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INVERSE SCATTERING TRANSFORM FOR THE TODA HIERARCHY

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Abstract. We provide a rigorous treatment of the inverse scattering transform for the entire Toda hierarchy. In addition, we revisit the connection between trace formulas and conserved quantities from the view point of Krein's spectral shift theory.

1. Introduction

In 1967 Gardner et al. ([6]) presented a method for solving the Korteweg-de Vries equation which is presently known as inverse scattering transform (IST). Their method is based on the connection between the Korteweg-de Vries and the one-dimensional Schrödinger equation. This connection becomes most transparent using an approach due to Lax [10] which rewrites completely integrable nonlinear evolution equations as linear evolution equations for linear operators, viz.

$$\frac{d}{dt} H(t) = [P(t), H(t)], \tag{1.1}$$

where $[P,H] = PH - HP$ denotes the usual commutator. Under suitable conditions, (1.1) will imply existence of a unitary propagator $U(s,t)$ for $H(t)$, viz.

$$H(t) = U(t,s) H(s) U(s,t), \quad U(t,s)^* = U(t,s)^{-1} = U(s,t). \tag{1.2}$$

In particular, this implies that the operators $H(t)$, $t \in \mathbb{R}$ are unitarily equivalent and that the spectrum $\sigma(H(t))$ is independent of t. Now the general idea is to find suitable spectral data for $H(t)$ which uniquely determine $H(t)$. Then equation (1.1) can be used to derive linear evolution equations which are easier to solve.

As shown in [6], a suitable set of spectral data for $H(t)$ are the scattering data whose time evolution is explicitly solvable. Clearly this only gives a necessary form of solutions since existence has been assumed in the outset. It turns out that the remaining step, to verify that the constructed "solutions" are indeed solutions, is much harder to prove than the method itself. In fact, looking at [11], Section 4.2 (where a rigorous proof is indicated on 3 pages), shows that this remaining problem is in fact nontrivial and, in the case of the Korteweg-de Vries equation, imposes additional restrictions on the scattering data.

The goal of the present paper is to establish this step for the Toda equations and, at the same time, treat the entire Toda hierarchy. This is possible since, on the contrary to the Korteweg-de Vries equation, existence and uniqueness of solutions of the Toda equations can be easily proved (cf. [4], Proposition 1). Hence it (essentially) remains to verify that solutions whose initial conditions decay sufficiently fast at $\pm\infty$ decay sufficiently fast for all $t \in \mathbb{R}$. In case of the first equation of the

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Toda hierarchy this is also shown in [4] (Proposition 4). As our main new result we will extend this proof to the entire hierarchy.

2. The Toda hierarchy

In this section we introduce the Toda hierarchy using the standard Lax formalism ([10]). We first review some basic facts from [2].

We will only consider bounded solutions and hence require

Hypothesis H.2.1. *Suppose $a(t), b(t)$ satisfy*

$$a(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \;\; b(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \;\; a(n,t) \neq 0 \;\; (n,t) \in \mathbb{Z} \times \mathbb{R}, \tag{2.1}$$

and let $t \mapsto (a(t), b(t))$ be Fréchet differentiable in the Banach space $\ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$.

Associated with $a(t), b(t)$ is a Jacobi operator

$$H(t): \begin{array}{ccc} \ell^2(\mathbb{Z}) & \to & \ell^2(\mathbb{Z}) \\ f & \mapsto & \tau(t) f \end{array}, \tag{2.2}$$

where

$$\tau(t) f(n) = a(n,t) f(n+1) + a(n-1,t) f(n-1) + b(n,t) f(n) \tag{2.3}$$

and $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable (complex-valued) sequences over $\mathbb{Z}$. Moreover, choose constants $c_0 = 1$, c_j, $1 \leq j \leq r$, $c_{r+1} = 0$, set

$$\begin{aligned} g_j(n,t) &= \sum_{\ell=0}^{r} c_{j-\ell} \langle \delta_n, H(t)^\ell \delta_n \rangle, \\ h_j(n,t) &= 2a(n,t) \sum_{\ell=0}^{r} c_{j-\ell} \langle \delta_{n+1}, H(t)^\ell \delta_n \rangle + c_{j+1} \end{aligned} \tag{2.4}$$

