Coercive Inequalities for Kawasaki Dynamics: The Product Case

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Abstract

We prove the Generalized Nash and Logarithmic Nash inequalities for a product measure with Dirichlet form associated to the Kawaski dynamics.

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1 Introduction

Let (Ω, Σ) be a Polish space with its Borel σ-algebra and let $P_t \equiv e^{t\mathcal{L}}$ be a Markov semigroup on the space of continuous functions $\mathcal{C}(\Omega)$. Let μ be a probability measure on (Ω, Σ) which is P_t invariant.

- We will say that we have *decay to equilibrium in* $\mathbb{L}_2$ *sense* iff there is a positive function $\theta(t)$ decreasing to zero when $t \nearrow \infty$ and a functional A with a dense domain $\mathcal{D}(A) \subset \mathcal{C}(\Omega)$ such that for any $f \in \mathcal{D}(A)$ we have
$$\mu(P_t f - \mu f)^2 \leq \theta(t) \cdot A(f) \tag{1.1}$$

- We will say that we have *decay to equilibrium in entropy sense* iff there is a positive function $\theta(t)$ decreasing to zero when $t \nearrow \infty$ and a functional A with a dense domain $\mathcal{D}(A) \subset \mathcal{C}(\Omega)$ such that for any $f \geq 0$, $f^{\frac{1}{2}} \in \mathcal{D}(A)$ we have

$$\mu(P_t f \log \frac{P_t f}{\mu f}) \leq \theta(t) \cdot A(f^{\frac{1}{2}}) \tag{1.2}$$

In both cases we assume that the functional A is homogenous of degree two and vanishing on constants

If the generator $\mathcal{L}$ is self adjoint in $\mathbb{L}_2(\mu)$ and has nondegenerate ground state, then using spectral theory one can show the following estimate of $\mathbb{L}_2$ decay to equilibrium: for any positive function G, one has

$$\mu(P_t f - \mu f)^2 \leq \sup_{\lambda > 0}(e^{-\lambda t} G(\lambda)^{-1}) \cdot \mu\left(G(\mathcal{L})(f - \mu f)\right)^2 \tag{1.3}$$

provided $f \in \mathcal{D}(G(\mathcal{L}))$. In particular if the spectrum of the generator $\mathcal{L}$ is continuous at zero, to get a nontrivial information about the long time behaviour we have to choose a function G which is singular at zero. Then however $G(\mathcal{L})$ can be defined possibly only on a dense set of functions. Clearly we will have different decay to equilibrium for classes of functions with different spectral behaviour (i.e. behaviour of corresponding spectral measures in the neighborhood of zero). This gives us a general abstract way of describing the decay to equilibrium in $\mathbb{L}_2$ sense. Unfortunately in practice it is very difficult to get a detailed description of the spectral measures of a selfadjoint unbounded operator or characterize a domain of its unbounded functions. Also this abstract way based only on the spectral theory does not need to give the optimal estimate on the rate function.

Clearly given $\mathbb{L}_2$ decay to equilibrium and using the simple observation that

$$\mu(f - \mu f)^2 - t\mathcal{E}(f) \leq \mu(P_t f - \mu f)^2$$

where $\mathcal{E}$ is the corresponding Dirichlet form, one can easily see that the following coercive inequality, called later on the **General Nash** inequality, is true

$$\mu(f - \mu f)^2 \leq \inf_{t > 0}\left(t\mathcal{E}(f) + \theta(t) \cdot A(f)\right) \tag{1.4}$$

In the derivation of this inequality one uses the fact, (which follows simply from the spectral theory), that the first derivative of $\mu(P_t f - \mu f)^2$ is a monotone decreasing function of time t. Motivated by the above considerations one would like also to introduce the following entropy bound, called later on the **Logarithmic Nash** inequality

$$\mu(f\log\frac{f}{\mu f}) \leq \inf_{t>0}\left(t\mathcal{E}(f) + \theta(t)\cdot A(f^{\frac{1}{2}})\right) \tag{1.5}$$

although in general this coercive inequality does not need to follow from the decay in entropy sense (unless some extra conditions, as e.g. weak Bakry - Emery condition, implying the monotonicity of the derivative of the entropy are imposed).

The inequality (1.4) in case when $\theta(t) = a_\gamma t^{-\gamma}$, with some constants $a_\gamma \in (0,1)$ and $\gamma \in (0,\infty)$, can be written in the more familiar in the literature following form, (see e.g. [4, 3, 7, 11, 26]),

$$\mu(f-\mu f)^2 \leq \mathcal{E}(f)^\alpha \cdot A(f)^{1-\alpha} \tag{\textbf{GN}}$$

with $\alpha = \gamma/(1+\gamma)$. Similarly in the case of the entropy bound, one gets

$$\mu(f\log\frac{f}{\mu f}) \leq \mathcal{E}(f)^\alpha \cdot A(f^{\frac{1}{2}})^{1-\alpha} \tag{\textbf{LN}}$$

In particular if one takes $A(f) = (\mu|f-\mu f|)^2$ one obtains the Classical Nash inequality, which implies the ultracontractivity property of the corresponding semigroup [4]. Alternatively when $\alpha = 1$ and A is replaced by a constant, or when $A(f)$ equals to a constant multiplied by $\mu(f-\mu f)^2$, one obtains the Spectral Gap inequality. In the case of entropy bound the corresponding choices give us the Logarithmic Sobolev inequality, which is equivalent to the hypercontractivity property of the corresponding semigroup [8].

It is an interesting fact that the coercive inequalities (**GN**) and (**LN**) introduced above imply a corresponding decay to equilibrium, (even without symmetry assumption about the semigroup), provided that the corresponding functionals A satisfy additional properties. In the first case we need one of the following properties to be true:

- *A is a P_t-monotone functional iff* for any $t \geq 0$ we have $P_t\mathcal{D}(A) \subseteq \mathcal{D}(A)$ and
$$A(P_t f) \leq A(f)$$

- *A is a P_t-bounded functional iff* there is a functional B with dense domain satisfying $P_t\mathcal{D}(B) \subseteq \mathcal{D}(A)$ and

$$A(P_t f) \le B(f)$$

for any $t \ge 0$.

In case relevant to the Logarithmic Nash inequality one needs similar properties for $A(f^{\frac{1}{2}})$. Given these properties one can derive the corresponding decay to equilibrium by the classical method based on the differential inequalities for relevant quantities (and a property of the Dirichlet form in the case of entropy)

Clearly all the $\mathbb{L}_p$ norms, with $p \in [1,\infty]$, as well as functionals $f \longmapsto \mu(G(\mathcal{L})(f-\mu f))^2$, are monotone. In many particular examples it is possible to show existence of other interesting and useful monotone functionals, [10, 11, 1]. It is clear that if a functional A is P_t-bounded, then one can replace it by a monotone functional $\widetilde{A}(f) \equiv \sup_t A(P_t f)$. However such functional is highly nonlocal and from the practical point of view may be rather useless. In general it is rather a challenging problem how to characterize a set of all monotone or bounded functionals for a given semigroups. It is also an important one, due to its relevance to the study of decay to equilibrium via the General Nash and Logarithmic Nash inequalities.

In the present paper we consider the symmetric exclusion process and demonstrate on this simple example that there is a systematic strategy for proving General Nash and Logarithmic Nash inequalities. Moreover we show that in this case one can give a good description of the set of corresponding monotone functionals and thus our Nash inequalities imply a decay to equilibrium in $\mathbb{L}_2$ as well as entropy sense, respectively. Let us mention that the decay to equilibrium in the $\mathbb{L}_2$ sense for this model has been studied by a method strongly based on the use of couplings and attractivity property of the corresponding stochastic dynamics in [5], (see also references given there). Of course by general arguments this decay implies the corresponding Nash inequality. However our method is free of extra properties of this model and, as we demonstrate in the companion paper, it applies as well to the situations where a nontrivial interaction is present. We think that presentation of our idea in the relatively simple situation, free of technical complications, is certainly useful.

We would like to mention also a work [11] in which a General Nash inequality has been proven for a nearest particle dynamics of a one dimensional model with some long range interaction.

The results concerning the Logarithmic Nash inequality are new. This interesting inequality, which in certain sense is stronger than the General Nash inequality, certainly deserves further studies. As we show in a companion paper the Logarithmic Nash inequality is also true in nonproduct case.

The paper is organized as follows. In Section 2 we introduce a basic notation and formulate (a part of) results of this work. Section 3 and Section 4 is devoted to the proof of General Nash and Logarithmic Nash inequality, respectively. The necessary study of the set of monotone functionals is done in Section 5. In the last section we demonstrate that one can extend our strategy to prove also a faster decay to equilibrium for appropriate classes of functions (by which we perform a more detailed analysis of the corresponding spectral properties).

2 Notation and Results

The state space of the process is $\Omega := \{0,1\}^{\mathbb{Z}^d}$, which is endowed with the product topology. Its elements $\eta \equiv \{\eta(i) : i \in \mathbb{Z}^d\}$ will be called configurations of particles. We will say that a site $i \in \mathbb{Z}^d$ is occupied and empty iff $\eta(i) = 1$ and $\eta(i) = 0$, respectively. The space of continuous functions on Ω, denoted by $\mathcal{C}(\Omega)$, becomes a Banach space under the uniform norm

$$\|f\|_u := \sup_{\eta\in\Omega} |f(\eta)| \tag{2.1}$$

We will need also Lipschitz seminorms $|||f|||_q$, $q \in [1, \infty)$ defined as follows

$$|||f|||_q := \left(\sum_{i\in\mathbb{Z}^d} \|\nabla_i f\|_u^q\right)^{\frac{1}{q}} \tag{2.2}$$

where

$$\nabla_i f(\eta) := f(\eta^i) - f(\eta) \tag{2.3}$$

with $\eta^i \in \Omega$, being a configuration "flipped" at site $i \in \mathbb{Z}^d$, defined as follows

$$\eta^i(j) := \begin{cases} 1 - \eta(i) & \text{if} \quad j = i \\ \eta(j) & \text{otherwise} \end{cases} \tag{2.4}$$

Note that $|||f|||_q = 0$ iff f is constant (and thus $|||\cdot|||_q$ is only a seminorm).Let us also define

$$\mathcal{C}_q(\Omega) := \{f \in C(\Omega) \quad : \quad |||f|||_q < \infty\} \tag{2.5}$$

Since for $q \geq p$, with $q, p \in [1, \infty)$, we have

$$|||f|||_q \leq |||f|||_p \tag{2.6}$$

therefore also

$$\mathcal{C}_p(\Omega) \subset \mathcal{C}_q(\Omega). \tag{2.7}$$

Next we introduce an unbounded operator $\mathcal{L}$ in $\mathcal{C}(\Omega)$ defined on the dense domain $\mathcal{C}_1(\Omega)$ by

$$\mathcal{L}f(\eta) = \sum_{\langle i,j \rangle} \delta_{i,j} f(\eta) \tag{2.8}$$

where $\langle \cdot, \cdot \rangle$ indicates the nearest neighbor pairs in $\mathbb{Z}^d$ and

$$\delta_{i,j} f(\eta) := T_{i,j} f(\eta) - f(\eta)$$

with $T_{i,j} f(\eta) \equiv f(\eta^{i,j})$

$$\eta^{i,j}(k) := \begin{cases} \eta(i) & \text{if} \quad k = j \\ \eta(j) & \text{if} \quad k = i \\ \eta(k) & \text{otherwise} \end{cases} \tag{2.9}$$

i.e. $\eta^{i,j}$ is the configuration obtained from η by exchanging the coordinates at the points i and j. If the sites i, j are both occupied or both empty, the exchange of corresponding coordinates does not change the configuration. This implies the following useful relation between the operators $\delta_{i,j}$ and ∇_i

$$\delta_{i,j} = [\eta(i)(1 - \eta(j)) + \eta(j)(1 - \eta(i))] \cdot (\nabla_i - T_{i,j} \nabla_j) \tag{2.10}$$

By [10, I Thm. 3.9] the closure of $\mathcal{L}$ (denoted later on by the same symbol) generates a Markov semigroup P_t on $\mathcal{C}(\Omega)$. We remark that $\mathcal{C}_1(\Omega)$ is a core for the Markov generator.

For $\rho \in [0, 1]$, let μ_ρ be the Bernoulli measure on Ω with density ρ, i.e. the product measure with marginals satisfying $\mu_\rho\,(\eta(i)) = \rho$ and $\mu_\rho\,(\eta(i) - \rho)^2 = \rho(1 - \rho) \equiv \mathbf{v}_\rho$. It is not difficult to verify that μ_ρ is invariant and reversible for P_t. It is also known (see [10, VIII Thm. 1.44]) that the set of invariant measures for P_t consists of all convex linear combinations of μ_ρ , $\rho \in [0, 1]$.

We can then define the semigroup P_t also on the space $\mathbb{L}_2(\mu_\rho)$. By reversibility it is a self-adjoint contracting semigroup and the Dirichlet form associated to it is given by

$$\mathbf{D}_\rho(f) \equiv \frac{1}{2} \sum_{\langle i,j \rangle} \mu_\rho \ (\delta_{i,j} f)^2 \tag{2.11}$$

on some domain $\mathcal{Q}(\mathbf{D}_\rho)$. In particular it is bounded if $f \in \mathcal{C}_1(\Omega)$ and then we have

$$\mathbf{D}_\rho(f) = -\mu_\rho \ (f \mathcal{L} f)$$

Our first main result establishes the algebraic rate of convergence to the invariant measure in $\mathbb{L}_2(\mu_\rho \ )$.

Theorem 1 *For any $\rho \in [0,1]$ and $q \in [1,2)$ the following **Generalized Nash** inequality is true*

$$\mu_\rho \ (f - \mu_\rho \ f)^2 \leq \ \mathbf{D}_\rho(f)^\alpha \ \mathbf{A}_q(f)^{1-\alpha} \tag{N}$$

for all $f \in \mathcal{C}_q(\Omega)$, with

$$\alpha \equiv \alpha(q,d) \equiv \left(\frac{1}{q} - \frac{1}{2} \right) \left(\frac{1}{d} + \frac{1}{q} - \frac{1}{2} \right)^{-1} \tag{2.12}$$

and a P_t-monotone functional

$$\mathbf{A}_q(f) \equiv a_q \ \vert\!\vert\!\vert f \vert\!\vert\!\vert_q^2 \tag{2.13}$$

with some constant $a_q \equiv a_q(d,\rho) \in (0,\infty)$. Therefore for any $t > 0$, we have

$$\mu_\rho \ (P_t f - \mu_\rho \ f)^2 \leq \left(\frac{\gamma}{2} \right)^\gamma \ \frac{\mathbf{A}_q(f)}{t^\gamma} \tag{2.14}$$

with $\gamma \equiv d(\frac{1}{q} - \frac{1}{2})$.

Remark. The decay rate is the best possible. To see, for example when $q = 1$, this it is enough to notice that $\mathcal{L}$ defined in (2.8) preserves the subspace of linear functions. It is thus possible to compute explicitly $P_t f$ when $f = \eta(0)$. In this case the inequality (2.14) is saturated.

