Relativistic quantum models for two bosons with interaction in the Schrödinger picture

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Abstract

Relativistic quantum models are given, in the Schrödinger picture, describing two bosons with interaction in four space-time dimensions. More precisely we start from the free model, involving functions $f(p_1, p_2)$ of two momenta $p_1, p_2 \in \mathbb{R}^3$ in the Hilbert space $L^2(\mathbb{R}^6, \sigma_2)$, where σ_2 is the Lorentz invariant measure. We add to the free Hamiltonian and the free Lorentz generators new interaction terms, such that the commutation rules of the Poincaré algebra remain satisfied. We get a large set of interaction terms solution to this problem. By integration we obtain unitary representations of the Poincaré group, in the same space $L^2(\mathbb{R}^6, \sigma_2)$. The existence of asymptotic states and of bound states assures that these models describe two particles with effective interaction.

1. Introduction

A new family of unitary representations of the Poincaré group in four space-time dimensions is given, which describe two boson systems with interaction. The construction of scattering states confirms that they really concern two particle systems while the existence of bound states confirms the effective interaction.

These models are obtained by perturbations of the Hamiltonian and the Lorentz generators of the two free boson model, in the Schrödinger picture (i.e. the Hilbert state space is simply made of functions of two momenta). A similar modification of the free model was proposed by Dirac in the last 40's [1], but in the classical case. His conclusion was that we get to too complicated equations. More recently, Currie [2] has obtained, under additional natural assumptions, that such modifications cannot generate an interaction (this is known as the 'non-interaction theorem'). The present paper shows that in the quantum case, these difficulties can be solved.

The result given here is the generalization of a similar result in two space-time dimensions [3], which was itself suggested by an original approch of the bound state problem in two dimensional space-time Quantum Field Theory (QFT). Let us describe this new approach.

The mathematical construction of bosonic QFT models in two space-time dimensions (the weakly coupled $P(\varphi)_2$ models) by Glimm, Jaffe, Spencer [4], has made possible a rigorous approach of the relativistic bound state problem. In this framework the Bethe-Salpeter method has been rigorously established and extensively studied [5]. This method provides a clear and precise theory for relativistic quantum particles with interaction at low energy, which gives satisfaction both at the physical and at the mathematical point of view. However this theory is incomplete because it neglects the construction of the eigenstates of the bound states; moreover the hole mathematical construction fails down at four dimensional space-time and for gauge interactions.

Before trying to grapple with these difficulties, it should be natural road to improve first the bound state description of the $P(\varphi)_2$ models, in order to develop intuition and new ideas. However, such a improvement requires unavoidably the construction of the eigenspaces of the bound states. But the standard strategy of QFT was precisely to avoid working with states of interacting particles (the vacuum and asymptotic states are the only states which are used). Thus a new method is needed, adapted to QFT models in which the existence of bound states is already known (from the Bethe-Salpeter method). Such a method has been developed by combining a variational and a perturbation approachs ([6], [7]). Let us explain in few words how it works. First, explicit expressions for the involving operators (Hamiltonian, etc...) on a sufficiently large set of vectors (the zero-time vectors) are calculated (at first perturbation orders only, but with perfect control of the remainders). Then the minimization of the Rayleigh quotient $(\psi, M\psi)\ \|\psi\|^{-2}$, where M is the mass operator, is performed over the largest combination ψ of such vectors. The vectors exhibited by the minimization can be considered as perturbative calculations of eigenvectors. This gives a method to construct approximations of the eigenspace, once the existence of a bound state is known.

This methods has brought several surprises. First, it leads to an eigenvalue equation for the bound state mass, which is very similar to the Schrödinger equation (and which can be considered as an approximed 'relativistic Schrödinger equation') [6], [8]. Second, a subspace is found which is stable under the Poincaré transformation (at first perturbation orders) and nevertheless carries interaction [7, Section 3], [8]. Such representations of the Poincaré group can be studied for themselves, without reference to QFT. The third surprise is that such representations can be constructed non-perturbatively [3]. They provide a set of models for two particles with effective interaction in the Schrödinger picture (because the time variable is not introduced, due to the initial restriction to zero-time vectors). Moreover, the complete set of such representations can be given, in the weak coupling regime. They are all constructed from an interaction kernel which, in the centre-of-mass frame, can be chosen arbitrarily [3].

The last surprise is that generalizations are possible, to $N > 2$ particle systems [9] and to higher dimensional space-time. The case of two particles in three space-time dimensions has been obtained for special interactions (involving the projection on the s-waves) [10]. With such an interaction, the direct connection with QFT fails. The present paper concerns the generalization to four space-time dimensions, for two particle systems. The difficulty comes from the highter dimension of the Poincaré group and non-integrability of some involving kernels. However, provided the interaction satisfies again the special property given above, unitary representations of the Poincaré group can be constructed, which describe two particles with interaction in four space-time dimensions. They give the first examples, to our knowledge, of relativistic quantum models for a finite number of particles with interaction in four space-time dimensions. Moreover, these models admit a rigorous mathematical construction which is remarkably simple.