and consider the Lax operator

$$P_{2r+2}(t) = -H(t)^{r+1} + \sum_{j=0}^{r} (2a(t) g_j(t) S^+ - h_j(t)) H(t)^{r-j} + g_{r+1}(t), \tag{2.5}$$

where $S^\pm f(n) = f(n \pm 1)$. Restricting to the two-dimensional nullspace $\mathrm{Ker}(\tau(t) - z)$, $z \in \mathbb{C}$ of $\tau(t) - z$, we have the following representation of $P_{2r+2}(t)$

$$P_{2r+2}(t)\Big|_{\mathrm{Ker}(\tau(t)-z)} = 2a(t) G_r(z,t) S^+ - H_{r+1}(z,t), \tag{2.6}$$

where $G_r(z,n,t)$ and $H_{r+1}(z,n,t)$ are monic polynomials in z of the type

$$\begin{aligned} G_r(z,n,t) &= \sum_{j=0}^{r} z^j g_{r-j}(n,t), \\ H_{r+1}(z,n,t) &= z^{r+1} + \sum_{j=0}^{r} z^j h_{r-j}(n,t) - g_{r+1}(n,t). \end{aligned} \tag{2.7}$$

A straightforward computation shows that the Lax equation

$$\frac{d}{dt} H(t) - [P_{2r+2}(t), H(t)] = 0, \quad t \in \mathbb{R} \tag{2.8}$$

is equivalent to

$$\mathrm{TL}_r(a(t),b(t))_1 = \dot{a}(t) - a(t)\Big(g_{r+1}^+(t) - g_{r+1}(t)\Big) = 0, \tag{2.9}$$

$$\mathrm{TL}_r(a(t),b(t))_2 = \dot{b}(t) - \Big(h_{r+1}(t) - h_{r+1}^-(t)\Big) = 0, \tag{2.10}$$

where the dot denotes a derivative with respect to t and $f^{\pm}(n) = f(n \pm 1)$. Varying $r \in \mathbb{N}_0$ yields the Toda hierarchy (TL hierarchy)

$$\mathrm{TL}_r(a,b) = (\mathrm{TL}_r(a,b)_1, \mathrm{TL}_r(a,b)_2) = 0, \quad r \in \mathbb{N}_0. \tag{2.11}$$

Finally we recall the basic existence and uniqueness theorem for the Toda hierarchy ([4], Proposition 1, see also [13], Theorem 2.4)

Theorem 2.2. *Suppose $(a_0, b_0) \in M = \ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$. Then there exists a unique integral curve $t \mapsto (a(t), b(t))$ in $C^\infty(\mathbb{R}, M)$ of the Toda equations, that is, $\mathrm{TL}_r(a(t),b(t)) = 0$, such that $(a(0), b(0)) = (a_0, b_0)$.*

3. Inverse scattering transform

We start with the trivial solution of the Toda equations

$$a_0(n,t) = a_0 = \frac{1}{2}, \qquad b_0(n,t) = b_0 = 0, \tag{3.1}$$

The sequences

$$\psi_\pm(z,n,t) = k^{\pm n} \exp\Big(\frac{\pm\alpha_r(k)t}{2}\Big), \quad z = \frac{k + k^{-1}}{2}, \tag{3.2}$$

where

$$\alpha_r(k) = 2\Big(kG_{0,r}(z) - H_{0,r+1}(z)\Big) = (k - k^{-1})G_{0,r}(z) \tag{3.3}$$

satisfy

$$\begin{aligned} H_0(t)\psi_\pm(z,n,t) &= z\psi_\pm(z,n,t), \\ \frac{d}{dt}\psi_\pm(z,n,t) &= P_{0,2r+2}(t)\psi_\pm(z,n,t) \\ &= 2a_0 G_{0,r}(z)\psi_\pm(z,n+1,t) - H_{0,r+1}(z)\psi_\pm(z,n,t) \end{aligned} \tag{3.4}$$

(we omit n, t in the arguments of $G_{0,r}, H_{0,r+1}$ since these quantities do not depend on n,t). Note $\alpha_r(k) = -\alpha_r(k^{-1})$. Explicitly we have

$$\begin{aligned} \alpha_0(k) &= k - k^{-1}, \\ \alpha_1(k) &= \frac{k^2 - k^{-2}}{2} + c_1(k - k^{-1}), \\ &\text{etc.}\ . \end{aligned} \tag{3.5}$$

Now we turn to the general case (cf. [4], Proposition 1 for the special case $r = 0$).