Our second main result establishes the algebraic rate of convergence to equilibrium in the entropy sense

Theorem 2 *For any $\rho \in [0,1]$ and any $q \in [1,2)$ the following **Logarithmic Nash** inequality is true*

$$\mu_\rho \ (f \log \frac{f}{\mu_\rho \ f}) \leq \ \mathbf{D}_\rho(f^{\frac{1}{2}})^{\overline{\alpha}} \widetilde{\mathbf{A}}_q(f^{\frac{1}{2}})^{1-\overline{\alpha}} \qquad \textbf{(LN)}$$

for all $f \in \mathcal{C}_q(\Omega)$, $f \geq 0$, with

$$\overline{\alpha}(q) \equiv \left(\frac{1}{q} - \frac{1}{2}\right)\left(\frac{1}{d} + \frac{1}{q}\right)^{-1} \qquad (2.15)$$

and

$$\widetilde{\mathbf{A}}_q(f^{\frac{1}{2}}) \equiv \overline{a}_q \ \mathbf{A}_q(f^{\frac{1}{2}}) \qquad (2.16)$$

with some constant $\overline{a}_q \equiv \overline{a}_q(d,\rho) \in (0,\infty)$. Moreover if $f > \varepsilon > 0$ for some constant $\varepsilon > 0$, then

$$\mu_\rho \ (f \log \frac{f}{\mu_\rho \ f}) \leq \ \mathbf{D}_\rho(f^{\frac{1}{2}})^{\alpha} \mathbf{B}_q(f^{\frac{1}{2}})^{1-\alpha} \qquad \textbf{(LN')}$$

with a monotone functional

$$\mathbf{B}_q(f^{\frac{1}{2}}) \equiv \frac{a'_q}{4 \inf f} |||f|||_q^2 \qquad (2.17)$$

defined with some constant $a'_q \equiv a'_q(d,\rho) \in (0,\infty)$ and $\alpha = \alpha(q,d)$ given by (2.12). Then for any $t > 0$, we have

$$\mu_\rho \ (P_t f \log \frac{P_t f}{\mu_\rho f}) \leq \left(\frac{\gamma}{4}\right)^\gamma \ \frac{\mathbf{B}_q(f)}{t^\gamma} \qquad (2.18)$$

with $\gamma \equiv \alpha/(1-\alpha)$ for any $t > 0$ and strictly positive function $f \in \mathcal{C}_q(\Omega)$

Remark 1 *If $f \in \mathbb{L}_2(\mu_\rho)$, $f \geq 0$, from Theorem 1 we can conclude similar decay in the entropy sense, but otherwise if $f \in \mathbb{L}_p(\mu_\rho)$, $p \in [1,2)$, and $f \geq \varepsilon > 0$, $f \in \mathcal{C}_q(\Omega) \setminus \mathcal{C}_1(\Omega)$, $q > 1$, Theorem 2 give us a nontrivial information.*

3 Proof of Generalized Nash Inequality

The proof will use finite dimensional approximations. Therefore we need to introduce some more notation.

For $\Lambda \subset \mathbb{Z}^d$ we denote by $\mu_{\Lambda,\rho}$ the product measure on $\Omega_\Lambda := \{0,1\}^\Lambda$ with density ρ, (i.e. $\mu_{\Lambda,\rho}\eta(i) = \rho$, for any $i \in \Lambda$). Accordingly, for a function f on Ω_Λ, we introduce the finite volume Dirichlet form by

$$\mathbf{D}_{\Lambda,\rho}(f) := \frac{1}{2} \sum_{\langle i,j\rangle \subset \Lambda} \mu_{\Lambda,\rho} \, (\delta_{ij} f)^2$$

For $k = 0, \cdots, |\Lambda|$, we introduce the canonical measures $\mu_\Lambda^k(\cdot)$ with k particles in a finite volume Λ, defined on Ω_Λ by requiring the following equality for the conditional expectation $\mu_{\Lambda,\rho} \, (\cdot \, |N_\Lambda)$ associated to the measure $\mu_{\Lambda,\rho}$ given the total number of particles $N_\Lambda := \sum_{i\in\Lambda} \eta(i)$ in Λ

$$\mu_{\Lambda,\rho} \, (f \, |N_\Lambda)(\eta) \equiv \sum_{k=0}^{|\Lambda|} \mu_\Lambda^k(f)\chi_k(\eta) \tag{3.1}$$

where $\chi_k(\eta) \equiv \chi\,(N_\Lambda(\eta) = k)$ denotes the characteristic function. Note that the measures μ_Λ^k on the right hand side of (3.1) is independent on ρ.

It is known, [8], that the probability measures $\mu_{\Lambda,\rho}$, $\Lambda \subset \mathbb{Z}^d$, satisfies the following useful inequality

Standard Spectral Gap Inequality : *For any* $\Lambda \subseteq \mathbb{Z}^d$ *we have*

$$\mu_{\Lambda,\rho} \, \left(g - \mu_{\Lambda,\rho} \, g\right)^2 \leq M(\rho)^{-1} \mathcal{E}_{\Lambda,\rho}(g) \qquad (\mathbf{SSG}(\mu_{\Lambda,\rho}))$$

for any function g *for which the standard Dirichlet form*

$$\mathcal{E}_{\Lambda,\rho}(g) \equiv \sum_{i\in\Lambda} \mu_{\Lambda,\rho} \, |\nabla_i g|^2$$

is finite, with a constant $M(\rho)^{-1} = \mathbf{v}_\rho$.

It is also known , (see [16, 6, 12]), that the measures μ_Λ^k satisfy the following

Finite Volume Spectral Gap Inequality : *For any cube* $\Lambda \subseteq \mathbb{Z}^d$ *we have*

$$\mu_\Lambda^k \left(f - \mu_\Lambda^k f\right)^2 \le c_0 |\Lambda|^{2/d} \mathbf{D}_\Lambda^k(f) \qquad (\mathbf{SG}(\mu_\Lambda^k))$$

with a constant $c_0 \in (0,\infty)$ *independent of* Λ *and* $k = 0, 1, ..., |\Lambda|$ *and* f*; by* $\mathbf{D}_\Lambda^k$ *we denote the following Dirichlet form corresponding to the measure* μ_Λ^k

$$\mathbf{D}_\Lambda^k(f) := \frac{1}{2} \sum_{\langle i,j\rangle \subset \Lambda} \mu_\Lambda^k (\delta_{ij} f)^2 \tag{3.2}$$

Our first technical ingredient necessary for the proof of Theorem 1 is the following (finite volume) bound.

Lemma 3 *For any* $\rho \in [0,1]$*, any* $q \in [1,2)$ *and any cube* $\Lambda \subset \mathbb{Z}^d$*, we have*

$$\mu_{\Lambda,\rho} \left(f - \mu_{\Lambda,\rho} f\right)^2 \le c_0 |\Lambda|^{2/d} \mathbf{D}_{\Lambda,\rho}(f) + M(\rho)^{-1} |\Lambda|^{-(\frac{2}{q}-1)} \left(\sum_{i\in\Lambda} \|\nabla_i f\|_u^q \right)^{\frac{2}{q}} \tag{3.3}$$

for all functions f *on* Ω_Λ.

Proof. It is sufficient to show the statement for $q = 1$, because then the general case follows from the following Holder inequality for any $q \in [1,\infty)$

$$\sum_{i\in\Lambda} \|\nabla_i f\|_u \le |\Lambda|^{1-\frac{1}{q}} \left(\sum_{i\in\Lambda} \|\nabla_i f\|_u^q \right)^{\frac{1}{q}} \tag{3.4}$$

Let $p_{\Lambda,\rho}(k) := \mu_{\Lambda,\rho}(\chi_k)$. Recalling the definition of μ_Λ^k, we have

$$\mu_{\Lambda,\rho} f = \sum_{k=0}^{|\Lambda|} p_{\Lambda,\rho}(k) \cdot \mu_\Lambda^k f \tag{3.5}$$

Hence

$$\mu_{\Lambda,\rho} (f - \mu_{\Lambda,\rho} f)^2 = \sum_{k=0}^{|\Lambda|} p_{\Lambda,\rho}(k) \mu_\Lambda^k (f - \mu_\Lambda^k f)^2 + \mu_{\Lambda,\rho} \left(\mu_{\Lambda,\rho}(f \mid N_\Lambda) - \mu_{\Lambda,\rho}\, \mu_{\Lambda,\rho}(f \mid N_\Lambda)\right)^2 \tag{3.6}$$

Using the spectral gap inequality $\mathbf{SG}(\mu_\Lambda^k)$ we can estimate the first part of the above sum as follows

$$\sum_{k=0}^{|\Lambda|} p_{\Lambda,\rho}(k)\mu_\Lambda^k(f-\mu_\Lambda^k f)^2 \leq c_0|\Lambda|^{2/d}\mathbf{D}_{\Lambda,\rho}(f) \tag{3.7}$$

We next bound the second part of the right hand side of (3.6) using $\mathbf{SSG}(\mu_{\Lambda,\rho})$. We have

$$\mu_{\Lambda,\rho}\ \left(\mu_{\Lambda,\rho}\ \left(f\ |N_\Lambda\right) - \mu_{\Lambda,\rho}\ \mu_{\Lambda,\rho}\ \left(f\ |N_\Lambda\right)\right)^2 \leq M(\rho)^{-1}\sum_{i\in\Lambda}\mu_{\Lambda,\rho}\left|\nabla_i\mu_{\Lambda,\rho}\left(f\ |N_\Lambda\right)\right|^2 \tag{3.8}$$

To estimate the right hand side of (3.8) we use the following simple lemma proven in the Appendix.

Lemma 4 *For any real function F and any finite set $\Lambda \subset \mathbb{Z}^d$ we have*

$$\sum_{i\in\Lambda}\mu_\rho\left|\nabla_i F\left(\mu_{\Lambda,\rho}\left(f\ |N_\Lambda\right)\right)\right|^2 = \frac{1}{\rho}\sum_{k=1}^{|\Lambda|}\left|F\left(\mu_\Lambda^k\left(f\right)\right) - F\left(\mu_\Lambda^{k-1}\left(f\right)\right)\right|^2 k\cdot\mu_\rho\ \chi_k \tag{3.9}$$

Hence

$$\sum_{i\in\Lambda}\mu_\rho\ \left|\nabla_i F\left(\mu_{\Lambda,\rho}\left(f\ |N_\Lambda\right)\right)\right|^2 \leq |\Lambda|\cdot\sup_{k=1,\ldots,|\Lambda|}\left|F\left(\mu_\Lambda^k\left(f\right)\right) - F\left(\mu_\Lambda^{k-1}\left(f\right)\right)\right|^2 \tag{3.10}$$

Now to finish the proof we need only to observe that the following inequality is true, [12],

Lemma 5

$$\left|\mu_\Lambda^{k+1}f - \mu_\Lambda^k f\right| \leq \frac{1}{|\Lambda|}\sum_{i\in\Lambda}\|\nabla_i f\|_u \tag{3.11}$$

The inequality (3.11) is a straightforward consequence of the identity

$$\mu_\Lambda^k f - \mu_\Lambda^{k+1} f = \frac{1}{k+1}\sum_{i\in\Lambda}\mu_\Lambda^{k+1}\left\{\nabla_i f\cdot\eta(i)\right\} \tag{3.12}$$

which is proven simply by noticing that

$$\mu_\Lambda^k f = \frac{1}{k+1}\sum_{i\in\Lambda}\mu_\Lambda^{k+1}\left\{f(\eta^i)\cdot\eta(i)\right\} \tag{3.13}$$

and the fact that by definition $\mu_\Lambda^{k+1}\{\eta(i)\} = \frac{k+1}{|\Lambda|}$.

This ends the proof of the inequality (3.3) for $q = 1$ and thus the proof of Lemma 3.

By using a martingale decomposition we next deduce from Lemma 3 the following (infinite volume) bound

Proposition 6 *For any $q \in [1,2)$ and $L \in \mathbb{N}$*

$$\mu_\rho\ \left(f - \mu_\rho\ f\right)^2 \le c_0 L^2 \mathbf{D}_\rho(f) + M(\rho)^{-1} L^{-d(\frac{2}{q}-1)} \|\!|f|\!\|_q^2 \tag{3.14}$$

for any $f \in \mathcal{C}_q(\Omega)$.

Proof. Let $\Lambda_0 = \{1,\cdots,L\}^d$ be a reference cube of side L in $\mathbb{Z}^d$. For $i \in \mathbb{Z}^d$, let $\Lambda_i := \Lambda_0 + iL$ be the translate of Λ_0 by a vector $iL \in \mathbb{Z}^d$. The family $\{\Lambda_i\}_{i\in\mathbb{Z}^d}$ defines then a partition of $\mathbb{Z}^d$. It will be more convenient to label its elements by a natural number; we thus obtain a family of cubes $\{\Lambda_\ell\}_1^\infty$such that for $\ell \neq \ell'$, $\Lambda_\ell \cap \Lambda_{\ell'} = \emptyset$ and $\bigcup_\ell \Lambda_\ell = \mathbb{Z}^d$. Let $\{Y_\ell\}_0^\infty$ be the increasing sequence defined by $Y_0 := \emptyset$, $Y_\ell := \bigcup_{\ell'\le\ell}\Lambda_{\ell'}$, if $\ell \ge 1$.

We associate to $\{Y_\ell\}_0^\infty$a family $\{E_\ell\}_0^\infty$ of conditional expectations defined by $E_0 f = f$ and for $\ell > 0$, $E_\ell f := \mu_{Y_\ell,\rho} f$, for any $f \in \mathcal{C}(\Omega)$.

We then have

$$f - \mu_\rho\ f = \sum_{\ell=1}^{\infty}(f_{\ell-1} - f_\ell) \tag{3.15}$$

where $f_\ell := E_\ell f = \mu_{\Lambda_\ell,\rho} f_{\ell-1}$. The series on the right hand side of (3.15) is convergent in $\mathbb{L}_2(\mu_\rho)$ (and is actually a finite sum when f is local (i.e.

depends only on a finite number of coordinates) and is absolutely convergent if $f \in \mathcal{C}_1(\Omega)$).

Since the sequence f_ℓ has orthogonal increments, we get

$$\mu_\rho\ (f - \mu_\rho\ f)^2 = \sum_{\ell=1}^{\infty} \mu_\rho\ (f_{\ell-1} - f_\ell)^2 = \sum_{\ell=1}^{\infty} \mu_\rho\ \mu_{\Lambda_\ell,\rho}(f_{\ell-1} - \mu_{\Lambda_\ell,\rho} f_{\ell-1})^2 \quad (3.16)$$

Keeping the variables outside Λ_ℓ fixed, we apply Lemma 3 to the cube Λ_ℓ and the function $f_{\ell-1}$. We get

$$\mu_{\Lambda_\ell,\rho}\left(f_{\ell-1} - \mu_{\Lambda_\ell,\rho} f_{\ell-1}\right)^2 \leq c_0 L^2 \mathbf{D}_{\Lambda_\ell,\rho}(f_{\ell-1}) + M(\rho)^{-1} L^{-d(\frac{2}{q}-1)} \left(\sum_{i\in\Lambda_\ell} \|\nabla_i f_{\ell-1}\|_u^q\right)^{\frac{2}{q}} \quad (3.17)$$

This bound can be simplified if we observe that for $i,j \in \Lambda_\ell$, by Cauchy-Schwartz inequality, one has

$$\mu_{\Lambda_\ell,\rho}\left(\mu_{Y_{\ell-1},\rho} f(\eta^{i,j}) - \mu_{Y_{\ell-1},\rho} f(\eta)\right)^2 \leq \mu_{Y_\ell,\rho}\left(f(\eta^{i,j}) - f(\eta)\right)^2 \quad (3.18)$$

and that

$$\|\nabla_i f_{\ell-1}\|_u \leq \|\nabla_i f\|_u \quad (3.19)$$

Hence, using (3.17) and (3.16), we get

$$\begin{aligned} &\mu_\rho\ \left(f - \mu_\rho\ f\right)^2 \leq \qquad (3.20) \\ &\leq \sum_{\ell=1}^{\infty} \left\{ c_0 L^2 \frac{1}{2} \sum_{\langle i,j\rangle \subset \Lambda_\ell} \mu_\rho\ (\delta_{i,j} f)^2 + M(\rho)^{-1} L^{-d(\frac{2}{q}-1)} \left(\sum_{i\in\Lambda_\ell} \|\nabla_i f\|_u^q\right)^{\frac{2}{q}} \right\} \\ &\leq c_0 L^2 \mathbf{D}_\rho(f) + M(\rho)^{-1} L^{-d(\frac{2}{q}-1)} \left(\sum_{i\in\mathbb{Z}^d} \|\nabla_i f\|_u^q\right)^{\frac{2}{q}} \end{aligned}$$

where in the last step we have used our assumption $q \in [1, 2)$. This ends the proof of (3.14).

