# COHERENT STATES MEASUREMENT ENTROPY

Jarosław Kwapień[1], Wojciech Słomczyński[2] and Karol Życzkowski[3]

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[1] Instytut Fizyki Jądrowej im. H. Niewodniczańskiego,

ul. Radzikowskiego 152, PL-31 305, Kraków, Poland

[2] Instytut Matematyki,

Uniwersytet Jagielloński, ul. Reymonta 4

PL-30 057, Kraków, Poland

[3] Instytut Fizyki im. M. Smoluchowskiego,

Uniwersytet Jagielloński, ul. Reymonta 4

PL-30 057, Kraków, Poland

e-mail: [1]kwapien@castor.if.uj.edu.pl [2]slomczyn@im.uj.edu.pl [3]karol@castor.if.uj.edu.pl

Coherent states (CS) quantum entropy can be split into two components. The dynamical entropy is linked with the dynamical properties of a quantum system. The measurement entropy, which tends to zero in the semiclassical limit, describes the unpredictability induced by the process of a quantum approximate measurement. We study the CS–measurement entropy for spin coherent states defined on the sphere discussing different methods dealing with the time limit $n \to \infty$. In particular we propose an effective technique of computing the entropy by iterated function systems. The dependence of CS–measurement entropy on the character of the partition of the phase space is analysed.

## I. INTRODUCTION

During the last decade a lot of attention has been paid to the analysis of quantum analogues of chaotic classical maps defined on a compact phase space. In particular quantum versions of the Baker map [?,?], the Arnold cat map [?] (torus) and the periodically kicked top [?,?,?] (sphere) become standard models often used in the study on *quantum chaology* [?,?,?]. The classical versions of these models can be called *chaotic* (for the kicked top under an appropriate choice of parameters), since the Kolmogorov - Sinai (KS) dynamical entropy of the systems is positive. The definition of KS–entropy cannot be adopted straightforwardly into quantum mechanics, as it is based on the concept of classical trajectory. Several methods of generalizing KS–entropy to quantum mechanics were proposed (see [?,?,?,?,?] for the complete bibliography), but most of them lead to zero entropy for all quantum systems represented on a finite–dimensional Hilbert space. This property is common to the definitions of quantum entropy due to Connes–Narnhoffer–Thirring [?], Gaspard [?], Alicki–Fannes [?], and Roepstorff [?]. Therefore these concepts of quantum entropy are not suitable to describe dynamical properties of the above mentioned quantum maps which, due to the compactness of the classical phase space, act on a finite–dimensional Hilbert space.

*Coherent states (CS) dynamical entropy*, introduced in [?], appears to be more adequate for a quantitative characterization of chaos in such quantum systems. This definition of quantum entropy takes into account the process of *sequential approximate measurement*. The notion of approximate, or unsharp, or fuzzy quantum measurement has been analysed in the last 25 years by Ali, Emchs and Prugovečki, Busch and Lahti, Davies and Lewis, Ozawa, Schroeck and many others in the framework of the operational approach to quantum mechanics, or stochastic (phase space) quantum mechanics (see monographs [?], [?], [?], and [?] for details and further references). In order to get some information about the localization of a state in a phase space one may perform double (or multiple, in a phase space of higher dimension) quantum measurement of canonically conjugated observables. Due to the uncertainty principle such a measurement cannot be sharp and has to be approximate. The definition of CS–entropy is therefore based on the modified postulate of wave function collapse. The original postulate of Lüders and von Neumann, corresponding to a single exact measurement, assumes that after a measurement the state undergoes a transition to an eigenstate of the observable [?]. The modified postulate, used in the description of an approximate multiple measurement, asserts that after the measurement the state is transformed into an appropriate mixture of coherent states, i.e., the coherent states are *a posteriori* states in the sense of Ozawa [?].

The CS-entropy with respect to a given partition can be divided into two parts: the *dynamical entropy* which describes the dynamical properties of a quantum system and the *measurement entropy* related to the unpredictability induced by the process of sequential quantum measurement. The proof of the fact that in the semiclassical limit the CS–measurement entropy tends to zero was sketched in [?]. In the same paper it was conjectured that *the CS–dynamical entropy tends to the KS–entropy of the corresponding classical system* if the unitary dynamics comes from an appropriate quantization procedure, and some results in this direction were obtained.

In this work we analyse in detail CS–measurement entropy for the SU(2) coherent states, where the phase space is the two–dimensional sphere. Such an example is of special physical interest, since it corresponds to an unsharp measurement of the spin components [?,?,?,?]. In [?] we proposed a general plan for studying the notion of CS–entropy. The results obtained here constitute the first step towards the realization of this scheme.

This paper is organized as follows. In sect. II we recall the definitions of CS–entropy, CS–measurement entropy and CS–dynamical entropy, and review some of their basic properties. In sect. III we summarize the standard facts on the SU(2) vector coherent states. Various methods of computing CS-measurement entropy for the partition of the sphere into two hemispheres are presented, and the semiclassical limit is discussed in sect. IV. The case of an arbitrary partition of the sphere is analysed in sect. V. In sect. VI we treat yet another method of calculating CS–entropy based on the notion of iterated function systems. The Rényi–type generalizations of CS–entropy are introduced in sect. VII. Finally, sect. VIII contains some concluding remarks.

## II. COHERENT STATES QUANTUM ENTROPY

### A. Kolmogorov-Sinai entropy

Coherent states entropy can be regarded [?] as a generalization of the classical Kolmogorov–Sinai entropy. Let us recall here the definition of the KS–entropy for a classical map $S : \Omega \to \Omega$ generating a discrete dynamical system. Let $\Omega$ be a compact phase space endowed with a probability measure $\mu$ and divided into $k$ disjoint measurable cells $E_1, \ldots, E_k$. The time evolution of classical trajectories during $n$ periods is described via probabilities

$$P^{cl}_{i_0,\ldots,i_{n-1}} = \mu(\{x \in \Omega : x \in E_{i_0}, S(x) \in E_{i_1}, \ldots, S^{n-1}(x) \in E_{i_{n-1}}\}), \tag{2.1}$$

of entering a given sequence of cells, where $i_l = 1, \dots, k;\ l = 0, \dots, n-1$. It is assumed that the initial points, determining uniquely each trajectory, are distributed uniformly in the phase space with respect to the measure $\mu$.

