Relative energies for non-Gibbsian states

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In memory of Roland Dobrushin

Abstract

We investigate the suggestion of R.L. Dobrushin, that for many examples of non-Gibbsian measures, a well-defined interaction potential can still be derived.
Such an interaction cannot be absolutely summable, uniformly in all configurations. Rather there will exist a typical set of configurations on which the interaction decays sufficiently.
We are showing how to complete this program, using quite elementary methods, for the example of the restriction of the Ising pure phases to one layer of the lattice. We also treat the Ising pure phases under Kadanoff transformation with block-size one and a toy model.
We sketch a general setup, showing that there is an intimate relation between the problem treated here and the decay of correlations in certain disordered systems.

keywords: Gibbsian formalism, relative energies, non-Gibbsian measures, renormalization-group pathologies.

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1 Introduction

Ever since the first appearance of 'natural' examples of non-Gibbsian measures [9, 12, 13] in simple models of statistical mechanics, there has been the feeling that in many cases the non-Gibbsianness is very weak. After all, all that matters is that one can make sense of relative energies somehow. These relative energies are indeed the objects most people concentrate on to parametrize or to investigate the effect of transformations of equilibrium states. This was especially emphasized by R.L. Dobrushin, who pointed out that the 'pathologies' of certain transformations [5] are not so worrisome as one might think at first. He explained that, in practice one would not see much of these pathologies, if the non-Gibbsian aspect is due only to special configurations of the system, which are very untypical anyway. In [4] he has shown (at least for one example [14]) how relative energies can be defined for typical configurations.

In this paper, we investigate his suggestion for three examples (including the model treated in [4]), where we show by quite elementary methods how to complete his program. The simplification of our method with respect to the one in [4] lies in the way we construct the interaction potential. For this we have benefitted from discussions with J. Bricmont, who together with A. Kupiainen [2] is using very similar methods to investigate the statistical mechanics of infinite dimensional dynamical systems.

As will become clear, an interesting relation (also observed by J. Bricmont) follows between non-Gibbsian aspects and certain singularities in disordered systems. These singularities show up if one considers atypical realizations of the disorder, not allowing a nice decay of the correlation functions uniformly in the set of all disorder realizations.

The paper is organized as follows. In section 2 we sketch a general setup and make the link to problems in disordered systems. Section 3 is devoted to the construction of an interaction for the projection of the Ising phases. In Section 4 we make a similar construction for the images of the Ising pure phases under Kadanoff transformation with block-size one. In Section 5 we study a toy model, which illustrates nicely the general philosophy.

2 General setup

The main problem of the paper is how to construct interactions for states that are obtained from Gibbs states via some transformation. For more details on the relationship between Gibbs states and interactions we refer to [5, 7].

Let μ be a probability measure on $\Omega_1 := \{+,-\}^\Lambda$ for some finite volume $\Lambda \subset \mathbb{Z}^d$. From μ we obtain another state ν on $\Omega_2 := \{+,-\}^V$ ($V \subset \Lambda$) via a stochastic transformation T. T is a stochastic matrix, i.e.:

$$T(\xi,\sigma) \geq 0, \qquad \sigma \in \Omega_1, \xi \in \Omega_2$$

$$\sum_{\xi\in\Omega_2} T(\xi,\sigma) = 1$$

and $\mu T = \nu$, i.e.:

$$\sum_{\sigma\in\Omega_1} T(\xi,\sigma)\mu(\sigma) = \nu(\xi)$$

Let H be some (energy) function on Λ and $\beta \geq 0$. Think of μ as a Gibbs measure with respect to the Hamiltonian H at inverse temperature β. Of course, for finite volumes Λ and V, ν will always be Gibbsian. The problem is then to see whether and in what sense the Gibbsian character is preserved as the volumes tend to infinity.
We assume that for all sets $A \subset V$ there is a set $B_A \subset \Lambda$ such that

$$\sum_{\sigma\in\Omega_1} T(\xi^A,\sigma)\mu(\sigma) = \sum_{\sigma\in\Omega_1} T(\xi,\sigma)\mu(\sigma)\exp[-\beta\Delta_{B_A} H(\sigma)] \tag{2.1}$$

where, for any sets $A \subset V$ and $B \subset \Lambda$

$$\begin{aligned} \xi^A(x) &= -\xi(x), \qquad x \in A \\ \xi^A(x) &= \quad \xi(x), \qquad x \notin A \\ \Delta_B H(\sigma) &= H(\sigma^B) - H(\sigma) \end{aligned}$$

the relative energy with respect to a spin-flip in B. (2.1) is satisfied whenever

$$\frac{\mu(\sigma^B)}{\mu(\sigma)} = \exp[-\beta\Delta_B H(\sigma)]$$
$$\text{and } T(\xi^A,\sigma) = T(\xi,\sigma^{B_A}) \tag{2.2}$$

In particular this is the case when T is a projection or decimation transformation, because then

$$T(\xi,\sigma) = \prod_{x\in V} \frac{1+\xi(x)\sigma(x)}{2} \tag{2.3}$$

and we can take $B_A = A$.
It holds also for (real) stochastic transformations like the Kadanoff transformation [11], where Λ is divided into disjoint blocks $B_x, x \in V$ such that $\cup_{x\in V} B_x = \Lambda$. The transformation is

$$T(\xi,\sigma) = \prod_{x\in V} \frac{\exp[p\xi(x)\sum_{y\in B_x}\sigma(y)]}{2\cosh[p\sum_{y\in B_x}\sigma(y)]} \tag{2.4}$$

$0 < p < \infty$. It is clear that we can take $B_A = \cup_{x\in A} B_x$.
Let $x, y \in V$, $x \neq y$. If we want to define an interaction $\tilde{H}$ for ν then the relative energy for a spin-flip at the site x in the configuration ξ should be given by

$$\Delta_x \tilde{H}(\xi) = -\log\frac{\nu(\xi^x)}{\nu(\xi)} \tag{2.5}$$

The effective temperature should be thought off as included in $\tilde{H}$. Formula (2.1) makes it possible to write

$$\begin{aligned}\Delta_x \tilde{H}(\xi) &= -\log \frac{\nu(\xi^x)}{\nu(\xi)} = -\log \langle \exp[-\beta \Delta_{B_x} H] \rangle_\xi \\ \Delta_x \tilde{H}(\xi^y) &= -\log \frac{\nu(\xi^{\{x,y\}})}{\nu(\xi^y)} = -\log \frac{\langle \exp[-\beta \Delta_{B_{\{x,y\}}} H] \rangle_\xi}{\langle \exp[-\beta \Delta_{B_y} H] \rangle_\xi},\end{aligned} \tag{2.6}$$

where

$$\langle f \rangle_\xi = \frac{\sum_{\sigma \in \Omega_1} f(\sigma) T(\xi,\sigma) \mu(\sigma)}{\sum_{\sigma \in \Omega_1} T(\xi,\sigma) \mu(\sigma)} \tag{2.7}$$

