(\lambda)=C(\bar\lambda)$.

The Hamiltonian of the model can be expressed in terms of $A(\lambda)+D(\lambda)$ by means of trace identities (Section VI.3 of [3]). The vacuum is eigenvector of the diagonal elements of $T(\lambda)$

$$A(\lambda)|0\rangle = a(\lambda)|0\rangle; \qquad D(\lambda)|0\rangle = d(\lambda)|0\rangle \tag{2.7}$$

$$\langle 0|A(\lambda) = a(\lambda)\langle 0|; \qquad \langle 0|D(\lambda) = d(\lambda)\langle 0|; \tag{2.8}$$

$$a(\lambda)=\exp\left\{-\frac{iL\lambda}{2}\right\}; \qquad d(\lambda)=\exp\left\{\frac{iL\lambda}{2}\right\} \tag{2.9}$$

Later we shall also use the function

$$r(\lambda) = \frac{a(\lambda)}{d(\lambda)} = e^{-i\lambda L} \tag{2.10}$$

The operator $C(\lambda)$ annihilates the vacuum vector and the operator $B(\lambda)$ annihilates the dual vacuum:

$$C(\lambda)|0\rangle = 0, \qquad \langle 0|B(\lambda) = 0. \tag{2.11}$$

The Hamiltonian of the model commutes with $A(\lambda) + D(\lambda)$ and they can be diagonalized simultaneously. The eigenvectors of the Hamiltonian are

$$\prod_{j=1}^{N} B(\mu_j)|0\rangle, \qquad \text{and} \qquad \langle 0|\prod_{j=1}^{N} C(\mu_j), \tag{2.12}$$

if $\mu_j$ satisfy Bethe Equations

$$\frac{a(\mu_j)}{d(\mu_j)} \prod_{\substack{k=1\\k\neq j}}^{N} \frac{f(\mu_j,\mu_k)}{f(\mu_k,\mu_j)} = 1, \qquad \text{or} \qquad \frac{a(\mu_j)}{d(\mu_j)} \prod_{k=1}^{N} \frac{h(\mu_j,\mu_k)}{h(\mu_k,\mu_j)} = (-1)^{N-1} \tag{2.13}$$

It is convenient to rewrite (2.13) in logarithmic form. For the ground state

$$\varphi_j + \pi \equiv L\mu_j + \sum_{k=1}^{N} i \ln\left(\frac{ic + \mu_j - \mu_k}{ic - \mu_j + \mu_k}\right) = 2\pi\left(j - \frac{N+1}{2}\right) \tag{2.14}$$

It is proven in Section I.2 of [3] that solutions $\mu_j$ of equation (2.14) are real.

The distribution of $\mu_j$ in the ground state in thermodynamic limit can be described by linear integral equation. The thermodynamic limit is defined in the following way: $N \to \infty, \quad L \to \infty$ and $N/L = D$ is fixed. In this limit $\mu_j$ condense ($\mu_{j+1} - \mu_j = \mathcal{O}(1/L)$) and fill the symmetric interval $[-q, q]$, where $q$ is the value of spectral parameter on the Fermi surface.

In the thermodynamic limit the function of local density $\rho(\mu)$ can be defined in the following way

$$\rho(\mu_j) = \lim \frac{1}{L(\mu_{j+1} - \mu_j)}. \tag{2.15}$$

The lim in the r.h.s. denotes the thermodynamic limit. This function satisfies the Lieb-Liniger integral equation

$$\rho(\mu) - \frac{1}{2\pi}\int_{-q}^{q} K(\nu,\mu)\rho(\nu)d\nu = \frac{1}{2\pi}. \tag{2.16}$$

Here

$$K(\nu,\mu) = \frac{2c}{c^2 + (\mu-\nu)^2}, \tag{2.17}$$

and

$$D = \frac{N}{L} = \int\limits_{-q}^{q} d\mu \rho(\mu). \tag{2.18}$$

In such a way we have described the ground state.

Now we can define the correlation function of the local fields

$$\langle \psi(0,0)\psi^\dagger(x,t)\rangle = \lim \frac{\langle 0| \prod\limits_{j=1}^{N} C(\mu_j)\psi(0,0)\psi^\dagger(x,t) \prod\limits_{j=1}^{N} B(\mu_j)|0\rangle}{\langle 0| \prod\limits_{j=1}^{N} C(\mu_j) \prod\limits_{j=1}^{N} B(\mu_j)|0\rangle}. \tag{2.19}$$

Here

$$\psi^\dagger(x,t) = e^{iHt}\psi^\dagger(x,0)e^{-iHt}. \tag{2.20}$$

We shall use the notation $\mu_j$ for the ground state only. The square of the norm of the ground state wave function (denominator of the correlation function) was found in [8],

$$\begin{aligned} \langle 0| \prod_{j=1}^{N} C(\mu_j) \prod_{j=1}^{N} B(\mu_j)|0\rangle \;\; &= \;\; c^N \left( \prod_{N\geq j>k\geq 1} g(\mu_j,\mu_k)g(\mu_k,\mu_j) \right) \\ &\times \left( \prod_{j=1}^{N}\prod_{k=1}^{N} h(\mu_j,\mu_k) \right) \det{}_N \frac{\partial \varphi_j}{\partial \mu_k}. \end{aligned} \tag{2.21}$$

Here $\partial\varphi_j/\partial\mu_k$ is $N\times N$ matrix

$$\frac{\partial \varphi_j}{\partial \mu_k} = \delta_{jk}\left[ L + \sum_{l=1}^{N} K(\mu_j,\mu_l)\right] - K(\mu_j,\mu_k). \tag{2.22}$$

Let us emphasize that $\det(\partial\varphi_j/\partial\mu_k) > 0$ (see Section I.2 of [3]). The thermodynamic limit of the square of the norm can be described by the following formula:

$$\lim \left( \frac{\det_N \frac{\partial \varphi_j}{\partial \mu_k}}{\prod_{j=1}^{N} 2\pi L \rho(\mu_j)} \right) = \det\left( \hat{I} - \frac{1}{2\pi}\hat{K} \right), \tag{2.23}$$

where $\hat{K}$ is an integral operator acting on some trial function $f(\lambda)$ as

$$(\hat{K}f)(\lambda) = \int\limits_{-q}^{q} K(\lambda,\mu) f(\mu)\, d\mu. \tag{2.24}$$

The proof can be found in [8] (see also Section X.4 of [3]).

In order to calculate the correlation function we shall also need a description of excited states. We need to consider excited states which have one more particle than in the ground state

$$\prod_{j=1}^{N+1} B(\lambda_j)|0\rangle, \qquad \text{and} \qquad \langle 0| \prod_{j=1}^{N+1} C(\lambda_j), \tag{2.25}$$

where $\lambda_j$ have to satisfy Bethe Equations

$$\frac{a(\lambda_j)}{d(\lambda_j)} \prod_{\substack{k=1\\k\neq j}}^{N+1} \frac{f(\lambda_j,\lambda_k)}{f(\lambda_k,\lambda_j)} = 1, \qquad \text{or} \qquad \frac{a(\lambda_j)}{d(\lambda_j)} \prod_{k=1}^{N+1} \frac{h(\lambda_j,\lambda_k)}{h(\lambda_k,\lambda_j)} = (-1)^N \tag{2.26}$$

We shall further assume that the number of particles in the ground state $N$ is even. In order to write the logarithmic form of the Bethe Equations it is convenient to introduce

$$\tilde{\varphi}_j \equiv L\lambda_j + \sum_{\substack{k=1\\k\neq j}}^{N+1} i \ln \left( \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic} \right). \tag{2.27}$$

The Bethe equations can now be written as

$$\tilde{\varphi}_j = 2\pi n_j, \tag{2.28}$$

where $n_j$ is an ordered set of different integer numbers $n_{j+1} > n_j$. One can prove that all $\lambda_j$ are real. In order to enumerate all the eigenstates in the sector with $N+1$ particles we have to consider all sets of ordered integers $n_j$. The square of the norm of the excited state is

$$\begin{aligned}\langle 0| \prod_{j=1}^{N+1} C(\lambda_j) \prod_{j=1}^{N+1} B(\lambda_j)|0\rangle &= c^{N+1} \left( \prod_{N+1\geq j>k\geq 1} g(\lambda_j,\lambda_k) g(\lambda_k,\lambda_j) \right) \\ &\times \left( \prod_{j=1}^{N+1} \prod_{k=1}^{N+1} h(\lambda_j,\lambda_k) \right) \det{}_{N+1} \frac{\partial \tilde{\varphi}_j}{\partial \lambda_k}.\end{aligned} \tag{2.29}$$

For the excited state $\det(\partial\tilde{\varphi}_j/\partial\lambda_k)$ is also positive. We shall also mention that the scattering matrix of elementary excitations can be found in Section I.4 of [3]. It depends strongly on momenta, this shows that the model is not free fermionic.

Now we can define the form factor in the finite volume

$$F_N = \langle 0| \prod_{j=1}^{N} C(\mu_j) \psi(0,0) \prod_{j=1}^{N+1} B(\lambda_j)|0\rangle. \tag{2.30}$$

We shall calculate it in the next section. We shall also need the conjugated form factor

$$\begin{aligned}&\langle 0| \prod_{j=1}^{N+1} C(\lambda_j) \psi^\dagger(x,t) \prod_{j=1}^{N} B(\mu_j)|0\rangle \\ &= e^{-iht} \cdot \exp\left[ it \left( \sum_{j=1}^{N+1} \lambda_j^2 - \sum_{k=1}^{N} \mu_k^2 \right) - ix \left( \sum_{j=1}^{N+1} \lambda_j - \sum_{k=1}^{N} \mu_k \right) \right] \cdot \overline{F}_N.\end{aligned} \tag{2.31}$$

Here we used the fact that the energy and momentum of the eigenstate are given by the expressions

$$E_{N+1} = \sum_{j=1}^{N+1} (\lambda_j^2 - h), \tag{2.32}$$

$$P_{N+1} = \sum_{j=1}^{N+1} \lambda_j. \tag{2.33}$$

# 3 Form Factor

The main purpose of the paper is to evaluate the correlation function. In the finite volume we shall use the notation

$$\langle \psi(0,0)\psi^\dagger(x,t)\rangle_N = \frac{\langle 0| \prod_{j=1}^N C(\mu_j)\psi(0,0)\psi^\dagger(x,t)\prod_{j=1}^N B(\mu_j)|0\rangle}{\langle 0|\prod_{j=1}^N C(\mu_j)\prod_{j=1}^N B(\mu_j)|0\rangle}. \tag{3.1}$$

We shall use the standard representation of correlation function in terms of the form factors

$$\begin{aligned}&\langle \psi(0,0)\psi^\dagger(x,t)\rangle_N \\ &= \sum_{\text{all } \{\lambda\}_{N+1}} \frac{\langle 0| \prod\limits_{j=1}^{N} C(\mu_j)\psi(0,0)\prod\limits_{j=1}^{N+1} B(\lambda_j)|0\rangle\langle 0|\prod\limits_{j=1}^{N+1} C(\lambda_j)\psi^\dagger(x,t)\prod\limits_{j=1}^{N} B(\mu_j)|0\rangle}{\langle 0|\prod\limits_{j=1}^{N+1} C(\lambda_j)\prod\limits_{j=1}^{N+1} B(\lambda_j)|0\rangle\langle 0|\prod\limits_{j=1}^{N} C(\mu_j)\prod\limits_{j=1}^{N} B(\mu_j)|0\rangle}.\end{aligned} \tag{3.2}$$

In order to calculate the form factor we need to know the action of the local field on the eigenvector. This can be found in [9] (see also Section XII.2 of [3]).

$$\psi(0,0)\prod_{j=1}^{N+1} B(\lambda_j)|0\rangle = -i\sqrt{c}\sum_{\ell=1}^{N+1} a(\lambda_\ell)\left(\prod_{\substack{m=1\\ m\neq\ell}}^{N+1} f(\lambda_\ell,\lambda_m)\right)\prod_{\substack{m=1\\ m\neq\ell}}^{N+1} B(\lambda_m)|0\rangle, \tag{3.3}$$

This permits us to represent form factor as follows

$$\begin{aligned} F_N &= -i\sqrt{c}\sum_{\ell=1}^{N+1} a(\lambda_\ell)\left(\prod_{\substack{m=1\\ m\neq\ell}}^{N+1} g(\lambda_\ell,\lambda_m)\right)\left(\prod_{\substack{m=1\\ m\neq\ell}}^{N+1} h(\lambda_\ell,\lambda_m)\right) \\ &\quad\times\langle 0|\prod_{j=1}^{N} C(\mu_j)\prod_{\substack{m=1\\ m\neq\ell}}^{N+1} B(\lambda_m)|0\rangle. \end{aligned} \tag{3.4}$$

Let us notice that the form factor is symmetric function of all the $\lambda_j$ because

$$[B(\lambda_j), B(\lambda_k)] = 0.$$

We now need to calculate the scalar product between the eigenvector and non-eigenvector

$$\langle 0|\prod_{j=1}^{N} C(\mu_j)\prod_{\substack{m=1\\ m\neq\ell}}^{N+1} B(\lambda_m)|0\rangle, \tag{3.5}$$

where $\mu_j$ satisfy the Bethe equations, but $\lambda_m$ do not. It can be done by the following theorem.

