POINCARE' RENORMALIZED FORMS

Giuseppe Gaeta*
I.H.E.S., 35 Route de Chartres
91440 Bures sur Yvette (France)
gaeta@ihes.fr

Summary. In Poincaré Normal Form theory, one considers a series of transformations generated by homogeneous polynomials obtained as solution of the homological equation; such solutions are unique up to terms in the kernel of the homological operator. Careful consideration of the higher order terms generated by polynomials differing for a term in this kernel leads to the possibility of further reducing the Normal Form expansion of a formal power series, in a completely algorithmic way. The algorithm is also applied to a number of concrete cases. An alternative formulation, conceptually convenient but computationally unpractical, is also presented, and it is shown that the discussion immediately extends to the Hamiltonian case and Birkhoff normal forms.

* Also at Centre Emile Borel, I.H.P., UMS 839 CNRS/UPMC. On leave from Department of Mathematics, Loughborough University, Loughborough LE11 3TU (England)

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Introduction, motivation, overview.

The theory of *Normal Forms* [1-3] was created by Poincaré in his Thesis, and is still a fundamental tool in our study and understanding of Nonlinear Dynamics.

In this note we will consider systems of ODEs in R^n defined by $\dot{x} = f(x)$ with $f(x)$ a (formal) vector power series having a zero in the origin, and (formal) coordinate transformations given by (formal) power series[1]. By a (generally, only formal) series of near-identity coordinate transformations, it is possible to transform a system to its Normal Form (NF in the following) up to any given order[2] m, and formally for $m = \infty$; thus, the study of the local behaviour of ODEs around a singular point can be reduced to the study of the local behaviour of ODEs which are in NF.

It should be stressed that A) the equivalence between the "original system" and its NF is, in general, only formal: thus, in general a system is *not* conjugated to its NF; however, for m sufficiently small (depending on the system), the system *is* conjugated to its partial NF of order m; B) For a given system, the reduction to NF – although obtained by means of a well defined algorithm – becomes computationally very difficult with the increasing of m: the required computations can be set in terms of linear algebra, but they require at order m to consider a basis of homogeneous monomials of degree m in the variables $x_1, ..., x_n$; calling $M(m,n)$ the dimension of this (this is the number of partitions of m as the sum of n nonnegative integers), we have to then to consider – and invert – matrices of order $[M(m,n)]^n$.

Thus, in practice, when we analyze a *given* system by means of NF techniques, we consider the partial NF of some order m, study the truncation of this at order m – which is by construction in NF – and then resort to other kinds of considerations, typically m-th order averaging, to ensure that the trajectories of the truncated and the full system are near enough (say with a distance less than ε) for long times (typically for $t < 1/\varepsilon^m$).

On the other side, if we want to consider the most general behaviour of a system, we can operate a first reduction to (Jordan) normal form for its linear part, i.e. for the matrix $A = (Df)(0)$, and then study the most general NF compatible with this linear part (the determination of this is also referred to as the problem of *unfolding* of the NF), and the behaviour of the systems given by these NFs.

It is well known that the NF corresponding to a given system is in general *not unique*[3]; correspondingly, the unfolding of NFs – which can be described as the most general *resonant* system[4] – is exhaustive, but in general not minimal: i.e., the Poincaré-Dulac theory provides a list of NFs such that any system can be (formally) reduced to one of the NFs of teh list, but we are not guaranteed that the NFs are pairwise not conjugated.

It is thus of interest to provide a reduced classification of NFs, i.e. a list of NFs (say, corresponding to a given linear part A) in which the redundancies in the Poincaré-Dulac classification, or at least some of them, have been eliminated.

This problem has been considered by several authors [4-9], mainly in the language of Lie algebra filtration; indeed, several theoretical result exist (see e.g. the section on "further normalization" in [6]), and it is actually also possible to define a *unique NF* [9]. However, all these results – to the best of my knowledge – are of difficult concrete implementation.

The purpose of this note is indeed to propose a procedure of "higher order normalization" (which for the formal case $m = \infty$ I call "renormalization") which is completely algorithmic, and not more difficult to implement than the standard Poincaré-Dulac normalization. As the proof of this will be based on consideration of the procedure itself, the discussion to follow will be completely constructive (as in the very spirit of perturbation theory [10]).

Actually, the procedure I propose below is nothing else than a direct generalization – or, more modestly, iteration (see sect.10) – of the one proposed by Poincaré; the main "new" ingredient will be to make use of the freedom in the choice of the generating functions h_k for the near-identity coordinate transformations which comes from the nontriviality of the kernel of $\mathcal{L}_0$, and a control of higher-order effects in this transformation.

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[1] This corresponds to the original Poincaré theory; for the generalizations of normal forms theory to different setting and/or more general class of transformations – e.g. $\mathcal{C}^k$ or topological ones – see [2].

[2] In the following, we will mean by NF the "infinite order" NF, and denote as "partial NF (of order m) the NF of order m.

[3] This is related to the fact that the homological operator $\mathcal{L}_0$ has a nontrivial kernel, see below.

[4] I.e. a system with nonlinear part in $\mathrm{Ker}(\mathcal{L}_0^+)$, see below.

Not surprisingly if one considers that – as mentioned above – previous results were obtained in the framework of Lie algebra, we will find it convenient to consider Lie-Poincaré [11,12] rather than standard Poincaré transformations (once again, details are given below). However, I will on purpose *not* discuss the Lie-theoretic side of the procedure described here: on the one side, the Lie-theoretic frame is clearly discussed by other authors; and on the other, I want to focus attention on the computational side and show how – as already stated – this higher order normalization is completely in the original frame of the Poincaré approach.

Plan of the paper. The paper is divided into three parts. In the first part (sections 1-4) we will recall the classical (Poincaré-Dulac) theory of NFs; this, although well known (see e.g. [1]), is discussed in some detail both to fix notation and to point out some features which are usually not discussed but which will be relevant for the discussion to follow. Thus, after briefly introducing the general setting of NFs theory (sect.1) and the detailed computation of the effect of a Poincaré transformation (sect.2), we will recall the Poincaré algorithm for reduction to NF (sect.3), and the formulation based on Lie-Poincaré, rather than standard Poincaré, transformations (sect.4). As this is perhaps less well known than the standard one, we also discuss it in more detail in the Appendix A.

In the second part (sections 5-11) we introduce the "higher normalization" (and the "renormalization" *tout court*), prove constructively – i.e. by giving a completely explicit algorithm – our main result, i.e. that any system can be transformed into m-th normalized form for any m (and formally into renormalized form, i.e. for $m = \infty$) by a formal series of Poincaré transformations, and discuss the relevance and limitation of this. In more detail, we first introduce "higher order homological operators" (sect.5) and discuss how these are related to the non-unicity of the standard NF (sect.6). We will then discuss as one can use "higher order effects" in the Poincaré transformations (i.e; effects on f_{k+m} for the transformation with generator h_k) to further simplify the NF; we will at first show explicitely and in some detail this for $m = 1$ and $m = 2$ (sect.7); this will provide an obvious motivation for the introduction of the required abstract functional setting (sect.8), in terms of which the generalization of the discussion of section 7 will be immediate, and we will give an explicit constructive – and easy – proof of the general result (sect.9). We will also discuss a slightly different approach, based on fully "iterate" the standard normalization several times (sect.10); this could provide perhaps an easier conceptual understanding, as it is an even more direct extension of the Poincaré procedure, but is computationally very inconvenient, as it is based on "successively sweeping all orders" several times, each time considering higher order effects. The discussion of several questions stemming from our construction, and of some of its advantages and limitations, is then given (sect.11).

The third part (sections 12-14) is devoted to examples; in all the examples we consider one is actually able to determine the *full* renormalized form (i.e. the unfolding can be determined up to $m = \infty$), and in many of them this is dramatically simpler than the standard Poincaré-Dulac NF. In particular, we consider the classical two-dimensional resonant problem with linear part corresponding to rotations (sect.12), i.e. with eigenvalues $\sigma = (\lambda, -\lambda)$; the more general two dimensional problem with eigenvalues $\sigma = (\lambda, -n\lambda)$ (sect.13); and a number of three-dimensional problems (sect.14).

We add then two appendices; in the first we discuss in more detail the Lie-Poincaré transformation (app.A), while the second is devoted to the extension of our approach and result to the Hamiltonian case and Birkhoff normal forms (app.B).

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1. General setting.

The Poincaré theory of Normal Forms [1-3] for dynamical systems, i.e. for first order autonomous smooth ODEs of the form

$$\dot{x} = f(x) \qquad , \; x \in R^n \; , \; f : R^n \to R^n \tag{1.1}$$

which we also call *dynamical systems*, or equivalently for vector fields

$$X = \sum_{i=1}^n f^i(x) \frac{\partial}{\partial x^i} \; , \tag{1.2}$$

is based on systematically employing near-identity changes of coordinates with homogeneous vector polynomial functions as generator.

One is interested in f being a formal power series, i.e.

$$f(x) = \sum_{k=0}^\infty f_k(x) \tag{1.3}$$

with $f_k(x)$ homogeneous of order $(k+1)$ in the x.

We denote by V the set of vector formal power series $f : R^n \to R^n$ which have the origin as a fixed point, and by $V_k \subset V$ the set of polynomial vector functions homogeneous of order $(k+1)$; obviously,

$$V \;=\; \sum_{k=0}^\infty {}^{\oplus} \; V_k \; . \tag{1.4}$$

It will be useful to define the bracket $\{.,.\} : V \times V \to V$ given by

$$\{f,g\} = (f \cdot \nabla) g - (g \cdot \nabla) f \equiv f^i \frac{\partial g}{\partial x^i} - g^i \frac{\partial f}{\partial x^i} \; ; \tag{1.5}$$

this expresses the Lie commutator of vector fields when we look at the component of vector fields in the x coordinates; that is, for $X = f^i \partial_i$ and $Y = g^i \partial_i$, we have $[X,Y] = h^i \partial_i$ with $h = \{f,g\}$. Notice that

$$\{.,.\} \; : \; V_k \times V_m \; \to \; V_{k+m} \; . \tag{1.6}$$

The (standard) homological operator $\mathcal{L}_0$ can be defined in terms of this bracket, as $\mathcal{L}_0(.) = \{f_0,.\}$; by (1.6), $\mathcal{L}_0 : V_k \to V_k$.