The contribution of $\xi(y)$ to $\Delta_x \tilde{H}(\xi)$ is given by

$$\begin{aligned}&\Delta_x \tilde{H}(\xi) - \Delta_x \tilde{H}(\xi^y) = \\ &\log \frac{\nu(\xi^{\{x,y\}})}{\nu(\xi^y)} - \log \frac{\nu(\xi^x)}{\nu(\xi)} = \log \frac{\langle \exp[-\beta \Delta_{B_{\{x,y\}}} H] \rangle_\xi}{\langle \exp[-\beta \Delta_{B_x} H] \rangle_\xi \langle \exp[-\beta \Delta_{B_y} H] \rangle_\xi}\end{aligned} \tag{2.8}$$

If H is a Hamiltonian for a strictly local interaction, then

$$\Delta_{B_{\{x,y\}}} H = \Delta_{B_x} H + \Delta_{B_y} H \tag{2.9}$$

whenever x and y are far enough apart and

$$\begin{aligned}&\Delta_x \tilde{H}(\xi) - \Delta_x \tilde{H}(\xi^y) = \log \left(1 + \frac{\langle f_x f_y \rangle_\xi - \langle f_x \rangle_\xi \langle f_y \rangle_\xi}{\langle f_x \rangle_\xi \langle f_y \rangle_\xi} \right) \\ &\text{where } f_z = \exp[-\beta \Delta_{B_z} H]\end{aligned} \tag{2.10}$$

We see then that the decay of the interaction for ν is closely related to the decay of correlations in the state $\langle \cdot \rangle_\xi$. To define ν as a Gibbs measure in the infinite volume case, one needs that the state $\langle \cdot \rangle_\xi$ has good decay of correlations uniformly in ξ, while Λ and V become infinite. But this is often too much to ask. We can compare the situation with the one in disordered systems where the decay of correlations is always proven only for a set of typical configurations of the random couplings and not for all configurations. There also will be 'bad' configurations spoiling a good decay of the correlations uniformly in the random couplings. We will therefore be interested in the decay of correlations (as in 2.10) for a typical set of configurations ξ. This allows to define an interaction potential for ν which behaves nicely (but not uniformly so) for a set of measure one.
When T is interpreted as a dynamics, the measure $\langle \cdot \rangle_\xi$ is seen to be the time reversal with respect to μ: it gives the probability for the state μ when the configuration in the next instant is known to be ξ.
Also note that the main result in [10] follows from the above analysis: if the correlations $\langle f ; g \rangle_\xi$ decay fast enough, uniformly in ξ, then ν can be defined as a bona-fide Gibbs

measure also in the limit $\Lambda, V \uparrow \infty$.
The way we presented the problem here allows to iterate the transformation. In particular, we don't need that μ is a Gibbs measure in the strong sense. All we need is a DLR-type equation as (2.1) and a sufficiently local character as in (2.9) for the associated relative energies.

3 The projection of the Ising model

For convenience, we restrict ourselves here to the two-dimensional case. This is however not essential. Consider the planar lattice $\mathbb{Z}^2$ for which each site $x \in \mathbb{Z}^2$ has coordinates (i, j). The integer lattice $\mathbb{Z}$ can be viewed as a one-dimensional sublattice containing the origin. That is, we take $x \in \mathbb{Z}$ iff $x = (i, 0)$.
An Ising spin configuration on $\mathbb{Z}^2$ is an element σ of $\Theta = \{-1, +1\}^{\mathbb{Z}^2}$.
Similarly, $\Omega = \{-1, +1\}^{\mathbb{Z}}$ contains all spin configurations $\xi = \{\xi(x), x \in \mathbb{Z}\}$ on $\mathbb{Z}$. We look at the Ising model on $\mathbb{Z}^2$. At low temperatures the restrictions $\nu^{\pm}$ to $\mathbb{Z}$ of the pure phases $\mu^{\pm}$ of the two-dimensional Ising model are known to be non-Gibbsian [14]. Our goal is to construct an interaction for ν^+ that decays exponentially fast for a set of typical configurations of ν^+. We will start by defining a relative energy for a spin-flip at the origin.
First we need some definitions.
We use the notation $\langle xy \rangle$ to indicate that two sites x and y are nearest neighbors in $\mathbb{Z}^2$. For a subset $\Lambda \subset \mathbb{Z}^2$, define Λ^c to be the complement of Λ and $\partial\Lambda = \{x \in \Lambda | \exists y \in \Lambda^c, \langle xy \rangle\}$. For $n \geq 1$, let $\Lambda_n = [-n, n]^2 \cap \mathbb{Z}^2$ and $V_n = \Lambda_n \cap \mathbb{Z}$. Let $\Gamma_n = \Lambda_n \cap \{(i, j) \in \mathbb{Z}^2 | j > 0\}$. Let μ_n^+ be the Ising measure on the volume Λ_n with $+$ boundary conditions:

$$H_n^+(\sigma) := -\sum_{\langle xy \rangle \subset \Lambda_n} (\sigma(x)\sigma(y) - 1) - \sum_{\substack{\langle xy \rangle \\ x \in \Lambda_n; y \in \Lambda_n^c}} (\sigma(x) - 1) \tag{3.1}$$

$$\mu_n^+(\sigma) := \frac{1}{Z_n^+(\beta)} \exp(-\beta H_n^+(\sigma)), \tag{3.2}$$

where $Z_n^+(\beta)$ is the usual normalizing partition function.
Denote with ν_n^+ the restriction of μ_n^+ to V_n.
The relative energy for a spin-flip at the origin is (see (2.5))

$$h_n^+(\xi) = \log \frac{\nu_n^+(\xi)}{\nu_n^+(\xi^o)} \tag{3.3}$$

A small calculation shows that

$$h_n^+(\xi) = 2\beta\xi(0)(\xi(1) + \xi(-1)) + 2 \log \frac{Z_n^{+,\xi}(\beta)}{Z_n^{+,\xi^o}(\beta)}, \tag{3.4}$$