The partial entropy of $S$ is

$$H_n = - \sum_{i_0,\dots,i_{n-1}=1}^{k} P^{cl}_{i_0,\dots,i_{n-1}} \ln P^{cl}_{i_0,\dots,i_{n-1}}, \tag{2.2}$$

the *KS–entropy with respect to the partition* $\mathcal{C} = \{E_1, \dots, E_k\}$ is given by

$$H_{KS}(S, \mathcal{C}) := \lim_{n\to\infty} \frac{1}{n} H_n, \tag{2.3}$$

and finally the *KS–entropy* of $S$ is defined as [?]

$$H_{KS}(S) := \sup_{\mathcal{C}} H_{KS}(S, \mathcal{C}). \tag{2.4}$$

In the above formula the supremum is taken over all possible finite partitions of the phase space. A partition for which the supremum is achieved is called *generating*. Knowledge of a $k$–element generating partition for a given map allows one to represent the time evolution of the system in a $k$–letters symbolic dynamics and to find the upper bound for the KS-entropy: $H_{KS}(S) \le \ln k$. For some classical systems, like the Baker map, it is straightforward to find a generating partition and to compute the KS-entropy. On the other hand, it is usually difficult to find a generating partition for an arbitrary classical map. Recent years brought some progress in this field: Christiansen and Politi found a good approximation for a generating partition for the standard map [?] and obtained a fair estimate for the KS-entropy (see also [?]).

The convergence to the limit in (2.3) is usually slow, not faster than $1/n$. It is therefore advantageous to consider the relative entropies $G_n$ defined as

$$G_n := H_n - H_{n-1}, \text{ for } n > 1;\ G_1 = H_1. \tag{2.5}$$

It is easy to show that the sequence $G_n$ also tends to $H_{KS}$ [?]. This limit is achieved usually much faster than the limit in (2.3). For example Misiurewicz and Ziemian [?,?] proved that for a certain class of maps from the unit interval onto itself this convergence is exponential (see also [?]). It seems that such a behaviour is typical for chaotic maps. We refer the reader to [?,?,?,?] for the review of recent results in this area. Note that the convergence of $\frac{1}{n}H_n = \frac{1}{n}\sum_{i=1}^{n} G_i$ is slower, since the terms of larger $i$ have to balance a poor precision of the approximation due to the initial terms [?].

### B. Approximate measurement and coherent states

The probabilities $P^{cl}$ entering the definition of classical KS–entropy (2.2) are meaningful under the assumption that during the time evolution of the system one can trace an individual trajectory and determine its localization in the phase space with infinite precision. This supposition, consistent with the principles of classical mechanics, is definitely not fulfilled in quantum mechanics.

Information concerning the time evolution of a quantum system may be obtained by the process of sequential measurement. The fundamental analysis of a single quantum measurement of a discrete observable $\hat{A}$, expanded in an orthonormal basis as $\hat{A} := \sum_{m=1}^{N} a_m |m\rangle\langle m|$, leads to the *collapse postulate* of Lüders and von Neumann. The canonical measurement of $\hat{A}$ yields with the probability $p_a = \sum_{a_m=a} \langle m|\hat{\rho}|m\rangle$ the state reduction [?]

$$\hat{\rho} \xrightarrow{\ measurement\ } \hat{\rho}' := \frac{\sum_{a_m=a} |m\rangle\langle m|\hat{\rho}|m\rangle\langle m|}{\sum_{a_m=a} \langle m|\hat{\rho}|m\rangle}, \tag{2.6}$$

provided the outcome is $a$, where $\hat{\rho}$ is a density matrix describing the state of the system before the measurement. If $|m\rangle$ is the only eigenstate of $\hat{A}$ corresponding to the eigenvalue $a_m$, then this formula simplifies and the act of measurement transforms $\hat{\rho}$ into a pure state $\hat{\rho}' = |m\rangle\langle m|$.

The measurement of a single observable does not provide sufficient information about the localization of the quantum state in the phase space. Such information can be acquired only in a simultaneous double (or multiple) approximate measurement of canonically conjugated observables. Let us consider an $N$–dimensional complex Hilbert space $\mathcal{H}$ which represents the *kinematics* of the system and a compact set $\Omega$ equipped with a probability measure $\mu$ (we shall write $dx$ for $d\mu(x)$) which forms a *phase space*, or in the other words, a space of experimental outcomes. A correspondence between both spaces can be established by introducing a *family of coherent states*, i.e., a continuous map $\Omega \ni x \longrightarrow |x\rangle \in \mathcal{H}$ satisfying the resolution of the identity $\int_\Omega |x\rangle\langle x|dx = I$ [?]. In this work we use the coherent states normalized as $\langle x|x\rangle = N$.

Following the ideas of Davies and Lewis [?,?] we assume, in a full analogy to (2.6), that a multiple approximate quantum measurement yields the state reduction

$$\hat{\rho} \xrightarrow{\quad measurement \quad} \hat{\rho}' := \frac{\frac{1}{N}\int_{E_i} |x\rangle\langle x|\hat{\rho}|x\rangle\langle x|dx}{\int_{E_i}\langle x|\hat{\rho}|x\rangle dx}, \tag{2.7}$$

provided the outcome is in the cell $E_i$, which occurs with the probability $P_i^{CS} = \int_{E_i}\langle x|\hat{\rho}|x\rangle dx$. Note that if one increases the precision of the measurement of a single variable (and simultaneously decreases the precision of the measurement of the canonically coupled variable) this postulate reduces in the limit to the standard collapse postulate of Lüders and von Neumann. Formally, one has to replace the coherent states $|x\rangle$ used in (2.7) by so–called *squeezed states*.

### C. CS–probabilities and CS–entropy

Our approach to quantum entropy is based on the assumption that the knowledge about the time evolution of a quantum state is obtained from a sequence of multiple approximate quantum measurements. The evolution of the system between every two subsequent measurements is governed by a unitary matrix $U$.

A scheme of the first three periods of the time evolution of the dynamical system is presented in Fig.1. Consider a quantum path encoded by the following sequence of cells $\{W, Y, Z, \ldots\}$. Let the initial state be proportional to the identity operator, i.e., $\hat{\rho}_0 = 1/N \cdot I$. The coherent states collapse postulate (2.7) allows us to calculate the probability that a given sequence of symbols occurs ( [?], see also [?]). Namely, we have

$$P_W^{CS} = \int_W dw \langle w|\hat{\rho}_0|w\rangle = \mu(W), \tag{2.8}$$

$$P_{WY}^{CS} = \int_W dw \int_Y dy \tfrac{1}{N}|\langle y|U|w\rangle|^2, \tag{2.9}$$

$$P_{WYZ}^{CS} = \int_W dw \int_Y dy \tfrac{1}{N}|\langle y|U|w\rangle|^2 \int_Z dz \tfrac{1}{N}|\langle z|U|y\rangle|^2. \tag{2.10}$$