In the following, we will need (linear) operators acting between the spaces V_k, and in particular we will have to consider the complementary sets of the ranges of such operators; it is thus convenient to introduce a scalar product in V (actually, in each of the V_k), so that we can consider the adjoint operators.

It turns out that the convenient scalar product is defined as follows [1,13,14]. First of all, we notice that each of the V_k is a finite dimensional vector space. In each of these, we can choose a basis $e_{\mu,j}(x)$ of functions which have components

$$e^i_{\mu,j}(x) = x^\mu \delta_{i,j} = (x^1)^\mu_1 ... (x^n)^\mu_n \delta_{i,j} \; ; \tag{1.7}$$

we define then a scalar product in V_k as

$$\left(e_{\nu,j}, e_{\mu,i} \right)_k \; = < \mu, \nu > \; \delta_{i,j} \; , \tag{1.8}$$

where $< .,. >$ is a scalar product in the space of monomials (in $x^1 ... x^n$); the customary – and convenient – choices are either $< \mu,\nu >= \delta_{\mu,\nu} \equiv \prod_{i=1}^n \delta_{\mu_i,\nu_i}$ (this is the standard choice [1], or the Bargmann [14,15] scalar product[5]

$$< \mu, \nu > = \; [\partial_\mu x^\nu]_{x=0} \; = \; \prod_{i=1}^n \; (\mu_i!) \; \delta_{\mu_i,\nu_i} \quad ; \tag{1.9}$$

[5] This has the advantage that, with $\mathcal{L}_0 = \{Ax,.\}$, the adjoint of this corresponds simply to the adjoint matrix, i.e. $\mathcal{L}_0^+ = \{A^+ x,.\}$.

The scalar product in V is then naturally defined in terms of these as $(f,g) = \sum_k (f_k, g_k)_k$.

2. Poincaré transformations.
One considers then near-identity changes of coordinates of the form

$$x = y + h_k(y) \qquad , \quad h_k \in V_k \ , \tag{2.1}$$

also called Poincaré transformations. We denote by Γ the jacobian of the change of coordinates, i.e. ${\Gamma^i}_j = \partial h_k^i / \partial y^j$. Under the change of coordinates (2.1), our system (1.1) is transformed into

$$\dot{y} = [I + \Gamma]^{-1} \ f\left(y + h_k(y)\right) \ . \tag{2.2}$$

For y – and therefore x – small enough, $\Lambda = (I+\Gamma)^{-1}$ does surely exist, and we can write it in a power series as $\Lambda \equiv (I+\Gamma)^{-1} = \sum_{m=0}^{\infty} [(-1)^m (\Gamma)^m]$.

Similarly, we can expand $f_m(y + h_k(y))$ as a power series; we write $J = (j_1, ..., j_n)$, $|J| = \sum_i j_i$. With this multiindex notation, $\partial_J := \partial_1^{j_1} ... \partial_n^{j_n}$, and similarly $h_k^J := (h_k^1)^{j_1} ... (h_k^n)^{j_n}$. We define the operators $\Phi_h^r = (1/r!) \sum_{|J|=r} \left(h^J \cdot \partial_J\right)$ (representing all the partial derivatives of order $|J|$), and in terms of these the system (2.2) can then be written as

$$\dot{y} \ = \ \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \left[(-1)^s \ \Gamma^s \Phi_{h_k}^{r-s} \right] \ f_m(y) \ . \tag{2.3}$$

Thus we see that, in a Poincaré transformation with generator $h_k \in V_k$ each term f_m is transformed into a new $\widetilde{f}_m$ given[6] by

$$\widetilde{f}_m \ = \ f_m \ + \ \sum_{p=1}^{[m/k]} \left[\sum_{s=0}^{p} (-1)^s \Gamma^s \Phi_{h_k}^{p-s} \right] f_{m-kp} \ . \tag{2.4}$$

One could, in principle, also obtain explicit formulas for terms of all degrees, but these become quickly too involved to be of practical use. We notice, however, that the terms of degree smaller than k are not changed at all,

$$\widetilde{f}_m = f_m \quad \forall m < k \ , \tag{2.5}$$

and the terms of degree $k \le m < 2k$ are changed according to

$$\widetilde{f}_{k+\nu} = f_{k+\nu} + [\Phi_h - \Gamma] f_\nu \quad (0 < \nu < k) \ . \tag{2.6}$$

3. Transformation to Poincaré normal form.
The transformation to Poincaré normal form is given by a well known algorithm, which is just the same if we consider the Poincaré or the Lie (see next section) form for the changes of coordinates. Indeed, in both cases we have that for the transformation with generator $h_k \in V_k$ it results $\widetilde{f}_m = f_m$ for $m < k$, and

$$\widetilde{f}_k = f_k + \{h_k, f_0\} = f_k - \mathcal{L}_0(h_k) \ ; \tag{3.1}$$

the expression for higher order terms are (at least for $m \ge 2k$) more involved, and differ slightly in the two approaches.

We consider then sequentially the terms in V_k for $k = 1, 2, 3, ...$ (up to any desired finite order m, or formally for $m \to \infty$), and choose suitable generators h_k; in this way we can normalize sequentially the terms f_k, and successive changes of coordinates of higher order leave these unchanged[7].

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[6] Here the square brackets in $[m/k]$ denotes integer part.

[7] In general, as mentioned in the Introduction, the series of changes of coordinates and of Poincaré transformations defined in this way is only formal (i.e. does not converge in any neighbourhood of the origin), even for finite – but sufficiently large – m; here we will not be concerned by this problem, as we deal with formal power series.

The "suitable" generator h_k mentioned here is obtained as solution to the homological equation; we first define the projection operator

$$P_0 : V \to \mathrm{Ran}(\mathcal{L}_0) = \left[\mathrm{Ker}(\mathcal{L}_0^+)\right]^\perp \tag{3.2}$$

(this could be properly defined in each of the V_k considering the restriction $\mathcal{L}_{0,k}$ of $\mathcal{L}_0$ to V_k, and the projection operators[8] $P_0^{(k)} : V_k \to \mathrm{Ran}\,(\mathcal{L}_{0,k})$; with these, $P_0 = \sum^\oplus P_{0,k}$), and then require that h_k solves the homological equation

$$\mathcal{L}_0(h_k) \ = \ P_0\,(f_k) \ . \tag{3.3}$$

The solutions to this are given by

$$h_k \ = \ \mathcal{L}_0^+\,(P_0 f_k) \ + \ \ell_k \ , \tag{3.4}$$

where $\ell_k(x)$ is an arbitrary function[9] in $\mathrm{Ker}(\mathcal{L}_0)$.

In this way, $\widehat{f}_k \in [\mathrm{Ran}(\mathcal{L}_0)]^\perp = \mathrm{Ker}\left(\mathcal{L}_0^+\right)$. As it is well known, proceeding in this way one finally arrives to a system $\dot{x} = f^*(x)$ which is in normal form (up to any given order n), i.e. such that all the $\widehat{f}_k(x)$ with $k \geq 1$ are in $\mathrm{Ker}(\mathcal{L}_0^+)$, and this means that all the nonlinear terms (up to any given order n) are resonant[10] with the linear part of the system.

4. Lie transformations.

A slightly different way of approaching Normal Forms is based on Lie – rather than Poincaré – transformations [11,12]. In this case, the change of coordinates is given by the time-one action[11] of a vector field H_k given by

$$H_k \ = \ h_k^i(x)\frac{\partial}{\partial x^i} \tag{4.1}$$

so that the change of coordinates is written as

$$\widetilde{x} \ := y \ = \ \left[e^{-\lambda H_k} x\right]_{\lambda=1} \ . \tag{4.2}$$

This has several advantages: first of all, we do not have to worry about the domain of existence of the inverse change of coordinates [11]; second, we are dealing with actions of vector fields and we can use Lie group theory; and finally, we have a representation of the vector field X in the new coordinates which is easier to handle.

In this way, X is transformed into

$$\widetilde{X} \ = \ \left[e^{\lambda H_k}\ X\ e^{-\lambda H_k}\right]_{\lambda=1} \ , \tag{4.3}$$

and this can be explicitely computed by the Baker-Campbell-Haussdorf formula [12,17].

We will just recall the final result, i.e. that $\widetilde{X}$ can be written as $\widetilde{X} = \widetilde{f}^i(x)\partial_i$, with

$$\widetilde{f} \ = \ f + \{h,f\} + \frac{1}{2}\{h,\{h,f\}\} + \frac{1}{6}\{h,\{h,\{h,f\}\}\} + ... \ . \tag{4.4}$$

More details of the derivation of this formula are given in the appendix.

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[8] clearly, the adjoint and the orthogonal space should be understood with respect to the scalar product defined in V.

[9] This freedom in choosing ℓ_k, i.e. h_k, does also come into play when we have to take into normal form not a single vector field, but a Lie algebra of vector fields [16].

[10] By "resonant" we mean precisely that they are in $\mathrm{Ker}(\mathcal{L}_0^+)$; however, denoting by λ_k ($k = 1, ..., n$) the eigenvalues of the matrix A corresponding to the linear part, and choosing as basis vectors in R^n the corresponding eigenvcetors, the resonant vectors $e_{\mu,j}$ (see sect.1) are those for which the *resonance relation* $\sum_k \mu_k \lambda_k = \lambda_j$ is satisfied, with μ_k nonnegative integers.

[11] That is, the time-one flow for the one-parameter group of diffeomorphisms of R^n generated by the vector field referred to.

¿From (4.4) it is easy to derive formulas for the decomposition of $\widetilde{f}$ into homogeneous factors, i.e. for $\widetilde{f} = \sum_m \widetilde{f}_m$. We introduce the notation $\mathcal{H}(.) = \{h, .\}$, and with this we have

$$\widetilde{f}_m \ = \ \sum_{s=0}^{[m/k]} \frac{1}{s!} \mathcal{H}^s \left(f_{m-sk} \right) \ . \tag{4.5}$$

Notice that we have written $[m/k]$ for the integer part of (m/k), and defined $\mathcal{H}^0(f) = f$.

5. The homological operators.

We will define a series of homological operators $\mathcal{L}_k$ associated to f; the usual homological operator, which we will denote by $\mathcal{L}_0$, will be the first of these. This definition will suit our way of proceeding, based on Poincaré-Lie transformations and thus on (4.4).