**Theorem 3.1** *The following determinant representation holds for such scalar products:*

$$\langle 0|\prod_{j=1}^{N} C(\mu_j)\prod_{j=1}^{N} B(\lambda_j)|0\rangle = \left\{\prod_{j=1}^{N} d(\mu_j)d(\lambda_j)\right\}$$
$$\times \left\{\prod_{N\geq j>k\geq 1} g(\lambda_j,\lambda_k)g(\mu_k,\mu_j)\right\}\left\{\prod_{j,k=1}^{N} h(\mu_j,\lambda_k)\right\}\det(M_{jk}), \tag{3.6}$$

*where*

$$M_{jk} = \frac{g(\mu_k,\lambda_j)}{h(\mu_k,\lambda_j)} - \frac{a(\lambda_j)}{d(\lambda_j)}\frac{g(\lambda_j,\mu_k)}{h(\lambda_j,\mu_k)}\prod_{m=1}^{N}\frac{f(\lambda_j,\mu_m)}{f(\mu_m,\lambda_j)}. \tag{3.7}$$

*Here the spectral parameters $\{\mu_j\}$ satisfy the Bethe Ansatz equations (2.13). The spectral parameters $\{\lambda_j\}$ are free and do not satisfy any equations.*

This theorem was proved in [10].

For the scalar product, which appears in the expression for the form factor we get

$$\begin{aligned}
&\langle 0|\prod_{j=1}^{N} C(\mu_j)\prod_{\substack{m=1\\ m\neq\ell}}^{N+1} B(\lambda_m)|0\rangle \\
&= \prod_{N\geq j>k\geq 1} g(\mu_j,\mu_k)\cdot \prod_{\substack{N+1\geq j>k\geq 1\\ j\neq\ell,\ k\neq\ell}} g(\lambda_k,\lambda_j)\cdot \prod_{j=1}^{N}\prod_{\substack{m=1\\ m\neq\ell}}^{N+1} h(\mu_j,\lambda_m) \\
&\times \prod_{j=1}^{N} d(\mu_j)\cdot \prod_{\substack{m=1\\ m\neq\ell}}^{N+1} d(\lambda_m)\cdot \det{}_N M^{(\ell)}.
\end{aligned} \tag{3.8}$$

Here the entries of the $N\times N$ matrix $M^{(\ell)}$ are

$$M^{(\ell)}_{jk} = t(\mu_k,\lambda_j) - r(\lambda_j)t(\lambda_j,\mu_k)\cdot\prod_{m=1}^{N}\frac{f(\lambda_j,\mu_m)}{f(\mu_m,\lambda_j)}, \qquad \begin{array}{c} j=1,\ldots,\ell-1,\ell+1,\ldots,N+1 \\ k=1,\ldots,N \end{array} \tag{3.9}$$

Let us recall that

$$t(\lambda,\mu) = \frac{(ic)^2}{(\lambda-\mu)(\lambda-\mu+ic)} \qquad \text{and} \qquad r(\lambda) = \frac{a(\lambda)}{d(\lambda)}.$$

Remember that the Bethe equations give:

$$r(\lambda_j) = \prod_{p=1}^{N+1} \frac{h(\lambda_p, \lambda_j)}{h(\lambda_j, \lambda_p)} \quad \text{and} \quad \prod_{m=1}^{N} \frac{f(\lambda_j, \mu_m)}{f(\mu_m, \lambda_j)} = (-1)^N \prod_{m=1}^{N} \frac{h(\lambda_j, \mu_m)}{h(\mu_m, \lambda_j)}, \tag{3.10}$$

or equivalently,

$$a(\lambda_\ell) \prod_{m=1}^{N+1} h(\lambda_\ell, \lambda_m) = d(\lambda_\ell) \prod_{m=1}^{N+1} h(\lambda_m, \lambda_\ell). \tag{3.11}$$

Expression (3.9) becomes

$$M_{jk}^{(\ell)} = t(\mu_k, \lambda_j) - t(\lambda_j, \mu_k) \left( \prod_{p=1}^{N+1} \frac{h(\lambda_p, \lambda_j)}{h(\lambda_j, \lambda_p)} \right) \cdot \left( \prod_{m=1}^{N} \frac{h(\lambda_j, \mu_m)}{h(\mu_m, \lambda_j)} \right). \tag{3.12}$$

Using the obvious equality

$$\begin{aligned} \prod_{\substack{m=1 \\ m \neq \ell}}^{N+1} g(\lambda_\ell, \lambda_m) &= \left( \prod_{m=1}^{\ell-1} g(\lambda_\ell, \lambda_m) \right) \left( \prod_{m=\ell+1}^{N+1} g(\lambda_\ell, \lambda_m) \right) \\ &= (-1)^{\ell-1} \left( \prod_{m=1}^{\ell-1} g(\lambda_m, \lambda_\ell) \right) \left( \prod_{m=\ell+1}^{N+1} g(\lambda_\ell, \lambda_m) \right), \end{aligned} \tag{3.13}$$

and substituting (3.8) into (3.4), we have

$$\begin{aligned} F_N = -i\sqrt{c} \sum_{\ell=1}^{N+1} (-1)^{\ell-1} & \prod_{N \geq j > k \geq 1} g(\mu_j, \mu_k) \prod_{N+1 \geq j > k \geq 1} g(\lambda_k, \lambda_j) \\ & \times \prod_{m=1}^{N+1} h(\lambda_m, \lambda_\ell) \prod_{j=1}^{N} \prod_{\substack{m=1 \\ m \neq \ell}}^{N+1} h(\mu_j, \lambda_m) \\ & \times \prod_{j=1}^{N} d(\mu_j) \prod_{m=1}^{N+1} d(\lambda_m) \cdot \det{}_N M^{(\ell)}. \end{aligned} \tag{3.14}$$

One can rewrite the determinant $\det_N M^{(\ell)}$ as

$$\det{}_N M^{(\ell)} = \left( \prod_{m=1}^{N} \prod_{\substack{j=1 \\ j \neq \ell}}^{N+1} \frac{1}{h(\mu_m, \lambda_j)} \right) \cdot \left( \prod_{p=1}^{N+1} \prod_{\substack{j=1 \\ j \neq \ell}}^{N+1} h(\lambda_p, \lambda_j) \right) \cdot \det{}_N S^{(\ell)}, \tag{3.15}$$

where

$$S_{jk}^{(\ell)} = t(\mu_k, \lambda_j) \frac{\prod_{m=1}^{N} h(\mu_m, \lambda_j)}{\prod_{p=1}^{N+1} h(\lambda_p, \lambda_j)} - t(\lambda_j, \mu_k) \frac{\prod_{m=1}^{N} h(\lambda_j, \mu_m)}{\prod_{p=1}^{N+1} h(\lambda_j, \lambda_p)}, \qquad \begin{array}{c} j = 1, \ldots, \ell-1, \ell+1, \ldots, N+1 \\ k = 1, \ldots, N \end{array} \tag{3.16}$$

Let us substitute (3.15) into (3.14),

$$
\begin{aligned}
F_N &= -i\sqrt{c} \prod_{N\geq j>k\geq 1} g(\mu_j,\mu_k) \prod_{N+1\geq j>k\geq 1} g(\lambda_k,\lambda_j) \prod_{m=1}^{N+1}\prod_{j=1}^{N+1} h(\lambda_m,\lambda_j) \\
&\quad \times \left(\sum_{\ell=1}^{N+1} (-1)^{\ell+1} \mathrm{det}_N S^{(\ell)}\right) \cdot \prod_{j=1}^{N} d(\mu_j) \prod_{m=1}^{N+1} d(\lambda_m).
\end{aligned}
\tag{3.17}
$$

In order to simplify this expression let us study

$$
\mathcal{M}i = \mathcal{M}i\,\{\lambda\} \equiv \sum_{\ell=1}^{N+1} (-1)^{\ell-1} \mathrm{det}_N S^{(\ell)}. \tag{3.18}
$$

Notice that $\mathcal{M}i$ is an antisymmetric function of all $\{\lambda_j\}$ because $F_N$ is symmetric and the product of functions $g(\lambda_k,\lambda_j)$ is antisymmetric. In particular,

$$
\mathcal{M}i\,\{\lambda\} = 0 \quad \text{if} \quad \lambda_j = \lambda_k. \tag{3.19}
$$

$\mathrm{det} S^{(\ell)}$ can be obtained from $\mathrm{det} S^{(N+1)}$ by replacing $\lambda_\ell$ and $\lambda_{N+1}$. This is a special case of a permutation

$$
(\lambda_1,\cdots,\lambda_\ell,\cdots,\lambda_N,\lambda_{N+1}) \longrightarrow (\lambda_1,\cdots,\lambda_{N+1},\cdots,\lambda_N,\lambda_\ell). \tag{3.20}
$$

Since $(-1)^{\ell-1}$ is the parity of this permutation,

$$
\begin{aligned}
\mathcal{M}i\{\lambda\} &= \sum_{\text{Permutation of all } \{\lambda_{N+1}\}} (-1)^P \prod_{j=1}^{N} S_{P(j)j} \\
&= \left(1+\frac{\partial}{\partial\alpha}\right) \mathrm{det}_N (S_{jk} - \alpha S_{N+1,k})\big|_{\alpha=0}\,.
\end{aligned}
\tag{3.21}
$$

Here $S_{jk}$ means $S_{jk}^{(N+1)}$ from (3.16) and $\mathrm{det} S_{jk}$ is the term $\ell = N+1$ in (3.18),

$$
-\frac{\partial}{\partial\alpha} \mathrm{det}_N (S_{jk} - \alpha S_{N+1,k})\big|_{\alpha=0}
$$

is the sum of $N$ terms where each of them differs from $\mathrm{det} S_{jk}$ by the replacement of the $\ell$-th line (corresponding to $\lambda_\ell$) by the $(N+1)$-th line. We can use the expression (3.21) to simplify the form factor (3.17)

$$
\begin{aligned}
F_N &= -i\sqrt{c} \prod_{N\geq j>k\geq 1} g(\mu_j,\mu_k) \prod_{N+1\geq j>k\geq 1} g(\lambda_k,\lambda_j) \prod_{m=1}^{N+1}\prod_{j=1}^{N+1} h(\lambda_m,\lambda_j) \\
&\quad \times \prod_{j=1}^{N} d(\mu_j) \prod_{m=1}^{N+1} d(\lambda_m) \cdot \mathcal{M}i\,\{\lambda\}\,.
\end{aligned}
\tag{3.22}
$$

The complex conjugate of form factor is

$$\overline{F}_N = \langle 0| \prod_{j=1}^{N+1} C(\lambda_j)\psi^\dagger(0,0) \prod_{j=1}^{N} B(\mu_j)|0\rangle. \tag{3.23}$$

Remember that $c$ and all $\lambda$, $\mu$ are real. Therefore complex conjugation gives

$$\begin{aligned} &\overline{g(\lambda,\mu)} = g(\mu,\lambda), \quad \overline{f(\lambda,\mu)} = f(\mu,\lambda), \quad \overline{h(\lambda,\mu)} = h(\mu,\lambda), \\ &\overline{t(\lambda,\mu)} = t(\mu,\lambda), \quad \overline{a(\lambda)} = d(\lambda) = a^{-1}(\lambda). \end{aligned} \tag{3.24}$$

So we have

$$\overline{S}_{jk} = -S_{jk},$$

and for even $N$

$$\overline{\mathcal{M}i\,\{\lambda\}} = (-1)^N \mathcal{M}i\,\{\lambda\} = \mathcal{M}i\,\{\lambda\}\,. \tag{3.25}$$

Hence for the complex conjugated form factor $\overline{F}_N$ we get

$$\begin{aligned} \overline{F}_N &= i\sqrt{c} \prod_{N\geq j>k\geq 1} g(\mu_k,\mu_j) \prod_{N+1\geq j>k\geq 1} g(\lambda_j,\lambda_k) \prod_{m=1}^{N+1}\prod_{j=1}^{N+1} h(\lambda_j,\lambda_m) \\ &\times \prod_{j=1}^{N} a(\mu_j) \prod_{m=1}^{N+1} a(\lambda_m)\mathcal{M}i\,\{\lambda\}. \end{aligned} \tag{3.26}$$

For the correlation function the quantity $|F_N|^2$ is important:

$$\begin{aligned} F_N\overline{F}_N &= c\left(\prod_{j=1}^{N} a(\mu_j)d(\mu_j)\right)\left(\prod_{m=1}^{N+1} a(\lambda_m)d(\lambda_m)\right)\cdot\left(\prod_{\substack{j=1,k=1\\ j\neq k}}^{N} g(\mu_j,\mu_k)\right) \\ &\times \left(\prod_{\substack{j=1,k=1\\ j\neq k}}^{N+1} g(\lambda_j,\lambda_k)\right)\cdot\left(\prod_{j=1}^{N+1}\prod_{k=1}^{N+1} h(\lambda_j,\lambda_k)\right)^2 (\mathcal{M}i\,\{\lambda\})^2. \end{aligned} \tag{3.27}$$

or,

$$\begin{aligned} &\frac{F_N\overline{F}_N}{\langle 0|\prod_{j=1}^{N} C(\mu_j)\prod_{j=1}^{N} B(\mu_j)|0\rangle\cdot\langle 0|\prod_{j=1}^{N+1} C(\lambda_j)\prod_{j=1}^{N+1} B(\lambda_j)|0\rangle} \\ &= c^{-2N}\frac{\left(\prod_{j=1}^{N+1}\prod_{k=1}^{N+1} h(\lambda_j,\lambda_k)\right)\cdot(\mathcal{M}i\,\{\lambda\})^2}{\left(\prod_{j=1}^{N}\prod_{k=1}^{N} h(\mu_j,\mu_k)\right)\det_N\frac{\partial\varphi_j}{\partial\mu_k}\det_{N+1}\frac{\partial\tilde{\varphi}_j}{\partial\lambda_k}} \end{aligned} \tag{3.28}$$