The paper is organized as follows. Section 2 presents the models, given by a family of operators which (formally) satisfy the commutation rules of the Poincaré algebra, provided some conditions on the 'interaction kernel' h hold. In particular, if h satisfies the special property given above, then these conditions reduce to a single 'fundamental equation' for h. In section 3 the complete set of solutions to this equation is given, in the weak coupling regime. Then we focus on a particular subset of solutions which satisfy all the analytic properties we need for the following proofs. Section 4 shows that the formal representations of the Poincaré algebra, we have just obtained, are well defined and can be integrated to give unitary representations of the Poincaré group. Section 5 gives the eigenvalue equation for the bound states (the 'relativistic Schrödinger equation') and shows the existence of a bound state in some examples, which assures the non-triviality of these models. In section 6 we sum up what we have found, a family of models for two particles (this last point is confirmed by the construction of scattering states) with effective interaction.

2. The models

The Lie algebra $\mathcal{G}$ of the Poincaré group $\mathcal{P}_+^\uparrow$ in four space-time dimensions is generated by the ten operators H (Hamiltonian), $\vec{P} = (P_1, P_2, P_3)$ (momentum), $\vec{J} = (J_1, J_2, J_3)$ (angular momentum) and $\vec{L} = (L_1, L_2, L_3)$ (generators of the Lorentz transformations), satisfying the 45 commutation rules

$$[P_j, P_k] = 0 \qquad [P_j, J_k] = i\varepsilon_{j,k,\ell} P_\ell \qquad [J_j, J_k] = i\varepsilon_{j,k,\ell} J_\ell \tag{1}$$

$$[P_j, H] = 0 \qquad [H, J_j] = 0 \tag{2}$$

$$[P_j, L_k] = i\delta_{j,k} H \qquad [J_j, L_k] = i\varepsilon_{j,k,\ell} L_\ell \tag{3}$$

$$[H, L_j] = iP_j \qquad [L_j, L_k] = -i\varepsilon_{j,k,\ell} J_\ell \tag{4}$$

for all $1 \leq j, k \leq 3$ where $\delta_{i,j}$ is the Kronecker tensor and $\varepsilon_{j,k,\ell} \in \{0, 1\}$ is the totally antisymetric tensor with $\varepsilon_{1,2,3} = 1$ (we have used the Einstein summation notation). This algebra admits two Casimir operators, the mass operator M, the square of which is given by

$$M^2 \; := \; H^2 \; - \vec{P}^2 \tag{5}$$

and the Pauli-Lubanski operator

$$W^2 \; := \; \left(\vec{P} \cdot \vec{J}\,\right)^2 \; - \; \left(-H\vec{J} + \vec{P} \wedge \vec{L}\,\right)^2 \tag{6}$$

where the dot "$\cdot$" and the wedge "$\wedge$" denote the ordinary scalar and vectorial products respectively.

We start from the representation describing one spinless particle of mass $m > 0$, in the Schrödinger picture, given by following choice of operators

$$\begin{aligned} \vec{P}\phi(\vec{p}) &= \vec{p}\,\phi(\vec{p}) \\ H\phi(\vec{p}) &= \omega(\vec{p})\phi(\vec{p}) \\ \vec{J}\phi(\vec{p}) &= -i\,\vec{p} \wedge \vec{\nabla}\phi(\vec{p}) \\ \vec{L}\phi(\vec{p}) &= -i\omega(\vec{p})\,\vec{\nabla}\phi(\vec{p}) \end{aligned} \tag{7}$$

for almost all $\vec{p} = (p_1, p_2, p_3) \in I\!R^3$ and suitable three-variable functions ϕ, where we have put $\omega(\vec{p}) := \sqrt{\vec{p}^{\,2} + m^2}$. We have used the notation $\vec{\nabla} := (\partial_{p_1}, \partial_{p_2}, \partial_{p_3})$. These operators define a representation of the algebra $\mathcal{G}$ which can be integrated to give a unitary, continuous and irreducible representation of the Poincaré group $\mathcal{P}_+^\uparrow$, in the function space $L^2(I\!R^3, \sigma)$, where σ is the measure $d\sigma(\vec{p}) := d\vec{p}\,[2\omega(\vec{p})]^{-1}$ (see Appendix A). The Casimir operators of the representation (7) are simply $M = m$ (times the identity operator) and $W = 0$.

The representation describing two identical bosons of mass $m > 0$ without interaction, in the Schrödinger picture, is given by the symmetrical tensor product of two copies of the representation (7) (the symmetry refers to the case where the particles are identical, but this restriction is not essential and could be removed). The action of the operators becomes

$$
\begin{aligned}
\vec{P}\ \phi(\vec{p}_1,\vec{p}_2) &= (\vec{p}_1+\vec{p}_2)\ \phi(\vec{p}_1,\vec{p}_2) \\
H_0\ \phi(\vec{p}_1,\vec{p}_2) &= (\omega(\vec{p}_1)+\omega(\vec{p}_2))\ \phi(\vec{p}_1,\vec{p}_2) \\
\vec{J}\ \phi(\vec{p}_1,\vec{p}_2) &= -i\left(\vec{p}_1\wedge\vec{\nabla}_1+\vec{p}_2\wedge\vec{\nabla}_2\right)\ \phi(\vec{p}_1,\vec{p}_2) \\
\vec{L}_0\ \phi(\vec{p}_1,\vec{p}_2) &= -i\left(\omega(\vec{p}_1)\vec{\nabla}_1+\omega(\vec{p}_2)\vec{\nabla}_2\right)\ \phi(\vec{p}_1,\vec{p}_2)
\end{aligned}
\tag{8}
$$

for almost all $(\vec{p}_1,\vec{p}_2)\in I\!R^6$ and suitable six variable functions ϕ, where $\vec{\nabla}_i := (\partial_{(\vec{p}_i)_1},\partial_{(\vec{p}_i)_2},\partial_{(\vec{p}_i)_3})$, $i\in\{1,2\}$. Because of the properties of the tensor product, (8) defines a representation of the algebra $\mathcal{G}$ which can be integrated to give a unitary and continuous representation of the Poincaré group $\mathcal{P}_+^\uparrow$, in the function space