For $f \in V$, $f = \sum f_k$, we define the Lie operator $\mathcal{F} : V \to V$ associated to f as $\mathcal{F} = \{f, .\}$; clearly we can write

$$\mathcal{F} \ = \ \sum_{k=0}^{\infty} \{f_k, .\} \ \equiv \ \sum_{k=0}^{\infty} \mathcal{L}_k \ . \tag{5.1}$$

The operators $\mathcal{L}_k = \{f_k, .\}$ defined in (5.1) are called the series of homological operators associated to f; the operator $\mathcal{L}_0$ coincides with the usual homological operator considered in Poincaré Normal Form theory. Notice that, by (1.6), $\mathcal{L}_k : V_m \to V_{m+k}$. We also denote by $\mathcal{L}_{k,m}$ the restriction of $\mathcal{L}_k$ to V_m.

Notice that when we operate a Poincaré transformation, the f_k – and thus the $\mathcal{L}_k$ – change. However, at each normalization stage we stabilize a new term f_s, and thus the corresponding $\mathcal{L}_s$.

It should be stressed that linear combinations of the homological operators do not permit to describe (4.4), or (4.5), in full generality: they are only related to the first nontrivial term in (4.5). However, it will turn out that, in the procedure we employ in the following, a suitable choice of the h permits to analyze iterated Poincaré-Lie transformations in terms of the $\mathcal{L}_k$ alone.

6. Non-unicity of Poincaré normal forms.

In the Poincaré procedure[12], shortly described above, one has no need to keep track of the effect of the transformation generated by $h_k \in V_k$ on terms of higher order: indeed, this will generate additional terms in V_s, in principle at all the higher orders $s > k$, but these can then be disposed of by the successive Poincaré transformations with generator h_s.

This point should be considered with some extra care: indeed, while the terms generated by $\mathcal{L}_0(h_s)$ are in $\mathrm{Ran}(\mathcal{L}_0) \cap V_s = \mathrm{Ran}\,(\mathcal{L}_{0,s})$, those appearing as "higher order terms" due to the transformation generated by h_k (with $k < s$) cannot be guaranteed to be (and in general, are not) in the same space; thus, not all of these can then be eliminated[13] by suitably choosing h_s.

However, this feature should not be necessarily seen as a drawback in the Poincaré procedure: if on the one side this shows that we could introduce resonant terms which were not initially present in our system, on the other side we could use the same mechanism to eliminate (some of the) resonant terms initially present, or generated as higher order terms by previous changes of coordinates. More in general, we could use these higher order effects of the Poincaré transformations to further normalize – we will use the term *renormalize* – the Poincaré normal form.

Indeed, as mentioned in the introduction, it is well known that the Poincaré normal form is by no means unique, i.e. that two different Poincaré normal forms can be conjugated. The idea of further normalization is also not new, and has been considered by several authors [4-9], mainly in the context of Lie algebras filtration; here we mention in particular the work of Broer [7], the "further normalization" of van der Meer [6], and the "unique normal forms" studied by Baider [9]. However, such an approach seems (at least to the present author) of difficult implementation for the study of concrete dynamical systems. Here, we want to study the same problem from a direct point of view, i.e. considering the properties of Poincaré transformations

[12] Here, by this we mean indifferently the usual Poincaré scheme, or the Poincaré-Lie one.

[13] If this was the case, the reduction to NF would amount to cancelling from the original system the nonresonant terms leaving the resonant ones unchanged; unfortunately, life is not so easy!

and the explicit higher order effects, see (2.4) and (4.5). This allows to give a well definite algorithm, which – once a basis is chosen in each of the V_k, e.g. the one given by the $e_{\mu,j}(x)$ considered in section 4 – only requires linear algebra computations. Moreover, these reduce to consideration of the action of (higher order) homological operators and to solution of relevant (higher order) homological equations, thus representing a natural and straightforward generalization of the Poincaré scheme.

7. Poincaré renormalization – I.

We can use the considerations of the previous section, and the general formulas obtained for the higher order action of a Poincaré transformation, to devise a scheme of Poincaré normalization which includes iterated normalization (renormalization) and leads to a simplified normal form unfolding. We choose to work in the Lie-Poincaré scheme; the reason for this preference will be clear in the following.

We sketch below the construction of such renormalized forms, first for orders two and three (to let the reader grasp an intuitive understanding, and at the same time provide motivation for the abstract setting introduced in the next section), and then giving the general algorithm.

It should be stressed that we give directly a convenient "open" algorithm, i.e. one which does not require to fix in advance the order to which we want to go in the renormalization procedure; a slightly different approach, in which we fix this in advance, is maybe a more immediate extension of the Poincaré procedure and could therefore provide a better conceptual understanding (but is computationally not convenient); this is discussed below in section 11.

We start from a system like (1.1), (1.3), and denote by $A = (Df)(0)$ the matrix corresponding to the linear part[14] of f (which will not be modified by our transformations). We rewrite the system, for further reference, as

$$\dot{x} \ = \ \sum_{k=0}^{\infty} f_k^{(0)}(x) \ ; \tag{7.1}$$

the upper index on f_k (and similar ones from now on) refers to the number of Poincaré transformation of order k applied so far.

We can then, by means of the usual Poincaré transformation with generator $h_1^{(0)}$ chosen as solution to the homological equation for $k = 1$, take the term $f_1^{(0)}$ into NF, i.e. transform it to

$$f_1^{(1)} \ = \ f_1^{(0)} - \mathcal{L}_0 \left(h_1^{(0)} \right) \ \in \ \mathrm{Ker}\left(\mathcal{L}_0^+ \right) \cap V_1 \ . \tag{7.2}$$

Let us now pass to consider the term in V_2 (which is now already changed, but that we still denote by $f_2^{(0)}$ for ease of notation); by the usual Poincaré procedure, i.e. choosing $h_2^{(0)}$ as solution to the homological equation at order $k = 2$, this is changed into

$$f_2^{(1)} \ = \ f_2^{(0)} - \mathcal{L}_0 \left(h_2^{(0)} \right) \ \in \ \mathrm{Ker}\left(\mathcal{L}_0^+ \right) \cap V_2 \ . \tag{7.3$'$}$$

However, we know that we can still change this by a transformation with generator $h_1^{(1)} \in V_1$. It is clear that now this $h_1^{(1)}$ cannot be completely arbitrary, if we want to keep the already normalized term $f_1^{(1)}$ in its present – and satisfactory – form; thus, to leave this untouched, we are obliged to require

$$h_1^{(1)} \ \in \ \mathrm{Ker}(\mathcal{L}_0) \cap V_1 \ . \tag{7.3$''$}$$

It is clear that in this way we can only eliminate terms which are in $\mathrm{Ran}(\mathcal{M}_1)$, where we have defined

$$\mathcal{M}_1 = [\mathcal{L}_1]_{\mathrm{Ker}(\mathcal{L}_0)} \ . \tag{7.4}$$

[14] We can, if we prefer, preliminarly operate a linear change of coordinates to take A into (real or complex) canonical form; however, this would not be of great use: even if A is semisimple, and thus can be diagonalized, our present considerations on higher order homological operators would not benefit much from this fact.

Indeed, notice that (7.3') implies that in (4.5) we are reduced to $\widetilde{f}_2 = f_2 + \{h_1, f_1\}$. It is easy to see that this same remark, suitably generalized, will also hold in general for our proposed procedure: the transformations will be given by action of the series of homological operators (this represents the main explicit advantage, in the present discussion, of considering Lie transformations).

Thus, this second normalization of the term in V_2 will transform $f_2^{(1)}$ into

$$f_2^{(2)} \ = \ f_2^{(1)} - \mathcal{L}_1 \left(h_1^{(1)} \right) \ = \ f_2^{(1)} - \mathcal{M}_1 \left(h_1^{(1)} \right) \ , \tag{7.5}$$

and by suitable choice of $h_2^{(1)}$ we can have

$$f_2^{(2)} \ \in \ \left[\mathrm{Ker}\left(\mathcal{L}_0^+ \right) \cap V_2 \right] \cap \mathrm{Ker}\left(\mathcal{M}_1^+ \right) \ . \tag{7.6$'$}$$

If we denote by P_1 the operator of projection on $\mathrm{Ran}(\mathcal{M}_1)$, this "suitable choice" amounts to

$$h_1^{(1)} \ = \ \mathcal{M}_1^+ \ P_1 \ f_2^{(1)} \quad . \tag{7.6$''$}$$

We could then repeat the same procedure for the term $f_3^{(0)}$, by applying Poincaré transformations with generators $h_3^{(0)}$ (solution of the homological equation at order $k = 3$), $h_2^{(1)}$, and $h_1^{(2)}$; this time, however, we should not only make sure that the $h_2^{(1)}$ does not modify the already established $f_2^{(2)}$, i.e. that

$$h_2^{(1)} \ \in \ \mathrm{Ker}\left(\mathcal{L}_0 \right) \ , \tag{7.7$'$}$$

but also that $h_1^{(2)}$ does not modify neither the $f_2^{(2)}$ nor the $f_1^{(1)}$, i.e. that

$$h_1^{(2)} \ \in \ \mathrm{Ker}\left(\mathcal{L}_0 \right) \cap \mathrm{Ker}\left(\mathcal{L}_1 \right) \ . \tag{7.7$''$}$$

Again, these requirements will guarantee that in (4.5) only the term $\mathcal{L}_1(h_2^{(1)})$ and $\mathcal{L}_2(h_1^{(2)})$ are effectively giving a contribution to the transformed f_3.

The general scheme of construction should at this point be quite clear, and we can pass to describe it in abstract terms.

8. Functional setting.

We will consider some chains of spaces and operators, together with the chain of spaces $V_k \subset V$ and the chain of operators $\mathcal{L}_k : V \to V$ introduced above. These definitions will be based on a formal power series $f = \sum f_k$ (as it was already the case for the $\mathcal{L}_k$).

We define the spaces $H^{(p)} \subseteq V$ by $H^{(0)} = V$ and, for $p \geq 1$,

$$H^{(p)} \ = \ \mathrm{Ker}(\mathcal{L}_0) \cap ... \cap \mathrm{Ker}(\mathcal{L}_{p-1}) \ = \ \bigcap_{s=0}^{p-1} \mathrm{Ker}\left(\mathcal{L}_s \right) \tag{8.1}$$

It is obvious that $H^{(p+1)} \subseteq H^{(p)}$ (so that the $H^{(p)}$ realize a filtration of V).

