Perturbation theory for the Fermi liquids in $d > 1$.

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Abstract: We develop a momentum space renormalization group formalism for the perturbative study of interacting Fermi systems in spatial dimensions greater than 1. We first prove improved bounds with respect to power counting for some Feynman graphs. These bounds show that among all graphs with 4 external legs, only some special ones, called direct and exchange graphs diverge, and they do so only at Cooper pairs or forward scattering configurations of external momenta. Using these bounds, we propose a novel renormalization scheme for the Fermi liquids, based on the physical idea of renormalizing only momentum configurations close (in a precise sense) to these.

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1 Introduction

We are concerned in this paper with the perturbation theory for a system of *spinless* fermions in d spatial dimensions at zero temperature interacting through a rotational invariant pair potential $\lambda_0(\vec{x}-\vec{y})$, which we will study in the grand-canonical ensemble. The (grand-canonical) Hamiltonian for the non-interacting system is $H_0 = T - \mu N$, where $T = \frac{1}{2m}\int d\vec{x}\, \vec{\partial}\psi^+_{\vec{x}} \cdot \vec{\partial}\psi^-_{\vec{x}}$ is the total kinetic energy, μ is the chemical potential and $N = \int d\vec{x}\, \psi^+_{\vec{x}}\psi^-_{\vec{x}}$ is the number of fermions operator. In the above formulae $\psi^{\pm}_{\vec{x}}$ are respectively creation and annihilation operators for fermions at position $\vec{x}$. The interaction part to be summed to H_0 is $\overline{\nu}_0 N + V$, where V, given by $V = \frac{1}{2}\int d\vec{x}\, d\vec{y}\, \lambda_0(\vec{x}-\vec{y})\ \psi^+_{\vec{x}}\psi^+_{\vec{y}}\psi^-_{\vec{y}}\psi^-_{\vec{x}}$, is the pair potential interaction and $\overline{\nu}_0$ is the chemical potential counterterm. The Fermi momentum p_F is given by $p_F = \sqrt{2m\mu}$.

Our long term objective is to understand in a mathematically rigorous way at any finite order in perturbation theory whether these systems are *normal* Fermi liquids in the sense of Landau theory, [1], or not. We present in this paper a new renormalization group formalism with physical ideas simplifying some aspects of the problem for $d > 1$. We hope to explore in forthcoming papers these simplifications.

Our formalism is an extension of the one invented by Benfatto and Gallavotti in [2], [3] to any spatial dimension d (theirs works only for odd values of d) incorporating some ideas already present in a non-rigorous work by Shankar [4], which shows that power counting bounds may be improved as a consequence of the different geometry of renormalizing around a surface, instead of around a point. In the rest of this introduction we present briefly our results and compare them to the works of Benfatto and Gallavotti and Shankar.

In section 2 we make our version of the formalism of Benfatto and Gallavotti, introducing a multiscale decomposition of the free-fermion propagator which, like theirs, has quasiparticles present from the beginning. Our multiscale decomposition has the desired properties only in momentum space, so we work in this space. Also in section 2 we present power counting bounds for the Feynman graphs.

Section 3 presents the main technical results of this work, showing improved bounds for some Feynman graphs with respect to power counting. Without these improvements, the novel renormalization scheme of section 4 would not be possible. First we state and show theorem 1, which exploits integration over angular variables on the Fermi surface to improve power counting for direct and exchange graphs, showing that they diverge only in some special configurations: Cooper pairs for direct and forward scattering for exchange graphs. We continue then with theorem 5, which shows improved bounds for other graphs. In particular, it shows that all the other 4-legged graphs which are neither direct, nor exchange do not diverge in any momentum configuration.

We devise in section 4 a new *momentum-dependent* and *graph-dependent* renormalization scheme which takes advantage of the improvements with respect to power counting to remove all divergences of the theory. This is another difference with respect to [2], which rely on power counting to define relevant and irrelevant couplings. The main idea behind our scheme is to renormalize *only* really divergent graphs, *only* at the momentum configurations in which they diverge. We see then that many of the terms Benfatto and Gallavotti thought were relevant are in fact irrelevant.

This fact was also noted by Shankar in [4], but in his non-rigorous approach he neglects all irrelevant terms, thus obtaining an oversimplified beta functional, too easy to analyze. On the contrary, the beta functional in the works of Benfatto and Gallavotti for $d = 3$ is too complicated. They are forced to make conjectures in order to simplify their beta functional and conclude, under these conjectures, that Fermi liquids in $d = 3$ are normal if the interaction is repulsive.

The complication of their beta functional comes from two sources. The first is the infinite number of running couplings, consequence of the fact that if $d > 1$ the Fermi surface consists of infinite points. This was overcome in Shankar's approach by neglecting the irrelevant terms. The second source of complication is the inclusion in the beta functional of unnecessary terms coming from graphs which seem relevant by power counting, but are in fact irrelevant if one uses better bounds.

In section 5 we give a definition of normal Fermi liquid and study the beta functional. Our beta functional is in some sense intermediate between the ones of Shankar and Benfatto and Gallavotti. For example, we also have the same problem of an infinite number of running couplings of the latter, but our renormalization scheme incorporating Shankar's ideas has introduced some simplifications:

1. Like in [2], we can also, by an angular momentum expansion, diagonalize the beta functional for the 4-legged couplings, not only to second order in perturbation theory, but to any order. The difference is due to non-inclusion in our beta functional of some graphs irrelevant by theorem 5 which begin to appear in [2] at third order.

2. Benfatto and Gallavotti neglect all 4-legged couplings, except Cooper pairs, without justifying it. Our renormalization scheme provides a factor strongly depleting non-Cooper-pairs and non-forward-scattering configurations, see (40).

3. When considering only the Cooper pairs configurations at second order, Benfatto and Gallavotti *neglect* the contribution of the exchange graph, showing that it is small. In our formalism we are able to cope with the exchange graphs and show that they give important contributions at *forward scattering* configurations. In order to separate in an efficient way contributions from direct and exchange graphs, we define two coupling functions: λ for direct graphs and μ for exchange ones. This is also in line with Shankar.

In spite of the simplifications, our beta functional is still very complicated, even at only second order, and a completely rigorous analysis of the flow of the running couplings has to be postponed to further work, where important questions to be addressed are the possibility of anomalous Fermi liquids [6] or even BCS superconductivity for repulsive interactions in $d = 3$ as suggested by Kohn and Luttinger, [7].

The case of non-spherical Fermi surface is technically more difficult, but should be similar to the spherical case we treat, as long as the Fermi surface is non-nested. In the same cited work Shankar extends to some of his results to this case.

Also interesting and more difficult is the case of attractive interactions, which are believed to produce BCS superconductivity. Attractive interactions are being rigorously studied by Feldman *et al* with different renormalization group methods and interesting results [8], [9], [10].

This same group has also implemented rigorously the same improvements beyond power counting of this paper in the more general case of non-spherical, non-nested Fermi surfaces [11]. With these bounds, they can also prove perturbative renormalizability of the theory, but they cannot answer yet about normality or anomaly. There are other interesting results of theirs, [12], showing that for the repulsive delta function interaction in $d = 2$ and spherical Fermi surface an instability occurs at third order in perturbation theory, driving the system to a superconducting state.

2 Multiscale decomposition and power counting

The free-fermion propagator is given by [2]

$$g(x-y) = \int \frac{dk}{(2\pi)^{d+1}} \frac{e^{ik_0(x_0-y_0)+i\vec{k}\cdot(\vec{x}-\vec{y})}}{ik_0 + e(\vec{k})} , \tag{1}$$

where $e(\vec{k}) = (\vec{k}^2 - p_F^2)/2m$. Due to its (infrared) singularity at the Fermi surface $k_0 = 0$, $|\vec{k}| = p_F$, some Feynman graphs may be divergent. More exactly, if G is some graph with $2n$ external legs, then its contribution to the effective potential is of the form

$$\int dk_1 \dots dk_{2n}\, \delta(k_1+\dots+k_n-k_{n+1}-\dots-k_{2n})\, f(k_1,\dots,k_{2n})\, \psi^+_{k_1}\dots\psi^+_{k_n}\psi^-_{k_{n+1}}\dots\psi^-_{k_{2n}} ,$$

where $k_1, \dots, k_{2n}$ are the momenta of the external legs and the function f, called *form factor* of the graph, can be infinite due to the singularity of the propagator on the Fermi surface. The propagator is also singular in the ultraviolet, but we will apply an UV cut-off in order to eliminate this technical problem which is expected to have nothing to do with the Fermi surface in the interacting case.