where

$$Z_n^{+,\xi}(\beta) := \sum_{\sigma(x)=\pm, x\in\Gamma_n} \exp(-\beta H_n^{+,\xi}(\sigma)) \tag{3.5}$$

with $H_n^{+,\xi}$ the Ising Hamiltonian on the volume Γ_n with + boundary conditions on $\Gamma_n^c \backslash \mathbb{Z}$ and ξ boundary conditions on $\mathbb{Z}$:

$$H_n^{+,\xi}(\sigma) := -\sum_{\langle xy\rangle\subset\Gamma_n} (\sigma(x)\sigma(y)-1) - \sum_{\substack{\langle xy\rangle \\ x\in\Gamma_n; y\in\Lambda_n^c}} (\sigma(x)-1) - \sum_{x\in V_n} (\xi(x)\sigma(x')-1) \tag{3.6}$$

where $\sigma(x')$ is the neighboring spin in Γ_n of $\xi(x)$. The relative energy (3.3) can be expanded using the telescopic identity (as in [2])

$$h_n^+(\xi) = \sum_{k=1}^{2n+1} (h_n^+(\xi_k) - h_n^+(\xi_{k-1})) + \Phi_0^n, \tag{3.7}$$

where $\xi_k(x) = \xi(x)$ if $x \in D_k$ and + otherwise and $D_k, k \in \mathbb{N}$ is defined by $D_{2l+1} = [-l, l] \cap \mathbb{Z}$ and $D_{2l} = [-l+1, l] \cap \mathbb{Z}$ except for $D_0 = \emptyset$.
The constant Φ_0^n in (3.7) is $h_n^+(+)$ with $+(x) = +, \forall x$. This is the vacuum or reference configuration for which we *a priori* know that $\lim_n h_n^+(+)$ makes sense. Now define the interaction potential

$$\Phi_k^n(\xi) := h_n^+(\xi_k) - h_n^+(\xi_{k-1}) \tag{3.8}$$

for $k \geq 1$. More explicitly,

$$\begin{aligned} &\Phi_1^n(\xi) = h_n^+(\xi_1) - h_n^+(+) = (\xi(0)-1)h_n^+(+) \\ &\Phi_2^n(\xi) + \Phi_3^n(\xi) = 2\beta\xi(0)(\xi(1)+\xi(-1)) - 4\beta\xi(0) + \\ &2\log\frac{Z_n^{+,\xi_2}(\beta)Z_n^{+,\xi_1^o}(\beta)}{Z_n^{+,\xi_2^o}(\beta)Z_n^{+,\xi_1}(\beta)} + 2\log\frac{Z_n^{+,\xi_3}(\beta)Z_n^{+,\xi_2^o}(\beta)}{Z_n^{+,\xi_3^o}(\beta)Z_n^{+,\xi_2}(\beta)} \end{aligned} \tag{3.9}$$

$$\Phi_k^n(\xi) = 2\log\frac{Z_n^{+,\xi_k}(\beta)Z_n^{+,\xi_{k-1}^o}(\beta)}{Z_n^{+,\xi_k^o}(\beta)Z_n^{+,\xi_{k-1}}(\beta)} \tag{3.10}$$

for all $k > 3$. Let $a_k = D_k \backslash D_{k-1}$. Observe that (3.10) equals zero whenever $\xi_k = \xi_{k-1}$, i.e. whenever $\xi(a_k) = +$.
As pointed out in the general setup, we can write the potential in terms of correlation functions. For any $\xi \in \Omega$ and any set $C \subset \Gamma_n$, let $\mu_C^{+,\xi}$ be the measure on C with ξ boundary conditions on V_n and + boundary conditions on the rest of ∂C. When $C = \Gamma_n$, we just write $\mu_n^{+,\xi}$. One readily sees that from (3.10) that for $k > 3$

$$\Phi_k^n(\xi) = (1-\xi(a_k))\log\frac{\mu_n^{+,\xi_k}(e^{-2\beta\xi(0)\sigma(0,1)}e^{2\beta\sigma(a_k,1)})}{\mu_n^{+,\xi_k}(e^{-2\beta\xi(0)\sigma(0,1)})\mu_n^{+,\xi_k}(e^{2\beta\sigma(a_k,1)})} \tag{3.11}$$

Our potential for ν^+ will then be $\{\Phi_k\}_k$

$$\Phi_k(\xi) = \lim_n \Phi_k^n(\xi) \tag{3.12}$$

The existence of this limit can be proven for all $\xi \in \Omega$ by a monotonicity argument. We want to show that there exists a set $\Omega^+ \subset \Omega$, with $\nu^+(\Omega^+) = 1$, such that for $\xi \in \Omega^+$, $\Phi_k^n(\xi)$ is exponentially small in k, uniformly in n. Explicitly, we choose

$$\begin{aligned} \Omega_l^+ &:= \{\xi \in \Omega | \forall k \geq l,\ \frac{1}{k} \sum_{x \in D_k} \xi(x) \geq 3/4\} \\ \Omega^+ &:= \bigcup_l \Omega_l^+ \end{aligned} \tag{3.13}$$

The 3/4 is of course arbitrary. The most important thing is that

Lemma 1 *If β is large enough, then*

$$\nu^+(\Omega^+) = 1$$

Proof:
The proof is easy using that for β large enough, there exists $\alpha = \alpha(\beta) > 0$ such that

$$\nu^+(\{\xi \in \Omega | \frac{1}{k} \sum_{x \in D_k} \xi(x) < 3/4\}) < e^{-\alpha k} \tag{3.14}$$

■

Although the definition of Ω_l^+ depends on the position of the boxes D_k, $k \geq l$, Ω^+ is translation invariant. Indeed, take the site m as reference point instead of 0. If $\xi \in \Omega_l$ for some l then also $\xi \in \Omega_{l'}^+(m) := \{\xi \in \Omega | \forall k \geq l',\ \frac{1}{k} \sum_{x-m \in D_k} \xi(x) \geq 3/4\}$ for some l'.