$$
\mathcal{H} \ := \ L^2_{Sym}(I\!R^6,\sigma_2), \qquad d\sigma_2(\vec{p}_1,\vec{p}_2) := \frac{d\vec{p}_1}{2\omega(\vec{p}_1)}\ \frac{d\vec{p}_2}{2\omega(\vec{p}_2)}
\tag{9}
$$

made of symmetrical functions (i.e. $\phi(\vec{p}_1,\vec{p}_2)=\phi(\vec{p}_2,\vec{p}_1)$). Note that $\sigma_2=\sigma\otimes\sigma$ and recall that $\omega(\vec{p}_i)=\sqrt{\vec{p}_i^{\,2}+m^2}$ for each $i\in\{1,2\}$. $\mathcal{H}$ describes two particles without interaction (therefore we have put an index 0 to H_0 and $\vec{L}_0$).

The Casimir operator of the representation (8) are given by the complicated expressions

$$
M_0^2\ \phi(\vec{p}_1,\vec{p}_2) = \left[(\omega(\vec{p}_1)+\omega(\vec{p}_2))^2 \ - \ (\vec{p}_1+\vec{p}_2)^2\right]\ \phi(\vec{p}_1,\vec{p}_2)
\tag{10}
$$

$$
\begin{aligned}
W_0^2\ \phi(\vec{p}_1,\vec{p}_2) = -&\Big[\left((\vec{p}_1\wedge\vec{p}_2)\cdot(\vec{\nabla}_1-\vec{\nabla}_2)\right)^2 \\
&- \left(\omega(\vec{p}_1)\vec{p}_2\wedge\vec{\nabla}_1+\omega(\vec{p}_2)\vec{p}_1\wedge\vec{\nabla}_2\right)^2\Big]\,\phi(\vec{p}_1,\vec{p}_2).
\end{aligned}
\tag{11}
$$

Now we want to modify these operators in order to introduce the interaction. To remain in the Schrödinger picture we keep $\vec{P}$ and $\vec{J}$ unchanged and modify only H_0 and $\vec{L}_0$. We introduce a symmetric operator $\mathcal{O}$, the *interaction operator*, on which we impose only, for the moment, the formal commutation properties

$$
[\mathcal{O},\vec{P}]=0 \qquad\qquad [\mathcal{O},\vec{J}]=0.
\tag{12}
$$

The interaction representation is defined as follows

$$
\begin{aligned}
&\vec{P},\ \vec{J} && \quad as\ \ in\ \ (8) \\
H &:= && H_0\ +\ \{\mathcal{O},H_0\} \\
\vec{L} &:= && \vec{L}_0\ +\ \{\mathcal{O},\vec{L}_0\}
\end{aligned}
\tag{13}
$$

where we have used the notation $\{A,B\}=AB+BA$. The particular form of the 'interaction parts' of (13) together with (12) have the following nice consequence.

Lemma 1 *The operators* (13) *satisfy formally the commutation rules* (1), (2) *and* (3) *of the algebra* $\mathcal{G}$.

The proof is only a straightforward algebraic calculation; (1) has not been touched; (2) is an immediate consequence of (12); (3) follows from a line of easy calculation. (Note the particular role of (3) : it gives the new operators H, L_1, L_2 after the change of L_3 only.) For more details on the proof, see [3] or [9]. Until now no condition on $\mathcal{O}$, except (12), were needed. But the last commutation rules (4) impose the following complicated equations

$$\begin{aligned} 0 &= [\{\mathcal{O}, H_0\}, L_{0,j}] + [H_0, \{\mathcal{O}, L_{0,j}\}] + [\{\mathcal{O}, H_0\}, \{\mathcal{O}, L_{0,j}\}] && (14) \\ 0 &= [\{\mathcal{O}, L_{0,j}\}, L_{0,k}] + [L_{0,j}, \{\mathcal{O}, L_{0,k}\}] + [\{\mathcal{O}, L_{0,j}\}, \{\mathcal{O}, L_{0,k}\}] && (15) \end{aligned}$$

for all $j \neq k \in \{1,2,3\}$. We get six equations for $\mathcal{O}$ of the same form (made of a linear term plus a bilinear term), which must hold simultaneously. We have now to find solutions to this system.

In order to have a better physical description we make the change of variables $(\vec{p}_1, \vec{p}_2) \to (\vec{P}, \vec{Q})$ given by

$$\begin{aligned} \vec{P} &= \vec{p}_1 + \vec{p}_2 \\ \vec{Q} &= \frac{1}{2}(\vec{p}_1 - \vec{p}_2) - \frac{1}{2}\,\vec{P}\,\frac{\omega(\vec{p}_1) - \omega(\vec{p}_2)}{M_0(\vec{p}_1,\vec{p}_2) + \omega(\vec{p}_1) + \omega(\vec{p}_2)} \end{aligned} \qquad (16)$$

where $M_0(\vec{p}_1,\vec{p}_2) = \left[(\omega(\vec{p}_1)+\omega(\vec{p}_2))^2 - \vec{P}^2\right]^{1/2}$ is the function appearing in (10). The variable $\vec{P}$ is the total momentum while $\vec{Q}$ is the relative momentum, defined as the momentum in a centre-of-mass frame (more precisely, $\vec{Q}$ is the transform of $\frac{1}{2}(\vec{p}_1 - \vec{p}_2)$ under a Lorentz transformation leading to a centre-of-mass frame, see Appendix B for more explanations).