The UV-cut-off propagator can be written in momentum space as

$$
\begin{aligned}
g_{uv}(k) &= \int_{1/4}^{\infty} d\alpha\,[-ik_0 + e(\vec{k})]\,e^{-\alpha(k_0^2+e(\vec{k})^2)} \\
&= \sum_{n=-\infty}^{0} \int_{2^{-2n-2}}^{2^{-2n}} d\alpha\,[-ik_0 + e(\vec{k})]\,e^{-\alpha(k_0^2+e(\vec{k})^2)} \equiv \sum_{n=-\infty}^{0} \hat{g}_n(k)\;. \qquad (2)
\end{aligned}
$$

As each $\hat{g}_n(k)$ is peaked around $|\vec{k}| = p_F$, then $\hat{g}_n(x)$ oscillates spatially with period p_F^{-1}, independently on n as explicitly seen in the formulae for $d = 1, 3$ in [2]. In order to have a good multiscale decomposition we further decompose the propagator over the Fermi surface, introducing *quasiparticle* fields similar to those of [2].

By some calculations we get to

$$
g_{uv}(x) = \sum_{n=-\infty}^{0} \int \frac{dk d\vec{\omega}}{(2\pi)^{d+1}}\, 2^{-n}\, g_n(2^{-n}k)\, e^{i[k_0 t + (k_1+p_F)\vec{\omega}\cdot\vec{x}]}\;, \qquad (3)
$$

with

$$
\begin{aligned}
g_n(k) = \Omega_d p_F^{d-1} \int_{1/4}^{1} & d\alpha\,\theta\left(k_1 + 2^{-n}p_F\right)\left[-ik_0 + \beta k_1(1 + \frac{2^{n-1}}{p_F}k_1)\right] \\
& (1 + \frac{2^n}{p_F}k_1)^{d-1} e^{-\alpha\left[k_0^2 + \beta^2 k_1^2\left(1+\frac{2^{n-1}}{p_F}k_1\right)^2\right]}\;, \qquad (4)
\end{aligned}
$$

being $\beta = p_F/m$ the Fermi velocity, Ω_d the surface of the d-dimensional unit sphere, integration in k means integration over real variables k_0 and k_1, $\vec{\omega}$ is a unitary vector in d dimensions, integration over $\vec{\omega}$ means averaging over the unit sphere and θ is the step function.

Notice from (3) that the $\vec{\omega}$-variables above introduced act like the direction of a momentum $(k_1 + p_F)\vec{\omega}$, which is near the Fermi sphere if $|k_1| \ll p_F$. So, if we decompose our original *particle* fields ψ_x as a sum

$$
\psi_x^{\pm} = \sum_{n=-\infty}^{0} \int \frac{dk d\vec{\omega}}{(2\pi)^{d+1}}\, \psi_{k,\vec{\omega}}^{\pm(n)}\, e^{\pm i[k_0 t + (k_1+p_F)\vec{\omega}\cdot\vec{x}]}\theta(k_1 + p_F)\;, \qquad (5)
$$

of new independent Grassmann fields $\psi_{k,\vec{\omega}}^{(n)}$ distributed with covariance $\langle \psi_{k,\vec{\omega}}^{-(n)} \psi_{k',\vec{\omega}'}^{+(n)} \rangle = \delta(\vec{\omega} - \vec{\omega}')\,\delta(2^{-n}(k - k'))\,2^{-3n} g_n((2^{-n}k))$, then we can interpret the $\psi_{k,\vec{\omega}}^{(n)}$ as *quasiparticle* fields representing momentum scale 2^n, $n \leq 0$, *around* the Fermi surface. As $g_n(k)$ asymptotically depends weakly on n, then all fields $\psi_{k,\vec{\omega}}^{(n)}$ are distributed, up to a trivial scaling factor, roughly the same, as required by renormalization group methods.

Differently from the analogous decomposition in [2], we cannot interpret the fields

$$\psi^{\pm(n)}_{x,\vec{\omega}} = \int \frac{dk}{(2\pi)^{d+1}} \, \psi^{\pm(n)}_{k,\vec{\omega}} \, e^{\pm i[k_0 t + k_1 \vec{\omega}\cdot\vec{x}]} \, \theta(k_1 + p_F)$$

as quasiparticle fields in position space at length scale 2^{-n}, for their covariance does not decay appropriately due to the θ-function singularity in (4). This fact shows that our decomposition (5) seems not to be suitable for renormalization group methods in position space.

The multiscale decomposition of the fields induces a factorization of the free grassmanian measure as $P_0(d\psi) = \prod_{n=-\infty}^{0} P_n(d\psi^{(n)})$, where $P_n(d\psi^{(n)})$ naturally means the grassmanian measure with covariance $\delta(\vec{\omega}-\vec{\omega}')\,\delta(2^{-n}(k-k'))\,2^{-3n} g_n((2^{-n}k))$.

We may now define the effective potential at scale h (more exactly at momentum scale 2^h, $h \leq 0$), as

$$V^{(h)}(\psi^{(\leq h)}) = -\log \int \prod_{n=h+1}^{0} P_n(d\psi^{(n)}) \, e^{-V^{(0)}(\psi^{(\leq h)}+\psi^{(h+1)}+\cdots+\psi^{(0)})} \, , \tag{6}$$

where $V^{(0)}$ is the interaction Hamiltonian rewritten in momentum space and Wick-ordered second-quantized form as

$$\begin{aligned} V^{(0)} &= \nu_0 \int dp_1 dp_2 \, \Big[: \psi^{+}_{k_1,\vec{\omega}_1} \psi^{-}_{k_2,\vec{\omega}_2} : \delta(p_1 - p_2) \theta(k_1^1 + p_F) \theta(k_2^1 + p_F) \Big] \\ &- \int dp_1 \ldots dp_4 \, [\lambda_0(k_1, \ldots, k_4, \vec{\omega}_1, \ldots, \vec{\omega}_4) : \psi^{+}_{k_1,\vec{\omega}_1} \psi^{+}_{k_2,\vec{\omega}_2} \psi^{-}_{k_3,\vec{\omega}_3} \psi^{-}_{k_4,\vec{\omega}_4} : \\ &\quad \delta(p_1 + p_2 - p_3 - p_4) \theta(k_1^1 + p_F) \ldots \theta(k_4^1 + p_F)] \, , \end{aligned} \tag{7}$$

with

$$\begin{aligned} \lambda_0(k_1^1, \ldots, k_4^1, \vec{\omega}_1, \ldots, \vec{\omega}_4) &= \frac{1}{4} [\hat{\lambda}_0(\vec{p}_1 - \vec{p}_4) - \hat{\lambda}_0(\vec{p}_1 - \vec{p}_3) \\ + \; \hat{\lambda}_0(\vec{p}_2 - \vec{p}_3) - \hat{\lambda}_0(\vec{p}_1 - \vec{p}_3)] \end{aligned} \tag{8}$$

In these formulae ν_0 can be easily obtained from $\overline{\nu}_0$, $\hat{\lambda}_0$ is the Fourier transform of the pair potential λ_0 and we use the abbreviation $\vec{p}_i$ for the spatial part of the momentum i, i.e. $(k_i^1 + p_F)\vec{\omega}_i$, p_i for $(k_0^i, \vec{p}_i)$ and dp_i for $dk_i d\vec{\omega}_i$.

In order to generate a formal perturbative expansion for the effective potentials $V^{(h)}$, one may use the algorithm of expansion in Gallavotti-Nicoló trees [13]. In the resulting scale-decomposed Feynman graphs there are two types of internal lines: *hard* and *soft*, see appendix C in [13]. To each internal line of frequency h_v in a scale decomposed graph is associated a propagator, which is $2^{-h_v} g_{h_v}(2^{-h}k)$ if the line is hard, or $2^{-h_v} g^s_{h_v}(2^{-h}k)$ if the line is soft, where

$$2^{-n} g^s_n(2^{-n}k) = \sum_{h=-\infty}^{n-1} 2^{-h} g_h(2^{-h}k) \, . \tag{9}$$

An important remark is that the function g_n^s also decays exponentially at infinity and, in spite of its singularity at $k = 0$, possesses an L_1-norm smaller than an n-independent constant. In the rest of this paper we will ignore soft lines, but the reader can verify that their above mentioned properties are sufficient to guarantee that all bounds that follow are unchanged if they are present.