Theorem 1
The projection ν_n^+ satisfies

$$\nu_n^+(\xi^0) = \nu_n^+(\xi) \exp[-\sum_{k=0}^{2n+1} \Phi_k^n(\xi)] \tag{3.15}$$

with interaction potential $\{\Phi_k^n\}_k$ defined in (3.9)-(3.10) satisfying:

1. $\lim_n \Phi_k^n(\xi) = \Phi_k(\xi)$ *exists for all $\xi \in \Omega$;*

and for β sufficiently large, uniformly as $n \uparrow \infty$

2. $|\Phi_k^n(\xi)| \leq const(\beta) e^{-\delta(\beta)k}$ *for all $\xi \in \Omega_l^+$ whenever $k > l$, $\delta(\beta) \uparrow \infty$ as $\beta \uparrow \infty$*

3. $\sum_{k=0}^{2n+1}|\Phi_k^n(\xi)| \le c(\xi) < \infty$ *for all* $\xi \in \Omega^+$;

The first statement of Theorem 1 (existence of the limit) was already observed after (3.11)-(3.12). The proof of 2,3 and 4 follows from combining (3.11) with

Proposition 1 *Take* $f_m(\sigma) = \exp[-2\beta\xi(m)\sigma(m,1)]$.
For β *large, there exists* $\delta = \delta(\beta)$ *(*$\delta \uparrow \infty$ *as* $\beta \uparrow \infty$*), such that for* $\xi \in \Omega_l^+$ *and* $2n+1 \ge k \ge l$

$$|\mu_n^{+,\xi_k}(f_0 f_{a_k}) - \mu_n^{+,\xi_k}(f_0)\mu_n^{+,\xi_k}(f_{a_k})| \le const(\beta)e^{-\delta k} \tag{3.16}$$

We introduce the following notation. For any two sets $A, B \subset \mathbb{Z}^2$, define the event $A \to B = \{(\sigma,\sigma') \in \Theta \times \Theta |$ there is a path π from A to B such that $(\sigma(x),\sigma'(x)) \neq (+,+)$ for all $x \in \pi\}$. A path from A to B is a sequence $x(0) \in A$, $x(1), \ldots, x(n) \in B$ of consecutive nearest neighbor sites.
Before proving Proposition 1 we need the following

Lemma 2 *If* β *is large enough there exists* $\delta = \delta(\beta)$ *(*$\delta \uparrow \infty$ *as* $\beta \uparrow \infty$*), such that for* $\xi \in \Omega_l$ *and* $k \ge l$

$$\mu_n^{+,\xi_k} \times \mu_n^{+,\xi_k}[(0,1) \to \Lambda_k^c] \le const(\beta)e^{-\delta k} \tag{3.17}$$

Proof:

First note that for ξ_k with $\xi \in \Omega_k^+$ only one spin out of eight can be minus in D_k and thus

$$\begin{aligned}\mu_n^{+,\xi_k} \times \mu_n^{+,\xi_k}[(0,1) \to \Lambda_k^c] &\le \exp[4\beta \sum_{x\in D_k}(1-\xi(x))]\, \mu_n^{+,+} \times \mu_n^{+,+}[(0,1) \to \Lambda_k^c] \\ &\le e^{\beta k}\, \mu_n^{+,+} \times \mu_n^{+,+}[(0,1) \to \Lambda_k^c] \end{aligned} \tag{3.18}$$

We can then use Proposition 2.4 in [3] (with some trivial modifications) to conclude that for β large

$$\mu_n^{+,\xi_k} \times \mu_n^{+,\xi_k}[(0,1) \to \Lambda_k^c] \le \mathrm{const}(\beta)e^{-\delta k} \tag{3.19}$$

and $\delta \uparrow \infty$ as $\beta \uparrow \infty$. ■

Let $\mu_k^{+,\xi,\eta}$ be the measure on Γ_k with ξ boundary condition on V_k and η boundary condition on $\partial\Lambda_k^c$.

Proof of Proposition 1:
For any configuration $\xi \in \Omega$ we have that

$$\begin{aligned}&|\mu_n^{+,\xi}(f_0 f_{a_k}) - \mu_n^{+,\xi}(f_0)\mu_n^{+,\xi}(f_{a_k})| = \\ &|\mu_n^{+,\xi}\left(\mu_n^{+,\xi}(f_{a_k}|\sigma = \eta \,\mathrm{on}\, \partial\Lambda_k^c)[\mu_k^{+,\xi,\eta}(f_0) - \mu_n^{+,\xi}(f_0)]\right)| \\ &\le e^{2\beta}\mu_n^{+,\xi}(|\mu_k^{+,\xi,\eta}(f_0) - \mu_n^{+,\xi}(f_0)|) \\ &= e^{2\beta}\mu_n^{+,\xi}(|\mu_k^{+,\xi,\eta} \times \mu_n^{+,\xi}(f_0 \times \mathbb{1} - \mathbb{1} \times f_0)|)\end{aligned} \tag{3.20}$$

The rest of the argument is similar to the ideas in [1]. Pick a configuration (σ,σ') from the distribution $\mu_k^{+,\xi,\eta}\times\mu_n^{+,\xi}$. If there is no path from $(0,1)$ to Λ_k^c on which $(\sigma,\sigma')\not\equiv(+,+)$, there exists a $*$-chain around $(0,1)$ separating it from Λ_k^c and on which $\sigma\equiv\sigma'$. This chain has a part in Γ_k on which $(\sigma,\sigma')\equiv(+,+)$ and a part in V_k on which lives the configuration ξ and $(\sigma,\sigma')\equiv(\xi,\xi)$. It follows that there exists a maximal $*$-chain Δ with this property inside Λ_k (maximal in the sense that it is contained in no other $*$-chain). Therefore

$$\mu_k^{+,\xi,\eta}\times\mu_n^{+,\xi}(f_0\times\mathbb{1}-\mathbb{1}\times f_0)= \tag{3.21}$$

$$\sum_{\substack{*\text{-chains }\Delta\\ \text{around }(0,1)}}\mu_k^{+,\xi,\eta}\times\mu_n^{+,\xi}(f_0\times\mathbb{1}-\mathbb{1}\times f_0|\Delta)\,\mu_k^{+,\xi,\eta}\times\mu_n^{+,\xi}(\Delta)+$$
$$\mu_k^{+,\xi,\eta}\times\mu_n^{+,\xi}(f_0\times\mathbb{1}-\mathbb{1}\times f_0|(0,1)\to\Lambda_k^c)\,\mu_k^{+,\xi,\eta}\times\mu_n^{+,\xi}[(0,1)\to\Lambda_k^c]$$