This formula gives us $|F_N|^2$ at $x=t=0$. Also it is easy to "switch" on space and time dependence using the formula (2.31). Note that $a(\lambda)d(\lambda)=1$ for nonlinear Schrödinger equation. Therefore the correlation function becomes

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_N=\left\{\frac{e^{-iht}c^{-2N}}{(\prod_{j=1}^{N}\prod_{k=1}^{N}h(\mu_j,\mu_k))\cdot\det_N\frac{\partial\varphi_j}{\partial\mu_k}}\right\}\sum_{\{\lambda\}}\frac{\Delta(\{\lambda\})}{\det_{N+1}\frac{\partial\tilde\varphi_j}{\partial\lambda_k}}. \tag{3.29}$$

Here we used the new notation

$$\Delta(\{\lambda\})=\left(\prod_{k=1}^{N+1}\prod_{j=1}^{N+1}h(\lambda_j,\lambda_k)\right)(\mathcal{M}i\,\{\lambda\})^2 e^{\left\{\sum_{j=1}^{N+1}\tau(\lambda_j)-\sum_{m=1}^{N}\tau(\mu_m)\right\}}, \tag{3.30}$$

where

$$\tau(\lambda)=it\lambda^2-ix\lambda. \tag{3.31}$$

## 4 The idea of summation with respect to $\lambda$

Now let us consider the sum with respect to all $\{\lambda\}_{N+1}$ in (3.29)

$$\sum_{\{\lambda\}_{N+1}}\frac{\Delta(\{\lambda\})}{\det_{N+1}\left(\frac{\partial\tilde\varphi_j}{\partial\lambda_k}\right)}$$

The idea of summation is the same as that which we used for impenetrable bosons (free fermionic case) [11] (see also Section XIII.5 of [3]). The factor $(\mathcal{M}i\,\{\lambda\})^2$ entering the r.h.s. of (3.30) contains $(N+1)!$ terms, all of them give the same contribution to the sum. So we can replace one of determinants $\mathcal{M}i\,\{\lambda\}$ by the product $\prod_{j=1}^{N}S_{jj}$

$$\begin{aligned}&\sum_{\{\lambda\}_{N+1}}\frac{\Delta(\{\lambda\})}{\det_{N+1}\left(\frac{\partial\tilde\varphi_j}{\partial\lambda_k}\right)}\\&=(N+1)!\sum_{\{\lambda\}_{N+1}}\prod_{m=1}^{N}e^{-\tau(\mu_m)}\cdot\left(\det_{N+1}\frac{\partial\tilde\varphi_j}{\partial\lambda_k}\right)^{-1}\\&\times\left(\prod_{k=1}^{N+1}\prod_{j=1}^{N+1}h(\lambda_j,\lambda_k)\right)\prod_{j=1}^{N+1}e^{\tau(\lambda_j)}\left(\prod_{j=1}^{N}S_{jj}\right)\mathcal{M}i\{\lambda\}.\end{aligned} \tag{4.1}$$

The sum with respect to all $\{\lambda\}_{N+1}$ means the sum with respect to all ordered sets of integers $\{n_j\}$ from (2.28). We also can admit $n_j=n_k$ because it leads to $\lambda_j=\lambda_k$ which does not contribute to (4.1) because of the antisymmetry of $\mathcal{M}i\{\lambda\}$. The factor $(N+1)!$ is absorbed as

$$(N+1)!\sum_{\{n_j\}}=\prod_{i=1}^{N+1}\sum_{n_i=-\infty}^{\infty}. \tag{4.2}$$

The correlation function becomes

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_N = \frac{e^{-iht}c^{-2N}\prod_{m=1}^N e^{-\tau(\mu_m)}}{(\prod_{j=1}^N\prod_{k=1}^N h(\mu_j,\mu_k))\cdot\det_N\frac{\partial\varphi_j}{\partial\mu_k}} \times \sum_{n_1\cdots n_{N+1}}\left(\frac{\widetilde{\Delta}(\{\lambda\})}{\det_{N+1}\frac{\partial\tilde{\varphi}_j}{\partial\lambda_k}}\right). \tag{4.3}$$

with

$$\widetilde{\Delta}(\{\lambda\}) = \left(\prod_{j=1}^{N+1}\prod_{k=1}^{N+1} h(\lambda_j,\lambda_k)\right)\prod_{j=1}^{N+1} e^{\tau(\lambda_j)} \times \left(1+\frac{\partial}{\partial\alpha}\right)\det_N(S_{jj}S_{jk}-\alpha S_{jj}S_{N+1\,k})\Big|_{\alpha=0}. \tag{4.4}$$

The main difference between the free fermionic case (coupling constant $c\to+\infty$) and the non-free fermionic case is that in the former it is possible to solve the Bethe equations (2.26) explicitly. On the contrary this is not possible for penetrable bosons.

Our approach is based on the formula

$$\sum_{n_1\cdots n_{N+1}}\left(\frac{\widetilde{\Delta}(\{\lambda\})}{\det_{N+1}\frac{\partial\tilde{\varphi}_j}{\partial\lambda_k}}\right) = \sum_{n_1\cdots n_{N+1}}\int_{-\infty}^{\infty} d^{N+1}\lambda\,\widetilde{\Delta}(\{\lambda\})\prod_{j=1}^{N+1}\delta(\tilde{\varphi}_j(\lambda)-2\pi n_j).$$

Remember that $\det\partial\tilde{\varphi}_j/\partial\lambda_k>0$. We shall also use the Poisson formula

$$\sum_{n=-\infty}^{\infty}\delta(x-2\pi n) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikx}. \tag{4.5}$$

So we have

$$\begin{aligned}\sum_{n_1\cdots n_{N+1}}\left(\frac{\widetilde{\Delta}(\{\lambda\})}{\det_{N+1}\frac{\partial\tilde{\varphi}_j}{\partial\lambda_k}}\right) &= \sum_{n_1\cdots n_{N+1}}\int_{-\infty}^{\infty} d^{N+1}\lambda\,\widetilde{\Delta}(\{\lambda\})\prod_{j=1}^{N+1}\delta(\tilde{\varphi}_j(\lambda)-2\pi n_j)\\ &= \sum_{n_1\cdots n_{N+1}}\left(\frac{1}{2\pi}\right)^{N+1}\int_{-\infty}^{\infty} d^{N+1}\lambda\,\widetilde{\Delta}(\{\lambda\})\prod_{j=1}^{N+1} e^{in_j\tilde{\varphi}_j(\lambda)}\\ &= \sum_{n_1\cdots n_{N+1}}\left(\frac{1}{2\pi}\right)^{N+1}\int_{-\infty}^{\infty} d^{N+1}\lambda\widetilde{\Delta}(\{\lambda\})\prod_{j=1}^{N+1} e^{iL\lambda_j n_j}\left(\prod_{k=1}^{N+1}\frac{h(\lambda_k,\lambda_j)}{h(\lambda_j,\lambda_k)}\right)^{n_j}.\end{aligned} \tag{4.6}$$

Thus we get the following representation for the correlation function

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_N = \frac{e^{-iht}c^{-2N}\prod_{m=1}^N e^{-\tau(\mu_m)}}{(\prod_{j=1}^N\prod_{k=1}^N h(\mu_j,\mu_k))\cdot\det_N\frac{\partial\varphi_j}{\partial\mu_k}} \times\sum_{n_1\cdots n_{N+1}}\left(\frac{1}{2\pi}\right)^{N+1}\int_{-\infty}^{\infty} d^{N+1}\lambda\widetilde{\Delta}(\{\lambda\})\prod_{j=1}^{N+1} e^{iL\lambda_j n_j}\left(\prod_{k=1}^{N+1}\frac{h(\lambda_k,\lambda_j)}{h(\lambda_j,\lambda_k)}\right)^{n_j}. \tag{4.7}$$

## 5 Quantum Dual Fields

In this section we introduce the auxiliary Fock space and auxiliary Bose fields $\phi_0(\lambda)$, $\phi_1(\lambda)$, $\phi_2(\lambda)$ $\phi_{A_j}(\lambda)$ and $\phi_{D_j}(\lambda)$ $(j=1,2)$. Further we shall call these operators dual fields [12] (see also Section IX.5 of [3]). Dual fields help us to rewrite double products in terms of single products.

By definition any operator $\phi_a(\lambda)$ $(a = 0,\ 1,\ 2,\ A_1,\ A_2,\ D_1,\ D_2)$ is the sum of two operators: "momentum" $p(\lambda)$ and "coordinate" $q(\lambda)$.

$$\begin{array}{ll} \phi_0(\lambda)=q_0(\lambda)+p_0(\lambda); & \\ \phi_{A_j}(\lambda)=q_{A_j}(\lambda)+p_{D_j}(\lambda); & \phi_{D_j}(\lambda)=q_{D_j}(\lambda)+p_{A_j}(\lambda); \\ \phi_1(\lambda)=q_1(\lambda)+p_2(\lambda); & \phi_2(\lambda)=q_2(\lambda)+p_1(\lambda). \end{array} \tag{5.1}$$

All operators "momenta" $p(\lambda)$ annihilate the vacuum vector $|0)$, all operators $q(\lambda)$ annihilate the dual vacuum $(0|$ :

$$p_a(\lambda)|0)=0, \quad (0|\,q_a(\lambda)=0, \qquad \text{for all } a, \qquad (0|0)=1.$$

The only nonzero commutation relations are

$$\left\{ \begin{array}{ll} [p_0(\lambda),q_0(\mu)]=\ln(h(\lambda,\mu)h(\mu,\lambda)); & \\ [p_{D_j}(\lambda),q_{D_k}(\mu)]=\delta_{jk}\ln h(\lambda,\mu); & [p_{A_j}(\lambda),q_{A_k}(\mu)]=\delta_{jk}\ln h(\mu,\lambda); \\ [p_1(\lambda),q_1(\mu)]=\ln\dfrac{h(\lambda,\mu)}{h(\mu,\lambda)}; & [p_2(\lambda),q_2(\mu)]=\ln\dfrac{h(\mu,\lambda)}{h(\lambda,\mu)}. \end{array} \right. \tag{5.2}$$

(We remind the reader that $h(\lambda,\mu)=(\lambda-\mu+ic)/ic$). The realization of these operators as linear combinations of canonical Bose fields is given in Appendix B.

It is easy to check that all dual fields commute with each other

$$[\phi_a(\lambda),\phi_b(\mu)]=0,$$

where $a,b$ run through the all possible indices. Using this property we can define functions of operators $\mathcal{F}(\{e^{\phi_a(\lambda)}\})$. One should understand such expression, for example as a power series over $\{e^{\phi_a(\lambda)}\}$. The following simple formulæ are useful:

$$e^{p_a(\lambda)}e^{q_a(\mu)}=e^{q_a(\mu)}e^{p_a(\lambda)}e^{[p_a(\lambda),q_a(\mu)]}, \tag{5.3}$$

$$(0|\prod_{j=1}^{M_1}e^{\alpha_j p_a(\lambda_j)}\prod_{k=1}^{M_2}e^{\beta_k q_a(\mu_k)}|0)=\prod_{j=1}^{M_1}\prod_{k=1}^{M_2}e^{\alpha_j\beta_k[p_a(\lambda_j),q_a(\mu_k)]}, \tag{5.4}$$

$$\prod_{j=1}^{M} e^{\beta_j p_a(\lambda_j)} \mathcal{F}\left(e^{\phi_a(\mu)}\right)|0) = \mathcal{F}\left(e^{\phi_a(\mu)} \prod_{j=1}^{M} e^{\beta_j [p_a(\lambda_j), q_a(\mu)]}\right)|0), \tag{5.5}$$

$$\mathcal{F}\left(e^{p_a(\mu)}\right) \prod_{j=1}^{M} e^{\beta_j \phi_a(\lambda_j)}|0) = \prod_{j=1}^{M} e^{\beta_j \phi_a(\lambda_j)}|0)\mathcal{F}\left(\prod_{j=1}^{M} e^{\beta_j [p_a(\mu), q_a(\lambda_j)]}\right). \tag{5.6}$$

Here $\{\lambda\}, \{\mu\}, \{\beta\}, \{\alpha\}$ are arbitrary complex numbers, $\mathcal{F}$ is a function. One can easily prove these formulæ.