For arbitrary $n$ measurements delivering results in the cells $\{E_{i_0}, E_{i_1}, \ldots, E_{i_{n-1}}\}$ the probabilities are equal to

$$P_{i_0,\ldots,i_{n-1}}^{CS} = \int_{E_{i_0}} dx_0 \cdots \int_{E_{i_{n-1}}} dx_{n-1} \prod_{u=1}^{n-1} K(x_{u-1}, x_u), \tag{2.11}$$

where $i_l = 1, \ldots, k;\ l = 0, \ldots, n-1$ and the kernel $K$ is given by

$$K(x, y) = \frac{1}{N}|\langle y|U|x\rangle|^2. \tag{2.12}$$

We call them *CS–probabilities*. Partition dependent, *coherent states (CS) entropy* $H^{CS}$ of a quantum map $U$ is defined likewise its classical counterpart (2.3)

$$H^{CS}(U, \mathcal{C}) := \lim_{n\to\infty} \frac{1}{n} H_n(U, \mathcal{C}), \tag{2.13}$$

where

$$H_n(U, \mathcal{C}) := -\sum_{i_0,\ldots,i_{n-1}=1}^{k} P_{i_0,\ldots,i_{n-1}}^{CS} \ln P_{i_0,\ldots,i_{n-1}}^{CS}, \tag{2.14}$$

and $\mathcal{C} = \{E_1, \ldots, E_k\}$. In the semiclassical limit the CS–entropy seems to tend to the KS–entropy, if the quantization procedure is *regular* [?], i.e., if some assumptions linking the family of quantum maps with the corresponding classical map are fulfilled.

Quantum CS-probabilities can be also used to define other quantities which measure the randomness of the system (for a recent account of such concepts see [?] and [?]) like Rényi–type entropy of order $\beta$ which we shall analyse in sect. VII. For some purposes, for instance, it might be useful to define *CS–inverse participation ratio* $\nu$

$$\nu(U, \mathcal{C}) := \sum_{i_0,\ldots,i_{n-1}=1}^{k} (P^{CS}_{i_0,\ldots,i_{n-1}})^2. \tag{2.15}$$

It is an analogue of a quantity often used in solid state physics to describe localization of a wave function [?], since its inverse gives the average number of occupied cells. It is linked to CS–Rényi entropy of order 2.

In the simplest case of the trivial dynamics the quantum map $U$ reduces to the identity operator $I$. Even in this case the quantum entropy $H^{CS}$ does not vanish, since the coherent states are not orthogonal and do overlap [?]. The *CS–measurement entropy* is given by [?,?]

$$H_{mes}(\mathcal{C}) := H^{CS}(U \equiv I, \mathcal{C}) \tag{2.16}$$

and depends on a family of coherent states in the phase space $\Omega$ and on a finite partition $\mathcal{C}$.

The *CS-dynamical entropy of a quantum map $U$ with respect to a partition $\mathcal{C}$* is defined as [?,?]

$$H_{dyn}(U, \mathcal{C}) := H^{CS}(U, \mathcal{C}) - H_{mes}(\mathcal{C}), \tag{2.17}$$

and partition independent *CS–dynamical entropy* as [?]

$$H_{dyn}(U) := \sup_{\mathcal{C}} H_{dyn}(U, \mathcal{C}). \tag{2.18}$$

In the present paper we study CS–measurement entropy and its dependence on a partition and the semiclassical parameter. This is a preliminary step to calculating CS–dynamical entropy, which is defined as the difference of two quantities. Moreover, the techniques we use in computing of CS–measurement entropy can be also applied in the general case.

### D. Properties of CS–measurement entropy

We now review some basic properties of CS–measurement entropy. Let us assume that a finite partition $\mathcal{C}$ of the phase space $\Omega$ is given. Let $H_n(\mathcal{C})$ be defined by (2.11)–(2.14) with $U = I$, and let $G_1(\mathcal{C}) = H_1(\mathcal{C})$; $G_n(\mathcal{C}) = H_n(\mathcal{C}) - H_{n-1}(\mathcal{C})$, for $n > 1$. Then, applying the general theory of entropy for random transformations [?,?], we obtain the following facts:

**(1)** the sequences $\frac{1}{n}H_n(\mathcal{C})$ and $G_n(\mathcal{C})$ decrease with $n$ to $H_{mes}(\mathcal{C})$;

**(2)** if a partition $\mathcal{C}'$ is finer than a partition $\mathcal{C}$, then $H_{mes}(\mathcal{C}') \geq H_{mes}(\mathcal{C})$.

Next, let us observe that the kernel $K$ which appears in (2.11) is bistochastic, i.e., $\int_\Omega K(x, \bar{y})dx = \int_\Omega K(\bar{x}, y)dy = 1$ for all $\bar{x}, \bar{y} \in \Omega$. Let us denote by $K_0$ the maximum of $K$. Then

**(3)** the CS–measurement entropy fulfills the following inequalities:

$$\frac{1}{n}H_n(\mathcal{C}) - \frac{1}{n}\ln K_0 \leq H_{mes}(\mathcal{C}) \leq \frac{1}{n}H_n(\mathcal{C}), \tag{2.19}$$

and, in consequence,

$$H_1(\mathcal{C}) - \ln K_0 \leq H_{mes}(\mathcal{C}) \leq H_1(\mathcal{C}) \tag{2.20}$$

(for the proof see Appendix A).

Note that $H_1(\mathcal{C})$ does not depend on the family of coherent states but only on the measure $\mu$ and it is just the entropy of the partition $\mathcal{C}$ with respect to the measure $\mu$. If $\Omega$ is a Riemannian manifold and $\mu$ is the Riemannian measure on $\Omega$, then one can deduce from (2) and (3) that $H_{mes}(\mathcal{C})$ can be arbitrarily large for a sufficiently fine partition $\mathcal{C}$.

It follows from (1) and (3) that

**(4)** if $H_1(\mathcal{C}) \neq H_{mes}(\mathcal{C})$, then the sequence $\frac{1}{n}H_n(\mathcal{C})$ converges to the entropy $H_{mes}(\mathcal{C})$ precisely as $n^{-1}$

(for the proof see Appendix B).

Another important property of CS–measurement entropy,

**(5)** $H_{mes}(\mathcal{C})$ tends to 0 in the semiclassical limit,

was proved in [?] for SU(2) (spin) coherent states. The decay seems to be rather slow. We shall try to evaluate its rate in the sequel.