Now since $\sigma=\sigma'$ on Δ

$$\mu_k^{+,\xi,\eta}\times\mu_n^{+,\xi}(f_0\times\mathbb{1}-\mathbb{1}\times f_0|\Delta)=0 \tag{3.22}$$

for every Δ so

$$|\mu_k^{+,\xi,\eta}\times\mu_n^{+,\xi}(f_0\times\mathbb{1}-\mathbb{1}\times f_0)|\le 2e^{2\beta}\mu_k^{+,\xi,\eta}\times\mu_n^{+,\xi}[(0,1)\to\Lambda_k^c] \tag{3.23}$$

Lemma 2 and (3.20),(3.23) now yield that

$$\begin{aligned}&|\mu_n^{+,\xi_k}(f_0f_{a_k})-\mu_n^{+,\xi_k}(f_0)\mu_n^{+,\xi_k}(f_{a_k})|\le\\ &e^{2\beta}\mu_n^{+,\xi_k}[|\mu_k^{+,\xi_k,\eta}\times\mu_n^{+,\xi_k}(f_0\times\mathbb{1}-\mathbb{1}\times f_0)|]\le\\ &2e^{4\beta}\mu_n^{+,\xi_k}\times\mu_n^{+,\xi_k}[(0,1)\to\Lambda_k^c]\le\text{const}(\beta)e^{-\delta k}\end{aligned} \tag{3.24}$$

■

Remarks:

- The proof of Lemma 2 works only for dimension $d=2$.
  The result is however more general. It can be proven with a little bit more effort, in a way analogous to Lemma 3 for the Ising model under Kadanoff transformation (see Section 4).

- The proof of the previous Proposition does not depend essentially on specific features of the Ising model. The important thing is to be able to prove Lemma 2, but this is expected to hold for a much wider class of measures, see chapter 6 in [7].
  A result that comes close to Lemma 2 is found in [8]. In fact, from [8] we know that for all temperatures $T<T_c$ the $(+,+)$ sites percolate in the $\mu^+\times\mu^+$ state.

- The same thing can of course be done for the minus-phase ν^- (obtained from μ^-). The result is an interaction Φ' with good decay on
$\Omega^- = \bigcup_l \{\xi \in \Omega | \forall k \geq l,\ \frac{1}{k}\sum_{x\in D_k} \xi(x) \leq -3/4\}$.
For any $\xi \in \Omega$ we have that $h_n^+(\xi) = h_n^-(-\xi)$. We don't know whether or not Φ' is physically equivalent to Φ.

- Observe that the set Ω^+ is not a closed set (for the usual product topology). It is therefore not trivial to translate this result in terms of almost sure continuity of specifications [5, 6, 7].

4 Kadanoff transformation for the Ising model

We consider here again the pure phases of the Ising model and apply the Kadanoff transformation with block size equal to 1, i.e. (see (2.4))

$$T(\xi,\sigma) = \prod_x \frac{\exp[p\xi(x)\sigma(x)]}{2\cosh p} \tag{4.1}$$

This is a transformation often used in image processing (see [5] for some references).
It is proven in [5] that for a given $0 < p < \infty$ there exists β_0 such that for all $\beta > \beta_0$ any Gibbs measure μ for the Ising model with zero field and inverse temperature β is transformed by T into a non-Gibbsian measure. This holds for any dimension $d \geq 2$. We restrict ourselves here to the case $d = 2$. We show that a potential can nevertheless be constructed, using the same methods as in the previous section.
Let μ_n^+ be as in the previous section and $\nu_n^+ = \mu_n^+ T$. The relative energy of ν_n^+ for a spin-flip at the origin is given by

$$h_n^+(\xi) = \log \frac{Z_n^{+,\xi}}{Z_n^{+,\xi^o}} \tag{4.2}$$

$$Z_n^{+,\xi} := \sum_{\sigma(x)=\pm, x\in\Lambda_n} \exp(-H_n^{+,\xi}(\sigma))$$

$$\begin{aligned} H_n^{+,\xi}(\sigma) := &-\beta \sum_{\langle xy\rangle\subset\Lambda_n} (\sigma(x)\sigma(y)-1) - \beta \sum_{\substack{\langle xy\rangle \\ x\in\Lambda_n; y\in\Lambda_n^c}} (\sigma(x)-1) - p\sum_{x\in\Lambda_n} \xi(x)(\sigma(x)-1) \\ &-p\sum_{x\in\Lambda_n} \xi(x) \end{aligned} \tag{4.3}$$

We now choose the volumes D_k as follows: $D_0 = \emptyset$, $D_1 = \{0\}$ and D_k is obtained from D_{k-1} by adding one of the points closest to the origin and not already contained in D_{k-1}. (The distance of the point (i,j) from the origin is given by $\max\{|i|,|j|\}$.) We call x_k the point contained in D_k but not in D_{k-1}. Again ξ_k is the configuration taking the values of ξ inside D_k and $+$ outside. We define $\Phi_0^n(\xi) := h_n^+(+)$ and for $k \geq 1$

$$\Phi_k^n(\xi) := h_n^+(\xi_k) - h_n^+(\xi_{k-1}) \tag{4.4}$$

$$\Phi_1^n(\xi) = h_n^+(+)(\xi(o) - 1) \tag{4.5}$$

and for $k > 1$

$$\Phi_k^n(\xi) = \frac{1 - \xi(x_k)}{2} \log \frac{\mu_n^{+,\xi_k}(e^{-2\beta\xi(o)\sigma(o)} e^{2\beta\sigma(x_k)})}{\mu_n^{+,\xi_k}(e^{-2\beta\xi(o)\sigma(o)})\mu_n^{+,\xi_k}(e^{2\beta\sigma(x_k)})} \tag{4.6}$$

where $\mu_n^{+,\xi}$ is the Gibbs state for $H_n^{+,\xi}$.
Following [3] we call a finite set $A \subset \mathbb{Z}^2$ enclosing if A and A^c are connected sets. For a connected set B we denote $\bar{B}$ for the union of B and the finite components of its complement. $\bar{B}$ is always enclosing. There is a one-to-one correspondence between enclosing sets and contours as they are usually defined for the Ising model (see e.g. [15]). Each contour borders an enclosing set, i.e. for each contour γ there is an enclosing set A such that $A = \text{Int}\gamma$ (the interior of the contour).
Let

$$\begin{aligned}
\Theta_l^+ &:= \{\xi \in \Theta | \forall k \geq l \text{ and all } V, \text{ connected sets with } \bar{V} \ni o, |V| \geq k : \\
&\qquad \frac{1}{|V|} \sum_{x \in V} \xi(x) \geq 1/2\} \\
\Theta^+ &:= \bigcup_l \Theta_l^+
\end{aligned} \tag{4.7}$$