Lemma 3.1. *Suppose $a(n,t)$, $b(n,t)$ is a solution of the Toda system satisfying (3.6) for one $t_0 \in \mathbb{R}$, then (3.6) holds for all $t \in \mathbb{R}$, that is,*

$$\sum_{n\in\mathbb{Z}} |n|(|1 - 2a(n,t)| + |b(n,t)|) < \infty. \tag{3.6}$$

Proof. Without loss of generality we choose $t_0 = 0$. Shifting $a \to a - \frac{1}{2}$ we can consider the norm

$$\|(a,b)\|_* = \sum_{n\in\mathbb{Z}} (1+|n|)(|1-2a(n)| + |b(n)|) \tag{3.7}$$

which remains finite at least locally. Next, we note that we have the estimate

$$\sum_{n\in\mathbb{Z}} (1+|n|)|g_r(n,t) - g_{0,r}| \leq C_r(\|H(0)\|)\|(a(t),b(t))\|_*, \tag{3.8}$$

$$\sum_{n\in\mathbb{Z}} (1+|n|)|h_r(n,t) - h_{0,r}| \leq C_r(\|H(0)\|)\|(a(t),b(t))\|_*, \tag{3.9}$$

where $C_r(\|H(0)\|)$ is some positive constant. It suffices to consider the case where $c_j = 0$, $1 \leq j \leq r$. In this case we infer from equations (3.15) and (3.25) of [12] that $g_j(n,t), h_j(n,t)$, $j \in \mathbb{N}_0$ can be computed from $g_0(n,t) = 1$, $h_0(n,t) = 0$ and

$$g_{j+1}(n,t) = \frac{h_j(n,t) + h_j(n-1,t)}{2} + b(n,t)g_j(n,t), \tag{3.10}$$

$$\begin{aligned} h_{j+1}(n,t) &= 2a(n,t)^2 \sum_{\ell=0}^{j} g_{j-\ell}(n,t)g_\ell(n+1,t) \\ &\quad - \frac{1}{2}\sum_{\ell=0}^{j} h_{j-\ell}(n,t)h_\ell(n,t). \end{aligned} \tag{3.11}$$

The claim now follows by induction (note that we have $g_i(n,t)g_j(m,t) - g_{0,i}g_{0,j} = (g_i(n,t) - g_{0,i})g_j(m,t) - g_{0,i}(g_j(m,t) - g_{0,j})$). Hence we infer from (2.9)

$$\begin{aligned} |a(n,t) - \frac{1}{2}| &\leq |a(n,0) - \frac{1}{2}| + \|H(0)\| \int_0^t |g_{r+1}(n,s) - g_{0,r+1}| \\ &\quad + |g_{r+1}(n+1,s) - g_{0,r+1}| ds, \end{aligned} \tag{3.12}$$

$$\begin{aligned} b(n,t)| &\leq |b(n,0)| + \int_0^t |h_{r+1}(n,s) - h_{0,r+1}| \\ &\quad + |h_{r+1}(n-1,s) - h_{0,r+1}| ds \end{aligned} \tag{3.13}$$

and thus

$$\|(a(t),b(t))\|_* \leq \|(a(0),b(0))\|_* + \tilde{C}\int_0^t \|(a(s),b(s))\|_* ds, \tag{3.14}$$

where $\tilde{C} = 2(1+\|H(0)\|)C_{r+1}(\|H(0)\|)$. The rest follows from Gronwall's inequality. □