We define then the operators $\mathcal{M}_k$ as the restriction of $\mathcal{L}_k$ to $\mathrm{Ker}(\mathcal{L}_0) \cap ... \cap \mathrm{Ker}(\mathcal{L}_{k-1})$, i.e. $\mathcal{M}_p = \mathcal{L}_p \big|_{H^{(p-1)}}$. Clearly,

$$\mathrm{Ker}\left(\mathcal{M}_p \right) \ = \ \bigcap_{s=0}^{p} \mathrm{Ker}\left(\mathcal{L}_s \right) \ = \ H^{(p+1)} \ . \tag{8.2}$$

Next, we define the spaces $F^{(p)}$ by $F^{(0)} = V$ and

$$F^{(p)} = \bigcap_{k=0}^{p} \left[\mathrm{Ran}\left(\mathcal{M}_k \right) \right]^\perp = \bigcap_{k=0}^{p} \mathrm{Ker}\left(\mathcal{M}_k^+ \right) \ ; \tag{8.3}$$

hence the $F^{(p)}$'s satisfy $F^{(p+1)} \subseteq F^{(p)}$ (and realize a filtration of V, as it was already the case for the $H^{(p)}$'s). It should be noticed that $F^{(p+1)} = F^{(p)} \setminus \left[\mathrm{Ran}\,(\mathcal{M}_p) \cap F^{(p)} \right]$.

We can also define projection operators for each of these spaces; we denote them by:

$$\begin{array}{ccc} \pi_s : V \to \mathrm{Ker}(\mathcal{L}_s) & , & \widehat{\pi}_s : V \to \mathrm{Ran}(\mathcal{L}_s^+) \\ \chi_s : V \to \mathrm{Ker}(\mathcal{L}_s^+) & , & \widehat{\chi}_s : V \to \mathrm{Ran}(\mathcal{L}_s) \ . \end{array} \tag{8.4}$$

It will be useful to define the composition of projection operators given by

$$\mu_s = \pi_{s-1} \circ \pi_{s-2} \circ \ldots \circ \pi_0 \qquad (s \geq 1) \tag{8.5}$$

(and $\mu_{s,k}$ as the restriction of this to V_k). Later on we will use the notation μ_0: this should be meant as the identity operator.

We also consider the projection on the range of $\mathcal{M}_s$, i.e.

$$P_s : V \to \mathrm{Ran}\,(\mathcal{M}_s) \tag{8.6}$$

In this way, we can redefine our spaces and operators using the projection operators; in particular we have $H^{(s)} = \mu_s\,[V]$ and $\mathcal{M}_s = \mathcal{L}_s \circ \mu_s$.

Remark 1. Notice that we could consider the decomposition of the spaces and operators introduced above according to (1.4), i.e. considering their intersection with, and restriction to, the spaces V_k. As in each of the V_k (considered as vector spaces) we have a finite dimensional basis, the relevant operators – in particular, the $\mathcal{M}_{p,k} = \mathcal{M}_p\big|_{V_k}$ – can be written in matrix form using these bases.

Thus, to implement the above remark, we define the intersections $H_k^{(p)} := H^{(p)} \cap V_k$; these do satisfy

$$H_k^{(p+1)} = H_k^{(p)} \cap \mathrm{Ker}(\mathcal{L}_{p,k}) \ . \tag{8.7}$$

We can then also consider the finite dimensional operators $\mathcal{M}_{p,k} : H_k^{(p)} \to V_{k+p}$ given by the restriction of $\mathcal{L}_{p,k}$ (recall this is the restriction of $\mathcal{L}_p$ to V_k) to $H_k^{(p)}$; clearly, $H_k^{(p+1)} = \mathrm{Ker}(\mathcal{M}_{p,k})$. The adjoints of these, defined of course with respect to the scalar product defined in the V_k and thus in V, satisfy $\mathcal{M}_{p,k}^+ : V_{k+p} \to H_k^{(p)}$. Finally, the intersections $F_k^{(p)} := F^{(p)} \cap V_k$ do also satisfy, for $k \geq p$,

$$F_k^{(p+1)} = F_k^{(p)} \cap \mathrm{Ker}(\mathcal{M}_{p,k-p}^+) \ . \tag{8.8}$$

9. Poincaré renormalization – II.

We can now, with the notation introduced in the previous section, describe in general and abstract terms the procedure sketched in section 7 above.

Definition 1. *The dynamical system $\dot{x} = f(x) = \sum f_k(x)$ (the vector field $X = f^i(x)\partial_i = \sum f_k^i(x)\partial_i$; the formal power series $f(x) = \sum f_k(x)$) is in* **Poincaré renormalized form** *up to order n if $f_k \in F_k^{(k)}$ for all $k \leq n$.*

Proposition 1. *Any dynamical system (vector field, formal power series) can be brought into Poincaré renormalized form up to any desired order n by means of a formal series of Lie-Poincaré transformations.*

`Proof.` We will prove constructively the above proposition by giving a well defined algorithm for the transformation to Poincaré renormalized form.

We operate sequentially for $k = 1, 2, ..., n$ in the following way. If $f_k^{(0)}$ is the term of order k after performing the required transformation at orders up to $k-1$, we operate then a series of Lie-Poincaré transformations with generators $h_k^{(0)}, h_{k-1}^{(1)}, ..., h_1^{(k-1)}$, where $h_p^{(s)} \in H_p^{(s)}$; this condition guarantees, see (4.5), that at each step the transformation generated by $h_{k-p}^{(p)}$ will transform $f_k^{(p)}$ into

$$f_k^{(p+1)} \ = \ f_k^{(p)} \ - \ \mathcal{M}_p \left(h_{k-p}^{(p)} \right) \ . \tag{9.1}$$

The $h_k^{(0)}$ is chosen as the solution to the standard homological equation (3.3), i.e. as

$$h_k^{(0)} = \mathcal{L}_0^+ \ P_0 \ f_k^{(0)} \equiv \mathcal{M}_0^+ \ P_0 \ f_k^{(0)} \ ; \tag{9.2}$$

the $h_{k-p}^{(p)}$ should be chosen as the projection on $H_{k-p}^{(p)}$ of the solution to the (higher order) homological equations

$$P_p f_k^{(p)} - \mathcal{M}_p \left(h_{k-p}^{(p)} \right) = 0 \ , \tag{9.3}$$

which means, explicitely,

$$h_{k-p}^{(p)} \ = \ \mu_p \circ \mathcal{M}_p^+ \circ P_p \ \left(f_k^{(p)} \right) \tag{9.4}$$

(with $\mu_0 = I$, the (9.2) is included in this formula as well).

Clearly, in this way we arrive in the end – i.e. after applying the procedure to terms of order $k = 1, 2, 3, ...n$ – at a system

$$\dot{x} \ = \ f^*(x) \ = \ \sum_{k=0}^{\infty} f_k^*(x) \tag{9.4}$$

in which, for $k \le n$,

$$f_k^*(x) \ = \ f_k^{(k)} \ \in \ F_k^{(k)} \ . \tag{9.5}$$

The proof is thus complete. $\odot$

Remark 2. We would like to point out that one can also work following a different order: once we have fixed the order n up to which we want to put the system in Normal Form, we can first proceed to the usual Poincaré normalization, i.e. consider the transformations generated on the $h_k^{(0)}$, and corresponding to the action of $\mathcal{L}_0$, for $k = 1, ..., n$; we can then consider the transformations, generated by $h_k^{(1)}$, corresponding to the action of $\mathcal{L}_1$ (or more precisely of $\mathcal{M}_1$), i.e. "second-normalize the normal form", and so on. This will be illustrated in section 11 below.

Remark 3. We would also like to remark explicitely that, although we have preferred to avoid a cumbersome notation, one could consider restriction of operators to the relevant V_k subspaces, and thus transform – by means of bases in each of the V_k – the above equations in algebraic ones. In order to do this, we should use the notation introduced at the end of section 8, and perform our computations using explicitely the basis $e_{\mu,i}$ introduced in section 1. In computer-assisted computations, it would be appropriate to proceed in this way.

10. Renormalization by iterated normalizations.

We have discussed so far a convenient construction of the renormalized forms; in this section we discuss a slightly different construction; this is computationally less convenient, but it has the advantage that it amounts literally to repeating several times the Poincaré procedure, each time using an higher – and stabilized at previous stages – homological operator $\mathcal{L}_k$ in lieu of the standard $\mathcal{L}_0$.

It should be stressed that although the result we can prove in this way is the same, and although the unfolding of renormalized form corresponding to the given linear part is exactly the same, following the two algorithms we can obtain a *different* renormalized form.

We will use the same notation introduced above[15]. Suppose now to have already transformed the initial system into its Poincaré-Dulac normal form, which we rewrite as

$$\dot{x} \ = \ f_0^{(0)}(x) + \sum_{k=1}^{\infty} f_k^{(1)}(x) \tag{10.1}$$

where $f_0^{(0)} = f_0$ and $f_k^{(1)} \in \mathrm{Ker}(\mathcal{L}_0^+) \cap V_k \ := \ F_k^{(1)}$.

We can now start to consider again Lie-Poincaré transformations with generators $h_k^{(1)} \in V_k$ (the upper index reminds that we have already operated one Poincaré normalization); in order to be sure these do not

[15] As the reader will immediately notice, the discussion will also be very similar to the previous one.

take the term f_k (of the same order) out of normal form, we must require that $h_k^{(1)} \in \mathrm{Ker}(\mathcal{L}_0)$: in this way no term in V_k is produced, but we still have terms produced at higher orders, i.e. we are changing f_m for $m > k$.

Let us at first concentrate in particular on the effects at order $k+1$. The term f_{k+1} changes, due to (4.5), according to

$$f_{k+1} \ \to \ f_{k+1} - \mathcal{L}_1(h_k^{(1)}) \ . \tag{10.2}$$

This means that we can change $f_2^{(1)}$ into a

$$f_2^{(2)} = f_2^{(1)} - \mathcal{L}_1(h_1^{(1)}) \tag{10.3}$$

without changing $f_1^{(1)}$ (because of $h_1^{(1)} \in \mathrm{Ker}(\mathcal{L}_0)$, as we required above).

Notice that in this way we do not generate nonresonant terms at higher orders, as the transformation operates by iterated brackets between functions in $\mathrm{Ker}(\mathcal{L}_0)$ [equivalently, commutators of vector fields which all commute with $X_0 = f_0(x)\partial_x$], see (4.4),(4.5), and this set is obviously closed under the bracket operation[16].