Again $\mu^+(\Theta^+) = 1$ for large enough β and

Theorem 2
The measure ν_n^+ satisfies

$$\nu_n^+(\xi^0) = \nu_n^+(\xi) \exp[-\sum_{k=0}^{(2n+1)^2} \Phi_k^n(\xi)] \tag{4.8}$$

with interaction potential $\{\Phi_k^n\}_k$ satisfying:

1. $\lim_n \Phi_k^n(\xi) = \Phi_k(\xi)$ *exists for all* $\xi \in \Theta$;

and for p sufficiently large, uniformly as $n \uparrow \infty$ and uniformly in β

2. $|\Phi_k^n(\xi)| \leq C(\beta, p)\ e^{-\lambda(p)\text{diam}D_k}$ *for all* $\xi \in \Theta_l^+$ *whenever* $\text{diam}D_k > l$ *(*$\text{diam}D_k \sim \sqrt{k}$ *is the diameter of the set D_k); ($\lambda(p) \uparrow \infty$ as $p \uparrow \infty$)*

3. $\sum_{k=0}^{(2n+1)^2} |\Phi_k^n(\xi)| \leq c(\xi) < \infty$ *for all* $\xi \in \Theta^+$;

The proof of Theorem 2 goes along the same lines as that of Theorem 1. It follows from

Proposition 2 *Take $f_x(\sigma) = \exp[-2\beta\xi(x)\sigma(x)]$. If p is large enough, there exists $\lambda = \lambda(p)$ ($\lambda \uparrow \infty$ as $p \uparrow \infty$), such that if $\xi \in \Theta_l^+$, we have that for $2n+1 \geq \mathrm{diam}D_k \geq l$*

$$|\mu_n^{+,\xi}(f_0 f_{x_k}) - \mu_n^{+,\xi}(f_0)\mu_n^{+,\xi}(f_{x_k})| \leq \mathrm{const}(\beta,p)e^{-\lambda \mathrm{diam} D_k} \tag{4.9}$$

We need the following

Lemma 3 *If p is large enough, there exists $\alpha = \alpha(p)$ ($\alpha \uparrow \infty$ as $p \uparrow \infty$), such that for $\xi \in \Theta_l^+$, $m \geq l$*

$$\mu_n^{+,\xi} \times \mu_n^{+,\xi}[(0,1) \to \Lambda_m^c] \leq \mathrm{const}(p)e^{-\alpha m} \tag{4.10}$$

Proof:

Denote $C(\sigma,\sigma')$ for the maximal connected set containing o on which $(\sigma,\sigma') \not\equiv (+,+)$. Denote with $\mathcal{E}(o)$ all enclosing sets containing the origin. Let $C \in \mathcal{E}(o)$.

$$\begin{aligned}
&\mu_n^{+,\xi} \times \mu_n^{+,\xi}(\bar{C}(\sigma,\sigma') = C) \leq \\
&\sum_{R\subset\partial C} \mu_n^{+,\xi} \times \mu_n^{+,\xi}(\sigma \equiv - \,\mathrm{on}\, R, \sigma \equiv + \,\mathrm{on}\, \partial C^c, \sigma' \equiv - \,\mathrm{on}\, \partial C \backslash R, \sigma' \equiv + \,\mathrm{on}\, \partial C^c) \\
&\sum_{R\subset\partial C} \mu_n^{+,\xi}(\sigma \equiv - \,\mathrm{on}\, R, \sigma \equiv + \,\mathrm{on}\, \partial C^c)\mu_n^{+,\xi}(\sigma' \equiv - \,\mathrm{on}\, \partial C \backslash R, \sigma' \equiv + \,\mathrm{on}\, \partial C^c)
\end{aligned} \tag{4.11}$$

We now need the following Lemma, which we will prove afterwards.

Lemma 4 *Let $C \in \mathcal{E}(o)$ and $R \subset \partial C$. If $\xi \in \Theta_l^+$ and ∂C contains at least $m \geq l$ sites, there exists $\delta = \delta(p)$ ($\delta \uparrow \infty$ as $p \uparrow \infty$), such that*

$$\mu_n^{+,\xi}(\sigma \equiv - \,on\, R, \sigma \equiv + \,on\, \partial C^c)\mu_n^{+,\xi}(\sigma' \equiv - \,on\, \partial C \backslash R, \sigma' \equiv + \,on\, \partial C^c) \leq const(p)e^{-\delta|\partial C|} \tag{4.12}$$

If we demand that there is a path from o to Λ_m^c then indeed $|\partial\bar{C}(\sigma,\sigma')| > m$. Using Lemma 4, we have that for $\xi \in \Theta_l^+$

$$\begin{aligned}
&\mu_n^{+,\xi} \times \mu_n^{+,\xi}(\bar{C}(\sigma,\sigma') = C) \leq \\
&2^{|\partial C|}\mathrm{const}(p)e^{-\delta|\partial C|} \leq \mathrm{const}(p)e^{-\lambda|\partial C|}
\end{aligned} \tag{4.13}$$

and $\lambda \uparrow \infty$ when $p \uparrow \infty$.