□

<u>**Conclusion of the Proof of Theorem**</u> 1*:*

To prove the Generalized Nash inequality we need only to optimize the bound in Proposition 6 with respect to the parameter L. For this we observe first that we have the following a'priori bound

Lemma 7 *For $q \in [1,2]$*

$$\mathbf{D}_\rho(f) \le 4\,d \cdot \mathbf{v}_\rho \cdot \; |\!|\!| f |\!|\!|_q^2 \tag{3.21}$$

Proof of Lemma 7: To see this, we use the definition (2.11) of the Dirichlet form and the identity (2.10) to get

$$\begin{aligned}
\mathbf{D}_\rho(f) &= \frac{1}{2}\sum_{\langle i,j\rangle} \mu_\rho \;\left(\left[\nabla_i f(\eta) - \nabla_j f(\eta^{i,j})\right]^2 \cdot [\eta(i)(1-\eta(j)) + \eta(j)(1-\eta(i))]\right)\\
&\le \sum_{\langle i,j\rangle} \left[\|\nabla_i f\|_u^2 + \|\nabla_j f\|_u^2\right]\; \mu_\rho \;\{\eta(i)(1-\eta(j)) + \eta(j)(1-\eta(i))\}\\
&\le 2\,d \sum_{i\in\mathbb{Z}^d} \|\nabla_i f\|_u^2 \; 2\cdot \mathbf{v}_\rho = 4\,d\cdot \mathbf{v}_\rho \cdot |\!|\!| f |\!|\!|_q^2
\end{aligned}$$

where in the last step we have used the property (2.6).

□

By (3.21) we have that

$$x := \left(\frac{4\,d\cdot \mathbf{v}_\rho \cdot |\!|\!| f |\!|\!|_q^2}{\mathbf{D}_\rho(f)}\right)^{\frac{1}{2+d(2/q-1)}} \ge 1 \tag{3.22}$$

and so $[x]^2 \le x^2$ and $[x]^{-d(\frac{2}{q}-1)} \le 2^{d(\frac{2}{q}-1)} x^{-d(\frac{2}{q}-1)}$. Thus setting $L = [x]$ in Proposition 6 (and the fact that $M(\rho)^{-1} = \mathbf{v}_\rho$), we obtain the Generalized Nash inequality (**N**) with

$$\alpha = \left(\frac{1}{q} - \frac{1}{2}\right)\left(\frac{1}{d} + \frac{1}{q} - \frac{1}{2}\right)^{-1} \tag{3.23}$$

and

$$a_q = {c_q}^{1/1-\alpha} \cdot \mathbf{v}_\rho \tag{3.24}$$

where

$$c_q \equiv c_0 \cdot (4\,d)^{1-\alpha} + 2^{d(\frac{2}{q}-1)} \cdot (4\,d)^{-\alpha}$$

The proof of the algebraic decay (2.14) given the inequality (**N**) follows by standard arguments (see e.g. [4, 3, 11]) once we know that $|\!|\!| \cdot |\!|\!|_q$ is monotone, i.e. that $|\!|\!| P_t f |\!|\!|_q \le |\!|\!| f |\!|\!|_q$. In the case $q = 1$ the desired monotonicity property is proven in [10, I Thm. 3.9(d)]. We shall prove a (more) general result in Section 5 below.

□

4 Proof of Logarithmic Nash Inequality

The idea of the proof of the Logarithmic Nash inequality is similar to the one used before to show the Nash Inequality. The important role in our proof is now played by the following Logarithmic Sobolev inequalities:

Standard Logarithmic Sobolev Inequality :

For any $\Lambda \subseteq \mathbb{Z}^d$

$$\mu_{\Lambda,\rho}\left(f \log \frac{f}{\mu_{\Lambda,\rho} f}\right) \leq C(\rho)\mathcal{E}_{\Lambda,\rho}(f^{\frac{1}{2}}) \qquad (\mathbf{SLS}(\mu_{\Lambda,\rho}))$$

for any $f \geq 0$ *for which the (standard) Dirichlet form*

$$\mathcal{E}_{\Lambda,\rho}(f^{\frac{1}{2}}) \equiv \sum_{i\in\Lambda} \mu_{\Lambda,\rho} \left|\nabla_i(f^{\frac{1}{2}})\right|^2$$

is finite, with a coefficient $C(\rho) \in (0,\infty)$ *independent of* f *and* Λ*; see* [8].

Finite Volume Logarithmic Sobolev Inequality $N_\Lambda = k$:

For any cube $\Lambda \subset \mathbb{Z}^d$

$$\mu_\Lambda^k\left(f \log \frac{f}{\mu_\Lambda^k f}\right) \leq \overline{c}_0 |\Lambda|^{\frac{2}{d}} \mathbf{D}_\Lambda^k(f^{\frac{1}{2}}) \qquad (\mathbf{LS}(\mu_\Lambda^k))$$

for any $f \geq 0$ *for which*

$$\mathbf{D}_\Lambda^k(f^{\frac{1}{2}}) \equiv \frac{1}{2}\sum_{\langle i,j\rangle\in\Lambda} \mu_{\Lambda,\rho}\left|\delta_{ij}(f^{\frac{1}{2}})\right|^2$$

is finite, with a coefficient $\overline{c}_0 \in (0,\infty)$ *independent of* f, Λ *and* $k = 0, 1, ..., |\Lambda|$*; see* [13].

Our first technical ingredient necessary for the proof of Theorem 2 is the following (finite volume) bound.

Lemma 8 *For any $\rho \in [0,1]$, any $q \in [1,2)$ and any cube $\Lambda \subset \mathbb{Z}^d$, we have*

$$\mu_{\Lambda,\rho}\left(f \log \frac{f}{\mu_{\Lambda,\rho} f}\right) \leq c_1 |\Lambda|^{1+\frac{2}{d}} \mathbf{D}_{\Lambda,\rho}(f^{\frac{1}{2}}) + c_2 |\Lambda|^{-(\frac{2}{q}-1)} \left(\sum_{i\in\Lambda} \|\nabla_i f^{\frac{1}{2}}\|_u^q\right)^{\frac{2}{q}} \tag{4.1}$$

for all functions f on Ω_Λ, $f \geq 0$, with the constants $c_1 \equiv \left(\overline{c}_0 + \frac{12C(\rho)}{\mathbf{v}_\rho} c_0\right)$ *and* $c_2 \equiv 12C(\rho)$, *both independent of f and Λ.*

Remark 2 *As we indicate in the proof below, for $f > \varepsilon > 0$, the similar inequality is true with the last factor on the right hand side of (4.1) replaced by* $\left(\sum_{i\in\Lambda} \|\nabla_i f^{\frac{1}{2}}\|_u^q\right)^{\frac{2}{q}} / (4 \inf f)$ *and the same exponent as in Lemma 3 used for the proof of Generalized Nash inequality.*

Proof. Similarly as in the proof of the Nash inequality it is sufficient to consider the case $q = 1$.We have

$$\mu_{\Lambda,\rho}\left(f \log \frac{f}{\mu_{\Lambda,\rho} f}\right) = \mu_{\Lambda,\rho}\left(\mu_{\Lambda,\rho}\left(f \ \log \frac{f}{\mu_{\Lambda,\rho}(f \mid N_\Lambda)}\middle| N_\Lambda\right)\right) +$$

$$+\mu_{\Lambda,\rho}\left(\mu_{\Lambda,\rho}(f \mid N_\Lambda) \log \frac{\mu_{\Lambda,\rho}(f \mid N_\Lambda)}{\mu_{\Lambda,\rho}\left(\mu_{\Lambda,\rho}(f \mid N_\Lambda)\right)}\right) \tag{4.2}$$

Since

$$\mu_{\Lambda,\rho}\left(\mu_{\Lambda,\rho}\left(f \ \log \frac{f}{\mu_{\Lambda,\rho}(f \mid N_\Lambda)}\middle| N_\Lambda\right)\right) = \sum_{k=0}^{|\Lambda|} p_{\Lambda,\rho}(k) \mu_\Lambda^k \left(f \log \frac{f}{\mu_\Lambda^k f}\right)$$

using the Finite Volume Logarithmic Sobolev inequality we can estimate the first part of the right hand side of (4.2) as follows

$$\mu_{\Lambda,\rho}\left(\mu_{\Lambda,\rho}\left(f \ \log \frac{f}{\mu_{\Lambda,\rho}(f \mid N_\Lambda)}\middle| N_\Lambda\right)\right) \leq \overline{c}_0 |\Lambda|^{2/d} \mathbf{D}_{\Lambda,\rho}(f^{\frac{1}{2}}) \tag{4.3}$$

To bound the second part of the right hand side of (4.2) we use first the Standard Logarithmic Sobolev inequality to get

$$\mu_{\Lambda,\rho}\left(\mu_{\Lambda,\rho}(f \mid N_\Lambda) \ \log \frac{\mu_{\Lambda,\rho}(f \mid N_\Lambda)}{\mu_{\Lambda,\rho}\left(\mu_{\Lambda,\rho}(f \mid N_\Lambda)\right)}\right) \leq C(\rho) \sum_{i\in\Lambda} \mu_{\Lambda,\rho} \left|\nabla_i \left(\mu_{\Lambda,\rho}(f \mid N_\Lambda)\right)^{\frac{1}{2}}\right|^2 \tag{4.4}$$

with the constant $C(\rho) \in (0, \infty)$ independent of Λ and the function f. To estimate the right hand side of (4.4) we use Lemma 4 with the function F equal to the square root, to get

$$\sum_{i \in \Lambda} \mu_{\Lambda,\rho} \left| \nabla_i \left(\mu_{\Lambda,\rho} \left(f \mid N_\Lambda \right) \right)^{\frac{1}{2}} \right|^2 \leq \frac{1}{\rho} \sum_{k=1}^{|\Lambda|} \left| \left(\mu_\Lambda^k (f) \right)^{\frac{1}{2}} - \left(\mu_\Lambda^{k-1} (f) \right)^{\frac{1}{2}} \right|^2 k \cdot \mu_{\Lambda,\rho} \chi_k \tag{4.5}$$

We have

$$\left| \left(\mu_\Lambda^k (f) \right)^{\frac{1}{2}} - \left(\mu_\Lambda^{k-1} (f) \right)^{\frac{1}{2}} \right| = \left| \mu_\Lambda^k (f) - \mu_\Lambda^{k-1} (f) \right| \cdot \left| \left(\mu_\Lambda^k (f) \right)^{\frac{1}{2}} + \left(\mu_\Lambda^{k-1} (f) \right)^{\frac{1}{2}} \right|^{-1} \tag{4.6}$$

Remark 3 *At this point, if $f > \varepsilon > 0$, we could estimate the right hand side by $\left| \mu_\Lambda^k (f) - \mu_\Lambda^{k-1} (f) \right| / (2 \inf f^{\frac{1}{2}})$ and using Lemma 5 arrive at the inequality indicated in the remark after Lemma 8*

Since

$$\left| \mu_\Lambda^k (f) - \mu_\Lambda^{k-1} (f) \right| = \left| \mu_\Lambda^k \otimes \widetilde{\mu}_\Lambda^{k-1} (f - \widetilde{f}) \right| = \left| \mu_\Lambda^k \otimes \widetilde{\mu}_\Lambda^{k-1} (f^{\frac{1}{2}} - \widetilde{f}^{\frac{1}{2}})(f^{\frac{1}{2}} + \widetilde{f}^{\frac{1}{2}}) \right| \leq$$

$$\leq 2 \left(\mu_\Lambda^k \otimes \widetilde{\mu}_\Lambda^{k-1} (f^{\frac{1}{2}} - \widetilde{f}^{\frac{1}{2}})^2 \right)^{\frac{1}{2}} \cdot \left(\left(\mu_\Lambda^k (f) \right)^{\frac{1}{2}} + \left(\mu_\Lambda^{k-1} (f) \right)^{\frac{1}{2}} \right), \tag{4.7}$$

where $\widetilde{f}^{\frac{1}{2}}$ is integrated with respect to the isomorphic copy $\widetilde{\mu}_\Lambda^{k-1}$ of μ_Λ^{k-1}, we have

$$\left| \left(\mu_\Lambda^k (f) \right)^{\frac{1}{2}} - \left(\mu_\Lambda^{k-1} (f) \right)^{\frac{1}{2}} \right| \leq 2 \left(\mu_\Lambda^k \otimes \widetilde{\mu}_\Lambda^{k-1} (f^{\frac{1}{2}} - \widetilde{f}^{\frac{1}{2}})^2 \right)^{\frac{1}{2}} \leq \tag{4.8}$$

$$\leq 2 \left\{ \left(\mu_\Lambda^k (f^{\frac{1}{2}} - \mu_\Lambda^k f^{\frac{1}{2}})^2 \right)^{\frac{1}{2}} + \left(\mu_\Lambda^{k-1} (f^{\frac{1}{2}} - \mu_\Lambda^{k-1} f^{\frac{1}{2}})^2 \right)^{\frac{1}{2}} + \left| \mu_\Lambda^k f^{\frac{1}{2}} - \mu_\Lambda^{k-1} f^{\frac{1}{2}} \right| \right\}$$

From (4.5) — (4.8) we obtain the following bound

$$\sum_{i \in \Lambda} \mu_{\Lambda,\rho} \left| \nabla_i \left(\mu_{\Lambda,\rho} \left(f \mid N_\Lambda \right) \right)^{\frac{1}{2}} \right|^2 \leq$$

$$\leq 12 \frac{1}{\rho} \sum_{k=1}^{|\Lambda|} \left[\mu_\Lambda^k (f^{\frac{1}{2}} - \mu_\Lambda^k f^{\frac{1}{2}})^2 + \mu_\Lambda^{k-1} (f^{\frac{1}{2}} - \mu_\Lambda^{k-1} f^{\frac{1}{2}})^2 \right] k \cdot \mu_{\Lambda,\rho} \chi_k +$$

$$+12\frac{1}{\rho}\sum_{k=1}^{|\Lambda|}\left|\mu_\Lambda^k f^{\frac{1}{2}} - \mu_\Lambda^{k-1} f^{\frac{1}{2}}\right|^2 k\cdot\mu_{\Lambda,\rho}\chi_k \tag{4.9}$$

Now we use the spectral gap inequality $\mathbf{SG}(\mu_\Lambda^k)$ to get

$$\frac{1}{\rho}\sum_{k=1}^{|\Lambda|}\left[\mu_\Lambda^k(f^{\frac{1}{2}} - \mu_\Lambda^k f^{\frac{1}{2}})^2 + \mu_\Lambda^{k-1}(f^{\frac{1}{2}} - \mu_\Lambda^{k-1} f^{\frac{1}{2}})^2\right] k\cdot\mu_{\Lambda,\rho}\chi_k \leq |\Lambda|^{1+\frac{2}{d}}\frac{c_0}{\mathbf{v}_\rho}\mathbf{D}_{\Lambda,\rho}(f^{\frac{1}{2}}) \tag{4.10}$$

The last part from the right hand side (4.9) can be estimated similarly as in Section 3, namely as follows

$$\frac{1}{\rho}\sum_{k=1}^{|\Lambda|}\left|\mu_\Lambda^k f^{\frac{1}{2}} - \mu_\Lambda^{k-1} f^{\frac{1}{2}}\right|^2 k\cdot\mu_\rho\ \chi_k \leq \frac{1}{|\Lambda|}\left(\sum_{i\in\Lambda}||\nabla_i f^{\frac{1}{2}}||_u\right)^2 \tag{4.11}$$

Combining (4.9) — (4.11) with (4.4), we arrive at the following estimate

$$\mu_{\Lambda,\rho}\left(\mu_{\Lambda,\rho}(f\ |N_\Lambda)\ \log\frac{\mu_{\Lambda,\rho}(f\ |N_\Lambda)}{\mu_{\Lambda,\rho}\left(\mu_{\Lambda,\rho}(f\ |N_\Lambda)\ \right)}\right) \leq$$

$$\leq 12C(\rho)\frac{|\Lambda|^{1+\frac{2}{d}}}{\mathbf{v}_\rho}c_0\mathbf{D}_{\Lambda,\rho}(f^{\frac{1}{2}}) + \frac{12C(\rho)}{|\Lambda|}\left(\sum_{i\in\Lambda}||\nabla_i f^{\frac{1}{2}}||_u\right)^2 \tag{4.12}$$

The inequality (4.12) together with (4.3) yield the following bound

$$\mu_{\Lambda,\rho}\left(f\log\frac{f}{\mu_{\Lambda,\rho}f}\right) \leq \left(\overline{c}_0 + \frac{12C(\rho)}{\mathbf{v}_\rho}c_0\right)|\Lambda|^{1+\frac{2}{d}}\mathbf{D}_{\Lambda,\rho}(f^{\frac{1}{2}}) + \frac{12C(\rho)}{|\Lambda|}\left(\sum_{i\in\Lambda}||\nabla_i f^{\frac{1}{2}}||_u\right)^2 \tag{4.13}$$

This ends the proof of Lemma 8 in case $q = 1$ from which the general case easily follows.