Now we turn to scattering theory for H (cf. [3], [8], [15]) and assume $a(n,t) > 0$ and (3.6). This implies

$$\sigma(H) = [-1,1], \quad \sigma_p(H) = \{\lambda_j\}_{j=1}^N \subseteq \mathbb{R}\backslash[-1,1], \tag{3.15}$$

where $N \in \mathbb{N}$ is finite, and the existence of the so called Jost solutions $f_\pm(k,n)$,

$$\Big(\tau - \frac{k+k^{-1}}{2}\Big) f_\pm(k,n,t) = 0, \quad \lim_{n\to\pm\infty} k^{\mp n} f_\pm(k,n,t) = 1, \quad |k| \leq 1. \tag{3.16}$$

Transmission $T(k,t)$ and reflection $R_\pm(k,t)$ coefficients are then defined via

$$T(k,t)f_\mp(k,n,t) = f_\pm(k^{-1},n,t) + R_\pm(k,t)f_\pm(k,n,t), \quad |k| = 1, \tag{3.17}$$

and the norming constants $\gamma_{\pm,j}(t)$ corresponding to $\lambda_j \in \sigma_p(H)$ are given by

(3.18)
$$\gamma_{\pm,j}(t)^{-1} = \sum_{n\in\mathbb{Z}} |f_\pm(k_j,n,t)|^2, \quad k_j = \lambda_j + \sqrt{\lambda_j^2 - 1} \in (-1,0)\cup(0,1),\ j \in J.$$

Clearly we are interested how the scattering data varies with respect to t.

Theorem 3.2. *Suppose $a(n,t)$, $b(n,t)$ is a solution of the Toda system satisfying (3.6) for one (and hence for all) $t_0 \in \mathbb{R}$. The functions*

$$\exp(\pm\alpha_r(k)t) f_\pm(k,n,t) \tag{3.19}$$

satisfy

$$H(t)u = zu, \quad \frac{d}{dt}u = P_{2r+2}(t)u \tag{3.20}$$

(weakly) with $z = (k+k^{-1})/2$. Here $f_\pm(k,n,t)$ are the Jost solutions and $\alpha_{r,\pm}(k)$ is defined in (3.3). In addition, we have

$$\begin{aligned} T(k,t) &= T(k,0), && (3.21)\\ R_\pm(k,t) &= R_\pm(k,0)\exp(\pm\alpha_r(k)t), && (3.22)\\ \gamma_{\pm,\ell}(t) &= \gamma_{\pm,\ell}(0)\exp(\mp 2\alpha_r(k_\ell)t), \quad 1 \le \ell \le N && (3.23) \end{aligned}$$

Proof. As in proof of [14], Theorem 5.1 one shows that $f_\pm(k,n,t)$ is continuously differentiable with respect to t and that $\lim_{n\to\pm\infty} k^{\mp n} f_\pm(k,n,t) \to 0$ (use the estimates (3.8) and (3.9)). Now let $(k+k^{-1})/2 \in \rho(H(t))$, then Lemmas 4.1 and 4.2 of [13] implies that the solution of (3.20) with initial condition $f_\pm(k,n,0)$ is of the form $C_\pm(t) f_\pm(k,n,t)$. Inserting this into (3.20), multiplying with $k^{\mp n}$ and evaluating as $n \to \pm\infty$ yields $C_\pm(t) = \exp(\pm\alpha_r(k)t)$. The general result for all $|k| < 1$ now follows from continuity. This immediately implies the formulas for $T(k,t), R_\pm(k,t)$. Finally, let $k = k_\ell$ then we have

$$\exp(\pm\alpha_r(k_\ell)t) f_\pm(k_\ell,n,t) = U(t,0) f_\pm(k_\ell,n,0), \tag{3.24}$$

(where $U(t,s)$ is the unitary propagator of P_{2r+2}) which implies

$$\frac{d}{dt}\frac{\exp(\mp 2\alpha_r(k_\ell)t)}{\gamma_{\pm,\ell}(t)} = \frac{d}{dt}\|U(t,0) f_\pm(k_\ell,.,0)\| = 0 \tag{3.25}$$

and concludes the proof. □

Thus the scattering data of $H(t)$ can be expressed in terms of those for $H(0)$ and $a(n,t), b(n,t)$ can be computed from $a(n,0), b(n,0)$ using the Gel'fand–Levitan–Marchenko equations ([8], Theorem 3). Since we have ensured the existence of a solution in the outset (Theorem 2.2 and Lemma 3.1) the sequences constructed by this procedure satisfy the Toda equations.