We can now continue as follows. Let W_s denote the number of enclosing sets bordered by a contour of length s and containing o. By the isoperimetric inequality on the square lattice we have that $W_s \leq (s^2/16)4^s$

$$\begin{aligned}
&\mu_n^{+,\xi} \times \mu_n^{+,\xi}[o \to \Lambda_m^c] = \\
&\mu_n^{+,\xi} \times \mu_n^{+,\xi}[C(\sigma,\sigma') \cap \Lambda_m^c \neq \emptyset] \leq \\
&\sum_{s=m}^{\infty} \sum_{\substack{C\in\mathcal{E}(o) \\ |\partial C|=s}} \mu_n^{+,\xi} \times \mu_n^{+,\xi}[\bar{C}(\sigma,\sigma') = C] \leq \\
&\mathrm{const}(p) \sum_{s=m}^{\infty} e^{-\lambda s}\frac{s^2}{16}4^s \leq \mathrm{const}(p)e^{-\alpha m}
\end{aligned} \tag{4.14}$$

for some $\alpha > 0$. ■

Proof of Lemma 4:

Let for any enclosing set $V \subset \mathbb{Z}^2$

$$\tilde{H_V}^{+,\xi}(\sigma) := -\beta \sum_{\langle xy \rangle \subset V} (\sigma(x)\sigma(y) - 1) - \beta \sum_{\substack{\langle xy \rangle \\ x \in V; y \in V^c}} (\sigma(x) - 1) - p \sum_{x \in V} \xi(x)(\sigma(x) - 1)$$

$$\tilde{H_V}^{-,\xi}(\sigma) := -\beta \sum_{\langle xy \rangle \subset V} (\sigma(x)\sigma(y) - 1) - \beta \sum_{\substack{\langle xy \rangle \\ x \in V; y \in V^c}} (-\sigma(x) - 1) - p \sum_{x \in V} \xi(x)(\sigma(x) + 1)$$

$$\tilde{Z}^{+,\xi}(V) := \sum_{\sigma(x); x \in V} \exp[\tilde{H_V}^{+,\xi}(\sigma)]$$

$$\tilde{Z}^{-,\xi}(V) := \sum_{\sigma(x); x \in V} \exp[\tilde{H_V}^{-,\xi}(\sigma)] \tag{4.15}$$

and observe that

$$\frac{\tilde{Z}^{-,\xi}(V)}{\tilde{Z}^{+,\xi}(V)} \leq \exp[2\beta |\partial V^c|] \exp[2p \sum_{x \in V} \xi(x)] \tag{4.16}$$

We have that

$$\begin{aligned} &\mu_n^{+,\xi}(\sigma \equiv - \text{ on } R, \sigma \equiv + \text{ on } \partial C^c)\mu_n^{+,\xi}(\sigma' \equiv - \text{ on } \partial C \backslash R, \sigma' \equiv + \text{ on } \partial C^c) \leq \\ &\mu_n^{+,\xi}(\sigma \equiv - \text{ on } R | \sigma \equiv + \text{ on } \partial C^c)\mu_n^{+,\xi}(\sigma' \equiv - \text{ on } \partial C \backslash R | \sigma' \equiv + \text{ on } \partial C^c) = \\ &\mu_C^{+,\xi}(\sigma \equiv - \text{ on } R)\mu_C^{+,\xi}(\sigma' \equiv - \text{ on } \partial C \backslash R) \leq \\ &\sum_{\substack{\{\gamma\},\{\gamma'\} \text{ inside } C \\ \{\gamma\} \text{ compatible with } R \\ \{\gamma'\} \text{ compatible with } \partial C \backslash R}} \prod_{\gamma} w_\xi(\gamma) \prod_{\gamma'} w_\xi(\gamma') \end{aligned} \tag{4.17}$$

The set $\{\gamma\}$ of contours is compatible with R if it is compatible with $\sigma \equiv -$ on R, $R \subset \cup_\gamma \text{Int}\gamma$ and $R \cap \text{Int}\gamma \neq \emptyset$ for each γ. The weight $w_\xi(\gamma)$ of a contour γ is given by

$$w_\xi(\gamma) := \exp[-2\beta|\gamma| - 2p \sum_{x \in \text{Int}\gamma} \xi(x)] \frac{\tilde{Z}^{-,\xi}(\text{Int}\gamma \backslash \partial \text{Int}\gamma)}{\tilde{Z}^{+,\xi}(\text{Int}\gamma \backslash \partial \text{Int}\gamma)} \tag{4.18}$$

Using (4.16) we see that

$$w_\xi(\gamma) \leq \exp[-2p \sum_{x \in \partial \text{Int}\gamma} \xi(x)] \tag{4.19}$$

Now regard the sets $\{\gamma\}, \{\gamma'\}$ in (4.17) as a single set of contours $\{\tilde{\gamma}\} = \{\gamma\} \cup \{\gamma'\}$. The contours $\tilde{\gamma}$ can overlap, but a site of ∂C can be at most inside two contours. Moreover, every site of ∂C is in the interior of at least one contour of $\{\tilde{\gamma}\}$ and $|\partial C| > m$.

Let $V := \cup_{\tilde\gamma}\partial\mathrm{Int}\tilde\gamma$. V is a connected set such that $\bar V \ni o$ and $|V| > m$. Because $\xi \in \Theta_l^+$, $m \geq l$,

$$\sum_{\tilde\gamma}\sum_{x\in\partial\mathrm{Int}\tilde\gamma}\xi(x) \geq \frac{1}{4}\sum_{\tilde\gamma}|\partial\mathrm{Int}\tilde\gamma| \tag{4.20}$$

This takes into account the worst situation where all $-$ spins are counted twice and all $+$ spins only once. Hence, in (4.17)

$$\prod_{\gamma} w_\xi(\gamma)\prod_{\gamma'} w_\xi(\gamma') \leq \prod_{\gamma} e^{-\frac{1}{2}p|\partial\mathrm{Int}\gamma|}\prod_{\gamma'} e^{-\frac{1}{2}p|\partial\mathrm{Int}\gamma'|} \tag{4.21}$$

Now we can proceed as in Lemma 2.5 of [3]. Enumerate now the set R as $\{x_1,\dots,x_q\}$. Each $x_i, i = 1,\dots q$ is enclosed by a contour of the set $\{\gamma\}$. The set $\{\gamma\}$ contains at most q contours. If $\{\gamma\}$ has s contours, let us call them $\gamma_{i_1},\dots,\gamma_{i_s}$, where x_{i_j} is the point of R with the smallest index, enclosed by γ_{i_j}. Put $L_{i_j} = |\gamma_{i_j}|$ for $j = 1,\dots,s$ and $L_k = 0$ if $k \in \{1,\dots,q\}$ but $k \neq L_{i_1},\dots,L_{i_s}$. Let $L = \sum_\gamma|\gamma|$. Then $L_1,\dots,L_q$ gives a partitioning of L into q nonnegative numbers. It is well known that the number of such partitions is smaller then $K\,4^L$ for some constant K. On the other hand, the number of contours passing a fixed point x and having length l is bounded by $l\,4^l \leq 8^l$. Therefore

$$\sum_{\substack{\{\gamma\},\{\gamma'\}\text{ inside } C\\ \{\gamma\}\text{ compatible with } R\\ \{\gamma'\}\text{ compatible with } \partial C\setminus R}}\prod_{\gamma} w_\xi(\gamma)\prod_{\gamma'} w_\xi(\gamma') \leq$$

$$\sum_{r\geq|R|} K\,32^r e^{-\frac{1}{2}pr}\sum_{s\geq|\partial C\setminus R|} K\,32^s e^{-\frac{1}{2}ps} \leq \mathrm{const}(p)e^{-\delta|\partial C|} \tag{4.22}$$

for some constant $\delta > 0$. ■

The proof of Proposition 2 is now analogous to that of Proposition 1.