Power counting arguments in momentum space similar to the position space ones, [2], can be used to bound the form factor of any graph. Let G be a scale decomposed Feynman graph of perturbative order m, originated by a certain tree T, with n^e external legs. Let v denote a cluster of T having frequency h_v. We will denote v' the cluster of T preceding v in the tree order, *i.e.* the first cluster of T we find if we descend from v towards the root. Let also v_0 denote the first non-trivial cluster of T, *i.e.*, the first branching point of T if we climb it beginning from the root. If we let $f_{h_{v_0}}$ denote the form factor of G, then its power counting bound is

$$|f_{h_{v_0}}(x_1,\ldots,x_{n^e},\vec{\omega}_1,\ldots,\vec{\omega}_{n^e})| \;<\; C^m\, M^m\, 2^{-h_{v'_0}\delta_{v_0}} \prod_{v\le v_0} 2^{-(h_v-h_{v'})\delta_v}\;, \qquad (10)$$

where

$$\delta_v \;=\; -2 \;+\; m_{2,v} \;+\; n_v^e/2 \qquad (11)$$

is called *dimension of the cluster* v, C and M are constants which do not depend neither on the graph nor on the frequencies attached to its clusters, $m_{2,v}$ is the number of vertices of type ν_0 contained in cluster v but not contained in any of its subclusters and n_v^e is the number of lines external to the cluster v. The complete proof of this bound can be found in [15].

As for each term of the product we have $h_v > h_{v'}$, then a graph having $\delta_v > 0$ for all its clusters v converges [3] and the bound for a graph in which some cluster v has $\delta_v \le 0$ diverges. Power counting is thus sufficient to guarantee that some classes of graphs converge. For example, any graph in which all clusters have 6 or more legs is convergent. On the other hand, we see that clusters with 2 or 4 legs may have non-positive dimension. Clusters with non-positive dimension will be either given an improved finite estimate, or renormalized.

3 Improvements beyond power counting

Up to now all we have done is equally valid for the Fermi liquids in $d = 1$ and $d > 1$, power counting being used also in bosonic problems to identify relevant and irrelevant couplings. In this section we are going to exploit the different "geometry" of the $d > 1$ case to prove the main technical results in this work, on which the renormalization scheme of section 4 is based.

[3] Of course the dimensional factor $2^{-h_{v'_0}\delta_{v_0}}$ diverges if $\delta_{v_0} < 0$, but the contribution to the *dimensionless* effective potential is convergent.

Consider the graph of figure 1, for which we have a divergent power counting bound. Its form factor is given by

$$f_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) = \int \frac{dk_5 d\vec{\omega}_5}{(2\pi)^{d+1}} \lambda_0(1,2,5,6)\,\lambda_0(5,6,3,4)$$
$$2^{-h} g_h(2^{-h}k_5)\, 2^{-h}\overline{g}_h(2^{-h}k_6)\;, \qquad (12)$$

where $\lambda_0(i_1,\ldots,i_4)$ is a short notation for $\lambda_0(\vec{p}_{i_1},\ldots,\vec{p}_{i_4})$ and the variables k_6 and $\vec{\omega}_6$ are to be considered as functions of momenta 1,2 and 5 given by

$$\begin{aligned} k_6^0 &= k_1^0 + k_2^0 - k_5^0 \\ k_6^1 &= |(k_1^1+p_F)\vec{\omega}_1 + (k_2^1+p_F)\vec{\omega}_2 - (k_5^1+p_F)\vec{\omega}_5| - p_F \\ \vec{\omega}_6 &= \frac{(k_1^1+p_F)\vec{\omega}_1 + (k_2^1+p_F)\vec{\omega}_2 - (k_5^1+p_F)\vec{\omega}_5}{k_6^1+p_F} \end{aligned} \qquad (13)$$

because of the integration of one momentum-conservation δ-function. Integration of this δ-function also changes the propagator of line 6 to $2^{-h}\overline{g}_h(2^{-h}k)$, where the new function $\overline{g}$ is defined as

$$\overline{g}_n(k) = \int_{1/4}^{1} d\alpha \left[-ik^0 + \beta k^1 (1 + \frac{2^{n-1}k^1}{p_F}) \right] e^{-\alpha\left[(k^0)^2 + \beta^2 (k^1)^2 \left(1+\frac{2^{n-1}k^1}{p_F}\right)^2 \right]}\;, \qquad (14)$$

and is similar to $g_n(k)$. In a general graph there is a *spanning tree*, [2], of lines having $\overline{g}$-propagators, which are all bounded by constants in order to obtain power counting. If we exploit the $\vec{\omega}$ dependence of the propagator we can obtain better bounds. Before stating the theorem which proves it, we introduce some definitions necessary for its full formulation.

A graph is called *direct* if it has 4 external legs, if these external legs originate as pairs from only 2 vertices, if both external legs at the same vertex have arrows pointing in the same direction and if it has no loops sharing an internal line. In figure 2 we show some examples of direct graphs.

In figure 3 we show some 4-legged graphs which are not direct.

A graph is called an *exchange graph*, see figure 4 for the prototype of them, if it has 4 external legs, if these legs originate as pairs from only 2 vertices, if the external legs originating from the same vertex have arrows pointing in opposite directions and if it has no loops sharing an internal line. Other examples may be obtained from direct graphs by reversing some arrows.

If G is a direct graph let us number as 1 and 2 the external lines exiting from one of the two vertices of G from which external lines exit, see figure 1. We define some important parameters:

$$\begin{aligned} \Gamma_0 &= k_1^0 + k_2^0 \\ \vec{\Gamma} &= (k_1^1+p_F)\vec{\omega}_1 + (k_2^1+p_F)\vec{\omega}_2 \\ \Gamma &= |\Gamma_0| + |\vec{\Gamma}|\;. \end{aligned} \qquad (15)$$

Analogously for exchange graphs: if 1 and 3 are external lines exiting from the same vertex, see figure 4, we define

$$\Delta \;=\; |\Delta_0| + |\vec{\Delta}| \;, \tag{16}$$

where $\Delta_0 = k_1^0 - k_3^0$ and $\vec{\Delta} = (k_1^1 + p_F)\vec{\omega}_1 - (k_3^1 + p_F)\vec{\omega}_3$.

We can now state one of our main results:

Theorem 1 *Let G be a direct graph with all its internal lines having frequency h, f_h be its form factor, δ its dimension (11) and Γ be defined as in (15). Then, if $d > 1$, given any constant ρ such that $0 < \rho < 2p_F$, there exist constants C, C' and N_0 dependent on ρ, but independent of G, Γ and h such that:*

1. *For $2^h N_0 < \Gamma < 2p_F - \rho$ we have*
$$|f_h| \;<\; C^m\, M^m\, 2^{-\delta h}\, \frac{2^h}{\Gamma} \;; \tag{17}$$
2. *For $2p_F - \rho < \Gamma < 2p_F + \rho$ and $2^{-h}\Gamma > N_0$ we have*
$$|f_h| \;<\; C^m M^m\, 2^{-\delta h}\, 2^{\frac{1}{2}h} \;; \tag{18}$$
3. *For $\Gamma > 2p_F + \rho$ and $2^{-h}\Gamma > N_0$ we have*
$$|f_h| \;<\; C^m M^m\, 2^{-\delta h}\, 2^h\, e^{-C' 2^{-2h}\Gamma^2} \;. \tag{19}$$

In the above formulas, m stands for the number of vertices in the graph and M is an estimate for the coupling constants.

A completely analogous result holds for exchange graphs by substituting Γ by Δ.

The theorem shows, for sufficiently small h (sufficiently small is here a Γ-dependent concept) and for some external-momenta configurations, *i.e.* $\Gamma \neq 0$, an improvement with respect to power counting for some 4-legged graphs already sufficient to transform a divergent bound into convergent. If $\Gamma = 0$, we get no improvement at all and are left only with the power counting estimate.

Notice that $\Gamma = 0$ means a configuration of equal and opposite external momenta 1 and 2 and, by total momentum conservation, also equal and opposite momenta 3 and 4; in other words a scattering process in which two fermions of equal and opposite momenta, *i.e.* a Cooper pair, are annihilated and another Cooper pair is created. We call $\Gamma = 0$ a *Cooper pairs* configuration.

A completely analogous result holds for exchange graphs with Δ in the place of Γ. Notice that $\Delta = 0$ means a *forward scattering* configuration.