□

By using a martingale decomposition we next deduce from Lemma 8 the following (infinite volume) bound

Proposition 9 *For any $q \in [1,2)$ and $L \in \mathbb{N}$*

$$\mu_\rho\ \left(f\log\frac{f}{\mu_\rho\ f}\right) \leq c_1 L^{d+2}\mathbf{D}_\rho(f^{\frac{1}{2}}) + c_2 L^{-d(\frac{2}{q}-1)}|||f^{\frac{1}{2}}|||_q^2 \tag{4.14}$$

for any $f \geq 0$, such that $f^{\frac{1}{2}} \in \mathcal{C}_q(\Omega)$.

Proof. . Let $\{Y_\ell\}_0^\infty$ be the increasing sequence defined by $Y_0 := \emptyset$, and $Y_\ell := \bigcup_{\ell' \leq \ell} \Lambda_{\ell'}$, if $\ell \geq 1$, where $\Lambda_\ell := \Lambda_0 + i_\ell L$, $i_\ell \in \mathbb{Z}^d$ and $\Lambda_0 = \{1, \cdots, L\}^d$. Let $\{E_\ell\}_0^\infty$ be a family of conditional expectations defined by $E_0 f = f$ and for $\ell > 0$, $E_\ell f := \mu_{Y_\ell,\rho} f$, for any $f \in \mathcal{C}(\Omega)$.

We then have

$$\mu_\rho \left(f \log \frac{f}{\mu_\rho \; f} \right) = \sum_{\ell=1}^{\infty} \mu_\rho \; \left(\mu_{\Lambda_\ell,\rho} \; \left(f_{\ell-1} \log \frac{f_{\ell-1}}{\mu_{\Lambda_\ell,\rho} \; f_{\ell-1}} \right) \right) \tag{4.15}$$

where $f_\ell := E_\ell f = \mu_{\Lambda_\ell,\rho} \; f_{\ell-1}$. For the fixed variables outside Λ_ℓ, we apply Lemma 8 to the cube Λ_ℓ and the function $f_{\ell-1}$. We get

$$\mu_{\Lambda_\ell,\rho} \; \left(f_{\ell-1} \log \frac{f_{\ell-1}}{\mu_{\Lambda_\ell,\rho} \; f_{\ell-1}} \right) \leq c_1 L^{d+2} \mathbf{D}_{\Lambda_\ell,\rho}((f_{\ell-1})^{\frac{1}{2}}) + c_2 L^{-d(\frac{2}{q}-1)} \left(\sum_{i \in \Lambda_\ell} \| \nabla_i (f_{\ell-1})^{\frac{1}{2}} \|_u^q \right)^{\frac{2}{q}} \tag{4.16}$$

This bound can be simplified if we observe that for $i, j \in \Lambda_\ell$, one has

$$\mu_{\Lambda_\ell,\rho} \; \left((\mu_{Y_{\ell-1},\rho} f)^{\frac{1}{2}} (\eta^{i,j}) - (\mu_{Y_{\ell-1},\rho} f)^{\frac{1}{2}} (\eta) \right)^2 \leq \mu_{Y_\ell,\rho} \left(f^{\frac{1}{2}} (\eta^{i,j}) - f^{\frac{1}{2}} (\eta) \right)^2 \tag{4.17}$$

and that

$$\| \nabla_i (f_{\ell-1})^{\frac{1}{2}} \|_u \leq \| \nabla_i f^{\frac{1}{2}} \|_u \tag{4.18}$$

Hence, using (4.15) — (4.18), we get

$$\mu_\rho \; \left(f \log \frac{f}{\mu_\rho \; f} \right) \leq \sum_{\ell=1}^{\infty} \left\{ c_1 L^{d+2} \frac{1}{2} \sum_{\langle i,j \rangle \subset \Lambda_\ell} \mu_\rho \; \left(\delta_{i,j} f^{\frac{1}{2}} \right)^2 + c_2 L^{-d(\frac{2}{q}-1)} \left(\sum_{i \in \Lambda_\ell} \| \nabla_i f^{\frac{1}{2}} \|_u^q \right)^{\frac{2}{q}} \right\}$$

$$\leq c_1 L^{d+2} \mathbf{D}_\rho (f^{\frac{1}{2}}) + c_2 L^{-d(\frac{2}{q}-1)} \left(\sum_{i \in \mathbb{Z}^d} \| \nabla_i f^{\frac{1}{2}} \|_u^q \right)^{\frac{2}{q}} \tag{4.19}$$

where in the last step we have used our assumption $q \in [1, 2)$. This ends the proof of (4.14).

$\square$

Conclusion of the Proof of Logarithmic Nash Inequality*:*

To prove the Logarithmic Nash inequality from Theorem 2 we need only to optimize the bound in Proposition 9 with respect to the parameter L. For this we use our a'priori bound Lemma 7

$$\mathbf{D}_\rho(f^{\frac{1}{2}}) \leq 4\,d \cdot \mathbf{v}_\rho \cdot |||f^{\frac{1}{2}}|||_q^2 \tag{4.20}$$

Thus for

$$x := \left(\frac{4\,d \cdot \mathbf{v}_\rho \cdot |||f^{\frac{1}{2}}|||_q^2}{\mathbf{D}_\rho(f^{\frac{1}{2}})} \right)^{\frac{1}{2+d2/q}} \geq 1$$

we have $[x]^{d+2} \leq x^{d+2}$ and $[x]^{-d(\frac{2}{q}-1)} \leq 2^{d(\frac{2}{q}-1)} x^{-d(\frac{2}{q}-1)}$. Using this and setting $L = [x]$ in Proposition 9, we obtain

$$\mu_\rho \left(f \log \frac{f}{\mu_\rho\ f} \right) \leq c_1 x^{d+2} \mathbf{D}_\rho(f^{\frac{1}{2}}) + c_2 2^{d(\frac{2}{q}-1)} x^{-d(\frac{2}{q}-1)} |||f^{\frac{1}{2}}|||_q^2 \tag{4.21}$$

from which (**LN**) easily follows with the characteristic exponent

$$\overline{\alpha} \equiv \left(\frac{1}{q} - \frac{1}{2} \right) \left(\frac{1}{d} + \frac{1}{q} \right)^{-1} \tag{4.22}$$

and with

$$\widetilde{\mathbf{A}}_q(f^{\frac{1}{2}}) \equiv \overline{a}_q \cdot |||f^{\frac{1}{2}}|||_q^2 \tag{4.23}$$

where

$$\overline{a}_q^{1-\alpha} \equiv c_1 \cdot (4\,d \cdot \mathbf{v}_\rho)^{1-\overline{\alpha}} + c_2 2^{d(\frac{2}{q}-1)} \cdot (4\,d \cdot \mathbf{v}_\rho)^{-\overline{\alpha}} \tag{4.24}$$

This ends the proof of the Logarithmic Nash Inequality (**LN**).

The proof of the inequality (**LN'**) proceeds in a similar way with the use of the finite volume inequality indicated in the remark after Lemma 8 and an observation that because

$$(\delta_{ij} f^{\frac{1}{2}})^2 \leq \frac{(\delta_{ij} f)^2}{4 \inf f} \tag{4.25}$$

and Lemma 7 we have

$$\mathbf{D}_\rho(f^{\frac{1}{2}}) \leq 4 d \mathbf{v}_\rho \frac{|||f|||_q^2}{4 \inf f} \tag{4.26}$$

□

Algebraic Decay to Equilibrium in the Entropy Sense

To proof of the algebraic decay of entropy given the inequality (**LN'**) we note first that, if $f \geq \varepsilon > 0$, then also $f_t \equiv P_t f \geq \varepsilon$ and we see that

$$\mathbf{B}_q(f) \equiv \frac{a'_q}{4\inf(f)} |||f|||_q^2 \tag{4.27}$$

is a monotone functional defined on a total set in $\mathcal{C}_q(\Omega)$ of strictly positive functions.

Remark 4 *Note that we have*

$$\widetilde{\mathbf{A}}_q(f_t^{\frac{1}{2}}) \leq C \cdot \mathbf{B}_q(f) \tag{4.28}$$

with some positive constant C independent of f.

Now if (**LN'**) is satisfied then we have also

$$\mu_\rho \left(f_t \log \frac{f_t}{\mu_\rho f} \right) \leq \mathbf{D}_\rho(f_t^{\frac{1}{2}})^\alpha \mathbf{B}_q(f)^{1-\alpha} \tag{4.29}$$

and so

$$\left(\frac{\mu_\rho \left(f_t \log \frac{f_t}{\mu_\rho f} \right)}{\mathbf{B}_q(f)^{1-\alpha}} \right)^{1/\alpha} \leq \mathbf{D}_\rho(f_t^{\frac{1}{2}}) \tag{4.30}$$

Using this we get

$$\frac{d}{dt}\mu_\rho \left(f_t \log \frac{f_t}{\mu_\rho f} \right) = \mu_\rho \left((\mathcal{L} f_t) \log \frac{f_t}{\mu_\rho f} \right) \leq$$

$$\leq -4\mathbf{D}_\rho(f_t^{\frac{1}{2}}) \leq -4 \left(\frac{\mu_\rho \left(f_t \log \frac{f_t}{\mu_\rho f} \right)}{\mathbf{B}_q(f)^{1-\alpha}} \right)^{1/\alpha} \tag{4.31}$$

where in the second step we have used a general inequality for Dirichlet forms. Solving this differential inequality we arrive at the following bound

$$\mu_\rho \left(f_t \log \frac{f_t}{\mu_\rho f} \right) \leq \left(\frac{\gamma}{4} \right)^\gamma \frac{\mathbf{B}_q(f)}{t^\gamma} \tag{4.32}$$

where $\gamma \equiv \alpha/(1-\alpha)$. This ends the proof of the algebraic decay to equilibrium in the entropy sense.

□

5 Monotonicity of seminorms

In this Section we identify two families of seminorms which are contracted by the semigroup P_t. These families are relevant to control the algebraic decay to equilibrium, but the results may be of its own interest.

We denote by $\mathcal{F} \equiv \mathcal{F}(\mathbb{Z}^d)$ and $\mathcal{F}_n \equiv \mathcal{F}_n(\mathbb{Z}^d)$ the collection of all finite subsets of $\mathbb{Z}^d$ and the collection of the subsets $Y \subset \mathbb{Z}^d$ of cardinality n, respectively. For $Y \in \mathcal{F}$ we define an operator

$$\nabla_Y := \prod_{j \in Y} \nabla_j \tag{5.1}$$

For $n \in \mathbb{N}$, $p, q \in [1, \infty]$, we define the following seminorms

$$\mathbf{V}^n_{q,p}(f) := \left(\sum_{Y \in \mathcal{F}_n} \left(\mu_\rho \, |\nabla_Y f|^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \tag{5.2}$$

Note that if $p \geq p'$, $q' \geq q$, then

$$\mathbf{V}^n_{q,p}(f) \leq \mathbf{V}^n_{q',p'}. \tag{5.3}$$

On the other hand for $n \neq \tilde{n}$ it is easy to check that $\mathbf{V}^n_{q,p}$ and $\mathbf{V}^{\tilde{n}}_{q,p}$ are non equivalent in the sense there exists f such that $\mathbf{V}^n_{q,p}(f) < \infty$ and $\mathbf{V}^{\tilde{n}}_{q,p}(f) = \infty$.

One can see that the seminorms $|||f|||_{n,p}$, $n \in \mathbb{N}$, $p \in [1, \infty]$, given by

$$|||f|||_{n,p} := \left(\sum_{X \in \mathcal{F}_n} \|\nabla_X f\|_u^p \right)^{1/p}, \tag{5.4}$$

(which we have used before), can be expressed as follows

$$|||f|||_{n,p} = \mathbf{V}^n_{\infty,p}(f) \tag{5.5}$$

We will show the following result

Theorem 10 *For any $n \in \mathbb{N}$, $p, q \in [1, \infty]$, the seminorm $\mathbf{V}^n_{q,p}$ is contracted by P_t, i.e. for any $t \geq 0$ we have*

$$\mathbf{V}^n_{q,p}(P_t f) \leq \mathbf{V}^n_{q,p}(f) \tag{5.6}$$

This results generalizes [10, I Thm. 3.9(d)] in which it is proven that $\mathbf{V}^{(1)}_{\infty,1}$ is contracted by P_t.