Proceeding in this way for $k = 1, 2, ...$, we can, by a suitable choice of the $h_k^{(1)}$, change the $f_m^{(1)}$ with $m \geq 2$ and eliminate any term in the image of $\mathrm{Ker}(\mathcal{L}_0)$ under $\mathcal{L}_1$, i.e. in the range of $\mathcal{M}_1$. In this way we arrive to a "second normalized form" for our system:

$$\dot{x} \ = \ f_0^{(0)} + f_1^{(1)} + \sum_{k=2}^{\infty} f_k^{(2)} \quad . \tag{10.4}$$

Notice that here $f_0^{(0)} \in V_0$ is still the original one, $f_1^{(1)} \in \mathrm{Ker}(\mathcal{L}_0^+)$ is the one obtained with the first normalization, and all the $f_k^{(2)}$ are in $\mathrm{Ker}(\mathcal{M}_1^+)$.

We can now repeat the same procedure choosing $h_k^{(2)} \in \mathrm{Ker}(\mathcal{L}_0) \cap \mathrm{Ker}(\mathcal{L}_1)$ and concentrating on the effect of the Poincaré transformation generated by this on f_{k+2}, changing all terms with $k \geq 3$ and producing in the end a "third normalized form"

$$\dot{x} \ = \ f_0^{(0)} + f_1^{(1)} + f_2^{(2)} + \sum_{k=3}^{\infty} f_k^{(3)} \quad , \tag{10.5}$$

and so on.

The general procedure is now clear; it is also clear that in this way we can, operating n successive "higher order" normalizations, take the system into renormalized form up to order n. We will also state the general result that can be obtained in this way as follows.

Definition 2. *The dynamical system $\dot{x} = \sum_{k=0}^{\infty} f_k$ (the vector field $X = f^i(x)\partial_i$, the formal power series $f(x) = \sum f_k(x)$) is said to be in* n-th normal form *if $f_k \in F_k^{(p)}$ for all k, with $p = \min(k, n)$. When this condition is satisfied for all n (with no upper limit), we say that the system is in* Poincaré renormalized form.

Notice that a system in n-th normal form is also in renormalized form up to order n.

Proposition 2. *Given a dynamical system $\dot{x} = \sum_{k=0}^{\infty} f_k^{(0)}$ (a vector field $X = f^i(x)\partial_i$, a formal power series $f(x) = \sum f_k(x)$), it is always possible to reduce it, by a sequence of n formal Poincaré normalizations, into the n-th normal form*

$$\dot{x} \ = \ \sum_{k=0}^{n} f_k^{(k)} \ + \ \sum_{s=n+1}^{\infty} f_s^{(n)} \ ,$$

where $f_k^{(k)} \in F_k^{(k)}$ for all $k \leq n$.

We can also say formally, considering infinite sequences of Poincaré normalizations, that a system can always be taken into renormalized form.

[16] The same kind of considerations apply when considering higher order normalizations; this is strictly related to the Lie algebra filtrations mentioned in the Introduction, and is a substantial advantage of the Lie-Poincaré approach in this context.

11. Discussion.

We will now briefly discuss some points related to the results obtained in this paper.

Remark 4. First of all, we notice that the procedure introduced here is not only completely constructive and algorithmic, but it can also be implemented on a computer in the same way as the standard Poincaré normalization. Thus, at least at the formal level (i.e. without considering the convergence properties for the series entering the procedure), taking a system into renormalized form is not really more difficult than taking it into standard Poincaré normal form.

Remark 5. Notice also that the main computational difficulties in the computer implementation of (formal) standard Poincaré normalization arise from the large size of involved matrices; this size is however the same, if we stop at the same order, in the renormalization proposed here, so that the latter is also not more demanding than the standard one in what concerns computational resources.

Remark 6. On the other side, the iteration of Poincaré normalizat ion allows to obtain a reduction in the normal form expansion, which could be quite significant as it will be shown by some of the examples given below. In this way, both the normal form classification and the study of systems of ODEs (vector fields, power series) via their normal form can be considerably simplified.

Remark 7. As briefly mentioned above, this "further simplification " of normal forms is closely related to the well known problem of the non-uniqueness of Poincaré-Dulac normal forms for resonant systems: indeed, our procedure amounts to obtain a classification of normal forms in which some redundant ones have been eliminated; also, the procedure can be seen as amounting to a careful choice of the terms $\delta h_k \in \mathrm{Ker}(\mathcal{L}_{0,k})$ which are not selected by the standard Poincaré procedure. It should be stressed that in general *neither the n-th normalized form, nor the renormalized form, will be unique*: that is we remove only partially the non-uniqueness inherent to the Poincaré procedure.

Remark 8. In this respect, as already mentioned in the introduction and in section 6, it is known that one can define a *unique* normal form (see [4-8], and particularly [9]); thus one could wonder where is the advantage in considering instead the renormalized, or the n-th normalized, forms introduced here; the answer is that the theory of unique normal forms is not easily implemented in concrete computations, while the method proposed here is completely constructive and moreover goes through computations which are of the same kind as those required by the standard Poincaré normalization. Thus, in a sense, these renormalized – or n-th normalized – forms represent a useful "compromise" between the simplicity of the algorithm leading to the standard Poincaré-Dulac normal form (at the price of a redundant classification), and the simplicity inherent to having a unique normal form (at the price of undergoing a difficult construction).

Remark 9. It can also be remarked that in the course of our construction, which aims at obtaining $f_k^* \in F_k^{(k)}$, the f_k – and thus the $\mathcal{L}_k$ – change: this means that the whole series of the $H^{(p)}$, $\mathcal{M}_p$ and in particular of the $F^{(p)}$ does also change, so that we are "aiming at a moving target". However, at each normalization stage we stabilize a new term f_s, and thus the $H^{(p)}$ and $F^{(p)}$ get successively stabilized, and at this point we can successfully attain our goal of attaining an $f_k^* \in F_k^{(k)}$.

Remark 10. In our discussion, we have supposed to have fixed a scalar product in each of the V_k and thus – considering these as orthogonal subspaces – in V. It should be stressed that the precise form of this is not essential to our construction (but obviously it is when we have to perform explicit computations).

Remark 11. We would like to recall that all of the above discussion is – as customary also in standard normal forms theory – conducted at a purely formal level[17], i.e. without considering the convergence of the power series determining the transformation into renormalized form. It should be mentioned that recent results [18-20] allow to infer the convergence of the (standard) normalizing transformation from suitable symmetry properties of the system, or of a system which is formally equivalent to it; thus, if the renormalized form unfolding (which is in some cases easy to determine, see the examples below) displays appropriate symmetry properties, it can be used to guarantee the convergence of the standard normalization.

[17] This however does not prevent it from being useful: even in the standard theory, in most concrete cases one performs the normalization up to some finite order N and studies the system truncated at order N, resorting then to different considerations to ensure the equivalence between the truncated system and the full one.

12. Planar vector fields with rotations as linear part (1:1 resonance).

We will consider, as a first and meaningful example, the unfolding of Normal Forms for vector fields in R^2 having linear part $f_0(x) = Ax$ with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{12.1}$$

As it is well known, the Poincaré Normal Forms corresponding to this can be written in the form

$$f(x) = Ax + \sum_{k=1}^{\infty} \left(x_1^2 + x_2^2\right)^k [a_k I + b_k A]\, x , \tag{12.2}$$

where the a_k, b_k are arbitrary real constants. Writing $r^2 = (x_1^2 + x_2^2)$, this reads

$$f_k(x) = \begin{cases} 0 & \text{for } k \text{ odd} \\ r^{2m}\,[a_m I + b_m A]\, x & \text{for } k = 2m. \end{cases} \tag{12.3}$$

We will see the the renormalized normal form unfolding is remarkably simpler. In order to do this, and to avoid trivial steps, we will consider the renormalized form of a system which is already in Poincaré normal form.

Thus, let us consider an f which has already be taken into Poincaré normal form, and proceed to its renormalization; we will consider only the nontrivial transformations (but keep the indices notation introduced above).

Thus, let us first consider the term $f_2^{(0)}$; since it is in normal form (and since for the same reason $f_1 = 0$, i.e. $\mathcal{L}_1 \equiv 0$), we cannot modify it by our algorithm, i.e. it will remain in the form given above,

$$f_2^{(2)} = f_2^{(0)} = r^2 (a_1 I + b_1 A) x . \tag{12.4}$$

We then have $f_3 = 0$, and

$$f_4^{(0)} = r^4 (a_2 I + b_2 A) x ; \tag{12.5}$$

the generator $h_2^{(2)}$ of the transformation

$$f_4^{(2)} = f_4^{(0)} - \mathcal{M}_2 \left(h_2^{(2)}\right) \tag{12.6}$$

must be, for $h_2^{(2)} \in \mathrm{Ker}(\mathcal{L}_0)$, of the form

$$h_2^{(2)} = r^2 (\alpha I + \beta A) x , \tag{12.7}$$

and thus

$$\mathcal{M}_2 \left(h_2^{(2)}\right) = 4\,(a_1 \beta - b_1 \alpha)\, Ax ; \tag{12.8}$$

it is then clear that, unless $a_1 = b_1 = 0$, we can always choose α, β so that

$$f_4^{(2)}(x) = f_4^{(4)}(x) = r^4 a_2 x . \tag{12.9}$$

Let us now consider f_6; now $h_4^{(2)}(x) = r^4 (\alpha I + \beta A) x$ (again for $h \in \mathrm{Ker}(\mathcal{L}_0)$), and

$$\mathcal{L}_2 \left(h_4^{(2)}\right) = [(-2 a_1 \alpha) I + (2 a_1 \beta - 4 b_1 \alpha) A]\, x . \tag{12.10}$$

Thus, if $a_1 \neq 0$, we can eliminate completely f_6 in this way.

It is quite easy to get convinced that, under the same condition $a_1 \neq 0$, the same holds for all the f_{2m}. Indeed, for

$$h_{2(k-1)}^{(3)}(x) = r^{2(k-1)}\, [\alpha I + \beta A]\, x \tag{12.11}$$

(again, $h^{(3)}_{2(k-1)} \in \mathrm{Ker}(\mathcal{L}_0)$ requires this form for h), we have

$$\mathcal{L}_2 \left(h^{(3)}_{2(k-1)} \right) \ = \ - \left[(2(k-2) a_1 \alpha) \, I + (2(k-1) a_1 \beta - 2 b_1 \alpha) \, A \right] x \ ; \tag{12.12}$$

thus, we can eliminate completely all the f_{2m}.