5 A toy model

In this section we treat a toy model to show more clearly where problems may arise in applying this method to other examples and to illustrate in a simple example the main philosophy of the method.

Consider independent spins $(\sigma(x), x \in \mathbb{Z})$ with identical distribution μ, $\mu(\sigma(x) = +) = p \neq 1/2$, $0 < p < 1$. Define the transformation

$$\xi(x) = \sigma(x)\sigma(x+1), \qquad x \in \mathbb{Z} \tag{5.1}$$

The $(\xi(x), x \in \mathbb{Z})$ form a one-dependent random field. In particular, the induced measure on the spins on the even (odd) sites is a product measure. Still the measure ν on the $(\xi(x), x \in \mathbb{Z})$ is strongly non-Gibbsian. To see this, we compute the conditional distributions. We observe that for $l \geq 1$

$$\xi(0) = \sigma(-l)\xi(-l)\cdots\xi(-1)\xi(1)\cdots\xi(l)\sigma(l+1) \tag{5.2}$$

Hence, for $\alpha_i = \pm, i = -l, \ldots, -1, 1, \ldots, l$,

$$\nu[\xi(0) = +|\xi(-l) = \alpha_{-l} \cdots, \xi(-1) = \alpha_{-1}, \xi(1) = \alpha_1, \cdots \xi(l) = \alpha_l] =$$
$$\mu[\sigma(-l)\sigma(l+1) = \prod_{\substack{j=-l \\ j\neq 0}}^{l} \alpha_j] = \begin{cases} p^2 + (1-p)^2 & \text{if } \prod_{\substack{j=-l \\ j\neq 0}}^{l} \alpha_j = + \\ \\ 2p(1-p) & \text{if } \prod_{\substack{j=-l \\ j\neq 0}}^{l} \alpha_j = - \end{cases} \tag{5.3}$$

This implies that no matter how large we take l,

$$\begin{aligned} &|\nu[\xi(0) = +|\xi(-l) = \alpha_{-l} \cdots, \xi(-1) = \alpha_{-1}, \xi(1) = \alpha_1, \cdots \xi(l) = +] - \\ &\quad \nu[\xi(0) = +|\xi(-l) = \alpha_{-l} \cdots, \xi(-1) = \alpha_{-1}, \xi(1) = \alpha_1, \cdots \xi(l) = -]| \\ &\quad = (2p-1)^2 > 0 \end{aligned} \tag{5.4}$$

and this for every configuration $\xi(-l) = \alpha_{-l} \cdots \eta(l-1) = \alpha_{l-1}$. In other words, the conditional probabilities are nowhere continuous (this observation is due to J. van den Berg). The measure μ_p is therefore strongly non-Gibbsian.
This phenomenon occurs, because the severe constraint (5.1) makes it possible to transmit information directly over arbitrary long distances. A reconstruction of the potential as in the previous section fails. There is no decay whatsoever. Indeed, let ν_n be the restriction of ν to $[-n, n] \cap \mathbb{Z}$ and

$$h_n(\xi) = \log \frac{\nu_n(\xi)}{\nu_n(\xi^0)} \tag{5.5}$$

It is easily seen that for $|x| < n$

$$h_n(\xi^x) - h_n(\xi) = \log \frac{2p(1-p)}{p^2 + (1-p)^2} \prod_{\substack{j=-n \\ j\neq 0}}^{n} \xi(j) \tag{5.6}$$

does not decay with $|x|$.
If the constraint is taken to be less severe, things work out better. Consider therefore a modified transformation on the σ-spins. $(\sigma(x), x \in \mathbb{Z})$ are mapped to a pair of variables $((\xi(x), \zeta(x)), x \in \mathbb{Z})$ both taking the values $\pm$.

$$\begin{aligned} &\zeta(x) = - \text{ with probability } q,\ 0 < q < 1 \\ &\xi(x) = \sigma(x) \text{ if } \zeta(x) = - \\ &\xi(x) = \sigma(x)\sigma(x+1) \text{ if } \zeta(x) = + \end{aligned} \tag{5.7}$$

Consider

$$h_l(\alpha, \zeta) = \log \frac{\nu[\xi(-l) = \alpha_{-l}, \cdots \eta(0) = \alpha_0, , \cdots, \xi(l) = \alpha_l, \zeta]}{\nu[\xi(-l) = \alpha_{-l}, \cdots, \xi(0) = -\alpha_0, \cdots \xi(l) = \alpha_l, \zeta]} \tag{5.8}$$

and let C_0 denote the connected set of sites containing 0 on which $\zeta \equiv +$ together with the right endpoint of this +-cluster. We have that

$$h_l(\xi^x, \zeta) - h_l(\xi, \zeta) = \begin{cases} \log \frac{1-p}{p} \prod_{j \in C_0 \cap [-l,l]} \xi(j) & \text{if } x \in C_0 \cap [-l, l] \\ \\ 0 & \text{otherwise} \end{cases} \tag{5.9}$$

Formally, the measure ν on the variables (ξ, ζ) is of the form

$$\nu(d\xi, d\zeta) = \exp[-H_\zeta(\xi)]\lambda_{1/2}(d\xi)\lambda_q(d\zeta) \tag{5.10}$$

where λ_q is the product measure with density q.
The Hamiltonian H_ζ is not behaving well uniformly in ζ. $H_\zeta(\xi^x) - H_\zeta(\xi)$ is a function of all $\xi(y)$ for which $\zeta(z) = +$ for all z between x and y and depends strongly on all of these $\xi(y)$. Therefore we are unable to define the range of the interaction uniformly in the ζ. For typical ζ however, there is only a finite number of sites in the +-cluster of x, so that $H_\zeta(\xi^x) - H_\zeta(\xi)$ is a strictly local function of ξ. Therefore the situation is even better than in the former examples, although ν is not Gibbsian.

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