Proof: We will show that the contractivity property can obtained as a result of some more general construction. For this we need to consider an extended configuration space $\Omega_n := \mathcal{F}_n \times \Omega$; we shall denote the points in this space by (Y, η). In $\mathcal{C}(\Omega_n)$ we can distinguish a subset of cylinder functions, i.e. functions which depend only on a finite number of coordinates η_j, $j \in \mathbb{Z}^d$, and vanishing if Y is not contained in some ball. In $\mathcal{C}(\Omega_n)$ we introduce an operator $\mathcal{L}_n$ densely defined on the set of cylinder functions $F = F(Y, \eta)$, by the following formula

$$\mathcal{L}_n F(Y,\eta) := \sum_{\langle i,j\rangle : \langle i,j\rangle \cap Y = \emptyset} \left[F(Y, \eta^{i,j}) \; - F(Y,\eta) \right] \tag{5.7}$$

$$+ \sum_{i \in Y} \sum_{j \notin Y : |j-i|=1} \left[F((Y \setminus i) \cup j, \eta^{i,j}) \; - F(Y,\eta) \right]$$

It is immediate to check that $\mathcal{L}_n$ is a Markov pregenerator, i.e. it satisfies the maximum principle. One can construct the corresponding semigroup $P^{(n)}_t$ on cylinder function as a limit

$$P^{(n)}_t F(Y,\eta) = \lim_{L\to\infty} P^{(n,L)}_t F(Y,\eta) \tag{5.8}$$

with $P^{(n,L)}_t \equiv e^{t\mathcal{L}_{n,L}}$ where $\mathcal{L}_{n,L}$ is a generator with periodic boundary conditions on the cube Λ_L of side L centered at the origin, which is defined in a similar way as in (5.7) except that $Y \subset \Lambda$ and the nearest neighbors pairs of points are understood as on the corresponding torus $\mathbb{T}^d_L$. The construction can be carried out in a similar way as in e.g. [18] (one needs then to introduce additional gradient operator $\partial_{X,\langle ij\rangle} F(Y,\eta) = F((Y \setminus i) \cup j, \eta^{i,j}) \; - F(Y,\eta)$, if $Y = X$ and zero otherwise). We remark that the restriction of $\mathcal{L}_n$ to the functions F independent of η equals precisely to the generator Δ_n of the simple symmetric exclusion process on the lattice $\mathbb{Z}^d$ with n particles.

The idea of considering such objects is based on the observation that for $\widetilde{F}(Y,\eta) = \nabla_Y f\,(\eta)$, $(Y,\eta) \in \Omega_n$, one has

$$\nabla_Y \mathcal{L} f(\eta) = \mathcal{L}_n \widetilde{F}(Y,\eta) \tag{5.9}$$

and therefore

$$\nabla_Y P_t f(\eta) = P^{(n)}_t \widetilde{F}(Y,\eta) \tag{5.10}$$

Thus, if for $p, q \in [1, \infty]$ we introduce Banach space $\mathbb{X}_{q,p}$ of functions on Ω_n, with a norm defined by

$$\|F\|_{q,p} := \left(\sum\nolimits_{Y \in \mathcal{F}_n} \left(\mu_\rho |F(Y, \cdot)|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \tag{5.11}$$

and we show that $P_t^{(n)}$ extends to a contractive semigroup in this space, then to get our result it is sufficient to note that for $\widetilde{F}(Y, \eta) := \nabla_Y f\,(\eta)$ we have

$$\mathbf{V}^n_{q,p}(f) = \|\widetilde{F}\|_{q,p} \tag{5.12}$$

(The idea of using a relation similar to (5.9) with a different purpose has been exploited in an interesting way e.g. in [2] in the context of Glauber dynamics.)

We note that $\mathbb{X}_{2,2}$ equals to the space $\mathbb{L}_2(dY \times d\mu_\rho)$, where dY denotes the counting measure on $\mathcal{F}_n$, with an inner product defined by

$$\langle F, G \rangle := \sum_{Y \in \mathcal{F}_n} \int d\mu_\rho(\eta) F(Y, \eta)\, G(Y, \eta) \tag{5.13}$$

A simple computation shows that $\mathcal{L}_n$, as defined in (5.7), is essentially self–adjoint and non positive on $\mathbb{L}_2(dY \times d\mu_\rho)$ and the similar statement is true for the generator $\mathcal{L}_{n,L}$ with periodic boundary conditions (on functions localized in Λ_L).

Given the finite volume approximation property (5.8), it is now sufficient to show the necessary contraction properties for $P_t^{(n,L)} \equiv e^{t\mathcal{L}_{n,L}}$ for the subspaces of functions localized in Λ_L. For this we will apply Hille–Yosida Theorem to show that of $\mathcal{L}_{n,L}$ generates a contractive semigroup on the Banach space $\mathbb{X}_{q,p}$, for any $q, p \in \{1, \infty\}$. It is sufficient to show that for a function $G \in \mathbb{X}_{q,p}$ satisfying

$$\lambda G - \mathcal{L}_{n,L} G = F \tag{5.14}$$

for some $\lambda > 0$ and a given $F \in \mathbb{X}_{q,p}$, $q, p \in \{1, \infty\}$, localized in Λ_L, we have

$$\lambda \|G\|_{q,p} \leq \|F\|_{q,p} \tag{5.15}$$

The case $(q, p) = (\infty, \infty)$ is easy and therefore will be omitted, but we note that from the considerations in $\mathbb{X}_{\infty,\infty}$ follows that the corresponding semigroup $P_t^{(n,L)}$ is positivity preserving.

Next we consider the case $(q,p) = (\infty, 1)$. Let $\bar\eta = \bar\eta(Y)$ be such that $\sup_\eta |G(Y,\eta)| = G(Y,\bar\eta)$. (Since Ω is compact, eventually changing signs of F and G, one can see that such $\bar\eta$ exists.) Recalling the definition $\mathcal{L}_{n,L}$ we have

$$\begin{aligned}
\lambda \|G(Y,\cdot)\|_u &= \lambda G(Y,\bar\eta) = F(Y,\bar\eta) + \sum_{i\in Y} \sum_{j\notin Y\,:\,|j-i|=1} \left[G((Y\setminus i)\cup j, \bar\eta^{i,j}) \; - G(Y,\bar\eta) \right] \\
&\le \|F(Y,\cdot)\|_u + \sum_{i\in Y} \sum_{j\notin Y\,:\,|j-i|=1} \left[\|G((Y\setminus i)\cup j, \cdot)\,\|_u - \|G(Y,\cdot)\|_u \right]
\end{aligned}$$

and summing over $Y \in \mathcal{F}_n(\Lambda_L)$ we get (5.15) in the considered case $(q,p) = (\infty, 1)$, (due to translation invariance of our generator the second parts on the right hand side of the above inequality adds up to zero). The bound (5.15) then implies that the resolvent of $\mathcal{L}_{n,L}$ satisfies $\|(\lambda - \mathcal{L}_{n,L})^{-1}\| \le \lambda^{-1}$ for all $\lambda > 0$, uniformly in L. Therefore, by the Hille–Yosida Theorem, $\mathcal{L}_{n,L}$ generates a contractive semigroup on $\mathbb{X}_{\infty,1}$.

To get the corresponding result for $(q,p) = (1,\infty)$, we note that there is $\overline{Y} \in \mathcal{F}_n(\Lambda_L)$ such that for the function G

$$\sup_{Y\in\mathcal{F}_n(\Lambda_L)} |\mu_\rho G(Y,\cdot)| = \mu_\rho G(\overline{Y},\cdot); \tag{5.16}$$

(if necessary we change the signs of F and G). Now we have

$$\begin{aligned}
\lambda \|G\|_{1,\infty} &= \lambda \mu_\rho G(\overline{Y},\eta) = \\
&= \mu_\rho F(\overline{Y},\eta) + \sum_{i\in \overline{Y}} \sum_{j\notin \overline{Y}\,:\,|j-i|=1} \left[\mu_\rho G((\overline{Y}\setminus i)\cup j, \eta^{i,j}) \; - \mu_\rho G(\overline{Y},\eta) \right] \\
&= \mu_\rho F(\overline{Y},\eta) + \sum_{i\in \overline{Y}} \sum_{j\notin \overline{Y}\,:\,|j-i|=1} \left[\mu_\rho G((\overline{Y}\setminus i)\cup j, \eta) \; - \mu_\rho G(\overline{Y},\eta) \right] \le \\
&\le \mu_\rho F(\overline{Y},\eta) \le \|F\|_{1,\infty}
\end{aligned} \tag{5.17}$$

here $\|\cdot\|_{1,\infty}$, means the norm associated with $\Omega_{n,L} \equiv \mathcal{F}_n(\Lambda_L) \times \Omega$.

To consider the case of $\mathbb{X}_{1,1}$, taking F nonnegative, we see that if for some $\lambda > 0$ the function G satisfies $\lambda G + \mathcal{L}_{n,L} G = F$, it is also nonnegative and we have

$$\lambda \|G\|_{1,1} = \lambda \sum_Y \mu G(Y,\cdot) = \sum_Y \mu F(Y,\cdot) + \sum_Y \mu \mathcal{L}_{n,L} G(Y,\cdot) = \sum_Y \mu F(Y,\cdot) = \|F\|_{1,1} \tag{5.18}$$

where in the last step we have used the symmetry of $\mathcal{L}_{n,L}$ in $\mathbb{X}_{2,2}$ space and the fact that $\mathcal{L}_{n,L}1 = 0$. This implies that the corresponding semigroup is also contractive in $\mathbb{X}_{1,1}$.

Since for the given $q \in [1,\infty]$, respectively $p \in [1,\infty]$, the family $\{\mathbb{X}_{q,r} : r \in [1,\infty]\}$, respectively $\{\mathbb{X}_{r,p} : r \in [1,\infty]\}$, forms a complex interpolating family, using the interpolation theorem we arrive at

$$\|P_t^{(n,L)} F\|_{q,p} \leq \|F\|_{q,p} \tag{5.19}$$

for every $p, q \in [1,\infty]$. Using this and the finite volume approximation property for cylinder functions we get the desired contractivity property for a dense set of functions in every space $\mathbb{X}_{q,p}$. By continuity this implies the general result and together with (5.12) implies the statement of the theorem. □

For $Y \in \mathcal{F}(\mathbb{Z}^d)$ and $\rho \in (0,1)$ we define

$$\sigma_Y := \mathbf{v}_\rho^{-|Y|/2} \prod_{i \in Y} (\eta_i - \rho) \tag{5.20}$$

We note that $\{\sigma_Y\}$, $Y \in \mathcal{F}_n(\mathbb{Z}^d)$, $n = 0, 1, \ldots$ forms an orthonormal basis in $\mathbb{L}_2(\mu_\rho)$. Therefore for any $f \in \mathbb{L}_2(\mu_\rho)$ we have the following expansion

$$f \equiv \sum_{n=0}^{\infty} f^{(n)} = \sum_{n=0}^{\infty} \sum_{Y \in \mathcal{F}_n} \mathrm{f}_Y^{(n)} \cdot \sigma_Y \tag{5.21}$$

Later on the subspace of $\mathbb{L}_2(\mu_\rho)$ of function satisfying $f = f^{(n)}$ will be denoted by $\mathcal{H}_n$. Using the expansion (5.21) one can see that in particular the following useful relation is true

$$\mu_\rho(\sigma_Y f) = (-2)^{-|Y|} \mu_\rho(\sigma_Y \nabla_Y f) \tag{5.22}$$

Now for $n \in \mathbb{N}$, $a \geq 0$ and $p \in [1,\infty]$ we define the Sobolev–type seminorms $\mathbf{H}^n_{-a,p}$ as follows

$$\mathbf{H}^n_{-a,p}(f) := 2^{-n} \sup_{0 \neq \varphi \in \ell_0(\mathcal{F}_n)} \frac{\sum_{Y \in \mathcal{F}_n} \varphi(Y) \mu_\rho(\sigma_Y \nabla_Y f)}{\|(-\Delta_n)^{a/2} \varphi\|_q} \tag{5.23}$$

where $q^{-1} + p^{-1} = 1$,the operator Δ_n is defined as the restriction of $\mathcal{L}_n$ to functions independent of $\eta \in \Omega$ and $\ell_0(\mathcal{F}_n)$ denotes the space of finite sequences over $\mathcal{F}_n$ and $\|\cdot\|_q$ denotes the norm in the space $\ell_q(\mathcal{F}_n)$ of sequences absolutely summable in the power q. Since Δ_n is self–adjoint in $\ell_2(\mathcal{F}_n)$, the operator $(-\Delta_n)^{a/2}$ is well defined via the spectral theorem. We note that, using (5.22), one has

$$\begin{aligned}\mathbf{H}^n_{-a,p}(f) &= \sup_{0\neq\varphi\in\ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n}\varphi(Y)\mu_\rho(\sigma_Y f)}{\|(-\Delta_n)^{a/2}\varphi\|_q} = \\ &= \sup_{0\neq\varphi\in\ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n}\varphi(Y)\mathrm{f}^{(n)}_Y}{\|(-\Delta_n)^{a/2}\varphi\|_q} \equiv \|(-\Delta_n)^{-a/2}\mathrm{f}^{(n)}\|_p \qquad (5.24)\end{aligned}$$

where $\mathrm{f}^{(n)} \equiv \{\mathrm{f}^{(n)}_Y\}_{Y\in\mathcal{F}_n}$. We shall denote the corresponding Sobolev type space by $W_{-a,p}(\mathcal{F}_n(\mathbb{Z}^d))$.

Theorem 11 *For any $n \in \mathbb{N}$, $p \in [1,\infty]$, $a \geq 0$, the seminorm $\mathbf{H}^n_{-a,p}$ is contracted by P_t, i.e. for any $t \geq 0$ we have*

$$\mathbf{H}^n_{-a,p}(P_t f) \leq \mathbf{H}^n_{-a,p}(f) \qquad (5.25)$$

Proof: The proof is similar to that of Theorem 10. For $Y \in \mathcal{F}_n$, consider $F(Y,\eta) := 2^{-n}\sigma_Y\nabla^n_Y f$.Then using the relation (5.22) we see that

$$\mathbf{H}^n_{-a,p}(f) = \sup_{0\neq\varphi\in\ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n}\varphi(Y)\mu_\rho(\sigma_Y f)}{\|(-\Delta_n)^{a/2}\varphi\|_q} = \sup_{0\neq\varphi\in\ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n}\varphi(Y)\mu_\rho(F(Y,\cdot))}{\|(-\Delta_n)^{a/2}\varphi\|_q} \qquad (5.26)$$

For such F we have (similarly as in the proof of Theorem 10)

$$\sigma_Y\nabla_Y\mathcal{L}f\,(\eta) = \mathcal{L}_n F\,(Y,\eta)$$

where $\mathcal{L}_n$ is the operator defined in (5.7). Therefore

$$\sigma_Y\nabla_Y P_t f(\eta) = P^{(n)}_t F\,(Y,\eta)$$

Since $P_t^{(n)}$ is a self adjoint semigroup on $\mathbb{L}_2(dY \times \mu_\rho)$, we obtain

$$\sum_{Y \in \mathcal{F}_n} \varphi(Y)\mu_\rho(P_t^{(n)}F(Y,\eta)) = \langle \varphi, P_t^{(n)}F\rangle = \langle P_t^{(n)}\varphi, F\rangle = \langle Q_t^{(n)}\varphi, F\rangle$$

because φ does not depend on η and on those function $P_t^{(n)}$ coincides with $Q_t^{(n)} = \exp\{t\Delta_n\}$.Thus, for $p^{-1} + q^{-1} = 1$, with $F(Y,\eta) = \sigma_Y \nabla_Y f$, we have

$$\begin{aligned}
\mathbf{H}^n_{-a,p}(P_t f) &= \sup_{\varphi \in \ell_0(\mathcal{F}_n)} \frac{\langle Q_t^{(n)}\varphi, F\rangle}{\|(-\Delta_n)^{a/2}\varphi\|_q} \\
&= \sup_{\varphi \in \ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n} Q_t^{(n)}\varphi(Y)\cdot \mu_\rho(\sigma_Y f)}{\|(-\Delta_n)^{a/2}Q_t^{(n)}\varphi\|_q} \cdot \frac{\|(-\Delta_n)^{a/2}Q_t^{(n)}\varphi\|_q}{\|(-\Delta_n)^{a/2}\varphi\|_q} \\
&\le \sup_{\varphi \in \ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n} Q_t^{(n)}\varphi(Y)\cdot \mu_\rho(\sigma_Y f)}{\|(-\Delta_n)^{a/2}Q_t^{(n)}\varphi\|_q} \le \mathbf{H}^n_{-a,p}(f)
\end{aligned}$$

where we used that, by ℓ_q–contractivity of the semigroup, $Q_t^{(n)} = \exp\{t\Delta_n\}$,

$$\|(-\Delta_n)^{a/2}Q_t^{(n)}\varphi\|_q \le \|(-\Delta_n)^{a/2}\varphi\|_q$$

□

6 Proof of faster decay

In this Section we pursue further our spectral analysis of the product case. We take an advantage of the fact that our stochastic dynamics is locally conservative in the sense that for any set $\Lambda \in \mathcal{F}$ the null subspace $\mathcal{I}_{\Lambda,\rho} \subseteq \mathbb{L}_2(\mu_{\Lambda,\rho})$ of the Dirichlet form $\mathbf{D}_{\Lambda,\rho}$ is nontrivial (i.e. contains nonconstant elements). In our case one knows that $\mathcal{I}_{\Lambda,\rho}$ is spanned by the following orthonormal functions

$$\mathbf{S}_\Lambda^{(n)} \equiv |\mathcal{F}_n(\Lambda)|^{-\frac{1}{2}} \cdot \sum_{Y\in\mathcal{F}_n(\Lambda)} \sigma_Y, \qquad n = 0, ..., |\Lambda| \tag{6.1}$$