Sketch of proof for theorem 1 :

We will be referring to the simplest direct graph in figure 1 as an example, but at the end the reader can convince himself that the same reasoning will extend trivially to each loop of the other graphs, being it possible to gain one extra improvement per loop.

We begin from the definition (12) of the form factor of the graph. If G_h represents either g_h or $\overline{g}_h$, then it can be easily seen from their definitions that $|G_h(k)| < Ce^{-\gamma\left[(k^0)^2+\frac{1}{4}\beta^2(k^1)^2\right]}$, where C and γ are positive constants independent of h. Using this bound for both g functions in (12), bounding each of the running couplings by a constant M, rescaling $k_5' = 2^{-h}k_5$ and integrating k_5^0, we arrive at

$$|f_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4)| \leq C^2M^2e^{-\gamma 2^{-h}|\Gamma_0|}\int dk_5^1 e^{-\frac{1}{4}\gamma\beta^2(k_5^1)^2}I(k_5^1)\;, \tag{20}$$

where

$$I(k_5^1) = \int d\vec{\omega}_5\, e^{-\frac{1}{4}\gamma\beta^2 2^{-2h}(|\vec{\Gamma}-(2^h k_5^1+p_F)\vec{\omega}_5|-p_F)^2}\;. \tag{21}$$

In the last formula, integration over $\vec{\omega}_5$ means integration over the $d-1$ angles that localize the unitary vector $\vec{\omega}_5$. But the integrand depends only on the angle θ between $\vec{\omega}_5$ and $\vec{\Gamma}$. We can then consider without loss of generality only the case $d=2$, the difference with other dimensions being only a numerical factor, which we can incorporate into the constant C and a factor $\sin^{d-2}\theta$ that we bound by 1.

If we define

$$N = \frac{1}{4}\gamma\beta^2 2^{-2h}|\vec{\Gamma}|^2 \tag{22}$$

$$p(\theta) = \frac{1}{|\vec{\Gamma}|^2}(|\vec{\Gamma}-r\vec{\omega}_5|-p_F)^2 \tag{23}$$

$$r = 2^h k_5^1 + p_F\;, \tag{24}$$

then

$$I(k_5^1) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-Np(\theta)}\;. \tag{25}$$

Asymptotic values when $N\to\infty$, *i.e.* $h\to-\infty$, of integrals of this kind can be calculated by Laplace's method [16]. There are three different cases, corresponding to the three cases in theorem 1: *1.* If $|\vec{\Gamma}| < p_F + r$, then p has two points of minimum and $p(\theta)$ is zero in these minima; *2.* If $|\vec{\Gamma}| = p_F + r$, then the unique point of minimum is $\theta = 0$ and p is still zero at this minimum; *3.* If $|\vec{\Gamma}| > p_F + r$, then again $\theta = 0$ is the unique point of minimum and now the value of p at this minimum is greater than zero.

Let us define θ_0 to be the position of the minimum of $p(\theta)$ (if there are two points of minimum we choose θ_0 to be the one with positive sine). If we apply Laplace's method in the first case we get to

$$I(k_5^1) \overset{N\to\infty}{\sim} \frac{2\pi^{1/2}}{\left[\frac{1}{2}Np''(\theta_0)\right]^{1/2}} = \frac{4\pi^{1/2}p_F}{\gamma^{1/2}\beta r\,|\sin\theta_0|}\,\frac{2^h}{|\vec{\Gamma}|}\;. \tag{26}$$

By definition of asymptoticity, this implies that there exists N_0 such that for $2^{-h}|\vec{\Gamma}| > N_0$ we have

$$I(k_5^1) < 2\,\frac{4\pi^{1/2}p_F}{\gamma^{1/2}\beta r\,|\sin\,\theta_0|}\,\frac{2^h}{|\vec{\Gamma}|}\ .$$

If we could use this result in (20), then the proof of the first assertion in the theorem would follow easily. The problem is that in using "normal" Laplace's method to obtain the last formula, we have tacitly assumed that $|\vec{\Gamma}|$ and r are fixed, so that N_0 in principle depends on them. As we still have to integrate on k_5^1, which is related to r by (24) and we want N_0 to be independent of $|\vec{\Gamma}|$, we need a sort of Laplace's method *uniform* in $|\vec{\Gamma}|$ and r.

We will proceed proving the first statement in theorem 1 and in the end comment on the differences in applying the same method to the other two statements. Imposing

$$0 \,<\, |\vec{\Gamma}| \,<\, 2p_F - \rho \quad \text{and} \quad p_F - \frac{1}{2}|\vec{\Gamma}| \,<\, r \,<\, p_F + \frac{1}{2}|\vec{\Gamma}| \tag{27}$$

we guarantee that $p(\theta)$ has two minima $\pm\theta_0$ with θ_0 bounded away from zero. This is the key feature in the first part of the theorem.

The proof goes as follows. Under conditions (27) we prove in lemma 2 that $p(\theta)$ has some uniformity properties in $|\vec{\Gamma}|$ and r. Then we use these properties in lemma 3 to show that $I(k_5^1)$ has a good bound if $|k_5^1| \,<\, \frac{1}{2}2^{-h}|\vec{\Gamma}|$. Finally, lemma 4 shows that these values of k_5^1 really give the dominant contribution and resumes the proof of the first part of the theorem for the special direct graph in figure 1.

Lemma 2 (Uniformity lemma) *Suppose conditions (27). Under these, given any $\epsilon > 0$, there exists $\delta > 0$, δ independent of $|\vec{\Gamma}|$ and r, such that for $|\theta - \theta_0| < \delta$ we have*

$$|p(\theta) - \frac{1}{2}p''(\theta_0)\,(\theta-\theta_0)^2| \,<\, \epsilon\,\frac{1}{2}p''(\theta_0)\,(\theta-\theta_0)^2\ . \tag{28}$$

Proof: Approximate $p(\theta)$ by its 2nd. order Taylor polynomial and use the differential formula for the remainder to estimate it. Conditions (27) are used to guarantee that each derivative of $p(\theta)$, in particular the third derivative, is bounded.

Lemma 3 (Uniform Laplace's method) *Under conditions (27) (which imply $|k_5^1| \,<\, \frac{1}{2}2^{-h}|\vec{\Gamma}|$), for any given $\epsilon > 0$ there exists a number N_0 independent of $|\vec{\Gamma}|$ and r such that for $N > N_0$ we have*

$$\left|I(k_5^1)\,-\,\frac{2\pi^{1/2}}{\left[\frac{1}{2}Np''(\theta_0)\right]^{1/2}}\right| \,<\, \epsilon\,\frac{2\pi^{1/2}}{\left[\frac{1}{2}Np''(\theta_0)\right]^{1/2}}\ .$$

In other words, $I(k_5^1)$ *is asymptotic to* $2\pi^{1/2}/\left[\frac{1}{2}Np''(\theta_0)\right]^{1/2}$*, asymptoticity being uniform in* $|\vec{\Gamma}|$ *and* r*.*

Proof: Just imitate the proof of the ordinary Laplace's method given in [16]. The only difference is that lemma 2 guarantees for any $\epsilon > 0$ existence of θ_1 independent of $|\vec{\Gamma}|$ and r such that for all $\theta \in [\theta_0, \theta_1]$ we have $|p(\theta) - \frac{1}{2}p''(\theta_0)\,(\theta-\theta_0)^2| < \epsilon\,\frac{1}{2}\,p''(\theta_0)\,(\theta-\theta_0)^2$. This will be used to estimate

$$\int_{\theta_0}^{\theta_1} d\theta\,\left(e^{-Np(\theta)} - e^{-\frac{1}{2}\,Np''(\theta_0)(\theta-\theta_0)^2}\right)$$

somewhere in the proof, guaranteeing uniformity.

Lemma 4 *If* $0 < \Gamma < 2p_F - \rho$*, where* ρ *is any positive number given, there exists a number* N_0 *independent of* h *and* Γ *such that the form factor of the graph in figure 1 satisfies*

$$|f_h| < C^2 M^2\,\frac{2^h}{\Gamma}\,. \tag{29}$$

Proof: Use lemma 3 to show that there exists N_0 such that for $2^{-h}|\vec{\Gamma}| > N_0$, $0 < |\vec{\Gamma}| < 2p_F - \rho$ and $|k_5^1| < \frac{1}{2}2^{-h}|\vec{\Gamma}|$ we have

$$I(k_5^1) < 2\,\frac{4\pi^{1/2}p_F}{\gamma^{1/2}\beta r|\sin\theta_0|}\,\frac{2^h}{|\vec{\Gamma}|} < C''\,\frac{2^h}{|\vec{\Gamma}|}\,,$$

with C'' independent of h, $|\vec{\Gamma}|$ and k_5^1.