We have thus shown that:

Lemma 1. *If in the Poincaré normal form (12.2) for $f(x)$ the constant a_1 is nonzero, then the corresponding Poincaré renormalized form is given by*

$$f^*(x) \ = \ Ax + r^2 \, (a_1 I + b_1 A) \, x + r^4 a_2 x$$

and thus its unfolding depends on three real parameters.

We can also analyze what happens if the nondegeneracy condition $a_1 \neq 0$ is not satisfied. We assume now that $a_1 = 0$ and $b_1 \neq 0$. ¿From (12.8), it appears that, choosing $\alpha = -b_2/(4b_1)$ in $h^{(2)}_2$, we can still reduce f_4 to $r^4 a_2 x$ (no reduction at all would be possible if $a_1 = b_1 = 0$).

When it comes to considering f_6, (12.10) shows that choosing $\alpha = -b_3/(4b_1)$ in $h^{(3)}_4$ we can arrive to $f^{(3)}_6 = r^6 a_3 x$. We can then proceed further in the renormalization; $\mathcal{L}_3 \equiv 0$, and thus the next – and last possible – step will be $f^{(5)}_6 = f^{(3)}_6 - \mathcal{L}_4 \left(h^{(4)}_2 \right)$, where $h^{(4)}_2 \equiv h = r^2 (\alpha I + \beta A) x$, due to the condition $h \in \mathrm{Ker}(\mathcal{L}_0)$. Recall however that we also have to ask $h \in \mathrm{Ker}(\mathcal{L}_2)$: this condition is readily see to be equivalent to $a_1 \beta = b_1 \alpha$; with our present assumptions, this means that $\alpha = 0$. Thus, we cannot eliminate $f^{(3)}_6$.

We could then check explicitely that the higher order terms, i.e. the f_{2k} with $k \geq 4$, can be completely eliminated.

Rather than going on with discussion of more and more degenerate cases, we will give a general criterion and an inductive proof of it.

Lemma 2. *Let the vector formal power series $f : R^2 \to R^2$ be given by $f(x) = Ax + \sum_{k=1}^{\infty} r^{2k} \left[a_k I + b_k A \right] x$ and let μ be the lowest number such that $a_\mu \neq 0$, and ν the lowest number such that $b_\nu \neq 0$, so that $f(x)$ can be written as*

$$f(x) \ = \ Ax \ + \ \sum_{k=\mu}^{\infty} r^{2k} a_k x \ + \ \sum_{k=\nu}^{\infty} r^{2k} b_k A x \ . \tag{12.13}$$

Then, the Poincaré renormalized form of f up to any given order n is given by

$$f^*(x) \ = \ Ax + r^{2\mu} a_\mu x + r^{2\nu} \beta A x + r^{4\mu} \alpha x \ , \tag{12.14}$$

where $a_\mu \neq 0$ is the same as in (12.13), and the α, β could (possibly, but not necessarily) vanish. In particular, if $\nu > \mu$, then $\beta = 0$.

`Proof.` To see that this is true, it is convenient to use the vector fields notation, with $X = f^i \partial_i = \sum_k X_k$, and $X_k = f^i_k \partial_i$.

It is useful to consider the vector fields D and R corresponding respectively to dilations and rotations in R^2, i.e.

$$D = x_1 \partial_1 + x_2 \partial_2 \quad , \quad R = -x_2 \partial_1 + x_1 \partial_2 \ ; \tag{12.15}$$

moreover, we consider the vector fields $Z_k = r^k D$ and $Y_k = r^k R$ (for k even); these satisfy

$$[a Z_k + b Y_k , \alpha Z_m + \beta Y_m] = (m-k) a \alpha Z_{(m+k)} + (m a \beta - k b \alpha) Y_{(m+k)} \ . \tag{12.16}$$

With the notation introduced above, the effect of $\mathcal{L}_k(h_m)$ with $h_m \in \mathrm{Ker}(\mathcal{L}_0)$ can be computed via

$$[a Z_k + b Y_k , \alpha Z_m + \beta Y_m] \ = \ (m-k) a \alpha Z_{k+m} \ + \ (m a \beta - k b \alpha) Y_{k+m} \ . \tag{12.17}$$

First of all, we notice that we can eliminate all the terms $a_p Z_p$ in f, except the one for $p = 2\mu$: indeed, it suffices to choose each time a $h_{p-\mu} = r^{2(p-\mu)} \, (\alpha I + \beta A) \, x$ with $\alpha = a_p / \left((p - 2\mu) a_\mu \right)$. Notice that by a suitable choice of β (in particular, $\beta = 0$ if $b_\mu = 0$) we can always manage to do this without modifying the term $b_p Y_p$. Let us then assume we eliminate first all the terms $a_p Z_p$ (except $p = \mu$ and possibly $p = 2\mu$) up to $p = n$.

Let us now look at the terms $b_p Y_p$ with p greater than the smaller of μ and ν: it is clear, again by (12.17), that these can be eliminated via the term $\mathcal{L}_\mu(h_{p-\mu})$ by choosing $\beta = b_p / ((p-\mu)a_\mu)$ (if $\mu < \nu$), or via the term $\mathcal{L}_\nu(h_{p-\nu})$ by choosing $\alpha = -b_p / ((p-\nu)b_\nu)$ (if $\nu < \mu$). Notice that if $\nu \leq \mu$, the term $b_\nu Y_\nu$ cannot be eliminated. ⊙

Remark 12. In the symplectic case, i.e. when all the a_k in (12.2) vanish, this corresponds to a classical result [21], recently extended [22].

13. A generalization (1:n resonances).

Consider, as a second example, the DS in R^2 – where we have coordinates (x_1, x_2) – with linear part $f_0(x) = Ax$ given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -n \end{pmatrix} \tag{13.1}$$

with n a positive integer; as we already studied the case $n = 1$, we consider only $n > 1$.

The resonance conditions for this A are

$$\mu_1 - n\mu_2 = 1 \quad \text{or} \quad \mu_1 - n\mu_2 = -n \tag{13.2}$$

which give resonant monomials of the form, respectively

$$\begin{pmatrix} c_1 (x^n y)^k x \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ c_2 (x^n y)^k y \end{pmatrix} \tag{13.3}$$

where c_1, c_2 are arbitrary real constants, and $k \geq 1$ a positive integer.

Thus, the Poincaré-Dulac normal form expansion is

$$f(x) = \begin{pmatrix} x \ + \ \sum_{k=1}^\infty \sigma_k (x^n y)^k x \\ -ny \ + \ \sum_{k=1}^\infty \rho_k (x^n y)^k y \end{pmatrix} \tag{13.4}$$

Let us now try to determine the corresponding PRF. It is clear that we only have to care about the terms f_m with $m = \nu k$, where $\nu = n+1$ and $k \in \mathbf{Z}_+$, as the other ones can be disposed of by the usual Poincaré normalization.

The first nonzero term $f_{\nu q}$ (i.e. the first nonzero resonant term) cannot be eliminated, and we will write it in the form

$$f_{\nu q}(x) = \begin{pmatrix} \alpha (x^n y)^q x \\ \beta (x^{n-1} y)^q y \end{pmatrix} \tag{13.5}$$

(notice that at least one of α, β must be nonzero).

The $H^{(0)} = \mathrm{Ker}(\mathcal{L}_0)$ is identified by (13.3) as well; in order to determine $H^{(\nu q)}$ we have to apply $\mathcal{L}_{\nu q}$ on terms of the form given by (13.3); we write these as

$$h(x) \equiv h_{\nu k}(x) = \begin{pmatrix} a_k (x^n y)^k x \\ b_k (x^n y)^k y \end{pmatrix} \tag{13.6}$$

Applying $\mathcal{L}_{\nu q} = \{f_{\nu q}, .\}$ on the $h_{\nu k} \in H^{(0)}$ we get

$$\mathcal{L}_{\nu q}(h_{\nu k}) \ = \ \begin{pmatrix} (a_k \beta k + a_k \alpha k n - \alpha b_k q - a_k \alpha n q)(x^n y)^{k+q} x \\ (b_k \beta k + \alpha b_k k n - b_k \beta q - a_k \beta n q)(x^n y)^{k+q} x \end{pmatrix} \tag{13.7}$$

Thus, in order to eliminate the term $f_{\nu(k+q)}$ via the action of $\mathcal{L}_{\nu q}$ on $h_{\nu k}$, we have to choose a_k, b_k in such a way that

$$\begin{pmatrix} \beta k + \alpha(k-q)n & -\alpha q \\ -\beta n q & \beta(k-q) + \alpha k n \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \ = \ \begin{pmatrix} \sigma_{k+q} \\ \rho_{k+q} \end{pmatrix} \tag{13.8}$$

This is a linear equation in (a_k, b_k), and can be solved whatever the values of σ_{k+q}, ρ_{k+q} if the determinant of the matrix M on the l.h.s. of (13.8) does not vanish.

This determinant is

$$\Delta = (\alpha n + \beta)^2 k(k-q) \tag{13.9}$$

and thus, unless $\beta = -\alpha n$, we can always eliminate all the higher order resonant terms at the exception of $f_{2\nu q}$.

Setting $k = q$ in M, it is immediate to observe that the terms with $\rho_{2q} = -n\sigma_{2q}$ are in the range of $\mathcal{L}_{\nu q}$ and thus can be eliminated; thus in this nondegenerate case (i.e. for $\beta \neq -\alpha n$) the system is reduced to

$$f(x) \ = \ \begin{pmatrix} x & +\alpha(x^n y)^q x & +n\gamma(x^n y)^{2q} x \\ -ny & +\beta(x^n y)^q y & +\gamma(x^n y)^{2q} y \end{pmatrix} \tag{13.10}$$

Let us consider the degenerate case in which $\Delta = 0$. When $\beta = \alpha n$, the kernel of M^+ corresponds to $\sigma = n\rho$, and thus – eliminating terms which are in the range of $\mathcal{M}_{\nu q}$ – we can still reduce to consider "second normalized" forms where the terms $f_{\nu p}$ with $p > q$ are of the form

$$f_{\nu p}(x) \ = \ \begin{pmatrix} n\rho_p (x^n y)^k x \\ \rho_p (x^n y)^k y \end{pmatrix} \tag{13.11}$$

Let now p be the smallest integer (greater than q) for which $\rho_p \neq 0$; to go beyond (13.11) we have to consider the action of $\mathcal{L}_{\nu p}$, but now we cannot consider all the $h \in H^{(0)}$ acceptable: we have to restrict ourselves to $\mathrm{Ker}(\mathcal{L}_0) \cap \mathcal{L}_{(\nu q)}$. From the above formulas it is easily seen that the $h_{\nu k} \in \mathrm{Ker}\,(\mathcal{L}_{\nu q})$ are those of the form

$$h_{\nu k} \ = \ \begin{pmatrix} a_k (x^n y)^k x \\ -n a_k (x^n y)^k y \end{pmatrix} \tag{13.12}$$

(i.e. $b_k = -n a_k$ in the notation used above).