(It is not difficult to see that the value of this function is constant on any set $\{\eta \in \Omega : N_\Lambda(\eta) = k\}$, $k = 0, ..., |\Lambda|$.) Let $\aleph_{\Lambda,\rho}$ denote the orthogonal complement of $\mathcal{I}_{\Lambda,\rho}$ in $\mathbb{L}_2(\mu_{\Lambda,\rho})$. It follows from the proof of Lemma 3 (see (3.6) and (3.7)) that, in case when $\Lambda \in \mathcal{F}$ is a cube of side $L \in \mathbb{N}$, then for any $f \in \aleph_{\Lambda,\rho}$, we have

$$\mu_{\Lambda,\rho}(f - \mu_{\Lambda,\rho} f)^2 \leq c_0 L^2 \cdot \mathbf{D}_{\Lambda,\rho}(f) \tag{6.2}$$

Given a partition $\{\Lambda_\ell\}_1^\infty$ of $\mathbb{Z}^d$ by disjoint cubes of side $L \in \mathbb{N}$ used in th proof of Proposition 6 and 9, we define the families $\{\wp_\ell\}_1^\infty$, $\{\mathcal{I}_\ell\}_1^\infty$ and $\{\aleph_\ell\}_1^\infty$ of orthogonal projectors respectively by setting

$$\wp_\ell f = E_{\ell-1} f - E_\ell f \tag{6.3}$$

(where $E_0 f \equiv f$ and $E_\ell f := \mu_{Y_\ell,\rho} f$, for $\ell \in \mathbb{N}$, with $\{Y_\ell\}_1^\infty$ being the increasing sequence defined by $Y_\ell := \bigcup_{\ell' \leq \ell} \Lambda_{\ell'}$),

$$\mathcal{I}_\ell f \equiv \sum_{m=1}^{|\Lambda|} \mathcal{I}_\ell^{(m)} f \equiv \sum_{m=1}^{|\Lambda|} \mu_{\Lambda_\ell,\rho} \left(\mathbf{S}_{\Lambda_\ell}^{(m)} \cdot \wp_\ell f \right) \cdot \mathbf{S}_{\Lambda_\ell}^{(m)} \tag{6.4}$$

and

$$\aleph_\ell f \equiv (\mathbf{1} - \mathcal{I}_\ell) \, \wp_\ell f \tag{6.5}$$

Using this notation it is easy to see that one has

$$\mu_\rho \left(f - \mu_\rho \ f \right)^2 = \sum_{\ell=1}^\infty \mu_\rho (\wp_\ell f)^2 = \sum_{\ell=1}^\infty \mu_\rho (\aleph_\ell f)^2 + \sum_{\ell=1}^\infty \mu_\rho (\mathcal{I}_\ell f)^2 \tag{6.6}$$

Since always $\mu_{\Lambda_\ell,\rho} \aleph_\ell f = 0$, the first part of the right hand side of (6.6) can be estimated with the help of (6.2) as follows

$$\sum_{\ell=1}^\infty \mu_\rho (\aleph_\ell f)^2 = \sum_{\ell=1}^\infty \mu_\rho \left(\mu_{\Lambda_\ell,\rho} (\aleph_\ell f)^2 \right) \leq c_0 L^2 \cdot \sum_{\ell=1}^\infty \mathbf{D}_{\Lambda_\ell,\rho} \left(\aleph_\ell f \right) \tag{6.7}$$

To analyze the second part on the right hand side of (6.6) we observe first that the function $\mathcal{I}_\ell f$ depends only on the variable in $\complement Y_\ell \equiv \mathbb{Z}^d \diagdown Y_\ell$ and as such can be expanded in the following series of orthogonal in $\mathbb{L}_2(\mu_\rho)$ elements

$$\mathcal{I}_\ell f = \mu_{\complement Y_\ell,\rho} \mathcal{I}_\ell f + \sum_{\ell' > \ell}^\infty \wp_{\ell'} \mathcal{I}_\ell f \tag{6.8}$$

Thus we have

$$\begin{aligned}\mu_\rho(\mathcal{I}_\ell f)^2 &= \sum_{\ell'>\ell}^{\infty} \mu_\rho\left(\wp_{\ell'}\mathcal{I}_\ell f\right)^2 + \mu_\rho(\mu_{\complement Y_\ell,\rho}\mathcal{I}_\ell f)^2 \qquad (6.9)\\ &= \sum_{\ell'>\ell}^{\infty} \mu_\rho\left(\aleph_{\ell'}\mathcal{I}_\ell f\right)^2 + \sum_{\ell'>\ell}^{\infty} \mu_\rho\left(\mathcal{I}_{\ell'}\mathcal{I}_\ell f\right)^2 + \mu_\rho(\mu_{\complement Y_\ell,\rho}\mathcal{I}_\ell f)^2\end{aligned}$$

Combining (6.6) - (6.9) we obtain the following inequality

$$\mu_\rho\left(f-\mu_\rho\ f\right)^2 \leq c_0 L^2 \cdot \sum_{\ell=1}^{\infty} \mathbf{D}_{\Lambda_\ell,\rho}\left(\aleph_\ell f\right) +$$

$$+\sum_{\ell=1}^{\infty}\sum_{\ell'>\ell}^{\infty} \mu_\rho\left(\aleph_{\ell'}\mathcal{I}_\ell f\right)^2 + \sum_{\ell=1}^{\infty}\sum_{\ell'>\ell}^{\infty} \mu_\rho\left(\mathcal{I}_{\ell'}\mathcal{I}_\ell f\right)^2 + \sum_{\ell=1}^{\infty} \mu_\rho(\mu_{\complement Y_\ell,\rho}\mathcal{I}_\ell f)^2 \qquad (6.10)$$

Applying this arguments inductively, we arrive at the following bound

$$\mu_\rho\left(f-\mu_\rho\ f\right)^2 \leq c_0 L^2 \cdot \{ \sum_{\ell(1)=1}^{\infty} \mathbf{D}_{\Lambda_{\ell(1)},\rho}\left(\aleph_{\ell(1)} f\right) + \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty} \mathbf{D}_{\Lambda_{\ell(2)},\rho}\left(\aleph_{\ell(2)}\mathcal{I}_{\ell(1)} f\right) + ...$$

$$+\sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty} .. \sum_{\ell(m)>\ell(m-1)}^{\infty} \mathbf{D}_{\Lambda_{\ell(m)},\rho}\left(\aleph_{\ell(m)}\mathcal{I}_{\ell(m-1)}..\mathcal{I}_{\ell(1)} f\right) + ...\} + \qquad (6.11)$$

$$+\{ \sum_{\ell(1)=1}^{\infty} \mu_\rho(\mu_{\complement Y_\ell,\rho}\mathcal{I}_\ell f)^2 + \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty} \mu_\rho\left(\mu_{\complement Y_{\ell(2)},\rho}\left(\mathcal{I}_{\ell(2)}\mathcal{I}_{\ell(1)} f\right)\right)^2 + ...$$

$$+\sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty} .. \sum_{\ell(m)>\ell(m-1)}^{\infty} \mu_\rho\left(\mu_{\complement Y_{\ell(m)},\rho}\left(\mathcal{I}_{\ell(m)}\mathcal{I}_{\ell(m-1)}..\mathcal{I}_{\ell(1)} f\right)\right)^2 + ...\}$$

We note that

$$\mathbf{D}_\rho(f) \geq \sum_{\ell(1)=1}^{\infty} \mathbf{D}_{\Lambda_{\ell(1)},\rho}\left(\aleph_{\ell(1)} f\right) + \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty} \mathbf{D}_{\Lambda_{\ell(2)},\rho}\left(\aleph_{\ell(2)}\mathcal{I}_{\ell(1)} f\right) + ... \qquad (6.12)$$

$$+\sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty} .. \sum_{\ell(m)>\ell(m-1)}^{\infty} \mathbf{D}_{\Lambda_{\ell(m)},\rho}\left(\aleph_{\ell(m)}\mathcal{I}_{\ell(m-1)}..\mathcal{I}_{\ell(1)} f\right) + ... \qquad (6.13)$$

Thus introducing the following notation

$$\mathcal{A}_\rho^{(L)}(f) \equiv \sum_{\ell(1)=1}^{\infty} \mu_\rho(\mu_{\complement Y_\ell,\rho}\mathcal{I}_\ell f)^2 + \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty} \mu_\rho\left(\mu_{\complement Y_{\ell(2)},\rho}\left(\mathcal{I}_{\ell(2)}\mathcal{I}_{\ell(1)}f\right)\right)^2 + \ldots$$

$$+ \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty} .. \sum_{\ell(m)>\ell(m-1)}^{\infty} \mu_\rho\left(\mu_{\complement Y_{\ell(m)},\rho}\left(\mathcal{I}_{\ell(m)}\mathcal{I}_{\ell(m-1)}..\mathcal{I}_{\ell(1)}f\right)\right)^2 + \ldots \tag{6.14}$$

we arrive at the following intermediate result

Proposition 12 *For any $L \in \mathbb{N}$, we have*

$$\mu_\rho\left(f - \mu_\rho f\right)^2 \leq c_0 L^2 \cdot \mathbf{D}_\rho(f) + \mathcal{A}_\rho^{(L)}(f) \tag{6.15}$$

for any function $f \in \mathbb{L}_2(\mu_\rho)$ for which the right hand side is finite.

The functional $\mathcal{A}_\rho^{(L)}$ is rather complicated and therefore we need to find a convenient bound for it. For this we remark that if $f \in \mathcal{H}_n$, i.e. when it is of the following form

$$f = \sum_{Y\in\mathcal{F}_n} \mathrm{f}_Y^{(n)} \cdot \sigma_Y \tag{6.16}$$

with some real numbers $\mathrm{f}_Y^{(n)}$, $Y \in \mathcal{F}_n$, then we have

$$\mathcal{A}_\rho^{(L)}(f) = \sum_{m=1}^{n} \sum_{\ell(1)<\ldots<\ell(m)} \sum_{\substack{k_1+..+k_m=n \\ 1\leq k_1,..,k_m\leq|\Lambda_0|}} \frac{1}{\prod_{j=1}^m |\mathcal{F}_{k(j)}(\Lambda_0)|} \left(\sum_{\substack{Y_1\subset\Lambda_{\ell(1)},..,Y_m\subset\Lambda_{\ell(m)} \\ |Y_1|=k_1,..,|Y_m|=k_m}} \mathrm{f}^{(n)}_{Y_1\cup..\cup Y_m} \right)^2$$

$$\leq \frac{\bar{b}_n}{|\mathcal{F}_n(\Lambda_0)|} \|\chi_L * f\|^2_{\ell_2(\mathbb{Z}^{nd})} \tag{6.17}$$

where

$$\bar{b}_n \equiv \max_{\substack{m=1,..,n \\ k_1+..+k_m=n \\ 1\leq k_1,..,k_m\leq|\Lambda_0|}} \frac{|\mathcal{F}_n(\Lambda_0)|}{\prod_{j=1}^m |\mathcal{F}_{k(j)}(\Lambda_0)|} \tag{6.18}$$

and where we have introduced a convolution type operator, by identifying $\mathcal{H}_n$ with a subspace $\ell_2(\mathcal{F}_n) \subset \ell_2(\mathbb{Z}^{dn})$, as follows

$$\chi_L * f(\mathbf{i}) \equiv \sum_{\mathbf{y}\in\mathbb{Z}^{dn}} \chi_L(\mathbf{y}-\mathbf{i}\cdot L)\cdot \mathrm{f}^{(n)}_{\mathbf{y}} \tag{6.19}$$

where $\mathbf{i} \equiv \{i_1, ..., i_n\}$ and $\mathbf{y} \equiv \{y_1, \ldots, y_n\}$, and

$$\chi_L(\mathbf{y}-\mathbf{i}\cdot L) \equiv \prod_{m=1}^{n} \mathbf{1}_{\Lambda_0}\left(y_m - i_m \cdot L\right) \tag{6.20}$$

with $\mathbf{1}_{\Lambda_0}$ being the characteristic function of a cube $\Lambda_0 \subset \mathbb{Z}^d$ of side L. Using (6.17) and the fact that $|\mathcal{F}_n(\Lambda_0)| = \frac{L^d...(L^d-n+1)}{n!}$, we see that for $f \in \mathcal{H}_n$, $1 \leq n \leq L^d$, we have

$$\mu_\rho\left(f-\mu_\rho f\right)^2 \leq c_0 L^2 \cdot \mathbf{D}_\rho(f) + \frac{b_n}{L^{dn}} \|\chi_L * f\|^2_{\ell_2(\mathbb{Z}^{nd})} \tag{6.21}$$

with some positive constant b_n. We will show the following estimate

Proposition 13 *For each $n, s \in \mathbf{N}$, $q \in [1,2]$ and $\gamma > 0$ there exist $L_0 = L_0(n,s,d,\gamma)$ and $b = b(n,s,q,d,\gamma)$ such that the following estimate*

$$L^{-nd}\left\|\chi_L * J\right\|^2_{\ell_2(\mathbb{Z}^{nd})} \leq b \cdot L^{-4s} \cdot L^{-nd(2/q-1)}\left\|J\right\|^2_{W_{-2s,q}(\mathcal{F}_n(\mathbb{Z}^d))} + \gamma\left\|J\right\|^2_{\ell_2(\mathcal{F}_n(\mathbb{Z}^d))} \tag{6.22}$$

holds for any $L > L_0$ and $J \in W_{-2s,q}(\mathcal{F}_n(Z^d))$.