Let us define the very important dual field $\psi(\lambda)$ as

$$\psi(\lambda) = \phi_0(\lambda) + \phi_{A_1}(\lambda) + \phi_{D_2}(\lambda) + \phi_2(\lambda). \tag{5.7}$$

**Theorem 5.1** *The correlation function (4.7) can be presented as the following vacuum expectation value in auxiliary Fock space*

$$\begin{aligned} \langle\psi(0,0)\psi^\dagger(x,t)\rangle_N &= \frac{e^{-iht}c^{-2N}}{\det_N \frac{\partial\varphi_j}{\partial\mu_k}}(0| \prod_{m=1}^{N}\left(e^{p_0(\mu_m)}e^{p_1(\mu_m)}\right) \\ &\times \frac{1}{(2\pi)^{N+1}} \sum_{n_1\cdots n_{N+1}} \int_{-\infty}^{\infty} d^{N+1}\lambda \left(\hat\gamma_1(\lambda_{N+1}) + \frac{\partial}{\partial\alpha}\right) \\ &\times \det{}_N\left(\widehat{S}_{jj}\widehat{S}_{jk}\hat\gamma_1(\lambda_j)\hat\gamma_2(\mu_j)\hat\gamma_2(\mu_k)\right. \\ &- \left.\alpha\widehat{S}_{jj}\widehat{S}_{N+1\,k}\hat\gamma_1(\lambda_j)\hat\gamma_1(\lambda_{N+1})\hat\gamma_2(\mu_j)\hat\gamma_2(\mu_k)\right)\Big|_{\alpha=0}|0). \end{aligned} \tag{5.8}$$

*where*

$$\widehat{S}_{jk} = t(\mu_k,\lambda_j)e^{-\phi_{D_1}(\lambda_j)} - t(\lambda_j,\mu_k)e^{-\phi_{A_2}(\lambda_j)}, \tag{5.9}$$

*and*

$$\begin{aligned} \hat\gamma_1(\lambda_j) &= e^{iL\lambda_j n_j + \tau(\lambda_j) + \psi(\lambda_j) + n_j\phi_1(\lambda_j)}, \\ \hat\gamma_2(\mu) &= e^{-\frac{1}{2}(\tau(\mu)+\psi(\mu))}. \end{aligned}$$

*Proof.* Let us move factors $\hat\gamma_1(\lambda)$ and $\hat\gamma_2(\mu)$ out of the determinant in (5.8). In the r.h.s. of (5.8) we get

$$\begin{aligned} &\left(1 + \frac{\partial}{\partial\alpha}\right)(0| \prod_{m=1}^{N}\left[e^{p_0(\mu_m)}e^{p_1(\mu_m)}\right] \\ \times &\prod_{m=1}^{N+1}\left[e^{iL\lambda_m n_m + \tau(\lambda_m) + \psi(\lambda_m) + n_m\phi_1(\lambda_m)}\right] \prod_{m=1}^{N}\left[e^{-\tau(\mu_m)-\psi(\mu_m)}\right] \\ \times &\det{}_N\left(\widehat{S}_{jj}\widehat{S}_{jk} - \alpha\widehat{S}_{jj}\widehat{S}_{N+1\,k}\right)\Big|_{\alpha=0}|0). \end{aligned} \tag{5.10}$$

Using (5.4) we find

$$
\begin{aligned}
&(0|\prod_{m=1}^{N}\left[e^{p_0(\mu_m)}e^{p_1(\mu_m)}\right]\prod_{m=1}^{N+1}\left[e^{\phi_0(\lambda_m)+n_m\phi_1(\lambda_m)+\phi_2(\lambda_m)}\right]\prod_{m=1}^{N}\left[e^{-\phi_0(\mu_m)-\phi_2(\mu_m)}\right]|0)\\
&\qquad=\frac{\prod_{j=1}^{N+1}\prod_{k=1}^{N+1}h(\lambda_j,\lambda_k)}{\prod_{j=1}^{N}\prod_{k=1}^{N}h(\mu_j,\mu_k)}\prod_{j=1}^{N+1}\prod_{k=1}^{N+1}\left(\frac{h(\lambda_k,\lambda_j)}{h(\lambda_j,\lambda_k)}\right)^{n_j}.
\end{aligned}
\tag{5.11}
$$

Using (5.5) we obtain

$$
\begin{aligned}
&(0|\prod_{m=1}^{N+1}\left[e^{\phi_{A_1}(\lambda_m)+\phi_{D_2}(\lambda_m)}\right]\prod_{m=1}^{N}\left[e^{-\phi_{A_1}(\mu_m)-\phi_{D_2}(\mu_m)}\right]\det_N\left(\widehat{S}_{jj}\widehat{S}_{jk}-\alpha\widehat{S}_{jj}\widehat{S}_{N+1\,k}\right)|0)\\
&\qquad=\det_N\left(S_{jj}S_{jk}-\alpha S_{jj}S_{N+1\,k}\right).
\end{aligned}
\tag{5.12}
$$

Combining (5.11) and (5.12) we get the r.h.s. of (4.7) with $\widetilde{\Delta}(\{\lambda\})$ defined in (4.4). The theorem is proved.

Now we can rewrite the r.h.s. for (5.8) as follows

$$
\begin{aligned}
&\frac{1}{(2\pi)^{N+1}}\sum_{n_1\cdots n_{N+1}}\int_{-\infty}^{\infty}d^{N+1}\lambda\left(\hat{\gamma}_1(\lambda_{N+1})+\frac{\partial}{\partial\alpha}\right)\\
&\times\det_N\left(\widehat{S}_{jj}\widehat{S}_{jk}\hat{\gamma}_1(\lambda_j)\hat{\gamma}_2(\mu_j)\hat{\gamma}_2(\mu_k)-\alpha\widehat{S}_{jj}\widehat{S}_{N+1\,k}\hat{\gamma}_1(\lambda_j)\hat{\gamma}_1(\lambda_{N+1})\hat{\gamma}_2(\mu_j)\hat{\gamma}_2(\mu_k)\right)\Big|_{\alpha=0}\\
&=\left(\frac{1}{2\pi}\sum_{n_{N+1}}\int_{-\infty}^{\infty}d\lambda_{N+1}\hat{\gamma}_1(\lambda_{N+1})+\frac{\partial}{\partial\alpha}\right)\\
&\times\det_N\left(\frac{1}{2\pi}\sum_{n_j}\int_{-\infty}^{\infty}d\lambda_j\widehat{S}_{jj}\widehat{S}_{jk}\hat{\gamma}_1(\lambda_j)\hat{\gamma}_2(\mu_j)\hat{\gamma}_2(\mu_k)\right.\\
&\left.-\alpha\frac{1}{4\pi^2}\sum_{n_j,n_{N+1}}\int_{-\infty}^{\infty}d\lambda_{N+1}d\lambda_j\widehat{S}_{jj}\widehat{S}_{N+1\,k}\hat{\gamma}_1(\lambda_j)\hat{\gamma}_1(\lambda_{N+1})\hat{\gamma}_2(\mu_j)\hat{\gamma}_2(\mu_k)\right)\Bigg|_{\alpha=0}.
\end{aligned}
\tag{5.13}
$$

In this formula we can perform the summation over integer $\{n_j\}$. Indeed, $n_j$ enters only in function $\hat{\gamma}_1(\lambda_j)$:

$$
\hat{\gamma}_1(\lambda_j)=e^{iL\lambda_j n_j+\tau(\lambda_j)+\psi(\lambda_j)+n_j\phi_1(\lambda_j)}.
$$

Recall that all dual fields commute with each other:

$$
[\phi_a(\lambda),\phi_b(\mu)]=0.
$$

This means that we can treat operators $\phi_a(\lambda)$ as diagonal operators. Due to the formula (B.8) from Appendix B we can consider operator $i\phi_1(\lambda)$ as real function of $\lambda$. Hence we can use formula (4.5) to sum up with respect to $n_j$

$$
\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{in(L\lambda-i\phi_1(\lambda))}=\sum_{n=-\infty}^{\infty}\delta(L\lambda-i\phi_1(\lambda)-2\pi n).
\tag{5.14}
$$

It means that $\lambda = \lambda_n$, where $\lambda_n$ is a root of the equation

$$L\lambda_n - 2\pi n = i\phi_1(\lambda_n). \tag{5.15}$$

The expression (5.15) is an operator equality, which is defined only on vectors of the form $\prod_m e^{\phi_2(\lambda_m)}|0)$. Therefore one should understand this equation in the sense of mean value:

$$(0|(L\lambda_n - i\phi_1(\lambda_n) - 2\pi n)\prod_m e^{\phi_2(\lambda_m)}|0) = 0, \tag{5.16}$$

where $\{\lambda_m\}$ are arbitrary real parameters. Then we can rewrite equation (5.16):

$$L\lambda_n - 2\pi n = i\sum_m \ln\frac{h(\lambda_m,\lambda_n)}{h(\lambda_n,\lambda_m)} = i\sum_m \ln\left(\frac{\lambda_m - \lambda_n + ic}{\lambda_n - \lambda_m + ic}\right). \tag{5.17}$$

The r.h.s. of (5.17) is a real bounded function of $\lambda_n$. Moreover it is a decreasing function of $\lambda_n$, because

$$\frac{\partial}{\partial\lambda_n} i\sum_m \ln\left(\frac{\lambda_m - \lambda_n + ic}{\lambda_n - \lambda_m + ic}\right) = -\sum_m \frac{2c}{(\lambda_n - \lambda_m)^2 + c^2} < 0.$$

The l.h.s. of (5.17) is a linear increasing function of $\lambda_n$, hence equation (5.17) has one real solution and this solution is unique. Also we have

$$(0|(L - i\phi_1'(\lambda_n))\prod_m e^{\phi_2(\lambda_m)}|0) = L + \sum_m \frac{2c}{(\lambda_n - \lambda_m)^2 + c^2} > 0. \tag{5.18}$$

Therefore, one can write

$$\delta(L\lambda - i\phi_1(\lambda) - 2\pi n) = \frac{\delta(\lambda - \lambda_n)}{L - i\phi_1'(\lambda)}, \tag{5.19}$$

where $\lambda_n$ is solution of equation (5.15).

Later we shall use notation

$$2\pi\hat{\rho}(\lambda) = 1 - \frac{i}{L}\phi_1'(\lambda). \tag{5.20}$$

We now arrive at the following formula for the correlation function in finite volume

$$\begin{aligned}\langle\psi(0,0)\psi^\dagger(x,t)\rangle_N \quad = \quad & \frac{e^{-iht}c^{-2N}}{\det_N\left(\frac{\partial\varphi_j}{\partial\mu_k}\right)}(0|\prod_{m=1}^{N}\left(e^{p_0(\mu_m)}e^{p_1(\mu_m)}\right) \\ & \times\left(G_N(x,t) + \frac{\partial}{\partial\alpha}\right)\cdot\det_N(\widehat{U}_{jk} - \alpha\widehat{Q}_j\widehat{Q}_k)|0)\Big|_{\alpha=0},\end{aligned} \tag{5.21}$$

where

$$G_N(x,t) = \frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{1}{2\pi\hat{\rho}(\lambda_n)}e^{\psi(\lambda_n)+\tau(\lambda_n)}, \tag{5.22}$$

and

$$\widehat{U}_{jk} = \frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{e^{\psi(\lambda_n)+\tau(\lambda_n)}e^{-\frac{1}{2}(\psi(\mu_j)+\psi(\mu_k)+\tau(\mu_j)+\tau(\mu_k))}}{2\pi\hat{\rho}(\lambda_n)}$$
$$\times\left\{t(\mu_k,\lambda_n)e^{-\phi_{D_1}(\lambda_n)} - t(\lambda_n,\mu_k)e^{-\phi_{A_2}(\lambda_n)}\right\}$$
$$\times\left\{t(\mu_j,\lambda_n)e^{-\phi_{D_1}(\lambda_n)} - t(\lambda_n,\mu_j)e^{-\phi_{A_2}(\lambda_n)}\right\}, \tag{5.23}$$

$$\widehat{Q}_j = \frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{e^{\psi(\lambda_n)+\tau(\lambda_n)}e^{-\frac{1}{2}(\psi(\mu_j)+\tau(\mu_j))}}{2\pi\hat{\rho}(\lambda_n)}$$
$$\times\left\{t(\mu_j,\lambda_n)e^{-\phi_{D_1}(\lambda_n)} - t(\lambda_n,\mu_j)e^{-\phi_{A_2}(\lambda_n)}\right\}. \tag{5.24}$$

Formula (5.21) is the determinant representation for the quantum correlation function in a finite volume.