Under this change of variables the free representation (8) becomes

$$\begin{aligned} \vec{P}\,\phi(\vec{P},\vec{Q}) &= \vec{P}\,\phi(\vec{P},\vec{Q}) \qquad \text{(multiplication by the variable } \vec{P}) \\ H_0\,\phi(\vec{P},\vec{Q}) &= \Omega(\vec{P},\vec{Q})\,\phi(\vec{P},\vec{Q}) \\ \vec{J}\,\phi(\vec{P},\vec{Q}) &= \left(\vec{J}^{\vec{P}} + \vec{J}^{\vec{Q}}\right)\phi(\vec{P},\vec{Q}) \\ \vec{L}_0\,\phi(\vec{P},\vec{Q}) &= \left(-i\Omega(\vec{P},\vec{Q})\vec{\nabla}_{\vec{P}} - \frac{\vec{P}\wedge\vec{J}^{\vec{Q}}}{2\omega(\vec{Q}) + \Omega(\vec{P},\vec{Q})}\right)\phi(\vec{P},\vec{Q}) \end{aligned} \qquad (17)$$

for almost all $(\vec{P},\vec{Q}) \in I\!R^6$ and suitable functions ϕ, where we have put

$$\begin{aligned} \Omega(\vec{P},\vec{Q}) &:= \sqrt{\vec{P}^2 + 4\omega(\vec{Q})^2} = \sqrt{\vec{P}^2 + 4\vec{Q}^2 + 4m^2} \\ \vec{J}^{\vec{P}} &:= -i\vec{P}\wedge\vec{\nabla}_{\vec{P}} \\ \vec{J}^{\vec{Q}} &:= -i\vec{Q}\wedge\vec{\nabla}_{\vec{Q}} \end{aligned} \qquad (18)$$

(see the calculation in Appendix B). As for (8), these operators define a representation of the algebra $\mathcal{G}$ which can be integrated to give a unitary and continuous representation of the Poincaré group $\mathcal{P}_+^\uparrow$, in the function space (9), which in these variables becomes

$$\mathcal{H} = L^2_{ev.\vec{Q}}(I\!R^3 \times I\!R^3, \mu), \qquad d\mu(\vec{P},\vec{Q}) := \frac{d\vec{Q}\; d\vec{P}}{2\omega(\vec{Q})\Omega(\vec{P},\vec{Q})} \;=\; \frac{d\sigma(\vec{Q})\; d\vec{P}}{\Omega(\vec{P},\vec{Q})} \tag{19}$$

made of even functions w.r.t. $\vec{Q}$ (i.e. $\phi(\vec{P},\vec{Q}) = \phi(\vec{P},-\vec{Q})$), see Appendix B.

Let us discuss the representation (17). In the non-relativistic case, it is possible to find relative variables which are invariant under the proper Galilean transformations. In the relativistic case we have not managed to find such relative variables, and our Lorentz generators contain a term involving $\vec{J}^{\vec{Q}}$, i.e. acting on $\vec{Q}$. Note that only the angular variables of $\vec{Q}$ are concerned, because $\vec{Q}^2 = \frac{1}{4}\left[(\vec{p}_1-\vec{p}_2)^2 - (\omega(\vec{p}_1) \;-\; \omega(\vec{p}_2))^2\right]$ is an invariant quantity. This last point can also be seen in the Casimir operators, which are now only functions of $\vec{Q}^2$ and $(\vec{J}^{\vec{Q}})^2$:

$$\begin{aligned} M_0^2\; \phi(\vec{P},\vec{Q}) &= 4\omega(\vec{Q})^2\; \phi(\vec{P},\vec{Q}) && (20)\\ W_0^2\; \phi(\vec{P},\vec{Q}) &= 4\omega(\vec{Q})^2\; (\vec{J}^{\vec{Q}})^2\; \phi(\vec{P},\vec{Q}). && (21) \end{aligned}$$

To find solutions of (14), (15) we must write $\mathcal{O}$ more explicitly. The most general form compatible with (12) is

$$\begin{aligned} \mathcal{O}\; \phi(\vec{P},\vec{Q}') &= \int_{I\!R^6} d\mu(\vec{P}',\vec{Q}')\;\; \phi(\vec{P}',\vec{Q}')\; \frac{\eta(\vec{P},\vec{Q},\vec{P}',\vec{Q}')}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P}',\vec{Q}')}\; \delta(\vec{P}-\vec{P}') \\ &= \int_{I\!R^3} \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\; \phi(\vec{P},\vec{Q}')\; \frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}')} \qquad (22) \end{aligned}$$

where $h(\vec{P},\vec{Q},\vec{Q}') = \eta(\vec{P},\vec{Q},\vec{P},\vec{Q}')$ is an arbitrary kernel (for the moment) even w.r.t $\vec{Q}$ and $\vec{Q}'$, that is $h(\vec{P},\vec{Q},\vec{Q}') = h(\vec{P},-\vec{Q},\vec{Q}') = h(\vec{P},\vec{Q},-\vec{Q}')$, and satisfying the symmetry condition

$$h(\vec{P},\vec{Q},\vec{Q}')^* \;=\; h(\vec{P},\vec{Q}',\vec{Q}) \tag{23}$$

where the star * means complex conjugation, all these conditions being necessary for $\mathcal{O}$ to be a symmetric operator on $\mathcal{H}$. The Dirac δ pseudo-function assures that $\mathcal{O}$ (formally) commutes with $\vec{P}$ and the last condition of (12) leads (formally) to

$$\left(\vec{J}^{\vec{P}} + \vec{J}^{\vec{Q}} + \vec{J}^{\vec{Q}'}\right)\; h(\vec{P},\vec{Q},\vec{Q}') \;=\; 0. \tag{24}$$