Acting with $\mathcal{L}_{\nu p}$ on the $h_{\nu k}$ given by (13.12), we get, as it can be seen either by direct computation, either from (13.7) by changing q into p and writing $\alpha = n\beta$ and $b = -na$,

$$\mathcal{L}_{\nu p}(h_{\nu k}) = \sigma_p k(1+n^2)(h_{\nu k}) \sigma_p k(1+n^2) a \begin{pmatrix} 1 \\ -n \end{pmatrix} \quad . \tag{13.13}$$

Again by direct computation or considering the M^+ corresponding to the above M with q replaced by p, $b_k = -n a_k$ and $\alpha = n\beta$, we have that now $\mathrm{Ker}(\mathcal{M}^+_{\nu p}) \cap V_s$ reduces, for $s > \nu p$, to

$$\begin{pmatrix} nc(x^n y)^{2p} x \\ c(x^n y)^{2p} y \end{pmatrix} \quad . \tag{13.14}$$

We have thus proved that in the degenerate case $\beta = \alpha n$ the PRF is given by

$$f(x) \ = \ \begin{pmatrix} x & +n\beta(x^n y)^q x & +n\gamma(x^n y)^p x & +n\eta(x^n y)^{2p} x \\ -ny & +\beta(x^n y)^q y & +\gamma(x^n y)^p y & +\eta(x^n y)^{2p} y \end{pmatrix}$$

14. Some three-dimensional examples.

We will now consider a number of three-dimensional examples; in these, the Poincaré NF will be a finite polynomial, but nevertheless our procedure will produce a simplified PRF.

A) Let us consider the VFs with linear part given by $f_0(x) = Ax$, and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad . \tag{14.1}$$

One can easily check that the PNF depend on four arbitrary constants, and is given explicitely by

$$f(x) \ = \ \begin{pmatrix} x \\ 2y + c_1 x^2 \\ 5z + c_2 x y^2 + c_3 x^3 y + c_4 x^5 \end{pmatrix} \ . \tag{14.2}$$

We want now to show how this can be reduced by our procedure. Let us start by operating with $\mathcal{L}_1$ on $h \in H^{(1)} = \mathrm{Ker}(\mathcal{L}_0)$; now

$$f_1 = \begin{pmatrix} 0 \\ c_1 x^2 \\ 0 \end{pmatrix} \ ; \tag{14.3}$$

obviously we get nothing by acting on $h_1^{(1)} \in H^{(1)} \cap V_1$ (which would be a multiple of f_1), while acting on

$$h_2^{(1)} = \begin{pmatrix} 0 \\ 0 \\ a x y^2 \end{pmatrix} \quad \text{and} \quad h_3^{(1)} = \begin{pmatrix} 0 \\ 0 \\ b x^3 y \end{pmatrix} \tag{14.4}$$

we can – provided $c_1 \neq 0$ – eliminate f_3 and f_4 by choosing

$$a = \frac{c_3}{2c_1} \ , \ b = \frac{c_4}{c_1} \ . \tag{14.5}$$

When $c_1 = 0$, we can still act with $\mathcal{L}_2$; as

$$f_2 = \begin{pmatrix} 0 \\ 0 \\ c_2 x y^2 \end{pmatrix} \ , \tag{14.6}$$

acting on $h_1^{(1)}$ (written as above), we can – provided $c_2 \neq 0$ – eliminate f_3 by choosing $a = -c_3/(2c_2)$; notice that now we are not able to modify f_4 in any way, as $\mathcal{L}_1 = \mathcal{L}_3 = 0$, and $\mathcal{L}_2(h_2^{(1)}) = 0$.

Finally, if $c_1 = c_2 = 0$, we write

$$f_3 = \begin{pmatrix} 0 \\ 0 \\ c_3 x^3 y \end{pmatrix} \tag{14.7}$$

and $\mathcal{L}_3(h_1^{(1)}$ eliminates f_4, if $c_3 \neq 0$, by choosing $a = -c_4/c_3$.

We summarize our discussion as follows: we have shown that the PRF for $f(x)$ corresponding to the linear part Ax (A as above) is in one of the following forms:

$$\begin{array}{ll}
(0) & f(x) = \begin{pmatrix} x \\ 2y \\ 5z \end{pmatrix} \\
(1) & f(x) = \begin{pmatrix} x \\ 2y + \alpha x^2 \\ 5z + \beta x y^2 \end{pmatrix} \\
(2) & f(x) = \begin{pmatrix} x \\ 2y \\ 5z + \alpha x y^2 + \beta x^5 \end{pmatrix} \\
(3) & f(x) = \begin{pmatrix} x \\ 2y \\ 5z + \alpha x^5 \end{pmatrix}
\end{array}$$

where α, β are arbitrary constants, with $\alpha \neq 0$.

Obviously we could have a more compact notation unifying cases (0),(2) and (3), if desired: that is, we can write the PRF as

$$f(x) \ = \ \begin{pmatrix} x \\ 2y + a_1 x^2 \\ 5z + a_2 x y^2 + a_3 x^5 \end{pmatrix}$$

where at least one of the a_i can always be taken to be zero.

This means that if we are interested in the nonlinear behaviours compatible with the assigned linear part, we could study these two-parameters families rather than the five-parameters family of PNFs; needless to say, this is a much lighter task.

B) Let us consider the VFs with linear part given by $f_0(x) = Ax$, and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{pmatrix} \ . \tag{14.8}$$

One can easily check that the PNF depend on five arbitrary constants, and is given explicitely by

$$f(x) \ = \ \begin{pmatrix} x \\ 3y + c_1 x^3 \\ 9z + c_2 y^3 + c_3 x^3 y^2 + c_4 x^6 y + c_5 x^9 \end{pmatrix} \ . \tag{14.9}$$

Let us now consider the action of $\mathcal{L}_2$, corresponding to

$$f_2 = \begin{pmatrix} 0 \\ c_1 x^3 \\ c_2 y^3 \end{pmatrix} \ , \tag{14.10}$$

on $H^{(1)}$; we will write

$$h_2^{(1)} = \begin{pmatrix} 0 \\ a x^3 \\ b y^3 \end{pmatrix} \ , \ h_4^{(1)} = \begin{pmatrix} 0 \\ 0 \\ d x^3 y^2 \end{pmatrix} \ , \ h_6^{(1)} = \begin{pmatrix} 0 \\ 0 \\ e x^6 y \end{pmatrix} \ . \tag{14.11}$$

With this notation, we have

$$\mathcal{L}_2(h_2^{(1)}) = \begin{pmatrix} 0 \\ 0 \\ 3(bc_1 - ac_2) x^3 y^2 \end{pmatrix} \ ; \ \mathcal{L}_2(h_4^{(1)}) = \begin{pmatrix} 0 \\ 0 \\ 2dc_1 x^6 y \end{pmatrix} \ ; \ \mathcal{L}_2(h_6^{(1)}) = \begin{pmatrix} 0 \\ 0 \\ ec_1 x^9 \end{pmatrix} \ . \tag{14.12}$$

Thus, if $c_1 \neq 0$, we can eliminate f_4, f_6 and f_8 by suitably choosing a, b, d, e. If $c_1 = 0$ and $c_2 \neq 0$, we can still eliminate f_4, but neither f_6 nor f_8 (the latter can actually be eliminated with successive steps in our procedure, if f_6 does not vanish, see below).

When $c_1 = c_2 = 0$, we can still act with $\mathcal{L}_4$, corresponding to

$$f_4 = \begin{pmatrix} 0 \\ 0 \\ c_3 x^3 y \end{pmatrix} \ . \tag{14.13}$$

With the same notation as above for $h_2^{(1)}$, we have

$$\mathcal{L}_4(h_2^{(1)}) = \begin{pmatrix} 0 \\ 0 \\ -2ac_3 \end{pmatrix} \tag{14.14}$$

and thus we can – if $c_3 \neq 0$ – eliminate f_6 in this way; however $\mathcal{L}_4(h_4^{(1)}) = 0$ and f_8 cannot be eliminated.

Finally, if $c_1 = c_2 = c_3 = 0$, with

$$f_6 = \begin{pmatrix} 0 \\ 0 \\ c_4 x^6 y \end{pmatrix} \tag{14.15}$$

we have

$$\mathcal{L}_6(h_2^{(1)}) = \begin{pmatrix} 0 \\ 0 \\ -2ac_4 \end{pmatrix} \tag{14.16}$$

and thus can eliminate f_8 (notice this also applies to the case $c_1 = 0 \neq c_2$ considered above).

Summarizing, we have shown that the PRF correspond to one of the following possibilities:

$$f \ = \ f_0 \ + \ \begin{cases} 0 \\ f_2 \\ f_2 + f_6 & \text{(with } c_1 = 0) \\ f_2 + f_8 & \text{(with } c_1 = 0) \\ f_4 + f_8 \\ f_6 \\ f_8 \end{cases} \tag{14.17}$$

C) We want now to show that sometimes the procedure we propose here is not able to operate simplification on the usual Poincaré normal form. Let us consider the VFs with linear part given by $f_0(x) = Ax$, and

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix} . \tag{14.18}$$

One can easily check that the PNF depends on two arbitrary constants, and is given explicitely by

$$f(x) \ = \ \begin{pmatrix} 2x \\ 3y \\ 8z + c_1 x y^2 + c_2 x^4 \end{pmatrix} . \tag{14.19}$$

In this case, our procedure cannot modify the form (14.19) of the system, and thus we get no further simplification.

Similarly, let us consider the VFs with linear part given by $f_0(x) = Ax$, and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} . \tag{14.20}$$

One can easily check that again the PNF depends on two arbitrary constants, and is now given explicitely by

$$f(x) \ = \ \begin{pmatrix} 2x \\ 3y \\ 10z + c_1 x^2 y^2 + c_2 x^5 \end{pmatrix} . \tag{14.21}$$

As in the previous example, our procedure cannot modify the form (14.21) of the system, and thus we get no further simplification.