### E. Matrix form of CS–probabilities

Let $\mathcal{C} = \{E_1, \ldots, E_k\}$. We assume that the kernel $K$ entering formula (2.11) has the form

$$K(x,y) = \sum_{l,r=0}^{M} a_{lr} g_r(x) f_l(y), \quad \text{for} \quad x, y \in X, \tag{2.21}$$

where $a_{lr} \in I\!R$, $f_l, g_r : \Omega \to I\!R$ are continuous, for $l, r = 0, \ldots, M$, and $f_0 = g_0 \equiv 1$ (in fact we can always present $K$ in such a form if the family of coherent states comes from the canonical group-theoretic construction (see [?,?,?]) with the finite-dimensional Hilbert space $\mathcal{H}$; then $M$ is an increasing function of the dimension of the Hilbert space). Let us define matrices $A = [a_{lr}]_{l,r=0}^{M}$ and $B(i)_{rl} = \int_{E_i} g_r(x) f_l(x) dx$ for $l, r = 0, \ldots, M$, $i = 1, \cdots, k$. Then the CS–probabilities are given by the first element of the following matrix product:

$$P^{CS}_{i_0,\ldots,i_{n-1}} = \big(B(i_{n-1})AB(i_{n-2})A \cdots AB(i_0)\big)_{00} \tag{2.22}$$

(the proof will appear in [?]). Now one can show that the family of the CS–probabilities generates on the code space $\mathcal{S}^{I\!N}$, where $\mathcal{S} = \{1, \ldots, k\}$, a shift–invariant measure, which is *algebraic* in the sense of Fannes *et al.* (see [?]). Clearly, the decomposition of the kernel $K$ is not unique. Moreover, the assumption $f_0 = g_0 \equiv 1$ is too restrictive. In fact, to apply the matrix method, it is enough to know that the constant function 1 is a linear combination of the functions $f_0, \ldots, f_M$ [?].

The above formula makes the calculation of the CS–entropy much easier. Moreover, it is a starting point for the further investigation of entropy utilizing the theory of iterated function systems. We present in sect. VI some results in this direction. For a fuller treatment we refer the reader to [?].

## III. SPIN COHERENT STATES

The two–dimensional sphere $S^2$ can be considered as the phase space of the periodically kicked top. This classical dynamical system is known to exhibit chaos under a suitable choice of system parameters [?]. In order to study a quantum analogue of this system it is convenient to consider the operator of angular momentum $J$. Its three components $\{J_x, J_y, J_z\}$ are related to the infinitesimal rotations along three orthogonal axes $\{x, y, z\}$ in $I\!R^3$ and fulfill the standard commutation relations $[J_l, J_m] = i\varepsilon_{lmn} J_n$, where $l, m, n = x, y, z$ and $\varepsilon_{lmn}$ represents the antisymmetric tensor (from now on we put $\hbar = 1$). The operators $J_\pm = J_x \pm iJ_y$ and $J_z$ are generators of the compact Lie group $SU(2)$. The eigenvalues $j(j+1)$, $j = 0, 1/2, 1, 3/2, \ldots$, of the Casimir operator $J^2 = J_x^2 + J_y^2 + J_z^2$ determine the dimension $N = 2j + 1$ of the Hilbert spaces $\mathcal{H}_N$ carrying the representation of the group. Common eigenstates $|j, m\rangle$, $m = -j, \ldots, j$, of the operators $J^2$ and $J_z$ form an orthonormal basis in $\mathcal{H}_N$.

The $SU(2)$ (spin) coherent states were introduced by Radcliffe [?] and Arecchi *et al.* [?]. For a thorough discussion we refer the reader to [?,?,?,?,?,?]. The idea is the following. Each point on the sphere labeled by the spherical coordinates $(\vartheta, \varphi)$ corresponds to the $SU(2)$ *coherent state* $|j, \vartheta, \varphi\rangle$ generated by the unitary operator $R(\vartheta, \varphi) = \exp\big[i\vartheta(\sin\varphi J_x - \cos\varphi J_y)\big]$ acting on the reference state $|j, j\rangle$. The natural projection $SU(2) \to SO(3)$ relates with the operator $R(\vartheta, \varphi)$ the rotation by the angle $\vartheta$ around the axis directed along the vector $(\sin\varphi, -\cos\varphi, 0)$ normal to the $z$–axis and to the vector $(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ (see Fig. 2). The state $|j, j\rangle$, pointing towards the "north pole" of the sphere, enjoys the minimal uncertainty, i.e., the expression $\sum_{l=x,y,z} \Delta J_l^2$ takes in this state the minimal value $j$ (the other possible choice of the reference state is $|j, -j\rangle$). More precisely we put

$$|j, \vartheta, \varphi\rangle = \sqrt{2j+1} \cdot R(\vartheta, \varphi)|j, j\rangle. \tag{3.1}$$

Using the stereographical projection $\gamma = \tan(\vartheta/2)\exp(i\varphi)$ one can find a complex representation of the coherent state $|j, \gamma\rangle := |j, \vartheta, \varphi\rangle$

$$|j,\gamma\rangle = \frac{\sqrt{2j+1}}{(1+|\gamma|^2)^j}\exp[\gamma J_-]|j,j\rangle. \tag{3.2}$$

The prefactor $\sqrt{2j+1}$ introduced into the above formulae ensures the coherent states identity resolution in the form

$$\int_{S^2} |j,\vartheta,\varphi\rangle\langle j,\vartheta,\varphi|\; d\mu(\vartheta,\varphi) = I, \tag{3.3}$$

where the Riemannian measure $\mu$ on $S^2$ is given by $d\mu = \sin\vartheta d\vartheta d\varphi/4\pi$ and therefore does not depend on the quantum number $j$. The norm of the coherent states changes with $j$ as $|\langle j,\vartheta,\varphi|j,\vartheta,\varphi\rangle| = 2j+1$, which enables the respective *Husimi–like distribution* $S^2 \ni (\vartheta',\varphi') \longrightarrow |\langle j,\vartheta,\varphi|j,\vartheta',\varphi'\rangle|^2 \in \mathbb{R}$ of the coherent state $|j,\vartheta,\varphi\rangle$ to tend to the Dirac $\delta$–function as $j\to\infty$. Thus, after such a renormalization we can treat the limit $j\to\infty$ as the *semiclassical limit* [?] or, in the other words, as the *sharp-point limit* in the sense of Schroeck [?]. If we had transformed spin coherent states in a different way defining $||j,\sqrt{2j}\gamma\rangle := (1+|\gamma|^2)^{-j}\exp[\gamma J_-]|j,j\rangle$, we would have obtained the canonical (harmonic oscillator) coherent states in the limit $j\to\infty$. This kind of limit, however, is completely different from the semiclassical limit we use in the present paper.

To simplify the notation in the following sections we shall omit the number $j$ labelling coherent states $|\vartheta,\varphi\rangle$ or $|\gamma\rangle$. Note that $S^2$ is isomorphic to the coset space $SU(2)/U(1)$, where $U(1)$ is the maximal stability subgroup of $SU(2)$ with respect to the state $|j,j\rangle$, i.e., the subgroup of all elements of $SU(2)$ which leave $|j,j\rangle$ invariant up to a phase factor. Hence the above construction can be treated as a particular case of the general construction of group-theoretic coherent states.