The denominateur in (22) is for h to be the kernel of the interaction part of H. In fact, the Hamiltonian (13) can be written simply

$$H\; \phi(\vec{P},\vec{Q}') \;=\; \Omega(\vec{P},\vec{Q})\; \phi(\vec{P},\vec{Q}) \;+\; \int_{I\!R^3} \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\; \phi(\vec{P},\vec{Q}')\; h(\vec{P},\vec{Q},\vec{Q}') \tag{25}$$

(note that h has no physical dimension) while $\vec{L}$ gives the complicated formula

$$\begin{aligned}
\vec{L}\,\phi(\vec{P},\vec{Q}') &= \vec{L}_0\phi(\vec{P},\vec{Q}) + \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\left[\vec{L}_0\phi(\vec{P},\vec{Q}')\right]\frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')} \\
&- i\vec{P}\int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\,\phi(\vec{P},\vec{Q}')\left[\frac{Dh(\vec{P},\vec{Q},\vec{Q}')}{\|\vec{P}\|\Omega(\vec{P},\vec{Q}')} - \frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')^2}\right] \\
&+ \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\,\phi(\vec{P},\vec{Q}') \\
&\times \frac{\vec{P}\wedge\left[\dfrac{\vec{J}^{\vec{Q}}}{2\omega(\vec{Q})+\Omega(\vec{P},\vec{Q})} + \dfrac{\vec{J}^{\vec{Q}'}}{2\omega(\vec{Q}')+\Omega(\vec{P},\vec{Q}')}\right]}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}')}\,h(\vec{P},\vec{Q},\vec{Q}').
\end{aligned} \tag{26}$$

The substitution of (22) in (14), (15) leads to six non-linear, integro-differential equations for h. Six equations for one unknown function. We obtain easily non-trivial solutions $h \neq 0$ satisfying the special condition

$$\vec{J}^{\vec{Q}}\,h(\vec{P},\vec{Q},\vec{Q}') = 0 \tag{27}$$

which leads to $\vec{J}^{\vec{P}}h = \vec{J}^{\vec{Q}'}h = 0$ because of (21) and (22). So (27) is equivalent to the requirement that $h(\vec{P},\vec{Q},\vec{Q}')$ is function of the norms $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$ only. In this case the six equations (14), (15) gives six times the same equation, that we write now. We first introduce the differential operator D given by

$$Dh(\vec{P},\vec{Q},\vec{Q}') := \frac{\Omega(\vec{P},\vec{Q})\ \Omega(\vec{P},\vec{Q}')}{\Omega(\vec{P},\vec{Q})\ +\ \Omega(\vec{P},\vec{Q}')}\,\partial_{\|\vec{P}\|}\,h(\vec{P},\vec{Q},\vec{Q}'). \tag{28}$$

Appendix C shows that (14), (15) leads to the single equation

$$\begin{aligned}
0 &= 2Dh(\vec{P},\vec{Q},\vec{Q}') + \int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P},\vec{Q}'')^2}\left\{-\frac{\|\vec{P}\|}{\Omega(\vec{P},\vec{Q}'')}\,h(\vec{P},\vec{Q},\vec{Q}'')h(\vec{P},\vec{Q}'',\vec{Q}')\right. \\
&+ \left. Dh(\vec{P},\vec{Q},\vec{Q}'')h(\vec{P},\vec{Q}'',\vec{Q}') + h(\vec{P},\vec{Q},\vec{Q}'')Dh(\vec{P},\vec{Q}'',\vec{Q}')\right\}.
\end{aligned} \tag{29}$$

This is the fundamental equation which guarantiees the relativistic structure of the theory. Let us sum up what we have found in a proposition.

Proposition 2 *Let $h(\vec{P},\vec{Q},\vec{Q}')$ be a function of $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$ only which satisfies* (23) *and* (29)*. Then the operators* (13)*, with $\mathcal{O}$ given by* (22)*, are symmetric and satisfy algebraically the commutation rules* (1) *to* (4) *of the algebra $\mathcal{G}$.*

3. Existence of solutions of the fundamental equation

We give the complete set of solutions of (29) lying in a ball of a Banach space, neglecting for the moment the conditions (23) and (27). Then we study the analytic properties of a particular subset of solutions.

Definition 1 Let $\mathcal{K}$ be a compact set of $I\!R^3$ with Lebesgue-measure $|\mathcal{K}| \neq 0$. We denote by $\mathcal{B}$ the Banach space of continuous and bounded functions $h(\vec{P},\vec{Q},\vec{Q}')$ with support in $(\vec{P},\vec{Q},\vec{Q}') \in I\!R^3 \times \mathcal{K} \times \mathcal{K}$ for which $Dh(\vec{P},\vec{Q},\vec{Q}')$ exists and is also bounded and continuous, given the norm

$$|h|_{\mathcal{B}} \;:=\; \|h\|_\infty + \|Dh\|_\infty \tag{30}$$

where the differential operator D is given by (28).

The following result assures the existence of a large class of solutions of (29).