In order to finish the proof of the first part of the theorem we have to control the large values $|k_5^1| > \frac{1}{2}2^{-h}|\vec{\Gamma}|$ in the integration (20). This is easy because we can use the easy power counting bound $I(k_5^1) < 1$ and exploit the smallness of $e^{-\frac{1}{4}\gamma\beta^2(k_5^1)^2}$ for the large values. This ends the proof of the first part of theorem 1.

The proof of the remaining two parts of theorem 1 is now a mere repetition of the ideas used to prove the first part. We explain the differences.

The hypothesis $|\vec{\Gamma}| > 2p_F + \rho$ of the third part together with $r < p_F + \rho/2$ implies $\theta_0 = 0$ and $p(0)$ and $p''(0)$ bounded away from zero, so that we can prove an analog of lemma 2 approximating $p(\theta)$ near $\theta = 0$ by $p(0) + \frac{1}{2}p''(0)\theta^2$ *uniformly* in $|\vec{\Gamma}|$ and r. An analog of lemma 3, shows that $I(k_5^1)$ is uniformly asymptotic to $C\,2^h\,e^{-C'.2^{-2h}|\vec{\Gamma}|^2}$. Finally, the large values of k_5^1 such that $r \geq p_F + \rho/2$ are dealt with by an analog of lemma 4.

The proof of the second part of the theorem is a bit more subtle, for when $|\vec{\Gamma}|$ is in a neighborhood of $2p_F$ and r is close to the dominant value p_F, then both $p(\theta_0)$ and

$p''(\theta_0)$ are small. There exists $\sigma > 0$, σ dependent on ρ but independent of $|\vec{\Gamma}|$, such that for $|r - p_F| < \sigma$ both are also non-negative and the fourth derivative $p^{(IV)}(\theta_0)$ is positive and bounded away from zero. We prove an analog of lemma 2 asserting that for any $\epsilon > 0$ there exists $\delta > 0$, δ dependent on ρ but independent of $|\vec{\Gamma}|$ and r such that if $2p_F - \rho < |\vec{\Gamma}| < 2p_F + \rho$, $r < p_F + \rho/2$ and $|\theta - \theta_0| < \delta$, then $p(\theta)$ is *bounded below* by $(1-\epsilon)\,\frac{1}{4!}\,p^{(IV)}(\theta_0)(\theta-\theta_0)^4$. We can use then an analog of lemma 3 to show that $I(k_5^1)$ has an *upper bound* uniformly asymptotic to $C/N^{1/4} = O(2^{\frac{1}{2}h})$. Again, as in the other cases, an analog of lemma 4 controls the values of k_5^1 such that $|r - p_F| \geq \sigma$.

We will finish this section with one more improvement theorem, corollary of theorem 1, for some classes of graphs, whose prototype is the graph in figure 5, the simplest 4-legged one which is neither direct, nor exchange.

Theorem 5 (Loop improvements) *Let G be a graph in which all lines have the same frequency h and such that there is at least one internal line being shared by two loops. If δ is the dimension of the graph, M is an upper bound for the coupling constants, m is the number of vertices and $0 < \epsilon < 1$, then there exists a constant C independent of G such that the form factor f_h obeys*

$$|f_h| \,<\, C^m M^m \, 2^{-\delta h} \, 2^{\frac{1}{2}(1-\epsilon)h} \,. \tag{30}$$

In other words, for these graphs power counting can be improved by a factor $2^{\frac{1}{2}(1-\epsilon)h}$.

Sketch of proof: Using the notation of figure 5, the idea is to integrate internal lines 7 and 8 only in the end. While they are fixed, integration of 5 and 6 leads to the form factor of a direct graph with $\Gamma_0 = k_1^0 + k_7^0$, $\vec{\Gamma} = (k_1^1 + p_F)\vec{\omega}_1 + (k_7^1 + p_F)\vec{\omega}_7$, and $\Gamma = |\Gamma_0| + |\vec{\Gamma}|$, to which we can apply theorem 1, remembering that Γ depends on the integration variables $k_7, \vec{\omega}_7$. As the measure of the region in $k_7, \vec{\omega}_7$ where $\Gamma \leq 2^h N_0$ (no improvement in theorem 1) is $O(2^h)$, we can have an improvement independent of the external momenta configuration. The final result is an improvement of only $2^{\frac{1}{2}(1-\epsilon)h}$ because the second case in theorem 1 gives only $2^{\frac{1}{2}h}$ and a logarithmic correction due to integration of $1/\Gamma$ in the region where $\Gamma > 2^h N_0$ but Γ is still $O(2^h)$ small accounts for the $2^{-\epsilon h}$.

4 Renormalization

In the usual position-space version of the tree expansion [14], [2] renormalization is accomplished by separating the local (relevant) and non-local (irrelevant) parts in the fields of each contribution to the effective potential. In our momentum-space formalism, the same separation into irrelevant and relevant parts is achieved by splitting each form factor f_h in parts f_h^R (irrelevant) and f_h^L (relevant) such that

$f_h^R + f_h^L = f_h$, but local will not be anymore a synonym of relevant. The separation scheme should be such that summing on frequency attributions the R parts, one has no more divergences, these being "hidden" in the relevant parts f_h^L.

As in any scale h, $\mathcal{L}V^{(h)}$, the relevant part of $V^{(h)}$, has to be a term with the same form as $V^{(0)}$, we may write it explicitly as

$$\mathcal{L}V^{(h)} = \int dp_1 dp_2\, 2^h \left[1 - (1 - e^{-2^{-h}|k|})^2\right] \nu_h(k_1)\, \delta(p_1 - p_2) \,:\psi^+_{k_1,\vec{\omega}_1}\psi^-_{k_2,\vec{\omega}_2}: \\ + \int dp_1 \dots dp_4 \left[e^{-2^{-h}\Gamma}\lambda_h(k_1,\dots,k_4,\vec{\omega}_1,\dots,\vec{\omega}_4) + e^{-2^{-h}\Delta}\mu_h(k_1,\dots,k_4,\vec{\omega}_1,\dots,\vec{\omega}_4)\right] \\ \delta(p_1 + p_2 - p_3 - p_4) :\psi^+_{k_1,\vec{\omega}_1}\psi^+_{k_2,\vec{\omega}_2}\psi^-_{k_3,\vec{\omega}_3}\psi^-_{k_4,\vec{\omega}_4}: , \tag{31}$$

where we are using the same p_i convention used in (7).

The functions $\nu_h(k)$, $\lambda_h(k_1,\dots,k_4,\vec{\omega}_1,\dots,\vec{\omega}_4)$ and $\mu_h(k_1,\dots,k_4,\vec{\omega}_1,\dots,\vec{\omega}_4)$ appearing at the formula above as coefficients of the terms with 2 and 4 fields are the *running coupling functions*. In usual problems, these functions do not depend on k and $\vec{\omega}$ and are called running coupling *constants*. We will sometimes indicate collectively all running couplings at scale h by v_h. In the process of constructing the $\mathcal{R}$ and $\mathcal{L}$ operations separating the irrelevant and relevant parts of the effective potentials it will become clear why we have running functions instead of running constants, why we appended some strange factors to these functions and why we used 2 running coupling functions λ and μ to describe the 4-field interaction.

We begin now the task of defining the $\mathcal{L}$ and $\mathcal{R}$ operations. A very easy definition of f^L and f^R is the one for a cluster with 6 or more external legs. As these clusters are already convergent by power counting, then we can define for them $f_h^L = 0$ and $f_h^R = f_h$.

Let us now define the $\mathcal{L}$ and $\mathcal{R}$ operations for a direct graph with all internal lines at frequency h. If we use the pure power-counting prescription of Benfatto and Gallavotti [2] of taking as relevant only the local part in position space, then we know it will work in the sense $\sum_{h=h'+1}^{0} f_h^R$ is not diverging when $h' \to -\infty$, but we also know from theorem 1 that power counting may be treating as divergent some convergent configurations.