We first recall the Young's inequality for the convolution operators. For $\psi \in \ell_1(\mathbb{Z}^{nd}))$ and $L \in \mathbb{N}$, let ψ_L be a convolution operator defined as follows

$$\psi_L * J \equiv \sum_{\mathbf{y}\in\mathbb{Z}^{nd}} \psi_L(\mathbf{y}-\mathbf{i}\cdot L)\cdot J(\mathbf{y}) \tag{6.23}$$

Then we have

Lemma 14 *For each $p \in [1,\infty]$, $\psi \in \ell_1(\mathbb{Z}^{nd}))$ and $J \in \ell_p(\mathbb{Z}^{nd})$*

$$\|\psi_L * J\|_{\ell_p(\mathbb{Z}^{nd})} \leq \|\psi\|^{(p-1)/p}_{\ell_1(\mathbb{Z}^{nd})} \cdot \left(\sup_{\mathbf{y}\in\mathbb{Z}^{nd}} \sum_{\mathbf{i}\in\mathbb{Z}^{nd}} |\psi(\mathbf{y}-\mathbf{i}L)|\right)^{1/p} \cdot \|J\|_{\ell_p(\mathbb{Z}^{nd})} \tag{6.24}$$

Proof. The inequality (6.24) follows by applying the Riesz-Thorin interpolation theorem with use of the following two simple bounds

$$\|\psi_L * J\|_{\ell_\infty(\mathbb{Z}^d)} = \sup_{\mathbf{i}\in\mathbb{Z}^{nd}} \left| \sum_{\mathbf{y}\in\mathbb{Z}^{nd}} \psi(\mathbf{y}-\mathbf{i}L)\cdot J(\mathbf{y}) \right| \le \|\psi\|_{\ell_1(\mathbb{Z}^{nd})}\cdot \|J\|_{\ell_\infty(\mathbb{Z}^{nd})} \tag{6.25}$$

and

$$\|\psi_L * J\|_{\ell_1(\mathbb{Z}^{nd})} = \sum_{\mathbf{i}\in\mathbb{Z}^{nd}} \left| \sum_{\mathbf{y}\in\mathbb{Z}^{nd}} \psi(\mathbf{y}-\mathbf{i}L)\cdot J(\mathbf{y}) \right| \le \sup_{\mathbf{y}\in\mathbb{Z}^{nd}} \sum_{\mathbf{i}\in\mathbb{Z}^{nd}} |\psi(\mathbf{y}-\mathbf{i}L)|\cdot\|J\|_{\ell_1(\mathbb{Z}^{nd})}$$

□

Proof of Proposition 13. Using Lemma 14 we will show that, given $s, n \in \mathbb{N}$ and $\gamma > 0$, one can choose "a smooth" approximation ψ_L^ε of the operator χ_L so that

$$L^{-nd}\,\|(\chi_L - \psi_L^\varepsilon) * J\|^2_{\ell_2(\mathbb{Z}^{nd})} \le \frac{\gamma}{2}\,\|J\|^2_{\ell_2(\mathcal{F}_n(\mathbb{Z}^d))} \tag{6.26}$$

and with some constant $a \in (0,\infty)$ we have

$$L^{-nd}\,\|\psi_L^\varepsilon * J\|^2_{\ell_2(\mathbb{Z}^{nd})} \le \frac{a}{2}\cdot L^{-4s}\cdot L^{-nd(2/q-1)}\,\|J\|^2_{W_{-2s,q}(\mathcal{F}_n(\mathbb{Z}^d))} \tag{6.27}$$

for any $J \in \ell_2(\mathcal{F}_n(\mathbb{Z}^d)) \cap W_{-2s,q}(\mathcal{F}_n(\mathbb{Z}^d))$.

Step 1(Proof of (6.26)): For $\delta \in (0,1)$ we choose first $\psi_\delta \in C_0^\infty(\mathbb{R})$ such that $0 \le \psi_\delta(r) \le 1$, $\psi_\delta(r) = 1$ for $r \in [0,1]$ and $\psi_\delta(r) = 0$ for $r \ge 1+\delta$ and $r \le -\delta$. For $\mathbf{x} \in \mathbb{Z}^{nd}$ define

$$\psi_{\delta,L}(\mathbf{x}) := \prod_{k=1}^{nd} \psi_\delta\left(L^{-1}x_k\right) \tag{6.28}$$

We want to show that the convolution operator involving χ_L can be well approximated by the one defined with the smooth function $\psi_{\delta,L}$. Since using the definition of $\chi_L$we have

$$\psi_{\delta,L}(\mathbf{x})-\chi_L(\mathbf{x}) = \sum_{k=1}^{nd} \prod_{m<k} \psi_\delta\left(L^{-1}x_m\right)\cdot\left[\psi_\delta\left(L^{-1}x_k\right) - \mathbf{1}_{[0,1]}\left(L^{-1}x_n\right)\right]\cdot\prod_{m>n} \mathbf{1}_{[0,1]}\left(L^{-1}x_m\right)$$

and so we get the following bound

$$\left\|\psi_{\delta,L} - \chi_L\right\|_{\ell_1(\mathbb{Z}^{nd})} \le nd\cdot[(1+2\delta)L]^{nd-1}\,[(1+2\delta)L - L] \le \delta\cdot 2nd\cdot(3L)^{nd}$$

Also we have

$$\sup_{\mathbf{x}\in\mathcal{F}_n(\mathbb{Z}^d)}\sum_{\mathbf{i}\in\mathbb{Z}^{nd}}\left|\psi_{\delta,L}(\mathbf{x}-\mathbf{i}L)-\chi_L(\mathbf{x}-\mathbf{i}L)\right|\le\sup_{\mathbf{x}\in\mathcal{F}_n(\mathbb{Z}^d)}\sum_{\mathbf{i}\in\mathbb{Z}^{nd}}\psi_{\delta,L}(\mathbf{x}-\mathbf{i}L)\le 3^{nd}$$

Therefore applying Lemma 14 with $p=2$ we get

$$\left\|\left(\chi_L-\psi_{\delta,L}\right)*J\right\|^2_{\ell_2(\mathbb{Z}^{nd})}\le\delta\cdot 2nd\cdot 3^{2nd}\cdot L^{nd}\cdot\|J\|^2_{\ell_2(\mathcal{F}_n(\mathbb{Z}^d))}\qquad(6.29)$$

Now we modify further the function $\psi_{\delta,L}$ by introducing another cutoff ε. The purpose is to obtain a new truncation $\psi^\varepsilon_L\equiv\psi^\varepsilon_{\delta,L}$ with support bounded away from the hyperplanes $y_k=y_{k'}$, $k,k'\in\{1,\dots,n\}$ where the exclusion condition applies.

Given $\varepsilon\in(0,1)$ we pick $\zeta_\varepsilon\in C^\infty(\mathbb{R})$ such that $0\le\zeta_\varepsilon(r)\le 1$, $\zeta_\varepsilon(r)=1$ for $|r|\ge 2\varepsilon$ and $\zeta_\varepsilon(r)=0$ for $|r|\le\varepsilon$. For $(y_1,\dots,y_n)\in\mathbb{Z}^{nd}$ let us define

$$\psi^\varepsilon_L(y_1,\dots,y_n):=\psi_{\delta,L}(y_1,\dots,y_n)\prod_{(l,l')}\zeta_\varepsilon\left(\frac{|y_l-y_{l'}|}{L}\right)\qquad(6.30)$$

where the product runs over the pairs $(l,l')\subset\{1,\dots,n\}$ and for $y\in\mathbb{R}^d$ we have used the notation $|y|:=\sup_{1\le l\le d}|y_l|$.

It is immediate to verify that

$$\left\|\psi_{\delta,L}-\psi^\varepsilon_L\right\|_{\ell_1(\mathbb{Z}^{nd})}\le\frac{n(n-1)}{2}(4\varepsilon L)^d[(1+2\delta)L]^{d(n-1)}\le\varepsilon^d\cdot\frac{n(n-1)}{2}\cdot 4^{nd}L^{nd}$$

Since

$$\sup_{\mathbf{y}\in\mathcal{F}_n(\mathbb{Z}^d)}\sum_{\mathbf{i}\in\mathbb{Z}^{nd}}\left|\psi_{\delta,L}(\mathbf{y}-\mathbf{i}L)-\psi^\varepsilon_L(\mathbf{y}-\mathbf{i}L)\right|\le\sup_{\mathbf{y}\in\mathcal{F}_n(\mathbb{Z}^d)}\sum_{\mathbf{i}\in\mathbb{Z}^{nd}}\psi_{\delta,L}(\mathbf{y}-\mathbf{i}L)\le 3^{nd}$$

by applying again Lemma 14, we find

$$\left\|\left(\psi_{\delta,L}-\psi^\varepsilon_L\right)*J\right\|^2_{\ell_2(\mathbb{Z}^{nd})}\le\varepsilon^d\cdot\frac{n(n-1)}{2}\cdot 4^{2nd}L^{nd}\cdot\|J\|^2_{\ell_2(\mathcal{F}_k(\mathbb{Z}^d))}\qquad(6.31)$$

Thus combining (6.29) and (6.31) we arrive at (6.26) provided that we have chosen δ and ε so that

$$\delta\cdot 2nd\cdot 3^{2nd}+\varepsilon^d\cdot\frac{n(n-1)}{2}\cdot 4^{2nd}\le\frac{\gamma}{2}\qquad(6.32)$$

Step 2.(Proof of (6.27)): We now estimate the main contribution to $\|\chi_L * J\|_{\ell_2(\mathbb{Z}^d)}$ by taking advantage of the specific form of ψ_L^ε, which will allow us to get the correct dependence on L.

Let us first note that, by the definition (6.30), $\psi_L^\varepsilon(y_1,\ldots,y_n)$ vanishes if there exist $(j,j') \subset \{1,\ldots,n\}$ such that $|x_j - x_{j'}| \le \varepsilon L$. For $L \ge \varepsilon^{-1}$ we can consider ψ_L^ε as a function on $\mathcal{F}_n(\mathbb{Z}^d)$. We also note that, for any $s \in \mathbb{N}$, $s \le \varepsilon L$, we have

$$(-\Delta_n)^s\,\psi_L^\varepsilon(y_1,\ldots,y_n) = (-\Delta_{\mathbb{Z}^{nd}})^s\,\psi_L^\varepsilon(y_1,\ldots,y_n) \tag{6.33}$$

where $\Delta_{\mathbb{Z}^{nd}}$ is the Laplacian on $\mathbb{Z}^{nd}$. In fact, due to the above support property of ψ_L^ε, the exclusion condition in Δ_n does not appear.

The hypotheses $J \in W_{-2s,q}(\mathcal{F}_n(\mathbb{Z}^d))$ means that J is in the range of $(-\Delta_n)^s$ (as an operator on $\ell_q(\mathcal{F}_n(\mathbb{Z}^d))$). We therefore have

$$\begin{aligned}
\psi_L^\varepsilon * J\,(\mathbf{i}) &= \sum_{Y\in\mathcal{F}_n(\mathbb{Z}^d)} \psi_L^\varepsilon(Y-\mathbf{i})\,(-\Delta_n)^s\,(-\Delta_n)^{-s}\,J(Y) \\
&= \sum_{Y\in\mathcal{F}_n(\mathbb{Z}^d)} (-\Delta_{\mathbb{Z}^{nd}})^s\,\psi_L^\varepsilon(Y-\mathbf{i})\,(-\Delta_n)^{-s}\,J(Y)
\end{aligned} \tag{6.34}$$

where we used the self–adjointness of $-\Delta_n$ and (6.33).

We note that $\psi_L^\varepsilon(y_1,\ldots,y_n)$ is of the form

$$\psi_L^\varepsilon(y_1,\ldots,y_n) = \psi^\varepsilon\left(\frac{y_1}{L},\ldots,\frac{y_n}{L}\right)$$

for a smooth (i.e. in $C_0^\infty(\mathbb{R}^{nd})$) function ψ^ε. By Taylor's Theorem we thus have

$$\sup_{(y_1,\ldots,y_n)\in\mathbb{Z}^{nd}} |(-\Delta_{\mathbb{Z}^{nd}})^s\,\psi_L^\varepsilon(y_1,\ldots,y_n)| \le L^{-2s}\cdot \sup_{\mathbf{r}\in\mathbb{R}^{nd}} |(-\Delta_{\mathbf{R}^{nd}})^s\,\psi^\varepsilon(\mathbf{r})| \le C\cdot L^{-2s} \tag{6.35}$$

for some constant $C = C(n,s,\delta,\varepsilon)$. In the above $\Delta_{\mathbb{R}^{nd}}$ is the Laplacian on $\mathbb{R}^{nd}$.

On the other hand we note the support of the function $(-\Delta_{\mathbb{Z}^{nd}})^s\,\psi_L^\varepsilon(Y-\mathbf{i})$ is contained in the region $|y_j - i_j| \le (1+\delta)L + s$, for all $j = 1,\ldots,n$. This is because $(-\Delta_{\mathbb{Z}^{nd}})^s$ enlarges the support of s lattice units (one for each power). Hence, by (6.34) and (6.35), denoting $Y = \{y_1, \ldots, y_k\}$

$$\|\psi_L^\varepsilon * J\|^2_{\ell_2(\mathbb{Z}^{nd})} \leq \sum_{\mathbf{i}\in\mathbb{Z}^{nd}} \left[\sum_{Y\in\mathcal{F}_n(\mathbb{Z}^d)} \left|(-\Delta_{\mathbb{Z}^{nd}})^s \psi_L^\varepsilon(Y-\mathbf{i})\right| \left|(-\Delta_n)^{-s} J(Y)\right| \right]^2 \leq \tag{6.36}$$

$$\begin{aligned}
&\leq\ C^2 \cdot L^{-4s} \sum_{\mathbf{i}\in\mathbb{Z}^{nd}} \left[\sum_{Y\in\mathcal{F}_n(\mathbb{Z}^d)} \prod_{j=1}^n \mathbf{1}_{|y_j-i_j|\leq(1+\delta)L+s} \cdot \left|(-\Delta_n)^{-s} J(Y)\right| \right]^2 \\
&\leq\ C^2 \cdot L^{-4s} \cdot [4L+2s]^{nd(2-2/q)} \cdot \sum_{\mathbf{i}\in\mathbb{Z}^{nd}} \left[\sum_{Y\in\mathcal{F}_n(\mathbb{Z}^d)} \prod_{j=1}^n \mathbf{1}_{|y_j-i_j|\leq 3L} \cdot \left|(-\Delta_n)^{-s} J(Y)\right|^q \right]^{2/q} \\
&\leq\ C^2 \cdot L^{-4s} \cdot (6L)^{nd(2-2/q)} \cdot \left[\sum_{\mathbf{i}\in\mathbb{Z}^{nd}} \sum_{Y\in\mathcal{F}_n(\mathbb{Z}^d)} \prod_{j=1}^n \mathbf{1}_{|y_j-i_j|\leq 3L} \cdot \left|(-\Delta_n)^{-s} J(Y)\right|^q \right]^{2/q} \\
&\leq\ \frac{a}{2} \cdot L^{-4s} \cdot L^{nd(2-2/q)} \|J\|_{W_{-2s,q}\left(\mathcal{F}_n(\mathbb{Z}^d)\right)}
\end{aligned}$$

where we used Hölder inequality in the second step and $q \in [1,2]$ in the third.

This ends the proof of (6.27) and so of Proposition 13. □

As a consequence of our considerations we get the following result.