Proposition 3 *There exists* $0 < K_1 < \infty$ *such that, for all* $c \in C_0^0(\mathcal{K}\times\mathcal{K})$ *satisfying* $\|c\|_\infty < K_1$, *there exists one and only one* $h \in \mathcal{B}$ *satisfying*

1) $|h|_{\mathcal{B}} < 2K_1$
2) $h(0,\vec{Q},\vec{Q}') = c(\vec{Q},\vec{Q}')$ *for all* $\vec{Q},\vec{Q}' \in I\!R^3$
3) $h(\vec{P},\vec{Q},\vec{Q}')$ *satisfies* (29) *for all* $\vec{P},\vec{Q},\vec{Q}' \in I\!R^3$.

The proposition gives the complete set of solutions h of (29) lying in a certain ball of $\mathcal{B}$; they are uniquely defined by their restriction to $\vec{P} = 0$, which can be chosen arbitrarily in a certain ball of $C_0^0(\mathcal{K}\times\mathcal{K})$. This analytic condition is the only restriction on h for $\vec{P} = 0$, that is, in a centre-of-mass frame.

Proof. Let us introduce a bilinear operator b on suitable functions f, g

$$\begin{aligned} &b(g,h)(\vec{P},\vec{Q},\vec{Q}') \;:= \\ -\;&\frac{1}{2}\int_0^{\|\vec{P}\|} d\|\vec{\zeta}\| \left(\frac{1}{\Omega(\vec{\zeta},\vec{Q})} + \frac{1}{\Omega(\vec{\zeta},\vec{Q}')}\right) \int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{\zeta},\vec{Q}'')^2} \Big\{ Dg(\vec{\zeta},\vec{Q},\vec{Q}'')h(\vec{\zeta},\vec{Q}'',\vec{Q}') \\ &+ g(\vec{\zeta},\vec{Q},\vec{Q}'')Dh(\vec{\zeta},\vec{Q}'',\vec{Q}') \;-\; \frac{\|\vec{\zeta}\|}{\Omega(\vec{\zeta},\vec{Q}'')}\, g(\vec{\zeta},\vec{Q},\vec{Q}'')h(\vec{\zeta},\vec{Q}'',\vec{Q}') \Big\}. \end{aligned} \tag{31}$$

for all $(\vec{P},\vec{Q},\vec{Q}') \in I\!R^9$. By applying the derivation operator D (given by (28)), with right derivation at 0, we get simply

$$\begin{aligned} Db(g,h)(\vec{P},\vec{Q},\vec{Q}') \;&=\; -\frac{1}{2}\int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P},\vec{Q}'')^2} \Big\{ -\frac{\|\vec{P}\|}{\Omega(\vec{P},\vec{Q}'')}\, f(\vec{P},\vec{Q},\vec{Q}'')g(\vec{P},\vec{Q}'',\vec{Q}') \\ +\quad &Dg(\vec{P},\vec{Q},\vec{Q}'')h(\vec{P},\vec{Q}'',\vec{Q}') \;+\; g(\vec{P},\vec{Q},\vec{Q}'')Dh(\vec{P},\vec{Q}'',\vec{Q}') \Big\}. \end{aligned} \tag{32}$$

The fundamental equation (29) can now be written $Dh - Db(h,h) = 0$. By integration this equation becomes

$$h(\vec{P},\vec{Q},\vec{Q}') \;=\; c(\vec{Q},\vec{Q}') \;+\; b(h,h)(\vec{P},\vec{Q},\vec{Q}') \;:=\; A(h)(\vec{P},\vec{Q},\vec{Q}') \tag{33}$$

where $c(\vec{Q},\vec{Q}')$ is an arbitrary function (the integration 'constant'). This is a fixed point equation which can be solved by applying the Banach fixed point theorem [11, Sec 1.1] in the Banach space $\mathcal{B}$. Obviously $b : \mathcal{B} \to \mathcal{B}$ and satisfies

$$|b(g,h)|_{\mathcal{B}} \;<\; K_1' \; |g|_{\mathcal{B}} \; |h|_{\mathcal{B}}. \tag{34}$$

where

$$\begin{aligned} K_1' &= \frac{1}{2}\int_0^\infty \frac{2\, d\zeta}{\sqrt{\zeta^2+4m^2}} \int_{\mathcal{K}} \frac{d\sigma(\vec{Q}'')}{\zeta^2+4\vec{Q}''^2+4m^2} \;+\; \frac{1}{2}\int_{\mathcal{K}} \frac{d\sigma(\vec{Q}'')}{4(\vec{Q}''^2+m^2)} \\ &< \frac{|\mathcal{K}|}{4m^3}. \end{aligned} \tag{35}$$

Let us take $K_1 = (4K_1')^{-1}$ and $c(\vec{Q},\vec{Q}') \in \mathcal{B}$ with $|c|_{\mathcal{B}} = \|c\|_\infty < K_1$. Then the operator $A(h)$ sends the ball $B(2K_1) := \{g \in \mathcal{B} \mid |g|_{\mathcal{B}} < 2K_1\}$ into itself and is a strict contraction on $B(2K_1)$ (see [3] or [9] for more details). In such a case, it follows from the Banach fixed point theorem that the equation $h = A(h)$ has one and only one solution in $B(2K_1)$ given by the convergent limit

$$h \;=\; \lim_{n\to\infty} \; A^n(0). \qquad [] \tag{36}$$

The previous proof constructs the solutions h in terms of their restriction to $\vec{P} = 0$, denoted by c, as the limit of a convergent sequence in $\mathcal{B}$, provided $\|c\|_\infty < m^3/|\mathcal{K}|$ (because the proof shows that we can take $K_1 = m^3/|\mathcal{K}|$). Let us compute the first terms of the sequence (36):

$$\begin{aligned} A(0) &= c \\ A^2(0) &= c \;+\; b(A(0),A(0)) \;=\; c \;+\; b(c,c) \\ A^3(0) &= c \;+\; b(A^2(0),A^2(0)) \\ &= c \;+\; b(c,c) \;+ b(b(c,c),c) \;+\; b(c,b(c,c)) \;+\; b(b(c,c),b(c,c)) \\ &\mathit{etc...} \end{aligned} \tag{37}$$