A good prescription should use theorem 1 and contain the fact that only configurations with $\Gamma = 0$ cause divergence, but also the information that each $\Gamma \neq 0$ configuration for a fixed Γ behaves as if it were relevant while h is not small enough, *i.e.* $2^{-h}\Gamma < N_0$, and irrelevant when $2^{-h}\Gamma > N_0$. Physically, a configuration with $\Gamma \neq 0$ behaves as a Cooper pair, being thus relevant, if the scale 2^h of momenta considered is much greater than Γ, passing smoothly to an irrelevant non-Cooper-pairs configuration when $2^h \ll \Gamma$. We propose then to define

$$f_h^R = (1 - e^{-2^{-(h-1)}\Gamma})\, f_h\ . \tag{32}$$

In fact, we have $f_h^R \approx 0$ and $f_h^L \approx f_h$ if $2^{-h}\Gamma \ll N_0$ and $f_h^R \approx f_h$ and $f_h^L \approx 0$ if

$2^{-h}\Gamma \gg N_0$. We show now that this prescription does really exclude from f_h^R all divergences:

Proposition 6 *If f_h^R is the renormalized form factor of a direct graph with m vertices, all internal lines at frequency h and root h', then, if all running couplings are bounded by M,*

$$|\sum_{h=h'+1}^{0} f_h^R| < C^m M^m ,$$

where C is a constant independent of h', Γ and of the graph.

Sketch of proof: In order to simplify things, take $\rho = p_F$ in theorem 1, so for $0 < \Gamma < p_F$ we have an improvement in the bound for f_h of the form $\frac{2^h}{\Gamma}$, if $2^{-h}\Gamma > N_0$. If $\Gamma > p_F$, then, take $\epsilon = \frac{1}{2}$ and get an improvement of at least $2^{\frac{1}{4}h}$ proving the proposition in this case. Let us then concentrate on the less easy case $0 < \Gamma < p_F$.

Given Γ, let H be defined as the smallest integer such that $2^{-H}\Gamma \leq N_0$. If $h < H$, then the factor $1 - e^{-2^{-(h-1)}\Gamma}$ is almost equal to 1, but we can use the $\frac{2^h}{\Gamma}$ improvement; summing it from $h = -\infty$ up to $h = H$, we get $\frac{2^H}{\Gamma}$, which is $O(1)$. If $h \geq H$, we can use that the factor $1 - e^{-2^{-(h-1)}\Gamma}$ is bounded by $2^{-(h-1)}\Gamma$; summing from $h = H$ up to $h = 0$ we get again something of order $2^{-H}\Gamma$, which is again $O(1)$. This ends the proof.

Analogously, if f_h is the form factor of an *exchange* graph in which all lines are at the same frequency h, then we define its irrelevant part as

$$f_h^R = (1 - e^{-2^{-(h-1)}\Delta})\, f_h\ . \tag{33}$$

We are now able to explain some strange features of (31). First of all, we define two running couplings λ_h and μ_h, because we want to keep separated the contributions to the relevant part $\mathcal{L}V^{(h)}$ coming from direct and exchange graphs. So, λ_h is by definition formed from contributions of direct graphs and μ_h from contributions of exchange graphs. This separation means that instead of having only one type of 4-line vertex in the renormalized graphs, we now have two. Every time we draw a graph we must also specify for each 4-line vertex if that vertex is a λ or a μ.

Now the factors accompanying the vertices. As the relevant part of the form factor f_{h+1} of a direct graph is $f_{h+1}^L = e^{-2^{-h}\Gamma} f_{h+1}$, then it is natural to define the running coupling λ_h so that the sum of all these contributions is not λ_h, but $e^{-2^{-h}\Gamma}\lambda_h$. The same for exchange graphs.

But as we are talking of two different types of 4-line vertices, we must now correct our renormalization prescription for direct and exchange graphs to take into account not only the two different types of vertices but also the factors accompanying them. All our preceding estimates on direct and exchange graphs remain correct, because the factors accompanying the λ and μ functions are both less than 1, but we can improve them in some cases.

Suppose for example that in a direct graph like the one in figure 1 the vertex at the left is a λ and the other a μ. Accompanying the μ_h function of the right vertex we have a factor $\exp[-2^{-(h-1)}\Delta]$. As we have to integrate momentum of line 5, on which $\Delta = |k_3^0 - k_5^0| + |(k_3^1 + p_F)\vec{\omega}_3 - (k_5^1 + p_F)\vec{\omega}_5|$ depends, and as the factor containing Δ decreases very quickly when Δ is greater than $O(2^h)$, then the effective size of the integration region is reduced to $O(2^h)$ around $k_5^0 = k_3^0$ and $(k_5^1 + p_F)\vec{\omega}_5 = (k_3^1 + p_F)\vec{\omega}_3$. We can then gain an extra, momentum-independent, 2^h factor with respect to power counting and the graph can be considered as irrelevant and renormalized as $f_h^R = f_h$. An analogous result holds for exchange graphs with some λ-vertex.

If a graph has 4 external lines, all internal lines of the same frequency and is neither direct, nor exchange, then, by theorem 5, it is already convergent and we define $f_h^R = f_h$ and $f_h^L = 0$. Of course these contribute neither to λ-, nor to μ-couplings.

With all these observations we can now conclude that the only graphs with 4 external lines and all internal lines with the same frequency that need to be non-trivially renormalized are the *direct* ones in which *all vertices are* λ and the *exchange* ones in which *all vertices are* μ. For these graphs we use respectively prescriptions (32) and (33).

This means that we have already shown how to renormalize any graph with all internal lines of the same frequency and 4 or more legs. We must still learn how to do the same for 2-legged graphs with all lines at the same frequency and then how to deal with graphs having lines of different frequencies.

By (11), the dimension of a cluster with 2 external legs in which there is no 2-legged vertex inside (*i.e.* $m_{2,v} = 0$) is -1. It is usual in renormalization group to define all running couplings to be *dimensionless*. This means that the running coupling ν_h associated to the contribution of 2-lined graphs should be defined so that each graph contributes to $2^h\nu_h$. As the dimension of 4-legged vertices is 0, their running couplings λ_h and μ_h are already dimensionless.

Of course, when we are using power counting to bound renormalized graphs, in which vertices should be thought of as representing not the bare constants v_0 but the running couplings v_h, then it is necessary to take into account the extra 2^h factor accompanying all the 2-legged vertices. That can be easily done and the result is that the dimensions (11) of the clusters are changed to

$$\delta_v = -2 + \frac{1}{2}n_v^e , \tag{34}$$

now dependent only on the number of external legs of the cluster.

Notice however that this argument is up to now pure power counting. We showed for 4-legged clusters some improvements beyond power counting which proved that many of the configurations we thought were relevant are indeed irrelevant. In the case of 2-legged clusters, the (power counting) dimension -1 seems to imply that these clusters are relevant for any momentum configuration. We now show easily that 2-legged clusters are relevant only when their external momentum is $O(2^h)$ or smaller. More exactly, if $k = (k^0, k^1)$ is the external momentum variable of a 2-legged cluster, we define $|k| = |k^0| + |k^1|$ and the irrelevant part f_h^R of the form factor f_h of a 2-legged cluster is defined as

$$f_h^R(k) = \left(1 - e^{-2^{-(h-1)}|k|}\right)^2 f_h(k) . \tag{35}$$

A calculation analogous to the one in the proof of proposition 6 now shows that prescription (35) does really work. If we define H as the smallest integer such that $2^{-H}|k| \leq 1$, then

$$\begin{aligned} \Big| \sum_{h=h'+1}^{0} f_h^R \Big| &< C^m M^m C' \, 2^H = C^m M^m C' \, 2^{h'} \, 2^{-h'} |k| \frac{1}{2^{-H}|k|} \\ &< C^m M^m \, 2C' \, 2^{h'} \, 2^{-h'} |k| . \end{aligned}$$

In this formula $2^{h'}$ is the correct dimensional factor for the sum, as it is a contribution to the 2-field part in $V^{(h')}$, and the dimensionless factor $2^{-h'}|k|$ offers no problem, as it can be absorbed in the propagator of lines at scale h', defining an effective propagator $g_{h'}^{eff}(2^{-h'}k) = 2^{-h'}|k| \, g_{h'}(2^{-h'}k)$ at that scale which decays exponentially when k is $O(2^{h'})$, just as the plain propagator.