Expansion of a coherent state in the eigenbasis of $J^2$ and $J_z$ reads

$$|\gamma\rangle = \sqrt{2j+1}\sum_{m=-j}^{m=j}\gamma^{j-m}(1+\gamma\bar{\gamma})^{-j}\Big[\binom{2j}{j-m}\Big]^{1/2}|j,m\rangle. \tag{3.4}$$

In the spherical variables $(\vartheta,\varphi)$ this expansion takes the form

$$|\vartheta,\varphi\rangle = \sqrt{2j+1}\sum_{m=-j}^{m=j}\sin^{j-m}(\frac{\vartheta}{2})\cos^{j+m}(\frac{\vartheta}{2})\exp\Big(i(j-m)\varphi\Big)\Big[\binom{2j}{j-m}\Big]^{1/2}|j,m\rangle. \tag{3.5}$$

The expectation values of the components of $J$ are

$$\langle j,\vartheta,\varphi|J|j,\vartheta,\varphi\rangle = j(2j+1)\big(\sin\vartheta\cos\varphi,\sin\vartheta\sin\varphi,\cos\vartheta\big) \tag{3.6}$$

which establishes the link between the coherent state $|j,\vartheta,\varphi\rangle$ and the vector $(\vartheta,\varphi)$ oriented along the direction defined by a point on the sphere.

The infinite basis formed in the Hilbert space by the coherent states is overcomplete. Two different $SU(2)$ coherent states overlap unless they point towards two opposite poles on the sphere. Expanding two coherent states in the $|j,m\rangle$ basis (3.5) we can calculate their overlap as

$$|\langle\vartheta',\varphi'|\vartheta,\varphi\rangle|^2 = (2j+1)^2\Big(\frac{1+\cos\Xi}{2}\Big)^{2j}, \tag{3.7}$$

where $\Xi$ is the angle between two vectors on $S^2$ related to the coherent states $|\vartheta,\varphi\rangle$ and $|\vartheta',\varphi'\rangle$. Hence the transition kernel $K$ defined by (2.12) takes (for $U=I$) the form

$$K((\vartheta,\varphi),(\vartheta',\varphi')) = \frac{|\langle\vartheta',\varphi'|\vartheta,\varphi\rangle|^2}{2j+1} = \frac{2j+1}{2^{2j}}\Big[1+\cos\vartheta\cos\vartheta'+\sin\vartheta\sin\vartheta'\cos(\varphi-\varphi')\Big]^{2j}. \tag{3.8}$$

The overlap decreases to 0 with $j$ for $|\vartheta,\varphi\rangle \neq |\vartheta',\varphi'\rangle$ and sufficiently large $j$.

## IV. MEASUREMENT ENTROPY FOR TWO HEMISPHERES

We would like to compute the CS–measurement entropy for the case corresponding to the physical process of simultaneous approximate measurement of different spin components. Let us first consider the simplest case, where the classical phase space $\Omega$ equal to the two–dimensional sphere $S^2$ is divided into two hemispheres $E_+ = \{(\vartheta,\varphi) : \varphi\in[0,2\pi), \vartheta\in[0,\pi/2]\}$ and $E_- = \{(\vartheta,\varphi) : \varphi\in[0,2\pi), \vartheta\in(\pi/2,\pi]\}$. The result of any measurement $i=\pm1$ gives information about the orientation of the spin.

## A. Transition probabilities

The CS-transition probabilities $P^{CS}$ for the results $i_0, \ldots, i_{n-1}$ of $n$ consecutive measurements are obtained from (2.11) and (2.12) by setting the evolution operator $U$ to be the identity and taking the appropriate integration domains. The explicit integral reads

$$P^{CS}_{i_0,\ldots,i_{n-1}} = (4\pi)^{-n} \int_{E_{i_0}} \sin\vartheta_0 d\vartheta_0 d\varphi_0 \cdots \int_{E_{i_{n-1}}} \sin\vartheta_{n-1} d\vartheta_{n-1} d\varphi_{n-1} \prod_{u=1}^{n-1} K\big((\vartheta_{u-1},\varphi_{u-1}),(\vartheta_u,\varphi_u)\big) \tag{4.1}$$

where the kernel $K$ is given by (3.8), and $i_u = \pm 1$ for $u = 0, \ldots, n-1$. Straightforward integration allows one to obtain analytical results for low values of $n$ and $j$, collected in Tab. 1.

In spite of the trivial dynamics ($U \equiv I$) the result of the first measurement may differ from the second one, and consequently, all the transition probabilities are nonzero. In the semiclassical limit $j \to \infty$ the "mixed" transition probabilities (e.g. $P^{CS}_{+-} = P^{CS}_{-+}$) vanish, while the survival probabilities (e.g. $P^{CS}_{++}$, $P^{CS}_{+++}$) tend to 1/2. The geometric symmetry of reflection induces the invariance of the probabilities with respect to the interchange of signs ($+ \longleftrightarrow -$). Moreover, due to the time–reversal invariance, the CS–probability for any sequence of results equals the CS–probability of the same sequence written in the reverse order (e.g. $P^{CS}_{++-} = P^{CS}_{-++}$, $P^{CS}_{+++-} = P^{CS}_{-+++}$, $P^{CS}_{++-+} = P^{CS}_{+-++}$). Observe, that for a given number of measurements $n$, the probabilities for two sequences of results with the same number of transitions are similar (e.g. for one transition: $P^{CS}_{+---} \approx P^{CS}_{++--}$; for two transitions: $P^{CS}_{+--+} \approx P^{CS}_{++-+}$). Direct integration of (4.1) does not allow one to obtain the CS–probabilities for larger values of $j$ or $n$, which is necessary to estimate the CS–measurement entropy. For this purpose it is convenient to formulate integrals in matrix form.

## B. Matrix formulation of integrals

Computation of the CS-probabilities can be significantly simplified by applying the general method described in sect. II.E. This can be seen, especially, for the division of the sphere into several latitudinal components $E_1, \ldots, E_k$, where $E_i = \{(\vartheta,\varphi) : \varphi \in [0, 2\pi), \vartheta \in W_i\}$ for $i = 1, \ldots, k$, and $\{W_1, \ldots, W_k\}$ is a partition of the interval $[0, \pi]$. Performing the substitutions $t_i = \cos\vartheta_i$ and integrating over $\varphi_0, \ldots, \varphi_n$ we can simplify formula (4.1) writing

$$P^{CS}_{i_0,\ldots,i_{n-1}} = \int_{\tilde{W}_{i_0}} \frac{1}{2} dt_0 \;\cdots \int_{\tilde{W}_{i_{n-1}}} \frac{1}{2} dt_{n-1} \prod_{u=1}^{n-1} \tilde{K}\big(t_{u-1}, t_u\big), \tag{4.2}$$

where $\tilde{W}_i = \{\cos t : t \in W_i\}$ for $i = 1, \ldots, k$, and the reduced kernel $\tilde{K}$ is given by

$$\tilde{K}\big(t, s\big) = \frac{2j+1}{4^{2j}} \sum_{q=0}^{2j} \binom{2j}{q}^2 \big((1+t)(1+s)\big)^q \big((1-t)(1-s)\big)^{2j-q} = \sum_{l,r=0}^{2j} a_{lr} t^l s^r, \tag{4.3}$$