The limit can be seen as a convergent series in powers of c. More precisely, by replacing c by λc in the definition of A (33), the limit (36) becomes a power series of λ

$$h \;=\; \sum_{n=0}^{\infty} \lambda^n \; h_n \tag{38}$$

converging for $-1 < \lambda \le 1$, where h_n is the sum of all terms in (36) of degree n in c. By going on the calculation (37) we see that h_n is given by the sum of all different ways of writing $n-1$ bilinear functions b applied on n functions c. For example,

$$\begin{aligned} h_3 &= b(b(c,c),c) \;+\; b(c,b(c,c)) \\ h_4 &= b(c,b(c,b(c,c))) \;+\; b(c,b(b(c,c),c)) \;+\; b(b(c,c),b(c,c)) \\ &\qquad + b(b(c,b(c,c)),c) \;+\; b(b(b(c,c),c),c) \end{aligned} \tag{39}$$

and so on.

We have now to extract from this large set of solutions a subset which realize what we want to construct: unitary representations of the Poincaré group describing two particles with effective interaction. For that purpose we restrict us to the following class of functions.

Definition 2. Let $\mathcal{K}$ be the ball of $I\!R^3$ of centre 0 and radius $R > 0$. Let $\mathcal{V}$ be the set of functions $c(\vec{Q}, \vec{Q}') \in C_0^2(\mathcal{K} \times \mathcal{K})$ which satisfy $c(\vec{Q}, \vec{Q}')^* = c(\vec{Q}', \vec{Q})$, which depend only on the norms $\|\vec{Q}\|$ and $\|\vec{Q}'\|$, and such that

$$\|c\|_\infty \quad \leq \quad \frac{m^3}{2\,|\mathcal{K}|}. \tag{40}$$

Note that $\mathcal{V}$ depends on the two positive parameters m and R.

Each $c \in \mathcal{V}$ satisfies the hypothesis of proposition 3 and moreover satisfies

$$\|c\|_{L^2(I\!R^6)} \quad \leq \quad \frac{m^3}{2} \tag{41}$$

where $L^2(I\!R^6)$ refers to the ordinary Lebesgue measure (not the σ_2 measure). The following set of properties hold on $\mathcal{V}$.

Proposition 4 *Let $c \in \mathcal{V}$. The function h deduced from c by* (35) *(or* (36)*) satisfies:*

1) *h satisfies* (23)*,*

2) *$h(\vec{P}, \vec{Q}, \vec{Q}')$ depends only on the norms $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$,*

3) *for all $\vec{Q}, \vec{Q}' \in I\!R^3$, the function $\vec{P} \to h(\vec{P}, \vec{Q}, \vec{Q}')$ is a function of $\vec{P}^2$ analytic in a $\mathbb{C}$-neighbourhood of $I\!R_+$,*

4) *for all $\vec{P} \in I\!R^3$, the function $\vec{Q}, \vec{Q}' \to h(\vec{P}, \vec{Q}, \vec{Q}')$ has compact support and belongs to $C^2(I\!R^6)$,*

5) *the partial derivatives of $h(\vec{P}, \vec{Q}, \vec{Q}')$, of any order w.r.t. $\vec{P}^2$ and of order ≤ 2 w.r.t. $\vec{Q}, \vec{Q}'$, belong to $\mathcal{B}$,*

6) *h and Dh satisfy the estimates*

$$\begin{aligned}
\sup_{\vec{P} \in I\!R^3} \int_{I\!R^6} d\vec{Q} d\vec{Q}' |h(\vec{P}, \vec{Q}, \vec{Q}')|^2 \quad &\leq \quad \left(\frac{2}{3}\right)^2 m^6 \\
\sup_{\vec{P} \in I\!R^3} \int_{I\!R^6} d\vec{Q} d\vec{Q}' |Dh(\vec{P}, \vec{Q}, \vec{Q}')|^2 \quad &\leq \quad \left(\frac{1}{20}\right)^2 m^6.
\end{aligned} \tag{42}$$