In the case $r = 0$ the inverse scattering procedure was first worked out by Flaschka [5]. In addition, Flaschka also worked out the inverse procedure in the reflection-less case (i.e., $R_\pm(k,t) = 0$). His formulas clearly apply to the entire Toda hierarchy upon using the t dependence of the norming constants given in (3.23). In addition, these formulas are the same as the ones obtained using the double commutation method (cf. [13]). For an alternative method in case of the Toda lattice ($r = 0, 1$) on the half-line we refer the reader to [1].

Finally, we briefly comment on conserved quantities (see [5], [7], [15]). Set $\alpha(k) = T(k)^{-1}$. Then $\alpha(k)$ is holomorphic inside the unit circle with simple poles at k_j, $1 \le j \le N$ and we obtain by virtue of the Poisson–Jensen formula (for $|k| < 1$)

$$\alpha(k) = \Big(\prod_{j=1}^{N} \frac{k - k_j}{|k_j|(k - k_j^{-1})} \Big) \exp \left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln(1 - |R_\pm(\mathrm{e}^{\mathrm{i}\varphi}, t)|^2) \frac{k + \mathrm{e}^{\mathrm{i}\varphi}}{k - \mathrm{e}^{\mathrm{i}\varphi}} d\varphi \right), \tag{3.26}$$

In particular, $\alpha(k)$ has the expansion

$$\alpha(k) = \frac{1}{A} \sum_{m=0}^{\infty} K_m k^m, \quad A = \prod_{n\in\mathbb{Z}} 2a(n,t) \tag{3.27}$$

and the coefficients $K_0 = 1$, $K_1 = -2\sum_{n\in\mathbb{Z}} b(n)$, $\dots$ are conserved quantities. Moreover, one computes

$$\frac{d}{dk}\alpha(k) = \frac{1-k^2}{2k^2} \sum_{n\in\mathbb{Z}} \Big(\alpha(k) - f_+(k,n) f_-(k,n) \Big). \tag{3.28}$$

Rephrasing this equation as ($H - H_0$ is clearly trace class)

$$\begin{aligned} -\frac{d}{dz} \ln \Delta(z) &= \sum_{j\in\mathbb{Z}} \Big(G(z,n,n) - G_0(z,n,n) \Big) \\ &= \mathrm{tr}\Big((H-z)^{-1} - (H_0 - z)^{-1} \Big), \end{aligned} \tag{3.29}$$

identifies $\Delta(z) = A\alpha(k(z))$ as the perturbation determinant of the pair H, H_0 in the sense of Krein [9]. Here $G(z,n,n)$, $G_0(z,n,n)$ denotes the Green function of H, H_0, respectively. By [9], Theorem 1 this implies

$$\Delta(z) = \exp\Big(\int_{\mathbb{R}} \frac{\xi_\Delta(\lambda) d\lambda}{\lambda - z} \Big), \tag{3.30}$$

where

$$\xi_\Delta(\lambda) = \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Im} \ln \Delta(\lambda + \mathrm{i}\varepsilon) \tag{3.31}$$

is of compact support (in the above formula $\ln \Delta(z)$ is the branch which is holomorphic for $z > \|H\|$ and satisfies $\ln \Delta(\infty) = 0$). Moreover,

$$\mathrm{tr}\Big(H^\ell - (H_0)^\ell \Big) = \ell \int_{\mathbb{R}} \lambda^{\ell-1} \xi_\alpha(\lambda) d\lambda. \tag{3.32}$$

Comparing coefficients in the asymptotic expansions of (3.26) and (3.30) gives a rigorous justification of the well-known formula

$$\mathrm{tr}\Big(H^\ell - (H_0)^\ell \Big) = \frac{-1}{2\pi} \int_{-\pi}^{\pi} \ln(1 - |R_\pm(\mathrm{e}^{\mathrm{i}\varphi}, t)|^2) \cos(\ell \mathrm{e}^{\mathrm{i}\varphi}) d\varphi + \sum_{j=1}^{N} \frac{k_j^\ell - k_j^{-\ell}}{\ell} \tag{3.33}$$

under the assumption (3.6).

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