for $t, s \in [-1, 1]$. Thus the kernel $\tilde{K}$ is represented in the form (2.21) with $\tilde{\Omega} = [-1, 1]$, $d\tilde{\mu}(t) = \frac{1}{2} dt$, $f_l(t) = t^l$, $g_r(s) = s^r$ for $t, s \in \tilde{\Omega}$, and $M = 2j$. Note that $\{\tilde{W}_1, \ldots, \tilde{W}_k\}$ forms a partition of $\tilde{\Omega}$. Hence we can apply formula (2.22) for the CS–probabilities writing them in the matrix form

$$P^{CS}_{i_0,\ldots,i_{n-1}} = \langle (1, 0, \ldots, 0) \;|\big(B(i_{n-1}) A B(i_{n-2}) A \cdots A B(i_0)\big)| (1, 0, \ldots, 0) \rangle, \tag{4.4}$$

with $A = [a_{lr}]_{l,r=0}^{2j}$ (given by (4.3)) and $B(i)_{rl} = \frac{1}{2} \int_{\tilde{W}_i} t^{l+r} dt$ for $i = 1, \ldots, k$; $l, r = 0, \ldots, 2j$.

If we divide the sphere into two hemispheres, then $B(i)_{rl} = i^{l+r}/2(l + r + 1)$ for $i = \pm 1$; $l, r = 0, \ldots, 2j$. In this case (4.4) takes a particularly simple form for $j = 1/2$

$$P^{CS}_{i_0,\ldots,i_{n-1}} = \frac{1}{2^n} \big\langle (1, 0) \;| \begin{pmatrix} 1 & i_0/2 \\ i_0/2 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & i_1/2 \\ i_1/2 & 1/3 \end{pmatrix} \cdots \begin{pmatrix} 1 & i_{n-1}/2 \\ i_{n-1}/2 & 1/3 \end{pmatrix} | (1, 0) \big\rangle, \tag{4.5}$$

where the results of the measurements $i_u$ are equal to $-1$ or $+1$ for $u = 0, \ldots, n-1$.

### C. Limit $n \to \infty$

In the remainder of this section we assume that $\mathcal{C}$ is the partition of the sphere into two hemispheres, i.e., $\mathcal{C} = \{E_+, E_-\}$. Moreover, we set $H_{mes} := H_{mes}(\mathcal{C})$, $H_n := H_n(\mathcal{C})$ and $G_n := G_n(\mathcal{C})$. In Tab. 2 we present partial and relative entropies calculated for two different values of $j$ with the aid of the formulae (2.13), (2.14) and (4.4).

We assert in sect. II.D(4) that $H_n$ converges to $H_{mes}$ exactly as $\frac{1}{n}$. One can deduce from Tab. 2 that the convergence of $G_n$ to the same limit is much faster. In fact, it seems to be exponential. In sect. VI we give some arguments supporting this statement. Thus, to calculate the limiting value we use the extrapolations $H_n \sim H_{mes} + \alpha/n$ and $G_n \sim H_{mes} + \gamma c^n$. The outcomes are contained in Tab. 2. Let us observe that the rate of convergence decreases with $j$ and hence the method of computing the CS-measurement entropy based on formula (4.4) does not lead to satisfactory results in the semiclassical limit, i.e., for large quantum number $j$.

### D. Semiclassical regime $j >> 1$

The matrix formula for the CS-probabilities is useful in numerical calculations, but as it was mentioned above, do not allow us to compute the entropy for very large values of $j$. For two measurements, however, one can obtain some exact results. Applying (4.2) and (4.3) we get an analytical formula for the CS–probability valid for any $j$

$$P^{CS}_{+-} = \binom{4j+1}{2j} 2^{-4j-2} \tag{4.6}$$

(for the proof see Appendix C).

Due to symmetry $P^{CS}_{-+} = P^{CS}_{+-}$ and $P^{CS}_{++} = P^{CS}_{--} = 1/2 - P^{CS}_{+-}$. It is convenient to introduce a $j$–dependent coefficient $\tau_j = P^{CS}_{+-}/P^{CS}_{++}$, which tends to 0 in the semiclassical limit $j \to \infty$. Using formula (4.6) we obtain

$$\tau_j = \frac{\binom{4j+1}{2j}}{2^{4j+1} - \binom{4j+1}{2j}}. \tag{4.7}$$

In order to get an upper bound for the CS-measurement entropy we may compute the relative entropy $G_2 = H_2 - H_1$ (see sect. II.D). The partial entropy after one measurement $H_1$ equals $\ln 2$, independently of $j$. Summing over four possible paths $++, +-, -+, --$ one can compute the partial entropy $H_2$ obtaining finally a formula

$$G_2 = \ln(\tau_j + 1) - \frac{\tau_j}{\tau_j + 1} \ln(\tau_j), \tag{4.8}$$

which is symmetric with respect to an involution $\tau_j \to 1/\tau_j$.

Inserting the expression (4.7) into the above formula we get an explicit approximation for $H_{mes}$. It is represented by a solid line in Fig. 3, while circles denote the results obtained numerically for small $j$ with the help of the matrix method presented above. In the semiclassical range $j >> 1$ it is legitimate to apply the Stirling approximation of the factorial in (4.7), which gives

$$H_{mes} < G_2 \sim \ln\sqrt{2\pi j} - \big(1 - \frac{1}{\sqrt{2\pi j}}\big)\ln(\sqrt{2\pi j} - 1) \tag{4.9}$$

This formula, providing a fair approximation (and an upper bound) for the CS–measurement entropy, is characterized by the asymptotic behaviour

$$G_2 \sim \frac{\ln j}{2\sqrt{2\pi j}}. \tag{4.10}$$

It is worth to note that formula (4.8) can be obtained from the Markovian approximation of the CS–probabilities. Let us assume for a moment that the probabilities $P^{CS}_{i_0,\ldots,i_{n-1}}$ were generated by a Markov shift. It follows from the symmetry of the problem that its initial vector would be $(1/2, 1/2)$ and its transition matrix $Q$ would have the form

$$Q = \begin{pmatrix} \text{a} \\ \text{1-a} \\ \text{1-a} \\ \text{a} \end{pmatrix}, \qquad \text{where} \quad a = 2P^{CS}_{++} = \frac{1}{\tau_j + 1}. \tag{4.11}$$

In fact, our probabilities $P^{CS}_{i_0,\dots,i_{n-1}}$ are not generated by a Markov shift, nevertheless, one can consider the Markovian approximation as above. Then the approximate probabilities $P^{Mar}$ depending only on the number $L$ of "transitions" from one hemisphere to the other ($L = \frac{1}{2}(n-1-\sum_{q=1}^{n-1} i_q i_{q-1})$) are equal to

$$P^{Mar}_{i_0,\dots,i_{n-1}} = \frac{1}{2} Q_{i_0,i_1} \cdot \ldots \cdot Q_{i_{n-2},i_{n-1}} = \frac{1}{2}(1-a)^L a^{n-1-L} = \frac{\tau_j^L}{2(\tau_j+1)^{n-1}}. \tag{4.12}$$

In this approximation the probabilities form a geometric series with the same ratio $\tau_j$ for any number of measurements $n$. The precision of such a Markovian approximation can be directly checked in Tab. 1 containing the exact probabilities $P^{CS}$. For example for $n = 4$ and $j = 1/2$ the numerators $289, 167(161), 103(97)$ and $65$ form an approximately geometric series, however, probabilities with the same number of the transitions are not equal (e.g. $P^{CS}_{+-++} \neq P^{CS}_{+--+}$).