Appendix A (Poincaré-Lie transformations).

In this appendix, we shortly go over the Poincaré-Lie transformation, and the derivation of (4.4); we will follow the discussion given in [12].

We recall that in this case the change of coordinates is given by (4.2), and that this transforms X into $\widetilde{X}$ given by (4.3) [5].

As mentioned in section 4, $\widetilde{X}$ can now be explicitely computed by the Baker-Campbell-Haussdorf formula [12,17], as

$$\widetilde{X} = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} X^{(n)} \tag{A.1}$$

where the $X^{(n)}$ are determined recursively by $X^{(n+1)} = \left[X^{(n)}, H_k\right]$, with $X^{(0)} = X$.

We can thus consider a one-parameter family of vector fields X_λ, where $X_0 = X$ and $X_1 = \widetilde{X}$; this satisfies $dX_\lambda/d\lambda = [H_k, X_\lambda]$. Correspondingly, we write x_λ for $e^{-\lambda H_k}x$ (i.e. the transformed coordinates, see (4.2), corresponding to λ), and $X_\lambda = f^i_\lambda(x_\lambda)(\partial/\partial x^i_\lambda)$; the f^i_λ's satisfy then

$$\frac{df^i_\lambda}{d\lambda} \ = \ \{h_k, f_\lambda\}^i_\lambda \ , \tag{A.2}$$

where $\{.,.\}_\lambda$ is the bracket $\{.,.\}$ in the x_λ coordinates, i.e. $\{f,g\}_\lambda = f^j(\partial g/\partial x^j_\lambda) - g^j(\partial f/\partial x^j_\lambda)$.

If we consider the power series expansion of f, and writing for ease of notation $f(x,\lambda) = f_\lambda(x_\lambda)$ and $\{.,.\}$ for $\{.,.\}_\lambda$, we have

$$\frac{\partial f^i_m(x,\lambda)}{\partial\lambda} \ = \ \{h_k, f_{m-k}\}^i \ . \tag{A.3}$$

The $\widetilde{X} = X_1$ is then written in the $\widetilde{x}$ coordinates as $\widetilde{X} = f^i(x,1)(\partial/\partial\widetilde{x}^i) \equiv \widetilde{f}^i(\widetilde{x})(\partial/\partial\widetilde{x}^i)$; the $\widetilde{f}$ correspond to the solution of (A.2) for $\lambda = 1$.

These can be expressed by means of the BCH formula: indeed, from (A.1) and the recursion relation for $X^{(n)}$, we have immediately that

$$f(x,\lambda) \ = \ \sum_{n=0}^{\infty} \left[\frac{(-1)^n\lambda^n}{n!} \ \varphi^{(n)}(x)\right] \tag{A.4}$$

with $\varphi^{(0)}(x) = f(x,0)$ and $\varphi^{(n+1)} = \{\varphi^{(n)}, h_k\}$.

¿From this, we have indeed, with $\mathcal{H}(.) = \{h,.\}$,

$$f_\lambda \ = \ \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \ \mathcal{H}^n(f) \ \ , \tag{A.5}$$

and for $\lambda = 1$, i.e. for $\widetilde{f}(x) = f(x,1)$, this is just (4.4).

Appendix B (The Hamiltonian case)

It is well known that the Poincaré theory of normal forms has a counterpart in the Hamiltonian case, due to Birkhoff (and then extended to the resonant case by Gustavsson, in the same way as the Poincaré theory was extended by Dulac); in this case, one can deal directly with the Hamiltonian and its normal form, rather than with the hamiltonian vector field this generates.

It turns out that the procedure for the reduction of Poincaré NF presented here can also be applied, pretty much in the same way, in the case of Birkhoff NF; in this appendix we will shortly show this extension (details will not be showed, as they are similar to those for the discussion given above, provided the appropriate parallel are done).

We will consider Hamiltonians defined in a neighbourhood of the origin in the phase space $R^{2\ell} \equiv R^n$, in which we will take coordinates[18] $q_1, ..., q_\ell; p_1, ..., p_\ell$; we also denote these by $x_1, ..., x_n$, where it is meant that $x_i = q_i$ for $i \le \ell$, and $x_i = p_{i-\ell}$ for $i > \ell$, and having a critical point in the origin. Notice that, by adding an inessential constant $c = -H(0)$ to $H(x)$, we can always require that $H(0) = 0$.

We consider scalar functions on R^n, and denote by $\mathcal{S}$ the space of formal power series $s : R^n \to R$; we also denote by $\mathcal{S}_k$ the space of homogeneous scalar functions of degree $(k-2)$ on R^n.

We can then write any Hamiltonian in $\mathcal{S}$ which has a critical point in the origin, choosing $H(0) = 0$, as

$$H(q,p) \ = \ \sum_{k=0}^\infty H_k(q,p) \tag{B.1}$$

where $H_k \in \mathcal{S}_k$.

We define then in $\mathcal{S}$ the familiar antisymmetric Poisson bracket $\{.,.\} : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ as

$$\{F,G\} \ = \ \sum_{i=1}^\ell \ \left[\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \ - \ \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right] \ . \tag{B.2}$$

With the above notation for $\mathcal{S}_k$, we have

$$\{.,.\} \ : \ \mathcal{S}_k \times \mathcal{S}_m \to \mathcal{S}_{k+m} \ . \tag{B.3}$$

To any $H \in \mathcal{S}$ we associate an hamiltonian vector field $X_H : R^n \to \mathrm{T}R^n$, defined by

$$X_H \ = \ \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} \ - \ \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \tag{B.4}$$

so that $X_H(F) = \{F,H\}$. It is immediate to check that

$$\{F,G\} = K \iff [X_F, X_G] = -X_K \ . \tag{B.5}$$

To any $F \in \mathcal{S}$ we can associate its $\mathtt{ad}$-operator $\mathtt{ad}_F \equiv \{F,.\}$ and its Lie operator[19] $\mathcal{P}_F$, defined as the time-one action of the flow under $(-X_F)$; this is also written as

$$\mathcal{P}_F \ = \ \exp\left[\mathtt{ad}_F\right] \ = \ \sum_{r=0}^\infty \frac{1}{r!} \left(\mathtt{ad}_F\right)^r \ . \tag{B.6}$$

We will now consider changes of coordinates given by the action of $\mathcal{P}_F$ for $F \in \mathcal{S}$, similarly to what we have done in the Poincaré case. Notice that again, when we consider only first nontrivial order terms – i.e., the action on H_m if $F \in \mathcal{S}_m$ – this gives simply

$$q_i \to q_i + \partial F / \partial p_i \quad , \quad p_i \to p_i - \partial F / \partial q_i \ ; \tag{B.7}$$

[18] We prefer to write all the coordinate indices as lower ones, no matter if contravariant or covariant, to simplify the notation.

[19] This is usually denoted by $\mathcal{L}_F$, but we want to keep the notation similar to the one already employed and reserve $\mathcal{L}$ for the homological operators.

the advantage of considering Lie operators rather than the usual near-identity changes of coordinates lies in the fact that, being defined in terms of Poisson bracket, these are guaranteed to generate canonical transformations at all orders; moreover, we are guaranteed of the invertibility of some relevant operators [11].

Now, as it can be checked by direct computation [11,23], the effect of the change of coordinates given by $\mathcal{P}_F$ is given by

$$\begin{aligned} H \ \to \ \widetilde{H} \ =& \sum_{r=0}^{\infty}\sum_{r=0}^{\infty} \frac{1}{r!} \left(\mathtt{ad}_F\right)^r (H) = \\ =& H + \{F,H\} + \frac{1}{2}\{F,\{F,H\}\} + \frac{1}{6}\{F,\{F,\{F,H\}\}\} + ... \end{aligned} \qquad (B.8)$$

which is formally analogous to (4.4). If we insert in this the expansion (B.1) for H and consider an $F \in \mathcal{S}_k$, we get the exact analogue of (4.5), i.e.

$$\widetilde{H}_m \ = \ \sum_{r=0}^{[m/k]} \frac{1}{r!} \left(\mathtt{ad}_F\right)^r H_{m-rk} \quad . \qquad (B.9)$$

Then, to any series (B.1) we associate a series of homological operators $\mathcal{L}_k$ given by

$$\mathcal{L}_k \ = \ \{H_k, .\} \ \equiv \ \mathtt{ad}_{H_k} \qquad (B.10)$$

which have the same formal properties of the $\mathcal{L}_k$ considered before.

The computation will now easily proceed along the same lines, and what's more using the same formal properties, as in the Poincaré case; indeed, we could repeat word by word (with the same symbols – which have now a different definition) the discussion given above, arriving at the same results. We will not bore the reader with this repetition, but only give the final result; in this it is meant that again $F^{(k)} = \bigcap_{s=0}^{k-1} \mathrm{Ker}(\mathcal{L}_k)$.

Definition 3. *We say that the Hamiltonian $H = \sum_k H_k$ is in* **Birkhoff renormalized form** *up to order n if $H_k \in F_k^{(k)}$ for all $k \le n$.*

Proposition 3. *Any Hamiltonian can be brought into Birkhoff renormalized form up to any desired order n by means of a formal series of Lie transforms.*

It is maybe worth mentioning explicitely that we can also proceed by iterated Bikhoff normalizations, exactly as we proceeded by iterated Poincaré normalizations in sect.10, arriving at the same results. We would then have explicitely:

Definition 4. *A polynomial Hamiltonian $H = \sum_{k=0}^{\infty} H_k$ is said to be in* n-th normal form *if $H_k \in F_k^{(p)}$ for all k, with $p = \min(k,n)$. When this condition is satisfied for all k (with no upper limit), we say that the system is in* Birkhoff renormalized form.

Notice that a Hamiltonian in n-th normal form is also in renormalized form up to order n, as it was the case for dynamical systems (vector fields, formal power series).

Proposition 4. *Given a Hamiltonian $H \in \mathcal{S}$, $H = \sum_{k=0}^{\infty} H_k^{(0)}$, it is always possible to reduce it, by a sequence of n formal Birkhoff normalizations, into the n-th normal form*

$$H \ = \ \sum_{k=0}^{n} H_k^{(k)} \ + \ \sum_{s=n+1}^{\infty} H_s^{(n)} \ ,$$

where $H_k^{(k)} \in F_k^{(k)}$ for all $k \le n$. Formally, considering infinite sequences of Birkhoff normalizations, a Hamiltonian can always be taken into renormalized form.

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