## 6 Thermodynamic limit

In order to calculate the correlation function in the ground state one should consider the limit where the number of particles and the length of the box tend to infinity with fixed constant density: $N \to \infty, \quad L \to \infty, \quad N/L = D = const.$ In this limit the parameters $\{\lambda_n\}$ are described by distribution density $\hat{\rho}(\lambda)$

$$\hat{\rho}(\lambda) = \frac{1}{2\pi}\left(1 - \frac{i}{L}\phi_1'(\lambda)\right).$$

(see Appendix A). The sums in the expressions for $\widehat{U}_{jk}$ and $\widehat{Q}_j$ can be replaced by corresponding integrals. Let us introduce the new function $Z(\lambda,\mu)$

$$Z(\lambda,\mu) = \frac{e^{-\phi_{D_1}(\lambda)}}{h(\mu,\lambda)} + \frac{e^{-\phi_{A_2}(\lambda)}}{h(\lambda,\mu)}. \tag{6.1}$$

Then we can rewrite (5.23) and (5.24) as

$$\widehat{U}_{jk} = \frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{(ic)^2 e^{\psi(\lambda_n)+\tau(\lambda_n)}}{2\pi\hat{\rho}(\lambda_n)(\lambda_n-\mu_j)(\lambda_n-\mu_k)}$$
$$\times e^{-\frac{1}{2}(\psi(\mu_j)+\psi(\mu_k)+\tau(\mu_j)+\tau(\mu_k))}Z(\lambda_n,\mu_k)Z(\lambda_n,\mu_j), \tag{6.2}$$

$$\widehat{Q}(\mu) = \frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{ic e^{\psi(\lambda_n)+\tau(\lambda_n)}e^{-\frac{1}{2}(\psi(\mu)+\tau(\mu))}}{2\pi\hat{\rho}(\lambda_n)(\lambda_n-\mu)}Z(\lambda_n,\mu). \tag{6.3}$$

Here $\widehat{Q}_j = -\widehat{Q}(\mu_j), \quad \widehat{Q}_k = -\widehat{Q}(\mu_k)$. Using formula (A.25), we get

$$\widehat{U}_{jk} = L\delta_{jk}\frac{(ic)^2 2\pi\hat{\rho}(\mu_j)}{4\sin^2\frac{L}{2}\hat{\xi}(\mu_j)}Z^2(\mu_j,\mu_j)$$

$$
\begin{aligned}
&-\frac{(ic)^2}{2}\delta_{jk}\cot\frac{L}{2}\hat{\xi}(\mu_j)\frac{\partial}{\partial\mu_j}\left[e^{\frac{1}{2}(\psi(\mu_j)-\psi(\mu_k)+\tau(\mu_j)-\tau(\mu_k))}Z(\mu_j,\mu_k)Z(\mu_j,\mu_j)\right.\\
&\qquad\left.-\,e^{-\frac{1}{2}(\psi(\mu_j)-\psi(\mu_k)+\tau(\mu_j)-\tau(\mu_k))}Z(\mu_k,\mu_j)Z(\mu_k,\mu_k)\right]_{\mu_k=\mu_j}\\
&-\frac{(ic)^2}{2}\frac{1-\delta_{jk}}{\mu_j-\mu_k}\left[e^{\frac{1}{2}(\psi(\mu_j)-\psi(\mu_k)+\tau(\mu_j)-\tau(\mu_k))}Z(\mu_j,\mu_k)Z(\mu_j,\mu_j)\cot\frac{L}{2}\hat{\xi}(\mu_j)\right.\\
&\qquad\left.-\,e^{-\frac{1}{2}(\psi(\mu_j)-\psi(\mu_k)+\tau(\mu_j)-\tau(\mu_k))}Z(\mu_k,\mu_j)Z(\mu_k,\mu_k)\cot\frac{L}{2}\hat{\xi}(\mu_k)\right]\\
&\qquad+\frac{(ic)^2}{2\pi(\mu_j-\mu_k)}\,\fint_{-\infty}^{\infty}d\lambda\left(\frac{1}{\lambda-\mu_j}-\frac{1}{\lambda-\mu_k}\right)e^{\psi(\lambda)+\tau(\lambda)}\\
&\qquad\times e^{-\frac{1}{2}(\psi(\mu_j)+\tau(\mu_j)+\psi(\mu_k)+\tau(\mu_k))}Z(\lambda,\mu_k)Z(\lambda,\mu_j)+\mathcal{O}(1/L). \qquad (6.4)
\end{aligned}
$$

Here we denote the principal value by the symbol

$$
\fint_{-\infty}^{\infty}\frac{d\lambda(\cdot)}{\lambda-\mu}\equiv \text{V.P.}\int_{-\infty}^{\infty}\frac{d\lambda(\cdot)}{\lambda-\mu}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\lambda(\cdot)}{\lambda-\mu+i0}+\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\lambda(\cdot)}{\lambda-\mu-i0}
$$

Using (A.21) we calculate the sum (6.3):

$$
\begin{aligned}
\widehat{Q}(\mu) &= \frac{ic}{2\pi}\fint_{-\infty}^{\infty}\frac{d\lambda}{\lambda-\mu}e^{\psi(\lambda)+\tau(\lambda)}e^{-\frac{1}{2}(\psi(\mu)+\tau(\mu))}Z(\lambda,\mu)\\
&\quad -\frac{ic}{2}e^{\frac{1}{2}(\psi(\mu)+\tau(\mu))}Z(\mu,\mu)\cot\frac{L}{2}\hat{\xi}(\mu)+\mathcal{O}(1/L). \qquad (6.5)
\end{aligned}
$$

Function $\hat{\xi}(\mu)$ is defined in Appendix A as

$$
\hat{\xi}(\mu)=\mu-\frac{i}{L}\phi_1(\mu),
$$

(see (A.2)), and hence

$$
\begin{aligned}
\cot\frac{L}{2}\hat{\xi}(\mu) &= i\frac{e^{iL\mu+\phi_1(\mu)}+1}{e^{iL\mu+\phi_1(\mu)}-1},\\
\sin^2\frac{L}{2}\hat{\xi}(\mu) &= \frac{1}{4}\left(2-e^{iL\mu+\phi_1(\mu)}-e^{-iL\mu-\phi_1(\mu)}\right).
\end{aligned}
$$

Let us turn back to the formula (5.21). We can move all $e^{p_1(\mu_m)}$ to the right vacuum $|0)$. Then each operator $\phi_1$ entering into $\widehat{U}$ and $\widehat{Q}$ should be replaced by the rule (see (5.5))

$$
\prod_{m=1}^{N}e^{p_1(\mu_m)}e^{\phi_1(\mu_j)}=\prod_{m=1}^{N}\frac{h(\mu_m,\mu_j)}{h(\mu_j,\mu_m)}e^{\phi_1(\mu_j)}\prod_{m=1}^{N}e^{p_1(\mu_m)}.
$$

Taking into account Bethe equations (2.13)

$$
e^{iL\mu_j}\prod_{m=1}^{N}\frac{h(\mu_m,\mu_j)}{h(\mu_j,\mu_m)}=-1, \qquad (N\text{ — even}),
$$

we get

$$\prod_{m=1}^{N} e^{p_1(\mu_m)} \cot \frac{L}{2}\hat{\xi}(\mu_j) = i\frac{\omega_-(\mu_j)}{\omega_+(\mu_j)} \prod_{m=1}^{N} e^{p_1(\mu_m)},$$

$$\prod_{m=1}^{N} e^{p_1(\mu_m)} \sin^2 \frac{L}{2}\hat{\xi}(\mu_j) = \left(\frac{\omega_+(\mu_j)}{2}\right)^2 \prod_{m=1}^{N} e^{p_1(\mu_m)},$$

where

$$\omega_\pm(\mu) = e^{\phi_1(\mu)/2} \pm e^{-\phi_1(\mu)/2}. \tag{6.6}$$

Operator $2\pi\hat{\rho}(\lambda)$ also contains $\phi_1(\lambda)$, so it does not commute with $p_1(\mu)$:

$$\prod_{m=1}^{N} e^{p_1(\mu_m)} 2\pi\hat{\rho}(\mu_j) = 2\pi\hat{R}(\mu_j) \prod_{m=1}^{N} e^{p_1(\mu_m)}. \tag{6.7}$$

where

$$2\pi\hat{R}(\mu) = \left(1 + \frac{1}{L}\sum_{m=1}^{N} K(\mu,\mu_m) - \frac{i}{L}\phi_1'(\mu)\right).$$

Hence we have the new representation for the correlation function

$$\begin{aligned}\langle\psi(0,0)\psi^\dagger(x,t)\rangle_N &= \frac{e^{-iht}c^{-2N}}{\det_N \frac{\partial\varphi_j}{\partial\mu_k}} \\ &\times (0| \prod_{m=1}^{N} e^{p_0(\mu_m)} \left(G_N(x,t) + \frac{\partial}{\partial\alpha}\right) \\ &\times \det_N\Big(\widetilde{U}_{jk} - \alpha\widetilde{Q}(\mu_j)\widetilde{Q}(\mu_k) + \mathcal{O}(1/L)\Big)|0)\Big|_{\alpha=0},\end{aligned} \tag{6.8}$$

where

$$\begin{aligned}\widetilde{U}_{jk} &= L\delta_{jk}\frac{(ic)^2 2\pi\hat{R}(\mu_j)}{\omega_+^2(\mu_j)} Z^2(\mu_j,\mu_j) \\ &-i\frac{(ic)^2}{2}\delta_{jk}\frac{\omega_-(\mu_j)}{\omega_+(\mu_j)}\frac{\partial}{\partial\mu_j}\Big[e^{\frac{1}{2}(\psi(\mu_j)-\psi(\mu_k)+\tau(\mu_j)-\tau(\mu_k))} Z(\mu_j,\mu_k)Z(\mu_j,\mu_j) \\ &\qquad - e^{-\frac{1}{2}(\psi(\mu_j)-\psi(\mu_k)+\tau(\mu_j)-\tau(\mu_k))} Z(\mu_k,\mu_j)Z(\mu_k,\mu_k)\Big]_{\mu_k=\mu_j} \\ &-i\frac{(ic)^2}{2}\frac{1-\delta_{jk}}{\mu_j-\mu_k}\Big[e^{\frac{1}{2}(\psi(\mu_j)-\psi(\mu_k)+\tau(\mu_j)-\tau(\mu_k))} Z(\mu_j,\mu_k)Z(\mu_j,\mu_j)\frac{\omega_-(\mu_j)}{\omega_+(\mu_j)} \\ &\qquad - e^{-\frac{1}{2}(\psi(\mu_j)-\psi(\mu_k)+\tau(\mu_j)-\tau(\mu_k))} Z(\mu_k,\mu_j)Z(\mu_k,\mu_k)\frac{\omega_-(\mu_k)}{\omega_+(\mu_k)}\Big] \\ &+\frac{(ic)^2}{2\pi(\mu_j-\mu_k)} \fint_{-\infty}^{\infty} d\lambda \left(\frac{1}{\lambda-\mu_j} - \frac{1}{\lambda-\mu_k}\right) e^{\psi(\lambda)+\tau(\lambda)} \\ &\times e^{-\frac{1}{2}(\psi(\mu_j)+\tau(\mu_j)+\psi(\mu_k)+\tau(\mu_k))} Z(\lambda,\mu_k)Z(\lambda,\mu_j).\end{aligned} \tag{6.9}$$

$$\widetilde{Q}(\mu) = \frac{ic}{2\pi} \fint_{-\infty}^{\infty} \frac{d\lambda}{\lambda-\mu} e^{\psi(\lambda)+\tau(\lambda)} e^{-\frac{1}{2}(\psi(\mu)+\tau(\mu))} Z(\lambda,\mu)$$
$$+\frac{c}{2} e^{\frac{1}{2}(\psi(\mu)+\tau(\mu))} Z(\mu,\mu) \frac{\omega_-(\mu)}{\omega_+(\mu)}. \tag{6.10}$$

Let us simplify the formulæ (6.9) and (6.10). First, in (6.9) the term proportional to $1-\delta_{jk}$ is defined only for $j \neq k$. Let us continue this term for all $j$ and $k$ using the l'Hôpital's rule for $j=k$. Then

$$\widetilde{U}_{jk} = L\delta_{jk} \frac{(ic)^2 2\pi\rho_L(\mu_j)}{\omega_+^2(\mu_j)} Z^2(\mu_j,\mu_j)$$
$$-i\frac{(ic)^2}{2}\frac{1}{\mu_j-\mu_k}\Bigg[ e^{\frac{1}{2}(\psi(\mu_j)-\psi(\mu_k)+\tau(\mu_j)-\tau(\mu_k))} Z(\mu_j,\mu_k) Z(\mu_j,\mu_j) \frac{\omega_-(\mu_j)}{\omega_+(\mu_j)}$$
$$- e^{-\frac{1}{2}(\psi(\mu_j)-\psi(\mu_k)+\tau(\mu_j)-\tau(\mu_k))} Z(\mu_k,\mu_j) Z(\mu_k,\mu_k) \frac{\omega_-(\mu_k)}{\omega_+(\mu_k)} \Bigg]$$
$$+\frac{(ic)^2}{2\pi(\mu_j-\mu_k)} \fint_{-\infty}^{\infty} d\lambda \left(\frac{1}{\lambda-\mu_j} - \frac{1}{\lambda-\mu_k}\right) e^{\psi(\lambda)+\tau(\lambda)}$$
$$\times e^{-\frac{1}{2}(\psi(\mu_j)+\tau(\mu_j)+\psi(\mu_k)+\tau(\mu_k))} Z(\lambda,\mu_k) Z(\lambda,\mu_j), \tag{6.11}$$

where

$$2\pi\rho_L(\mu) = \left(1 + \frac{1}{L}\sum_{m=1}^{N} K(\mu,\mu_m)\right). \tag{6.12}$$

Using the Sokhodsky formula

$$\text{V.P.}\frac{1}{x} = \frac{1}{x \pm i0} \pm i\pi\delta(x),$$

one can rewrite the expressions (6.11) and (6.10) as follows

$$\widetilde{U}_{jk} = L\delta_{jk} \frac{(ic)^2 2\pi\rho_L(\mu_j)}{\omega_+^2(\mu_j)} Z^2(\mu_j,\mu_j)$$
$$+\frac{(ic)^2}{2\pi(\mu_j-\mu_k)} \int_{-\infty}^{\infty} \frac{d\lambda}{\omega_+(\lambda)} \left( \frac{e^{\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu_j+i0} + \frac{e^{-\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu_j-i0} - \frac{e^{\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu_k+i0} - \frac{e^{-\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu_k-i0} \right)$$
$$\times e^{\psi(\lambda)+\tau(\lambda)} e^{-\frac{1}{2}(\psi(\mu_j)+\tau(\mu_j)+\psi(\mu_k)+\tau(\mu_k))} Z(\lambda,\mu_k) Z(\lambda,\mu_j), \tag{6.13}$$

$$\widetilde{Q}(\mu) = \frac{ic}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\omega_+(\lambda)} \left( \frac{e^{\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu+i0} + \frac{e^{-\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu-i0} \right) e^{\psi(\lambda)+\tau(\lambda)} e^{-\frac{1}{2}(\psi(\mu)+\tau(\mu))} Z(\lambda,\mu). \tag{6.14}$$