Summing over all $2^n$ possible sequences we obtain the following approximate formula for the partial entropy:

$$H^{Mar}_n = -\sum_{L=0}^{n-1} \binom{n-1}{L} \frac{\tau_j^L}{(\tau_j+1)^{n-1}} \ln\Big[\frac{\tau_j^L}{2(\tau_j+1)^{n-1}}\Big] = \ln 2 + (n-1)\Big[\ln(\tau_j+1) - \frac{\tau_j}{\tau_j+1}\ln(\tau_j)\Big]. \tag{4.13}$$

Now dividing both sides by $n$ and performing the limit $n \to \infty$ we arrive at the relative entropy $G_2$ given by (4.8).

Let us recall that in the semiclassical limit ($j \to \infty$) the relative measurement entropy $G_2$ tends to zero as $\ln j/\sqrt{j}$. This defines the scale in which the quantum effects reveal. Unfortunately the precision of this approximation is not sufficient to conclude, whether the logarithmic prefactor describes correctly the decay of the measurement entropy $H_{mes}$ in the semiclassical limit, or whether its existence is an artifact introduced by the approximation.

## V. CS–MEASUREMENT ENTROPY FOR VARIOUS PARTITIONS

The CS–measurement entropy depends on the number of cells in a partition and on their shape. In this section we consider several partitions of a different type: into two cells: both connected, one connected and one disconnected, both disconnected, and into many cells. Figs. 4, 7, and 10 contain schemes for these partitions. In all the cases we compute the entropy using the matrix formulation introduced in sects. II.E and IV.B. As in the preceding section we assume that $\mathcal{C}$ denotes the respective partition of the sphere, putting $H_{mes} := H_{mes}(\mathcal{C})$, $H_n := H_n(\mathcal{C})$, and $G_n := G_n(\mathcal{C})$.

### A. Two connected cells

Let us split the sphere into two segments along a parallel $\Theta_c$. The northern segment $E_+$ contains points with $\vartheta \in [0, \Theta_c]$, while the southern $E_-$ those with $\vartheta \in (\Theta_c, \pi]$. This partition is shown schematically in Fig. 4.

Fig. 5 represents the dependence of the partial entropy $H_n/n$ on the variable $\cos\Theta_c$ for $n = 8$ measurements and several values of $j$. For each value of $n$ and $j$ the partial entropy achieves its maximum at $\cos\Theta_c = 0$, for the partition into two hemispheres. The solid horizontal line drawn at $\ln 2$ represents the maximal entropy admissible for the partition containing two cells. For increasing values of $j$ the partial entropy decreases and tends to zero for $j \to \infty$.

For any of these partitions the partial entropy $H_n/n$ approaches the limiting value $H_{mes}$ approximately as $1/n$ (see sect. II.D(4)). As in the previously discussed case of two hemispheres, we estimate the limiting value $H_{mes}$ by computing the relative entropy $G_n$. Fig. 6 shows a comparison of the partial entropies $H_n/n$ with $H_{mes}$ extrapolated in this way for $j = 1/2$, $n = 2$ (Fig. 6a) and $j = 5$, $n = 2$ or $n = 8$ (Fig. 6b). The difference increases with the spin length $j$.

### B. Two disconnected cells

Let us now analyse another two classes of partitions of the sphere into two cells. In the first case (Fig. 7a) we divide the sphere into three parts along parallels $\pi - \Theta_d$ and $\Theta_d$, and then join the lower and upper parts, thus obtaining two cells: a connected spherical zone and the disconnected union of two spherical segments. The CS–measurement entropy (Fig. 8) changes in this case from 0 (for $\Theta_d = 0$) to $\ln 2$ (for $\Theta_d = \pi/3$), which is the largest possible value for the CS–measurement entropy with respect to a two-element partition.

In the second case (Fig. 7b) we start from the splitting of the sphere into the lower and upper hemispheres. Next, we cut symmetrically two "pieces of cake" out of both hemispheres, and then join the four parts across. We get in this way two disconnected cells marked in black and white in Fig. 7b. The CS–measurement entropy (Fig. 9) changes from $\ln 2$ (for $\Phi_d = \pi$) to $0.6613\ldots$ (for $\Phi_d = 0$). The latter case relates to the partition of the sphere into two hemispheres studied in sect. IV.

It is worth pointing out that in both cases the entropy attains the maximal admissible value $\ln 2$ for some value of the parameter. We do not know whether one can generalize this observation and find a partition of an arbitrary number of cells $k$ giving the maximal allowed value of the entropy equal $\ln k$.

### C. Many cells

Let us consider $k$ disjoint zones created on the sphere by $k-1$ parallels. As in the case of two cells, represented in Fig. 4, the CS–measurement entropy seems to achieve its maximum, if the cells have the same volume $= 1/k$ (see Fig. 10). We computed the CS–measurement entropy $H_{mes}(\mathcal{C}_k)$ for the partitions $\mathcal{C}_k$ of the sphere into $k = 2, \ldots, 1000$ zones of the same volume. Note that for large $k$ even the second relative entropy $G_2(\mathcal{C}_k)$ provides a reliable estimate for $H_{mes}(\mathcal{C}_k)$. In Fig. 11 we present the CS-measurement entropy displayed for $j = 1/2$ as a function of the number of cells $k$ (circles). The solid line represents the function $\ln k$, which gives the upper bound for the entropy with respect to a partition consisting of $k$ cells.