As a consequence of (35), the relevant part of a graph with 2 external lines, root frequency $h-1$ and first non-trivial cluster at frequency h is then $f_h^L(k) = \chi(2^{-(h-1)}|k|) f_h$, where $\chi(k) = \left[1 - \left(1 - e^{-|k|}\right)^2\right]$. It is then natural to define the running couplings ν_{h-1} such that the sum of the contributions coming from all graphs like the one considered is $2^{h-1}\chi(2^{-(h-1)}|k|)\nu_{h-1}(k)$. This explains the strange form of the coefficient of the 2-field term in (31). As it happened to the running couplings related to 4-field interactions, ν_h is also a function, not a constant. It can be shown, by rotational invariance, that the ν are functions only of k and not of quasimomenta $\vec{\omega}$.

In order to complete the renormalization prescription, we now have to say how to renormalize graphs in which there are internal lines of different frequencies. The prescription is to begin by renormalizing the innermost clusters in the graphs, in which all lines have the same frequency. Erase then these clusters and substitute them for "effective" vertices with the same number of lines; in the resulting graph renormalize the innermost clusters again and continue the procedure until you arrive at the outermost cluster. With this idea it can be shown that all renormalized graphs are convergent.

As an example take the graph in figure 6, a contribution to $\mathcal{R}V^{(h)}$, generated by the tree in the same figure. We renormalize it as follows. If we erase the clusters of frequencies h_1 and h_2 and substitute them for effective vertices we get the graph in figure 7, which we call *the graph of figure 6 seen at frequency* h_3. As it is a direct graph, we may renormalize it at frequency h_3 as $f^R_{h_3} = (1 - e^{-2^{-(h_3-1)}\Gamma})\, f_{h_3}$, where f_{h_3} was obtained renormalizing clusters h_1 and h_2, *i.e.*

$$f_{h_3} = \int dk_7 d\vec{\omega}_7\, f^R_{h_1}(1,2,7,8)\, f^R_{h_2}(7,8,3,4)\, 2^{-2h_3}\, g_{h_3}(2^{-h_3}k_7)\, \overline{g}_{h_3}(2^{-h_3}k_8)\ , \tag{36}$$

with $f^R_{h_1} = (1 - e^{-2^{-(h_1-1)}\Gamma}) f_{h_1}$ and $f^R_{h_2} = h_{h_2}$.

5 The beta functional and properties of the Fermi surface

Having renormalized our theory, now the (renormalized) tree expansion gives us the running couplings at each scale h as a perturbative series in the running couplings at every preceding scale $l > h$, [14]. These evolution equations for the running couplings are called the beta functional.

As we said at the beginning of this work, our main concern is about normality or anomaly of the Fermi surface. The answer to this question depends on the behaviour of the running couplings predicted by the beta functional, as we now explain.

¿From the relation between the effective potentials and Schwinger functions, see [2] or [14], it can be easily shown that the 2-point Schwinger function is given as

$$S_{(\geq h)}(p) = \frac{1}{ip_0 - e(\vec{p})} + \left(\frac{1}{ip_0 - e(\vec{p})}\right)^2 \left[2^h \nu_h(k)\, \chi(2^{-(h-1)}|k|) + w_h(k)\right]\ , \tag{37}$$

where $S_{(\geq h)}$ is the Schwinger function evaluated with the infrared cut-off at frequency h, p is the momentum, related to the $k, \vec{\omega}$ variables by the convention explained after equation (8) and $e(\vec{p}) = (\vec{p}^2 - p_F^2)/(2m)$. The first term in the right-hand side is the *free* 2-point Schwinger function, whereas the second term is the contribution of the interaction. In this second term there are also two terms: the ν term is the relevant part of the 2-field term in $V^{(h)}$ and the w term, which we do not write explicitly, is the irrelevant part of the same 2-field term.

In order to have in the interacting case a Fermi surface with the same thermodynamical properties as the free case, it is necessary that the 2-point Schwinger function maintains the $[ip_0 - e(\vec{p})]^{-1}$ singularity. So, it seems reasonable to *define* a normal Fermi liquid as one in which the term with singularity proportional to $[ip_0 - e(\vec{p})]^{-2}$ in (37) vanishes. As $w_h(k)$ is the irrelevant part of the 2-field term in $V^{(h)}$, then it is proportional to $\left(1 - e^{-2^{-(h-1)}|k|}\right)^2$, which goes quadratically to 0

when $k \to 0$, causing then no singularity. The necessary condition for a Fermi liquid to be normal is then that the function $Z(k)$ defined by

$$Z(k) = \lim_{h\to-\infty} \left[2^h \nu_h(k)\, \chi(2^{-(h-1)}|k|)\right] \tag{38}$$

goes to zero when $k \to 0$ as $[ip_0 - e(\vec{p})]^n$, with $n \geq 1$. In particular, when $k = 0$ this implies that we must have

$$2^h \nu_h(0) \stackrel{h\to-\infty}{\to} 0\ . \tag{39}$$

Observe that this definition accounts for a possible wave-function renormalization if it happens that $n = 1$ above.

The counterterm ν_0 should be thought of as a constant function $\nu_0(k)$ fixed from the beginning so that the physical value of the Fermi momentum is p_F. But the natural definition of physical value of p_F is the position of the singularity in the 2-point Schwinger function[4], so that normality of the Fermi liquid depends on the possibility of fine-tuning ν_0 such that (39) is satisfied.

By standard renormalization group arguments, versions of the stable manifold theorem of Dynamical Systems, one can show existence of a unique λ_0-dependent value of ν_0 such that $\nu_h(0) \stackrel{h\to-\infty}{\to} 0$. This fulfills condition (39), which is part of the complete condition for normality of the Fermi surface. The rest of the normality definition should then be checked *a posteriori*.

If we want to believe in our perturbative results, then another important condition necessary for normality of the Fermi surface, also to be checked *a posteriori*, is that the running couplings do not grow unboundedly when $h \to -\infty$, *i.e. consistence* of perturbation theory, to which we now turn.

In figure 8 we show the renormalized tree expansion for $V^{(h-1)}$ to first order in perturbation theory. As usual, see [14], an end line of a tree attached to a frequency label l means $\mathcal{L}V^{(l)}$. In the formalism of Benfatto and Gallavotti, where a $\mathcal{R}$ applied over a $\mathcal{L}$ gives a zero result, all the trees other than the first are null. That does not happen in our formalism.

Arguments very similar to the proofs of proposition 6 and its analog for the renormalization of 2-legged graphs show that the sum over l in figure 8 is already convergent, so that it is natural to define the $\mathcal{L}$ operation at scale h for all the trees within the sum as 0; thus only the first tree will contribute to the running couplings v_{h-1}. As a consequence, we have as the first-order beta functional

$$\begin{aligned}
\nu_{h-1}(k) &= 2\,\chi(2^{-h}|k|)\,\nu_h(k) \\
\lambda_{h-1}(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) &= e^{-2^{-h}\Gamma}\,\lambda_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) \\
\mu_{h-1}(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) &= e^{-2^{-h}\Delta}\,\mu_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4)
\end{aligned} \tag{40}$$

[4] This is analogous to the definition of physical mass in Quantum Field Theory as the position of the pole of the 2-point Schwinger function.

¿From these, we see to first order that running couplings in configurations with non-zero k, Γ, Δ, respectively, are strongly depleted to zero as $h \to -\infty$. Of course that does not mean that the contribution of the trees within the sum in figure 8 to the irrelevant part of the effective potential is negligible. Furthermore, these irrelevant terms are still going to contribute for example at higher orders in the beta functional, *i.e.* to the relevant part of the effective potential.

As other terms depleting configurations with non-zero k, Γ, Δ are present also at higher orders, it seems natural to *conjecture* that ν, λ, μ for non-zero k, Γ, Δ, respectively, quickly converge to zero as $h \to -\infty$. If this conjecture above is true, then we are left only with $\nu_h(0)$, λ_h at Cooper pairs configurations and μ_h at forward scattering configurations to be concerned about. Of these three couplings, we already know that it is possible to fine tune the initial value ν_0 such that $\nu_h(0) \to 0$. We are then left with λ_h and μ_h at Cooper pairs and forward scattering, respectively.