Now let us move the term proportional to the length of the box $L$ out of determinant:

$$\det_N(\widetilde{U}_{jk} - \alpha\widetilde{Q}(\mu_j)\widetilde{Q}(\mu_k)) = \prod_{a=1}^{N} \left[ (ic)^2 2\pi\rho_L(\mu_a) L \left(\frac{Z(\mu_a,\mu_a)}{\omega_+(\mu_a)}\right)^2 \right]$$
$$\times \det_N \left( \frac{\widetilde{U}_{jk} - \alpha\widetilde{Q}(\mu_j)\widetilde{Q}(\mu_k)}{(ic)^2 2\pi\rho(\mu_k) L} \cdot \frac{\omega_+(\mu_j)\omega_+(\mu_k)}{Z(\mu_j,\mu_j) Z(\mu_k,\mu_k)} \right). \tag{6.15}$$

One can move the product $\prod_{a=1}^{N}\left(\frac{Z(\mu_a,\mu_a)}{\omega_+(\mu_a)}\right)^2$ to the left vacuum $(0|$ :

$$(0|\prod_{a=1}^{N} e^{p_0(\mu_a)}\left(\frac{Z(\mu_a,\mu_a)}{\omega_+(\mu_a)}\right)^2 = (0|\prod_{a=1}^{N} e^{p_0(\mu_a)}\left(\frac{e^{-p_{A_1}(\mu_a)}+e^{-p_{D_2}(\mu_a)}}{e^{p_2(\mu_a)/2}+e^{-p_2(\mu_a)/2}}\right)^2$$
$$\equiv (0|\prod_{a=1}^{N}\mathcal{P}(\mu_a). \tag{6.16}$$

Now let us move the product in the r.h.s. of (6.16) to the right vacuum. The only operator in the expressions for $\widetilde{U}$ and $\widetilde{Q}$ which does not commute with $\mathcal{P}(\mu_a)$ is $\psi(\lambda)=\phi_0(\lambda)+\phi_{A_1}(\lambda)+\phi_{D_2}(\lambda)+\phi_2(\lambda)$. In order to move $\prod\limits_{a=1}^{N}\mathcal{P}(\mu_a)$ through the determinant we use the following lemma.

**Lemma 6.1** *For arbitrary $M=1,2,\dots$ and arbitrary complex numbers $\lambda_1\dots,\lambda_M$, $\beta_1\dots,\beta_M$,:*

$$\mathcal{P}(\mu_a)\prod_{m=1}^{M} e^{\beta_m\psi(\lambda_m)}|0) = \prod_{m=1}^{M} e^{\beta_m\psi(\lambda_m)}|0) \tag{6.17}$$

*Proof.* The proof is straightforward:

$$\mathcal{P}(\mu_a)\prod_{m=1}^{M} e^{\beta_m\psi(\lambda_m)}|0) = e^{p_0(\mu_a)}\left(\frac{e^{-p_{A_1}(\mu_a)}+e^{-p_{D_2}(\mu_a)}}{e^{p_2(\mu_a)/2}+e^{-p_2(\mu_a)/2}}\right)^2\prod_{m=1}^{M} e^{\beta_m\psi(\lambda_m)}|0)$$

$$=\prod_{m=1}^{M} e^{\beta_m\psi(\lambda_m)}|0)\prod_{m=1}^{M}[h(\mu_a,\lambda_m)h(\lambda_m,\mu_a)]^{\beta_m}$$

$$\times\left(\frac{\prod\limits_{m=1}^{M}[h(\lambda_m,\mu_a)]^{-\beta_m}+\prod\limits_{m=1}^{M}[h(\mu_a,\lambda_m)]^{-\beta_m}}{\prod\limits_{m=1}^{M}\left[\frac{h(\lambda_m,\mu_a)}{h(\mu_a,\lambda_m)}\right]^{\beta_m/2}+\prod\limits_{m=1}^{M}\left[\frac{h(\mu_a,\lambda_m)}{h(\lambda_m,\mu_a)}\right]^{\beta_m/2}}\right)^2 = \prod_{m=1}^{M} e^{\beta_m\psi(\lambda_m)}|0).$$

This proves the lemma.

Since the determinant in r.h.s. of (6.15), being a function of operator $\psi$, is some linear combination of products of the type $\prod\limits_{m=1}^{M} e^{\beta_m\psi(\lambda_m)}$ (with different $M,\{\beta\}$ and $\{\lambda\}$), we can move $\prod_{a=1}^{N}\mathcal{P}(\mu_a)$ to the right vacuum without changing the matrix elements of the determinant (6.15). Therefore we have

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_N = \frac{e^{-iht}}{\det_N\frac{\partial\varphi_j}{\partial\mu_k}}$$
$$\times (0|\prod_{a=1}^{N}\Big(2\pi\rho_L(\mu_a)L\Big)\left(G_N(x,t)+\frac{\partial}{\partial\alpha}\right)\cdot\det_N\Big(W_{jk}+\mathcal{O}(1/L^2)\Big)|0)\Big|_{\alpha=0}, \tag{6.18}$$

where

$$W_{jk}=\delta_{jk}+\frac{1}{2\pi\rho_L(\mu_k)L}(V_{jk}-\alpha P(\mu_j)P(\mu_k)), \tag{6.19}$$

and

$$V_{jk}=\frac{\omega_+(\mu_j)\omega_+(\mu_k)}{2\pi(\mu_j-\mu_k)Z(\mu_j,\mu_j)Z(\mu_k,\mu_k)}$$
$$\times\int\limits_{-\infty}^{\infty}\frac{d\lambda}{\omega_+(\lambda)}\left(\frac{e^{\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu_j+i0}+\frac{e^{-\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu_j-i0}-\frac{e^{\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu_k+i0}-\frac{e^{-\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu_k-i0}\right)$$
$$\times e^{\psi(\lambda)+\tau(\lambda)}e^{-\frac{1}{2}(\psi(\mu_j)+\tau(\mu_j)+\psi(\mu_k)+\tau(\mu_k))}Z(\lambda,\mu_k)Z(\lambda,\mu_j), \qquad (6.20)$$

$$P(\mu)=\frac{\omega_+(\mu)}{2\pi Z(\mu,\mu)}\int\limits_{-\infty}^{\infty}\frac{d\lambda}{\omega_+(\lambda)}\left(\frac{e^{\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu+i0}+\frac{e^{-\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu-i0}\right)$$
$$\times e^{\psi(\lambda)+\tau(\lambda)}e^{-\frac{1}{2}(\psi(\mu)+\tau(\mu))}Z(\lambda,\mu). \qquad (6.21)$$

Recall that in the thermodynamic limit

$$\det{}_N\frac{\partial\varphi_j}{\partial\mu_k}\to\prod_{a=1}^{N}\Big(2\pi\rho(\mu_a)L\Big)\det(\hat{I}-\frac{1}{2\pi}\hat{K}),\qquad (\text{see (2.23)}), \qquad (6.22)$$

$$G_N(x,t)\to G(x,t)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}e^{\psi(\nu)+\tau(\nu)}\,d\nu, \qquad (6.23)$$

$$\rho_L(\lambda)\to\rho(\lambda),\qquad (\text{see (2.16)}),$$

The formula (6.18) contains the determinant of the matrix $W$. In the thermodynamic limit it will turn into a determinant of an integral operator. The simplest way to see this is to express $\det W$ in terms of traces of powers of the matrix $(V-\alpha PP)$. The replacement of summation by integration (in the limit) is straightforward and is explained in detail in Section XI.4 of [3]. Therefore the determinant tends to the Fredholm determinant. Now we arrive at the main theorem.

**Theorem 6.2** *In the thermodynamic limit, the time-dependent correlation function have the following Fredholm determinant formula.*

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle=e^{-iht}(0|\left(G(x,t)+\frac{\partial}{\partial\alpha}\right)\times\frac{\det\left(\hat{I}+\frac{1}{2\pi}\hat{V}_\alpha\right)}{\det\left(\hat{I}-\frac{1}{2\pi}\hat{K}\right)}|0)\Bigg|_{\alpha=0}. \qquad (6.24)$$

*Here the integral operator* $\hat{V}_\alpha$ *is given by*

$$\left(\hat{V}_\alpha f\right)(\lambda)=\int_{-q}^{q}\left(\hat{V}(\lambda,\mu)-\alpha\hat{P}(\mu)\hat{P}(\lambda)\right)f(\mu)d\mu, \qquad (6.25)$$

*where $q$ is the value of spectral parameter on the Fermi surface. Here the kernels $\hat{V}(\mu_1,\mu_2)$ and $\hat{P}(\mu)$ are given by*

$$\hat{V}(\mu_1,\mu_2) = \frac{\omega_+(\mu_1)\omega_+(\mu_2)}{2\pi(\mu_1-\mu_2)Z(\mu_1,\mu_1)Z(\mu_2,\mu_2)}$$
$$\times \int_{-\infty}^{\infty} \frac{d\lambda}{\omega_+(\lambda)} \left( \frac{e^{\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu_1+i0} + \frac{e^{-\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu_1-i0} - \frac{e^{\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu_2+i0} - \frac{e^{-\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu_2-i0} \right)$$
$$\times e^{\psi(\lambda)+\tau(\lambda)} e^{-\frac{1}{2}(\psi(\mu_1)+\tau(\mu_1)+\psi(\mu_2)+\tau(\mu_2))} Z(\lambda,\mu_2)Z(\lambda,\mu_1), \tag{6.26}$$

*and*

$$\hat{P}(\mu) = \frac{\omega_+(\mu)}{2\pi Z(\mu,\mu)} \int_{-\infty}^{\infty} \frac{d\lambda}{\omega_+(\lambda)} \left( \frac{e^{\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu+i0} + \frac{e^{-\frac{1}{2}\phi_1(\lambda)}}{\lambda-\mu-i0} \right)$$
$$\times e^{\psi(\lambda)+\tau(\lambda)} e^{-\frac{1}{2}(\psi(\mu)+\tau(\mu))} Z(\lambda,\mu), \tag{6.27}$$

*where*

$$\omega_+(\mu) = e^{\phi_1(\mu)/2} + e^{-\phi_1(\mu)/2},$$
$$Z(\lambda,\mu) = \frac{e^{-\phi_{D_1}(\lambda)}}{h(\mu,\lambda)} + \frac{e^{-\phi_{A_2}(\lambda)}}{h(\lambda,\mu)},$$
$$\tau(\lambda) = it\lambda^2 - ix\lambda.$$

*The integral operator $\hat{K}$ is given in* (2.24)*.*

**We want to emphasize that formula (6.24) is our main result.**

It is easy to show that it has the correct free fermionic limit. If $c \to +\infty$ (free fermionic case) then all commutators (5.2) of auxiliary "momenta" and "coordinates" go to zero because in this limit $h(\lambda,\mu) \to 1$. Hence one can put all dual fields $\phi_a(\lambda) = 0$. In particular

$$\psi(\lambda) = 0, \qquad \phi_1(\lambda) = 0,$$
$$\omega_+(\lambda) = 2, \qquad Z(\lambda,\mu) = 2.$$

whereby we have

$$\hat{V}(\mu_1,\mu_2) \overset{c\to\infty}{=} \frac{2}{\pi(\mu_1-\mu_2)} \fint_{-\infty}^{\infty} d\lambda \left( \frac{1}{\lambda-\mu_1} - \frac{1}{\lambda-\mu_2} \right) e^{\tau(\lambda)-\frac{1}{2}\tau(\mu_1)-\frac{1}{2}\tau(\mu_2)}, \tag{6.28}$$

$$\hat{P}(\mu) \overset{c\to\infty}{=} \frac{1}{\pi} \fint_{-\infty}^{\infty} d\lambda \frac{1}{\lambda-\mu} e^{\tau(\lambda)-\frac{1}{2}\tau(\mu)}. \tag{6.29}$$

$$\hat{K} \overset{c\to\infty}{=} 0, \tag{6.30}$$

$$G(x,t) \overset{c\to\infty}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu e^{\tau(\nu)}. \tag{6.31}$$

Substitution of these formulæ into (6.24) reproduces the result of [11]. In order to obtain Lenard's determinant formula [2] one should also put $t = 0$.

## Summary

The main result of the paper is formula (6.24). It represents the correlation function of local fields (in the infinite volume) as a mean value of a determinant of a Fredholm integral operator. In order to obtain this formula we introduced an auxiliary Fock space and auxiliary Bose fields (all of them belong to the same Abelian sub-algebra). This is the first step in description of the correlation function. In forthcoming publications we shall describe the Fredholm determinant by a completely integrable integro-differential equation. Then we shall solve this equation by means of the Riemann-Hilbert problem and evaluate its long-distance asymptotic.