Proposition 15 *For any $n, s \in \mathbb{N}$, $q \in [1,2]$ there exist $K_n \in \mathbb{N}$ and a constant $C' \in (0,\infty)$ such that for any $L \geq K_n$ we have*

$$\mu_\rho\left(f - \mu_\rho f\right)^2 \leq 2c_0 L^2 \cdot \mathbf{D}_\rho(f) + C' \cdot L^{-4s} \cdot L^{-nd(2/q-1)} \cdot \|f\|^2_{W_{-2s,q}(\mathcal{F}_n(\mathbb{Z}^d))} \tag{6.37}$$

for any $f \in \mathcal{H}_n$ for which the right hand side is finite

Proof of Proposition 15. We choose $\gamma = \frac{1}{2}b_n$ and $K_n \equiv \max\{L_0, n^{1/d}\}$ in Proposition 13 and note that clearly in our situation

$$\|f\|^2_{\ell_2(\mathcal{F}_n(\mathbb{Z}^d))} = \mu_\rho\left(f - \mu_\rho f\right)^2 \tag{6.38}$$

Thus taking into the account (6.21), one easily arrives at

$$\mu_\rho\left(f - \mu_\rho f\right)^2 \leq 2c_0 L^2 \cdot \mathbf{D}_\rho(f) + C' \cdot L^{-4s} \cdot L^{-nd(2/q-1)} \cdot \|f\|^2_{W_{-2s,q}(\mathcal{F}_n(\mathbb{Z}^d))} \tag{6.39}$$

for all $L \geq K_n$ with a constant $C' \equiv 2b_n \cdot b \in (0,\infty)$. $\square$

For a finite sequences $\mathbf{s} = \{-s(l) \in \mathbb{N}\}_{l=1,..,n}, \mathbf{q} = \{q(l) \in (2,1]\}_{l=1,..,n}$, we define a space $W_{\mathbf{s},\mathbf{q}} \equiv W_{\mathbf{s},\mathbf{q}}(\mathcal{F}(\mathbb{Z}^d))$ of functions for which the following seminorm is finite

$$\|f\|^2_{W\mathbf{s},\mathbf{q}} \equiv \sum_{l=1,..,n} \|f\|^2_{W_{s(l),q(l)}(\mathcal{F}_l(\mathbb{Z}^d))} \tag{6.40}$$

where $\|f\|^2_{W_{s(l),q(l)}(\mathcal{F}_l(\mathbb{Z}^d))}$ denotes the corresponding seminorm used before for the projection of f onto $\mathcal{H}_l$. As the spaces $\mathcal{H}_l$ are orthogonal from Proposition 15 we get the following corollary.

Corollary 16 *For any n and finite sequences $\mathbf{s}, \mathbf{q}$ such that*

$$2s(l) + nd\,(2/q(l) - 1) = Nd \tag{6.41}$$

for some number $N \in (0,\infty)$, there exist $L_N \in \mathbb{N}$ and a constants $B_N \in (0,\infty)$ such that for any $L \geq L_N$ we have

$$\mu_\rho\left(f - \mu_\rho f\right)^2 \leq 2c_0 L^2 \cdot \mathbf{D}_\rho(f) + B_N \cdot L^{-Nd} \cdot \|f\|^2_{W\mathbf{s},\mathbf{q}} \tag{6.42}$$

for any $f \in \oplus_{l=1,..,n}\mathcal{H}_l$ for which the right hand side is finite

Using this result we prove now the following theorem.

Theorem 17 *For any $\rho \in [0,1]$ and $N \in (0,\infty)$ the following* ***Generalized Nash*** *inequality is true*

$$\mu_\rho\left(f - \mu_\rho f\right)^2 \leq \mathbf{D}_\rho(f)^{\alpha_N} \mathbf{A}_{W\mathbf{s},\mathbf{q}}(f)^{1-\alpha_N} \tag{6.43}$$

with

$$\alpha_N \equiv \frac{Nd}{2+Nd} \tag{6.44}$$

for any $f \in \oplus_{l=1,..,n}\mathcal{H}_l$ for which the following functional is finite

$$\mathbf{A}_{W\mathbf{s},\mathbf{q}}(f) \equiv a_{\mathbf{s},\mathbf{q}} \cdot \|f\|^2_{W\mathbf{s},\mathbf{q}} \tag{6.45}$$

with some constant $a_{\mathbf{s},\mathbf{q}} \in (0,\infty)$ *and finite sequences* $\mathbf{s} = \{-s(l) \in \mathbb{N}\}_{l=1,..,n}$ *and* $\mathbf{q} = \{q(l) \in (1,2]\}_{l=1,..,n}$ *such that*

$$2s(l) + nd\,(2/q(l) - 1) = Nd \tag{6.46}$$

Then for any $t > 0$*, we have the following decay to equilibrium estimate*

$$\mu_\rho \left(P_t f - \mu_\rho f\right)^2 \leq \frac{\mathbf{A}_{W\mathbf{s},\mathbf{q}}(f)}{(\gamma_N t)^{\gamma_N}} \tag{6.47}$$

with $\gamma_N \equiv \alpha_N/(1-\alpha_N)$

Proof We observe first that for $f = \sum_{m\in\mathbb{N}} f^{(m)}$ with $f^{(m)} = \sum_{Y\in\mathcal{F}_m} \mathrm{f}_Y^{(m)} \sigma_Y$, we have

$$\mathbf{D}_\rho(f) = -\sum_{m\in\mathbb{N}} (\mathrm{f}^{(m)}, \Delta_m \mathrm{f}^{(m)})_{\ell_2(\mathcal{F}_m)} \tag{6.48}$$

Since the operator Δ_m is bounded in $\ell_2(\mathcal{F}_m)$, we have for any $-s \in \mathbb{N}$ and $q \in [1,2]$

$$(\mathrm{f}^{(m)}, -\Delta_m \mathrm{f}^{(m)})_{\ell_2(\mathcal{F}_m)} \leq \left\| (\Delta_m)^{-s+1/2} \right\| \cdot \|f\|^2_{W_{s,2}(\mathcal{F}_m)} \leq \left\| (\Delta_m)^{-s+1/2} \right\| \cdot \|f\|^2_{W_{s,q}(\mathcal{F}_m)} \tag{6.49}$$

Using this it is easy to see that for any finite sequence $\mathbf{s},\mathbf{q}$ there is a constant $C_{\mathbf{s},\mathbf{q}} \in (0,\infty)$ such that

$$\mathbf{D}_\rho(f) \leq C_{\mathbf{s},\mathbf{q}} \cdot \|f\|^2_{W\mathbf{s},\mathbf{q}} \tag{6.50}$$

for any $f \in \oplus_{l=1,..,n} \mathcal{H}_l$ for which the $W\mathbf{s},\mathbf{q}$ seminorm is finite. Using Corollary 16 we see that for any real number $x > L_N$, we have

$$\mu_\rho \left(f - \mu_\rho f\right)^2 \leq 2c_0 x^2 \cdot \mathbf{D}_\rho(f) + 2^{Nd} B_N \cdot x^{-Nd} \cdot \|f\|^2_{W\mathbf{s},\mathbf{q}} \tag{6.51}$$

Thus choosing

$$x = \left(\frac{C_{\mathbf{s},\mathbf{q}} \cdot \|f\|^2_{W\mathbf{s},\mathbf{q}}}{\mathbf{D}_\rho(f)} \right)^{\frac{1}{2+Nd}} \cdot L_N \tag{6.52}$$

we arrive at the desired Generalized Nash inequality with

$$\alpha \equiv \frac{Nd}{2+Nd} \tag{6.53}$$

and the corresponding functional $\mathbf{A}_{W\mathbf{s},\mathbf{q}}$ defined with the constant $a_{\mathbf{s},\mathbf{q}}$ given by

$$a_{\mathbf{s},\mathbf{q}}^{\frac{1}{1-\alpha}} \equiv 2c_0 L_N^2 \left(C_{\mathbf{s},\mathbf{q}}\right)^{1-\alpha} + 2^{Nd} B_N \cdot L_N^{-Nd} \cdot \left(C_{\mathbf{s},\mathbf{q}}\right)^{-\alpha} \tag{6.54}$$

Given our Generalized Nash inequality, the decay to equilibrium estimate follows by standard methods using the monotonicity of the seminorm $\|\cdot\|^2_{W\mathbf{s},\mathbf{q}}$ which has been proven in Section 5. This ends the proof of Theorem 17. $\square$

Remark 5 *.One can extend this results to show a faster decay also for any function $f \in \mathcal{H}_{>N_0} = \oplus_{n>N_0}\mathcal{H}_n$.*

*For a finite sequence $\mathbf{p} = \{p(n) \in [1,2), n = N_0+1, \ldots, 2N_0+1\}$ and $f \in \mathcal{H}_{>N_0}$ we introduce $|||f|||_{N_0,\mathbf{p}} := \sum_{n=N_0+1}^{2N_0+1} |||f|||_{n,p(n)}$, where $|||f|||_{n,p}$has been defined in (5.4). For any $\rho \in [0,1]$, $N_0 \in \mathbb{N}$ and $N \in (0,\infty)$ the following **Generalized Nash** inequality is true*

$$\mu_\rho \left(f - \mu_\rho f\right)^2 \leq \mathbf{D}_\rho(f)^{\alpha_N} \widetilde{\mathbf{A}}_{N_0,\mathbf{p}}(f)^{1-\alpha_N} \tag{6.55}$$

with α_N as defined in (6.44) for any $f \in \mathcal{H}_{>N_0}$ for which the following functional is finite $\widetilde{\mathbf{A}}_{N_0,\mathbf{p}}(f) \equiv \widetilde{a}_{\mathbf{p}} |||f|||_{N_0,\mathbf{p}}$ with some constant $\widetilde{a}_{\mathbf{p}} \in (0,\infty)$ and any finite sequence $\mathbf{p} = \{p(n) \in [1,2), n = N_0+1, \ldots, 2N_0+1\}$ such that $n[\frac{2}{p(n)} - 1] \geq N$ for each $n = N_0+1, \ldots, 2N_0+1$.

Remark 6 *We mention also the work [21] where a condition of faster decay has been exploited to prove the central limit theorem for additive functionals of the symmetric exclusion process.*

A Proof of Lemma 4

Lemma 18 *For any real function F and any finite set $\Lambda \subset \mathbb{Z}^d$ we have*

$$\sum_{i\in\Lambda} \mu_\rho \, \left|\nabla_i F\left(\mu_{\Lambda,\rho} \, \left(f \, |N_\Lambda\right)\right)\right|^2 = \frac{1}{\rho} \sum_{k=1}^{|\Lambda|} \left|F\left(\mu_\Lambda^k(f)\right) - F\left(\mu_\Lambda^{k-1}(f)\right)\right|^2 k \cdot \mu_\rho \, \chi_k \tag{A.1}$$

Hence

$$\sum_{i\in\Lambda} \mu_\rho \, \left|\nabla_i F\left(\mu_{\Lambda,\rho} \, \left(f \, |N_\Lambda\right)\right)\right|^2 \leq |\Lambda| \cdot \sup_{k=1,\ldots,|\Lambda|} \left|F\left(\mu_\Lambda^k(f)\right) - F\left(\mu_\Lambda^{k-1}(f)\right)\right|^2 \tag{A.2}$$

Proof: We note first that

$$\nabla_i F\left(\mu_{\Lambda,\rho}\ \ (f\ |N_\Lambda)\right)(\eta) = \sum_{k=0}^{|\Lambda|} F\left(\mu_\Lambda^k\,(f)\right)\nabla_i \chi_k(\eta) \tag{A.3}$$

where $\chi_k \equiv \chi(N_\Lambda = k)$. Since

$$\begin{aligned}\nabla_i \chi_k(\eta) &\equiv \chi_k(\eta^i) - \chi_k(\eta) = \\ &= \chi(\eta_i = 1)\left(\chi_k(\eta^i) - \chi_k(\eta)\right) + \chi(\eta_i = 0)\left(\chi_k(\eta^i) - \chi_k(\eta)\right)\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}\chi(\eta_i = 1)\chi_k(\eta^i) &= \left\{ \begin{array}{ll} \chi(\eta_i = 1)\chi_{k+1}(\eta) & \text{for } k = 0, \ldots, |\Lambda| - 1 \\ 0 & \text{for } k = |\Lambda| \end{array}\right. \\ \chi(\eta_i = 0)\chi_k(\eta^i) &= \left\{ \begin{array}{ll} \chi(\eta_i = 0)\chi_{k-1}(\eta) & \text{for } k = 1, \ldots, |\Lambda| \\ 0 & \text{for } k = 0 \end{array}\right.\end{aligned} \tag{A.5}$$

we get

$$\begin{aligned}&\nabla_i F\left(\mu_{\Lambda,\rho}\,(f\ |N_\Lambda)\right)(\eta) = \\ &= \sum_{k=0}^{|\Lambda|-1} F\left(\mu_\Lambda^k\,(f)\right)\chi(\eta_i = 1)\left(\chi_{k+1}(\eta) - \chi_k(\eta)\right) - F\left(\mu_\Lambda^{|\Lambda|}\,(f)\right)\chi(\eta_i = 1)\chi_{|\Lambda|}(\eta) + \\ &+ \sum_{k=1}^{|\Lambda|} F\left(\mu_\Lambda^k\,(f)\right)\chi(\eta_i = 0)\left(\chi_{k-1}(\eta) - \chi_k(\eta)\right) - F\left(\mu_\Lambda^0\,(f)\right)\chi(\eta_i = 0)\chi_0(\eta)\end{aligned} \tag{A.6}$$

Rearranging the sums, we obtain

$$\begin{aligned}\nabla_i F\left(\mu_{\Lambda,\rho}\,(f\ |N_\Lambda)\right)(\eta) &= \sum_{k=1}^{|\Lambda|} \left(F\left(\mu_\Lambda^{k-1}\,(f)\right) - F\left(\mu_\Lambda^k\,(f)\right)\right)\chi(\eta_i = 1)\chi_k(\eta) + \\ &+ \sum_{k=0}^{|\Lambda|-1} \left(F\left(\mu_\Lambda^{k+1}\,(f)\right) - F\left(\mu_\Lambda^k\,(f)\right)\right)\chi(\eta_i = 0)\chi_k(\eta)\end{aligned} \tag{A.7}$$

Thence we get

$$\sum_{i\in\Lambda} \mu_\rho \left|\nabla_i F\left(\mu_{\Lambda,\rho}\,(f\ |N_\Lambda)\right)\right|^2 = \sum_{i\in\Lambda} \mu_\rho \{ \sum_{k=1}^{|\Lambda|} \left|F\left(\mu_\Lambda^{k-1}\,(f)\right) - F\left(\mu_\Lambda^k\,(f)\right)\right|^2 \chi(\eta_i = 1)\chi_k(\eta) +$$

$$+\sum_{k=0}^{|\Lambda|-1}\left|F\left(\mu_\Lambda^{k+1}(f)\right)-F\left(\mu_\Lambda^k(f)\right)\right|^2\chi(\eta_i=0)\chi_k(\eta)\} \tag{A.8}$$

which after changing the order of summation gives

$$\begin{aligned}\sum_{i\in\Lambda}\mu_\rho\left|\nabla_i F\left(\mu_{\Lambda,\rho}(f\,|N_\Lambda)\right)\right|^2 &= \sum_{k=1}^{|\Lambda|}\left|F\left(\mu_\Lambda^{k-1}(f)\right)-F\left(\mu_\Lambda^k(f)\right)\right|^2 k\cdot\mu_\rho\chi_k + \\ &+\sum_{k=0}^{|\Lambda|-1}\left|F\left(\mu_\Lambda^{k+1}(f)\right)-F\left(\mu_\Lambda^k(f)\right)\right|^2(|\Lambda|-k)\cdot\mu_\rho\chi_k\end{aligned} \tag{A.9}$$

Until this place we did not use any specific assumptions about the measure μ_ρ or corresponding conditional measures. Now we observe that for $k = 0,...,|\Lambda|-1$, we have

$$(|\Lambda|-k)\mu_\rho\chi_k=\frac{(1-\rho)}{\rho}(k+1)\mu_\rho\chi_{k+1} \tag{A.10}$$

Using this together with the previous relation, after simple transformations, we arrive at

$$\sum_{i\in\Lambda}\mu_\rho\left|\nabla_i F\left(\mu_\Lambda^\rho(f\,|N_\Lambda)\right)\right|^2=\frac{1}{\rho}\sum_{k=1}^{|\Lambda|}\left|F\left(\mu_\Lambda^k(f)\right)-F\left(\mu_\Lambda^{k-1}(f)\right)\right|^2 k\cdot\mu_\rho\chi_k \tag{A.11}$$

□

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