At second order, the beta functional for the λ_h is

$$\begin{aligned}\lambda_{h-1}(1,\ldots,4) &= e^{-2^{-h}\Gamma}\lambda_h(1,\ldots,4)\\ &- e^{-2\,.\,2^{-h}\Gamma}\int dk_5\, d\vec{\omega}_5\, G_h(k_5,\vec{\omega}_5)\,\Lambda_h(1,2,5,6)\,\Lambda_h(5,6,3,4)\ ,\end{aligned} \tag{41}$$

with

$$G_h(k_5,\vec{\omega}_5) = \frac{4}{(2\pi)^{d+1}}\, 2^{-2h}\, \overline{g}_h(2^{-h}k_6)\, \left[g_h(2^{-h}k_5) + 2g_h^s(2^{-h}k_5)\right] \tag{42}$$

and

$$\Lambda_h = \lambda_h + \sum_{l=h+1}^{0} \left[(1-e^{-2^{-(l-1)}\Gamma})e^{-2^{-l}\Gamma}\lambda_l + (1-e^{-2^{-(l-1)}\Delta})e^{-2^{-l}\Delta}\mu_l\right]\ . \tag{43}$$

In (41) and (42), k_6 and $\vec{\omega}_6$ are functions of $k_5, \vec{\omega}_5$ given by (13).

Equation (41) is of course a complicated evolution, but it is remarkably simplified for Cooper pairs configurations. In order to simplify the notation, let us define a symbol for the λ in Cooper pairs configurations: $\lambda_h(1,3) = \lambda_h(1,-1,3,-3)$, where $-i$ means $k_i, -\vec{\omega}_i$.

As a first simplification, the contributions to Λ_h coming from renormalized λ-vertices are null, if $\Gamma = 0$. We may also neglect the contributions from renormalized μ-vertices, because the μ will be sizeable only if $\Delta = 0$ and, in this case, they will be null because of the factor $1 - e^{-2^{-(l-1)}\Delta}$.

Another simplification is that as λ_0 does not depend on the zero components k_i^0 of the k variables, then, for $\Gamma = 0$, it can be seen from (41) that $\lambda_{-1}(1,3)$ does not depend on k_1^0 and k_3^0. Continuing the inductive procedure, we have that to second order, *i.e.* evolution given by (41), the λ_h in Cooper pairs do not depend on the k^0 variables.

Another fact is that the imaginary part of the function $G_h(k_5, \vec{\omega}_5)$ appearing in (41) is odd in the variable k_5^0. As the λ_h in Cooper pairs do not depend on k_5^0, then the imaginary part of $G_h(k_5, \vec{\omega}_5)$ integrates to zero. As the λ_0 are real numbers, then this argument shows that the λ_{-1} are also real and, by induction, the λ_h in Cooper pairs are real for any value of h. Of course the argument does not hold for configurations which are not Cooper pairs. It can be seen that this property is also easily extensible to any finite order in perturbation theory.

We continue by noticing that for $d = 2$ we can expand the λ_h in Cooper pairs in a Fourier series

$$\lambda_h(1,3) \,=\, \sum_{l=-\infty}^{l=\infty} \lambda_h^l(k_1^1, k_3^1)\, \frac{e^{il(\theta_1-\theta_3)}}{\sqrt{2\pi}} \,, \tag{44}$$

where $\lambda_h^l(k_1^1, k_3^1)$ are the Fourier coefficients and, by rotational invariance, $\lambda_h(1,3)$ depends only on the difference of the angles θ_1, θ_3 which the $\vec{\omega}$'s make with some axis. There is an analogous expansion in Legendre polynomials in the case $d = 3$, see [2], section 13, and all the results we describe from now on remain valid also in this case.

By Fourier expanding we can evaluate the angular integral in (41), obtaining then that the evolution equations for the coefficients λ_h^l are diagonal in l:

$$\lambda_{h-1}^l(k_1^1, k_3^1) \,=\, \lambda_h^l(k_1^1, k_3^1) \,-\, \sqrt{2\pi} \int dk_5\, G_h(k_5)\, \lambda_h^l(k_1^1, k_5^1)\, \lambda_h^l(k_5^1, k_3^1) \,. \tag{45}$$

In order to arrive at this result, it has been used that k_6 becomes independent of $\vec{\omega}_5$, for Cooper pairs, and then G_h becomes independent of it, too, and can be taken off the angular integral.

This trick of Fourier expanding is very interesting, because the complicated dependence of (41) on the 3 angles θ_1, θ_3 and θ_5 disappears and we have the same evolution for any l without ever mixing different values.

But there is more we can say. As a consequence of the real-valuedness of $\lambda_h(1,3)$ it follows that $\overline{\lambda_h^l} \,=\, \lambda_h^{-l}$, where the bar means here complex conjugation. As $\lambda_0(1,3)$ is an even function of θ, then $\lambda_0^l = \lambda_0^{-l}$ and, by the evolution (45), $\lambda_h^l \,=\, \lambda_h^{-l}$ for all l. This shows in turn that $\lambda_h^l(k_1^1, k_3^1)$ is real for all l.

As the real part of $G_h(k_5)$ is positive, the last result shows, by the evolution (45), that for each set of values l, k_1^1, k_3^1, we have $\lambda_{h-1}^l(k_1^1, k_3^1) \,<\, \lambda_h^l(k_1^1, k_3^1)$ if $\lambda_h^l(k_1^1, k_3^1)$ has the same sign for all values of k_1^1, k_3^1.

This means that if we begin with an attractive potential $\lambda_0(\vec{x} - \vec{y})$, then all $\lambda_h^l(k_1^1, k_3^1)$ are going to be negative and explode in the limit $h \to -\infty$. Thus, attractive interactions cause an inconsistent perturbation theory already at second order and our methods fail in this case. This failure is in agreement with the BCS theory of superconductivity, which expects formation of *bounded* Cooper pairs even at arbitrarily weak attraction. We could not hope that our perturbative theory around the free Fermi gas could describe a superconductor, an object too different of the Fermi gas.

On the other hand, if the interaction $\lambda_0(\vec{x}-\vec{y})$ is repulsive then, if we manage to prove that the $\lambda^l_h(k^1_1, k^1_3)$ never become negative, then they tend to zero as $h \to -\infty$. In this case we have not only consistence to second order, but also asymptotic freedom in the infrared.

Stating it all again, we cannot solve the evolution equations (41) for the λ-couplings, yet, not even to second order, but it can be seen that the equations for the couplings at Cooper pairs configurations imply that they are always real numbers and that their evolution at second order can be simplified by an angular momentum expansion. They are also *compatible* with a failure of the perturbative expansion even for arbitrarily weak attractive potentials and asymptotic freedom for repulsive potentials, which by its turn is compatible with a normal Fermi liquid and Landau theory. Furthermore, it seems that the λ-couplings for all non-zero Γ, *i.e* non-Cooper pairs, go rapidly to zero when $h \to -\infty$, as well as all μ with non-zero Δ and all ν with non-zero k.

The situation for the μ-couplings with $\Delta = 0$ is similar. We can show exactly the same real-valuedness properties and diagonalization of the second order evolution equations by an angular momentum expansion. The only difference is that the function $\overline{G}_h$ analogous to the G_h appearing in (45), given by

$$\overline{G}_h(k_5, \vec{\omega}_5) \;=\; \frac{16}{(2\pi)^{d+1}}\, 2^{-2h}\, \overline{g}_h(2^{-h}k_6)\, \left[g_h(2^{-h}k_5) \,+\, 2g^s_h(2^{-h}k_5)\right] \,, \tag{46}$$

has not any more a positive real part at forward scattering configurations. This follows from the fact that at these configurations we have $k_6 = k_5$, instead of $k^1_6 = k^1_5$ and $k^0_6 = -k^0_5$ of the Cooper pairs. As a consequence, the evolution equations for the angular momentum coefficients do not have any more a clear behavior related to the sign of the interaction potential, but that does not rule out a bounded evolution of the μ_h at forward scattering.

Another important consequence of our different renormalization procedure is that even to third and higher orders in perturbation theory the λ_h and μ_h evolutions are given respectively by direct and exchange graphs. The simple form of these graphs implies, by use of symmetry properties of the form factors, see [15], section 5.4, that the diagonalization of the evolution at Cooper pairs or forward scattering holds also to any perturbative order. We think that this property will be of fundamental importance when trying to analyze further orders of the perturbative series.

At this point we can resume the discussion by noting that it seems *possible* that, to second order in perturbation theory, Fermi liquids at dimensions $d > 1$ are *normal* if the interaction potential is *repulsive*[5]. This follows from the facts that

[5]This conjecture is in contrast with the non-rigorous result of Kohn and Luttinger for $d = 3$, [7]. On the other hand, it is confirmed to second order and refuted at third in $d = 2$ for the particular model in [12]. Within our methods we do not see why there should be some difference between $d = 2$ and $d = 3$.

the ν_0 counterterm can be fine tuned so that one of the normality conditions hold and that the perturbative theory seems to be consistent for repulsive potentials. We hope to explore further this possibility in other works.

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