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## A Summation of singular expressions

Let us consider equation (5.15)

$$L\lambda_n - i\phi_1(\lambda_n) = 2\pi n. \tag{A.1}$$

where $i\phi_1(\lambda)$ is a real and bounded function for $\operatorname{Im}\lambda = 0$. Let us introduce a function $\hat{\xi}(\lambda)$

$$\hat{\xi}(\lambda) = \lambda - \frac{i}{L}\phi_1(\lambda). \tag{A.2}$$

Obviously

$$\hat{\xi}(\lambda_n) = \frac{2\pi n}{L}. \tag{A.3}$$

Comparing with (5.20) we get

$$2\pi\hat{\rho}(\lambda) = 1 - \frac{i}{L}\phi_1'(\lambda) = \hat{\xi}'(\lambda). \tag{A.4}$$

It follows from equation (A.1) that

$$|\lambda_{n+1} - \lambda_n| \leq \frac{2}{L}(\pi + M),$$

where

$$M = \sup_{-\infty<\lambda<\infty} |\phi_1(\lambda)|.$$

Hence, $|\lambda_{n+1} - \lambda_n| \to 0$ if $L \to \infty$ and we can make the following estimate

$$\frac{1}{L(\lambda_{n+1} - \lambda_n)} = \hat{\rho}(\lambda_n) + \mathcal{O}(1/L^2). \tag{A.5}$$

Due to (5.18) we have $2\pi\hat{\rho}(\lambda) > 0$.

During study of thermodynamic limit the following sums appeared

$$S = \frac{1}{L}\sum_{n=-\infty}^{\infty} \frac{f(\lambda_n)}{2\pi\hat{\rho}(\lambda_n)(\lambda_n - \mu)}. \tag{A.6}$$

Here $f(\lambda)$ is some smooth function, $\mu$ is some fixed point on the real axis. We shall be interested in the asymptotic of this sum when $L$ goes to infinity.

Let us present (A.6) as the sum of three summands

$$\begin{aligned} S = \frac{1}{2\pi L}\Bigg(&\sum_{n=-\infty}^{N_1-1} \frac{f(\lambda_n)}{\hat{\rho}(\lambda_n)(\lambda_n - \mu)} + \\ &+ \sum_{n=N_1}^{N_2} \frac{f(\lambda_n)}{\hat{\rho}(\lambda_n)(\lambda_n - \mu)} + \sum_{n=N_2+1}^{\infty} \frac{f(\lambda_n)}{\hat{\rho}(\lambda_n)(\lambda_n - \mu)}\Bigg). \end{aligned} \tag{A.7}$$

Here $N_1$ and $N_2$ are integers such that in the limit $L \to \infty$, the following properties are valid

$$0 < \mu - \lambda_{N_1} < \infty, \qquad 0 < \lambda_{N_2} - \mu < \infty. \tag{A.8}$$

Obviously the first and the third summands in (A.7) have no singularities in the domain of summation. The corresponding sums are integral sums, for example

$$\begin{aligned} S_1 = \frac{1}{L}\sum_{n=-\infty}^{N_1-1} \frac{f(\lambda_n)}{2\pi\hat{\rho}(\lambda_n)(\lambda_n - \mu)} &= \sum_{n=-\infty}^{N_1-1} \frac{f(\lambda_n)(\lambda_{n+1} - \lambda_n)}{2\pi(\lambda_n - \mu)} + \mathcal{O}(1/L^2) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\lambda_{N_1}} \frac{f(\lambda)}{\lambda - \mu}\, d\lambda + \mathcal{O}(1/L). \end{aligned} \tag{A.9}$$

An analogous formula is valid for $S_3$

$$S_3 = \frac{1}{L} \sum_{n=N_2+1}^{\infty} \frac{f(\lambda_n)}{2\pi\hat{\rho}(\lambda_n)(\lambda_n - \mu)} = \frac{1}{2\pi} \int\limits_{\lambda_{N_2}}^{\infty} \frac{f(\lambda)}{\lambda - \mu}\, d\lambda + \mathcal{O}(1/L). \tag{A.10}$$

Consider the second summand in (A.7)

$$S_2 = \frac{1}{L} \sum_{n=N_1}^{N_2} \frac{f(\lambda_n)}{2\pi\hat{\rho}(\lambda_n)(\lambda_n - \mu)}.$$

One can present $S_2$ in the following form

$$S_2 = S_2^{(1)} + S_2^{(2)},$$

where

$$\begin{aligned} S_2^{(1)} &= \frac{1}{L} \sum_{n=N_1}^{N_2} \left( \frac{f(\lambda_n)}{2\pi\hat{\rho}(\lambda_n)(\lambda_n - \mu)} - \frac{f(\mu)}{\hat{\xi}(\lambda_n) - \hat{\xi}(\mu)} \right), \\ S_2^{(2)} &= \frac{f(\mu)}{L} \sum_{n=N_1}^{N_2} \frac{1}{\hat{\xi}(\lambda_n) - \hat{\xi}(\mu)}. \end{aligned} \tag{A.11}$$

Due to (A.4) $S_2^{(1)}$ has no singularities in the domain of summation. Therefore it can be replaced by the corresponding integral

$$S_2^{(1)} = \frac{1}{2\pi} \int\limits_{\lambda_{N_1}}^{\lambda_{N_2}} \left( \frac{f(\lambda)}{\lambda - \mu} - \frac{2\pi\hat{\rho}(\lambda) f(\mu)}{\hat{\xi}(\lambda) - \hat{\xi}(\mu)} \right) d\lambda + \mathcal{O}(1/L). \tag{A.12}$$

Using (A.3) we can rewrite $S_2^{(2)}$ in the following form

$$S_2^{(2)} = \frac{f(\mu)}{2\pi} \sum_{n=N_1}^{N_2} \frac{1}{n - \frac{L}{2\pi}\hat{\xi}(\mu)}.$$

The last sum can be calculated explicitly in terms of the logarithmic derivative of the $\Gamma$-function:

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x).$$

We shall use the following properties of the $\psi$-function

$$\psi(x) + \frac{1}{x} = \psi(x+1), \tag{A.13}$$

$$\psi(x) - \psi(1-x) = -\pi \cot \pi x, \tag{A.14}$$

$$\psi(x) \to \ln x + \mathcal{O}(1/x), \qquad x \to +\infty. \tag{A.15}$$

Now using (A.13) we can write

$$S_2^{(2)} = \frac{f(\mu)}{2\pi}\left[\psi(N_2 - \frac{L}{2\pi}\hat{\xi}(\mu) + 1) - \psi(N_1 - \frac{L}{2\pi}\hat{\xi}(\mu))\right]. \tag{A.16}$$

The argument of the second $\psi$-function in (A.16) is negative. Using (A.14) one can flip the sign of this argument

$$\begin{aligned} S_2^{(2)} &= \frac{f(\mu)}{2\pi}\Big[\psi(N_2 - \frac{L}{2\pi}\hat{\xi}(\mu) + 1) \\ -\psi(\frac{L}{2\pi}\hat{\xi}(\mu) &- N_1 + 1) - \pi\cot\left(\frac{L}{2}\hat{\xi}(\mu)\right)\Big]. \end{aligned} \tag{A.17}$$

Remember now that $0 < \mu - \lambda_{N_1} < \infty$ and $0 < \lambda_{N_2} - \mu < \infty$ (see (A.8)). This means that the arguments of $\psi$-functions in (A.17) tend to infinity if $L \to \infty$. Therefore we can use the asymptotic formula (A.15)

$$S_2^{(2)} = \frac{f(\mu)}{2\pi}\left[\ln\left(\frac{N_2 - \frac{L}{2\pi}\hat{\xi}(\mu)}{\frac{L}{2\pi}\hat{\xi}(\mu) - N_1}\right) - \pi\cot\left(\frac{L}{2}\hat{\xi}(\mu)\right)\right] + \mathcal{O}(1/L). \tag{A.18}$$

Now let us turn back to (A.12). Let us present the r.h.s. as the difference of two integrals. Both of them should be understood in the sense of principal value (V.P.):

$$S_2^{(1)} = \frac{1}{2\pi}\fint_{\lambda_{N_1}}^{\lambda_{N_2}}\frac{f(\lambda)}{\lambda-\mu}\,d\lambda - \fint_{\lambda_{N_1}}^{\lambda_{N_2}}\frac{f(\mu)\hat{\rho}(\lambda)}{\hat{\xi}(\lambda)-\hat{\xi}(\mu)}\,d\lambda + \mathcal{O}(1/L). \tag{A.19}$$

Due to (A.4) one can compute the second term in (A.19) explicitly

$$\begin{aligned} \fint_{\lambda_{N_1}}^{\lambda_{N_2}} d\lambda\,\frac{f(\mu)\hat{\rho}(\lambda)}{\hat{\xi}(\lambda)-\hat{\xi}(\mu)} &= \frac{f(\mu)}{2\pi}\fint_{\lambda_{N_1}}^{\lambda_{N_2}}\frac{d\hat{\xi}(\lambda)}{\hat{\xi}(\lambda)-\hat{\xi}(\mu)} = \\ \frac{f(\mu)}{2\pi}\ln\left(\frac{\hat{\xi}(\lambda_{N_2})-\hat{\xi}(\mu)}{\hat{\xi}(\mu)-\hat{\xi}(\lambda_{N_1})}\right) &= \frac{f(\mu)}{2\pi}\ln\left(\frac{N_2 - \frac{L}{2\pi}\hat{\xi}(\mu)}{\frac{L}{2\pi}\hat{\xi}(\mu) - N_1}\right). \end{aligned} \tag{A.20}$$

Combining now (A.18), (A.19) and (A.20) we get

$$S_2 = \frac{1}{2\pi}\fint_{\lambda_{N_1}}^{\lambda_{N_2}}\frac{f(\lambda)}{\lambda-\mu}\,d\lambda - \frac{f(\mu)}{2}\cot\frac{L}{2}\hat{\xi}(\mu) + \mathcal{O}(1/L).$$

Finally, using (A.9) and (A.10) we find

$$\begin{aligned} S &= \frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{f(\lambda_n)}{2\pi\hat{\rho}(\lambda_n)(\lambda_n-\mu)} \\ &= \frac{1}{2\pi}\fint_{-\infty}^{\infty}\frac{f(\lambda)}{\lambda-\mu}\,d\lambda - \frac{f(\mu)}{2}\cot\frac{L}{2}\hat{\xi}(\mu) + \mathcal{O}(1/L). \end{aligned} \tag{A.21}$$

This formula describes the asymptotic behavior of the sum (A.6).

For the evaluation of the thermodynamic limit of our determinant representation it is necessary to consider a sum containing the second order pole

$$S' = \frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{f(\lambda_n)}{2\pi\hat{\rho}(\lambda_n)(\lambda_n-\mu)^2}. \tag{A.22}$$

Taking the derivative of (A.21) with respect to $\mu$, we get

$$\begin{aligned} S' &= L\frac{2\pi\hat{\rho}(\mu)f(\mu)}{4\sin^2\frac{L}{2}\hat{\xi}(\mu)} \\ &+ \frac{1}{2\pi}\frac{\partial}{\partial\mu}\fint_{-\infty}^{\infty}\frac{f(\lambda)}{\lambda-\mu}\,d\lambda - \frac{1}{2}f'(\mu)\cot\frac{L}{2}\hat{\xi}(\mu) + \mathcal{O}(1/L). \end{aligned} \tag{A.23}$$

We can use formulæ (A.21) and (A.23) to calculate thermodynamic limit of (6.2)

$$\begin{aligned} &\frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{f(\lambda_n|\mu_j,\mu_k)}{2\pi\hat{\rho}(\lambda_n)(\lambda_n-\mu_j)(\lambda_n-\mu_k)} \\ &= -\frac{1}{2(\mu_j-\mu_k)}\left[f(\mu_j|\mu_j,\mu_k)\cot\frac{L}{2}\hat{\xi}(\mu_j) - f(\mu_k|\mu_j,\mu_k)\cot\frac{L}{2}\hat{\xi}(\mu_k)\right] \\ &+\frac{1}{2\pi(\mu_j-\mu_k)}\fint_{-\infty}^{\infty}d\lambda f(\lambda|\mu_j,\mu_k)\left(\frac{1}{\lambda-\mu_j}-\frac{1}{\lambda-\mu_k}\right) + \mathcal{O}(1/L). \end{aligned} \tag{A.24}$$

One should understand the r.h.s. of this equality by l'Hôpital's rule if $j = k$. It is also useful to extract explicitly the term proportional to the length of the box $L$

$$\begin{aligned} \frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{f(\lambda_n|\mu_j,\mu_k)}{2\pi\hat{\rho}(\lambda_n)(\lambda_n-\mu_j)(\lambda_n-\mu_k)} &= \delta_{jk}L\frac{2\pi\hat{\rho}(\mu_j)f(\mu_j|\mu_j,\mu_j)}{4\sin^2\frac{L}{2}\hat{\xi}(\mu_j)} \\ &+\frac{1}{2\pi(\mu_j-\mu_k)}\fint_{-\infty}^{\infty}d\lambda f(\lambda|\mu_j,\mu_k)\left(\frac{1}{\lambda-\mu_j}-\frac{1}{\lambda-\mu_k}\right) \\ &-\frac{1-\delta_{jk}}{2(\mu_j-\mu_k)}\left[f(\mu_j|\mu_j,\mu_k)\cot\frac{L}{2}\hat{\xi}(\mu_j) - f(\mu_k|\mu_j,\mu_k)\cot\frac{L}{2}\hat{\xi}(\mu_k)\right] \\ &-\frac{\delta_{jk}}{2}\cot\frac{L}{2}\hat{\xi}(\mu_j)\frac{\partial}{\partial\mu_j}\left[f(\mu_j|\mu_j,\mu_k) - f(\mu_k|\mu_j,\mu_k)\right]_{\mu_j=\mu_k} + \mathcal{O}(1/L). \end{aligned} \tag{A.25}$$

We have used formulæ (A.21) and (A.25) in Section 6.