Since for any partition $\mathcal{C}$ and a quantum map $U$ the CS–dynamical entropy is defined by (2.17) as the difference of $H(U, \mathcal{C})$ and $H_{mes}(\mathcal{C})$, it is convenient to consider the quantity $H_{max}(\mathcal{C}_k) := \ln k - H_{mes}(\mathcal{C}_k)$, limiting the partition dependent dynamical entropy $H_{dyn}(U, \mathcal{C}_k)$ from the above. From (2.20) and (3.8) we know that $H_{max}(\mathcal{C}_k) \leq \ln(2j+1)$ as $H_1(\mathcal{C}_k) = \ln k$. Although using this method one can establish the finiteness of the partition independent dynamical entropy $H_{dyn}(U)$ given by (2.18), this upper bound seems to be rather crude. In fact $H_{max}(\mathcal{C}_k)$ decreases with $j$. It is interesting to observe that this quantity converges for $k \to \infty$. The limiting value depends on $j$ and is close to 0.06 for $j = 1/2$. The inset in Fig. 11 shows the dependence of $H_{max}(\mathcal{C}_k)$ on $k^{-2}$ for $k = 30, \ldots, 1000$; $j = 1/2$. The data displayed in this way are well fitted by a straight line, which allows us to postulate an approximate relation $H_{mes}(C_k) \approx \ln k - 0.05999745 + 0.1637/k^2$, found for $j = 1/2$.

## VI. CS-ENTROPY AND ITERATED FUNCTION SYSTEMS

In this section we establish a relationship between CS–entropy and iterated function systems (IFSs). Firstly, we show how to obtain an IFS from a bistochastic kernel and a partition of the phase space. Then, we use this system to get an integral formula for CS–entropy and propose a new method of computing CS–entropy based on the ergodic theorem for IFSs. For more information on IFSs see [**?**], [**?**], and [**?**].

### A. Iterated function systems and an integral formula for CS–entropy

We follow the notation of sects. II.D and II.E. With each cell $E_i$ ($i = 1, \ldots, k$) of the partition we associate an $(M+1) \times (M+1)$ – matrix $D(i) = B(i)A$. We consider functions $p_i : \mathbb{R}^M \to \mathbb{R}^+$ and partial maps $F_i : \mathbb{R}^M \to \mathbb{R}^M$ given by

$$p_i(\lambda) = (1, 0, \ldots, 0)(D(i)(1, \lambda)) \tag{6.1}$$

and

$$F_i(\lambda) = (D(i)(1, \lambda))/p_i(\lambda) \tag{6.2}$$

for $\lambda \in \mathbb{R}^M$, $i =, 1 \ldots, k$.

Let us suppose that the functions $g_0 \equiv 1, g_1, \ldots, g_M$ are lineary independent. Then one can show that

**(0)** $p_i(\lambda) \geq 0$ for $i = 1, \ldots, k$ and $\sum_{i=1}^{k} p_i = 1$, i.e., the functions $\{p_i\}_{i=1,\ldots,k}$ can be treated as place-dependent probabilities.

Moreover we shall assume that there exists a set $X \subset \mathbb{R}^M$ such that

**(a)** X is a compact set with $\lambda_0 := (\int_X g_1, \ldots, \int_X g_M) \in X$, and for every $i = 1, \dots, k$:

**(b)** $F_i(X) \subset X$,

**(c)** $p_i|_X > 0$,

**(d)** $F_i|_X$ is a Lipschitz function with the Lipschitz constant $c_i < 1$.

Then the following assertions hold:

**(1)** $\mathcal{F} = (F_i, p_i)_{i=1}^k$ is an *iterated function system* on $X$.

**(2)** The IFS $\mathcal{F}$ generates the following operator $V$ acting on $M(X)$ (the space of all probability measures on X):

$$(V\nu)(B) = \sum_{i=1}^{k} \int_{F_i^{-1}(B)} p_i(\lambda) d\nu(\lambda) \tag{6.3}$$

for $\nu \in M(X)$ and $B \in B(X)$, where $B(X)$ denotes the family of all Borel sets on $X$. This operator describes the *evolution of probability measures* under the action of $\mathcal{F}$. We shall denote by $(Z_n^\nu)_{n\in I\!N}$ the associated Markov stochastic process having the initial distribution $\nu$.

**(3)** There is a unique *invariant probability measure* $\mu$ for the IFS defined above fulfilling the equation $V\mu = \mu$. This measure is attractive, i.e., $V^n\nu$ converges weakly to $\mu$ for every $\nu \in M(X)$ as $n \to \infty$.

**(4)** The relative entropies $G_n$ are given by

$$G_n = \int_X h_k(p_1(\lambda), \dots, p_k(\lambda)) d(V^n \delta_{\lambda_0})(\lambda) \quad \text{for} \quad n \in I\!N, \tag{6.4}$$

where $h_k$ is the Shannon–Boltzmann entropy function given by $h_k(p_1, \dots, p_k) = -\sum_{i=1}^k p_i \ln p_i$ for any $p_i \geq 0$ such that $\sum_{i=1}^k p_i = 1$.

**(5)** The CS–entropy $H_{mes}$ is given by an *integral formula*

$$H_{mes} = \int_X h_k(p_1(\lambda), \dots, p_k(\lambda)) d\mu(\lambda). \tag{6.5}$$

Let us sketch briefly the proof of the above statements. Assertion (1) follows from (0) and assumption (b). The Markov processes generated by IFSs were analysed in [?] and [?]. Assertion (3) can be deduced from assumptions (c) and (d), and [65, Th.2.1]. Formulae (6.4) and (6.5) were proved by Fannes *et al.* in [?] for algebraic measures, i.e., under the assumption that the formula for probabilities analogous to (2.22) holds. They followed an earlier result of Blackwell [?] on the entropy of functions of a finite–state Markov chain. In these both papers, however, the authors did not refer to the theory of IFSs and assumed that the matrices $D(i)$ are positive. In spite of this, their proof can also be applied in our case. For more details we refer the reader to [?].

### B. Ergodic theorem and random algorithm for computing CS–entropy

The key point in our reasoning is to find a set $X$ fulfilling conditions (a)–(d) above. In all the cases we analysed this task was not too difficult to accomplish. We shall give some examples below. Utilising the results presented in [?] and [?] we can go even further and prove (under some additional assumptions) that $G_n$ tends to $H_{mes}$ exponentially. Moreover, applying the *Kaijser–Elton ergodic theorem for IFSs* (see [?] and [?]) we obtain the following formula:

$$H_{mes} = \lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} h(Z_l^\nu) \qquad \mu - \text{almost everywhere}, \tag{6.6}$$

where $h = h_k(p_1, \dots, p_k)$ and $\nu$ is an arbitrary initial distribution.

This formula gives another numerical method of computing CS–entropy. To obtain the value $H_{mes}$ it suffices to calculate Ceasaro means of the function $h$ along a trajectory of the stochastic process $(Z_l^\nu)_{l\in I\!N}$. This is a particular case of the general method which appeared under the name of Random Iterated Algorithm in [?]. The convergence in (6.6) seems to be rather slow, namely as $n^{1/2}$. Note, however, that here the time computational complexity grows with $k$ (the number of elements of the partition) lineary, whereas in the "matrix method" we considered in sects. II.E and IV.B it grows polynomially (as $k^n$). Hence the method based on formula (6.6) may be specially useful for large values of $k$.