# B Representation of dual fields

What is the relation between our dual fields and the canonical Bose fields. Canonical Bose fields $\psi_l(\lambda)$ can be characterized as follows

$$[\psi_l(\lambda), \psi_m^\dagger(\lambda)] = \delta_{lm}\delta(\lambda-\mu), \tag{B.1}$$

(do not confuse this $\psi_l(\lambda)$ with the dual field $\psi(\lambda)$) and

$$\psi_l(\lambda)|0) = 0, \qquad (0|\,\psi_l^\dagger(\lambda) = 0. \tag{B.2}$$

The dual fields which appeared in this paper have the form

$$\phi_a(\lambda) = q_a(\lambda) + p_a(\lambda), \tag{B.3}$$

where $p_a(\lambda)$ is the annihilation part of $\phi_a(\lambda)$ and $q_a(\lambda)$ is its creation part. Their commutation relations are

$$[p_a(\lambda), q_b(\mu)] = \alpha_{ab}(\lambda, \mu). \tag{B.4}$$

Here $\alpha_{ab}(\lambda, \mu)$ is some complex function.

$$p_a(\lambda)|0) = 0, \qquad (0|\,q_a(\lambda) = 0 \tag{B.5}$$

One can represent our $p_a$ and $q_b$ in terms of $\psi_l$ and $\psi_l^\dagger$, for example as

$$p_a(\lambda) = \psi_a(\lambda), \tag{B.6}$$

$$q_b(\mu) = \sum_c \int_{-\infty}^{\infty} d\nu \alpha_{cb}(\nu, \mu)\psi_c^\dagger(\nu). \tag{B.7}$$

This shows that the dual fields, which appear in this paper are linear combinations of the standard Bose fields.

Let us now consider a related issue. We can realize the dual fields $\phi_1(\lambda)$ and $\phi_2(\lambda)$ as

$$q_2(\lambda) = \psi_1^\dagger(\lambda), \qquad p_1(\lambda) = \psi_1(\lambda);$$
$$q_1(\lambda) = \int_{-\infty}^{\infty} \ln\frac{h(\nu,\lambda)}{h(\lambda,\nu)}\psi_2^\dagger(\nu)\,d\nu, \qquad p_2(\lambda) = \int_{-\infty}^{\infty} \ln\frac{h(\nu,\lambda)}{h(\lambda,\nu)}\psi_2(\nu)\,d\nu,$$

Here $\dagger$ means Hermitian conjugation, and

$$[\psi_1(\lambda), \psi_2^\dagger(\mu)] = [\psi_2(\lambda), \psi_1^\dagger(\mu)] = \delta(\lambda - \mu).$$

Other commutators are equal to zero. These commutation relations differ from (B.1) only by a trivial relabeling. Then

$$\phi_2(\lambda) = \psi_1^\dagger(\lambda) + \psi_1(\lambda), \qquad \phi_1(\lambda) = \int_{-\infty}^{\infty} \ln\frac{h(\nu,\lambda)}{h(\lambda,\nu)}(\psi_2^\dagger(\nu) + \psi_2(\nu))\,d\nu.$$

This means that $\phi_2(\lambda)$ and $i\phi_1(\lambda)$ are Hermitian operators:

$$\phi_2^\dagger(\lambda) = \phi_2(\lambda), \qquad (i\phi_1(\lambda))^\dagger = i\phi_1(\lambda), \qquad \text{for } \operatorname{Im}\lambda = 0 \tag{B.8}$$

After diagonalization they will turn into real functions.

# C Reduction of number of dual fields

We would like to reduce the number of dual fields in the determinant formula for the correlation function in (6.24). Here we shall show that

$$\phi_1(\lambda) = \phi_{D_1}(\lambda) - \phi_{A_2}(\lambda) \qquad \text{and} \qquad \phi_2(\lambda) = 0. \tag{C.1}$$

Recall the definition of the dual quantum fields (5.1) and (5.2)

$$\begin{array}{ll} \phi_0(\lambda) = q_0(\lambda) + p_0(\lambda); & \\ \phi_{A_j}(\lambda) = q_{A_j}(\lambda) + p_{D_j}(\lambda); & \phi_{D_j}(\lambda) = q_{D_j}(\lambda) + p_{A_j}(\lambda); \\ \phi_1(\lambda) = q_1(\lambda) + p_2(\lambda); & \phi_2(\lambda) = q_2(\lambda) + p_1(\lambda). \end{array} \tag{C.2}$$

$$\left\{ \begin{array}{ll} [p_0(\lambda), q_0(\mu)] = \ln(h(\lambda,\mu)h(\mu,\lambda)); & \\ [p_{D_j}(\lambda), q_{D_k}(\mu)] = \delta_{jk} \ln h(\lambda,\mu); & [p_{A_j}(\lambda), q_{A_k}(\mu)] = \delta_{jk} \ln h(\mu,\lambda); \\ [p_1(\lambda), q_1(\mu)] = \ln \dfrac{h(\lambda,\mu)}{h(\mu,\lambda)}; & [p_2(\lambda), q_2(\mu)] = \ln \dfrac{h(\mu,\lambda)}{h(\lambda,\mu)}. \end{array} \right. \tag{C.3}$$

Remember also the definition of the field $\psi(\lambda)$ (5.7)

$$\psi(\lambda) = \phi_0(\lambda) + \phi_{A_1}(\lambda) + \phi_{D_2}(\lambda) + \phi_2(\lambda).$$

Notice that $q_1(\lambda)$ and $p_2(\lambda)$ entering into $\phi_1(\lambda)$ do not commute only with $\psi(\mu)$:

$$\begin{aligned} [p_2(\lambda), \psi(\mu)] &= \ln \frac{h(\mu,\lambda)}{h(\lambda,\mu)}, \\ [\psi(\mu), q_1(\lambda)] &= \ln \frac{h(\mu,\lambda)}{h(\lambda,\mu)}. \end{aligned}$$

On the other hand $q_{D_1}(\lambda) - q_{A_2}(\lambda)$ and $p_{A_1}(\lambda) - p_{D_2}(\lambda)$ entering into $\phi_{D_1}(\lambda) - \phi_{A_2}(\lambda)$ also do not commute only with $\psi(\mu)$:

$$\begin{aligned} [(p_{A_1}(\lambda) - p_{D_2}(\lambda)), \psi(\mu)] &= \ln \frac{h(\mu,\lambda)}{h(\lambda,\mu)}, \\ [\psi(\mu), (q_{D_1}(\lambda) - q_{A_2}(\lambda))] &= \ln \frac{h(\mu,\lambda)}{h(\lambda,\mu)}, \end{aligned}$$

so we can identify

$$\phi_1(\lambda) = \phi_{D_1}(\lambda) - \phi_{A_2}(\lambda). \tag{C.4}$$

Then we can also put $\phi_2(\lambda) = q_2(\lambda) + p_1(\lambda) = 0$ because after the replacement (C.4), operators $q_2(\lambda)$ and $p_1(\lambda)$ commute with everything.

Such a replacement implies

$$\omega_+(\lambda) = e^{\frac{1}{2}(\phi_{D_1}(\lambda)+\phi_{A_2}(\lambda))} Z(\lambda,\lambda). \tag{C.5}$$

It means, that Fredholm determinant in (6.24) realy depends on three dual fields $\psi(\lambda)$, $\phi_{A_2}(\lambda)$ and $\phi_{D_1}(\lambda)$. We shall use this fact in our next publications.

# D Thermodynamics

The thermodynamics of the quantum nonlinear Schrödinger equation was described by C. N. Yang and C. P. Yang [13]. It involves few equations. The central equation is for an energy of excitation $\varepsilon(\lambda)$:

$$\varepsilon(\lambda) = \lambda^2 - h - \frac{T}{2\pi}\int_{-\infty}^{\infty} \frac{2c}{c^2+(\lambda-\mu)^2} \ln\left(1+e^{-\frac{\varepsilon(\mu)}{T}}\right) d\mu. \tag{D.1}$$

Other important functions are the local density (in momentum space) of particles $\rho_p(\lambda)$ and the total local density $\rho_t(\lambda)$ (it includes particles and holes). They satisfy equations:

$$2\pi\rho_t(\lambda) = 1 + \int_{-\infty}^{\infty} \frac{2c}{c^2+(\lambda-\mu)^2}\rho_p(\mu)\, d\mu, \tag{D.2}$$

$$\frac{\rho_p(\lambda)}{\rho_t(\lambda)} = \left(1+e^{\frac{\varepsilon(\lambda)}{T}}\right)^{-1} \equiv \vartheta(\lambda). \tag{D.3}$$

The global density $D = N/L$ can be represented as

$$D = \int_{-\infty}^{\infty} \rho_p(\lambda)\, d\lambda. \tag{D.4}$$

In order to obtain the determinant representation of the temperature correlation function we can use the following representation

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_T \equiv \frac{\mathrm{tr}\left(e^{-\frac{H}{T}}\psi(0,0)\psi^\dagger(x,t)\right)}{\mathrm{tr}\, e^{-\frac{H}{T}}} = \frac{\langle\Omega_T|\psi(0,0)\psi^\dagger(x,t)|\Omega_T\rangle}{\langle\Omega_T|\Omega_T\rangle}. \tag{D.5}$$

Here $|\Omega_T\rangle$ is one of eigenfunctions of the Hamiltonian, which is present in the state of thermo-equilibrium. It is proven in Section I.8 of [3] that the r.h.s. of (D.5) does not depend on the particular choice of $|\Omega_T\rangle$.

Now we have to recalculate thermodynamic limit. First we shall return to the determinant representation of the correlation function in a finite volume, see formulæ (6.18)–(6.21). We should also notice that the thermodynamic limit of square of the norm should be changed (comparing to (6.22)) as follows:

$$\det{}_N \frac{\partial \varphi_j}{\partial \mu_k} \to \prod_{a=1}^{N} \Big(2\pi \rho_t(\mu_a) L\Big) \det(\hat{I} - \frac{1}{2\pi}\hat{K}_T), \tag{D.6}$$

see Section X.4 of [3]. Here $\mu_j$ correspond to $|\Omega_T\rangle$. In the thermodynamic limit summation with respect to indicis $j$ and $k$ in (6.18) will be replaced by integration $\rho_p(\mu_k)\, d\mu_k$. Also $\rho_L(\mu)$ defined in (6.12) goes to $\rho_t$:

$$\rho_L(\mu) \to \rho_t(\mu).$$

After dividing $\rho_p(\mu_k)\, d\mu_k$ by $\rho_t(\mu_k)$ which appears in the denominator (6.19) we shall obtain an integration $\int\limits_{-\infty}^{\infty} \vartheta(\lambda)\, d\lambda(\cdot)$ insted of $\int\limits_{-q}^{q} d\lambda(\cdot)$. The details of these calculations are explained in Section XI.5 of [3].

The integral operator $\hat{K}_T$ can be defined by its kernel

$$K_T(\mu_1, \mu_2) = \left(\frac{2c}{c^2 + (\mu_1 - \mu_2)^2}\right) \sqrt{\vartheta(\mu_1)}\sqrt{\vartheta(\mu_2)}. \tag{D.7}$$

Now let us formulate the final formula for the representation of the temperature correlation function of local fields in the thermodynamic limit

$$\langle \psi(0,0)\psi^\dagger(x,t)\rangle = e^{-iht} (0| \left(G(x,t) + \frac{\partial}{\partial \alpha}\right) \times \left.\frac{\det\left(\hat{I} + \frac{1}{2\pi}\hat{V}_{\alpha,T}\right)}{\det\left(\hat{I} - \frac{1}{2\pi}\hat{K}\right)} |0)\right|_{\alpha=0}. \tag{D.8}$$

Here the integral operator $\hat{V}_{\alpha,T}$ is given by

$$\left(\hat{V}_{\alpha,T} f\right)(\lambda) = \int_{-\infty}^{\infty} \left(\hat{V}_T(\lambda,\mu) - \alpha \hat{P}_T(\mu)\hat{P}_T(\lambda)\right) f(\mu) d\mu. \tag{D.9}$$

Here the kernel of $\frac{1}{2\pi}\Big(\hat{V}_T(\mu_1,\mu_2) - \alpha\hat{P}_T(\mu_1)\hat{P}_T(\mu_2)\Big)$ differs from the zero temperature case by the measure and limits of integration:

$$\hat{V}_T(\mu_1,\mu_2) - \alpha \hat{P}_T(\mu_1)\hat{P}_T(\mu_2) = \left(\hat{V}(\mu_1,\mu_2) - \alpha\hat{P}(\mu_1)\hat{P}(\mu_2)\right)\sqrt{\vartheta(\mu_1)}\sqrt{\vartheta(\mu_2)}. \tag{D.10}$$

It acts on the whole real axis $-\infty < \mu < \infty$. Here $\hat{V}(\mu_1,\mu_2)$ is given exactly by (6.26), $\hat{P}(\mu)$ is given by formula (6.27) and $G(x,t)$ is given by (6.23).

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