Proof. First we establish the convergence of $h = \sum_{n=0}^{\infty} h_n$, where the h_n are introduced by (38), by another way. By the estimate on b given by (34) follows that $|h_n|_{\mathcal{B}} < \mathcal{N}_n K_1'^{n-1} |c|_{\mathcal{B}}^n$, where $\mathcal{N}_n$ is the number of terms of h_n. To get an estimate on $\mathcal{N}_n$ we consider the trivial equation : $x = u + x^2$ with $u \in \mathbb{C}$ and small $|u|$, which is a caricature of the equation (33). The fixed-point theorem leads to a solution $x = \sum_{n=1}^{\infty} \mathcal{N}_n u^n$, convergent for $|u| < 1/4$ and unique in the disc $|x| < 1/2$, where the $\mathcal{N}_n$ are given by $\mathcal{N}_1 = 1$ and $\mathcal{N}_n = \sum_{i=1}^{n-1} \mathcal{N}_i \mathcal{N}_{n-i}$ for all $n > 1$. On the other hand, an elementary calculation gives $x = \frac{1}{2}(1 - \sqrt{1-4u})$ for $|u| \le 1/4$ as the unique solution near the origin. By expansion of this solution in power series we obtain explicit expressions for $\mathcal{N}_n$, from which follows easily the estimate $\mathcal{N}_n < 4^n$ for all $n \ge 1$. By summing up these estimates we get

$$|h_n|_{\mathcal{B}} \ < \ {K'}_1^{n-1} \ \mathcal{N}_n \ |c|^n_{\mathcal{B}} \ < \ \frac{1}{K'_1}(4K'_1|c|_{\mathcal{B}})^n \tag{43}$$

which assures the convergence of $h = \sum_{n=0}^{\infty} h_n$ provided $4K'_1|c|_{\mathcal{B}} < 1$, in agreement with the proof of proposition 3.

Proof of 1). Because $A(0) = c$ we have $A(0)(\vec{Q}, \vec{Q}') = A(0)(\vec{Q}', \vec{Q})^*$. If $g \in \mathcal{B}$ satisfies $g(\vec{P}, \vec{Q}, \vec{Q}') = g(\vec{P}, \vec{Q}', \vec{Q})^*$, so does $b(g,g)$ and $A(g)$. By an induction argument it follows that $A^n(0)$ satisfies this symmetry for all n, and so does the uniform limit $lim_{n\to\infty} A^n(0)$.

Proof of 2). If $f(\vec{P}, \vec{Q}, \vec{Q}')$, $g(\vec{P}, \vec{Q}, \vec{Q}')$ are only function of $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$, so are $b(f,g)(\vec{P}, \vec{Q}, \vec{Q}')$ and $Db(f,g)(\vec{P}, \vec{Q}, \vec{Q}')$. Thus, if $c(\vec{Q}, \vec{Q}')$ is only function of $\|\vec{Q}\|$, $\|\vec{Q}'\|$, then $h_n(\vec{P}, \vec{Q}, \vec{Q}')$ is only function of $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$ for all n. This property passes obviously to the uniform limit $h = lim_{N\to\infty} \sum_{n=0}^{N} h_n$.

Proof of 3). Let us fix $(\vec{Q}, \vec{Q}') \in I\!R^6$ and put $x = \|\vec{P}\|$. In h_n, x appears only in kernels $\sqrt{x^2 + 4\omega(\vec{\xi})^2}$ or as limit of integrals $\int_0^x dy$ over a variable y appearing only in kernels $\sqrt{y^2 + 4\omega(\vec{\chi})^2}$ or as limit of integrals $\int_0^y dz$, and so on, for various variables $\vec{\xi}, \vec{\chi}$, etc... All these kernels are analytic in a strip $\{x \in \mathbb{C} | \ |Imx| < 2m\}$. Moreover the derivation of $(x^2 + 4\omega(\vec{\xi})^2)^{k/2}$, for any $k \in \mathbb{Z}$, gives a function bounded by a constant times $(x^2 + 4\omega(\vec{\xi})^2)^{k/2}$. Thus the integrability is preserved. From all these properties follows that $h_n(\vec{P}, \vec{Q}, \vec{Q}')$ is an analytic function of $\|\vec{P}\|$ in the strip given above, for all $(\vec{Q}, \vec{Q}') \in I\!R^6$.

To study the convergence for $n \to \infty$ we note that $\partial^j_{\|\vec{P}\|} h_n$ satisfies the bound (43) multiplied by a constant due to the derivations of the kernels, and by the factor $(n-1)^j$ for the number of terms caused by j derivations, according to the Leibniz formula. These new factors have no consequence on the convergence, and thus $\partial^j_{\|\vec{P}\|} h_n$ converges to an element of $\mathcal{B}$, which is nothing else than $\partial^j_{\|\vec{P}\|} h$ because of the uniform convergence.

The analyticity is obtained by giving a small fixed imaginary part to $\|\vec{P}\|$, taking $\|\vec{P}\| + i\eta$ in the definition of b (31), with $|\eta|$ small enough to have again the estimate (34). The same argument as before, based on (43) which is still valid, shows the convergence of h_n and $\partial^j_{\|\vec{P}\|} h_n$, for all n, uniformly in η.

To see that h is an analytic function of $\vec{P}^2$ (and not only on $\|\vec{P}\|$) we note that, if $f(\vec{P}, \vec{Q}, \vec{Q}')$, $g(\vec{P}, \vec{Q}, \vec{Q}') \in \mathcal{B}$, functions of $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$ only, are even in $\|\vec{P}\|$, so is $b(f,g)(\vec{P}, \vec{Q}, \vec{Q}')$. A glance to (37) suffices to convince ourselves that all the terms of h_n, and thus the limit h itself, are even functions of $\|\vec{P}\|$.

Proof of 4). The functions h_n can be written as

$$h_n(\vec{P}, \vec{Q}, \vec{Q}') \ = \ \int d\sigma(\vec{Q}''_1) \cdots d\sigma(\vec{Q}''_{n-1}) \ c(\vec{Q}, \vec{Q}''_1) \ c(\vec{Q}''_1, \vec{Q}''_2) \cdots$$

$$\cdots c(\vec{Q}''_{n-1},\vec{Q}')\ \Theta_n(\vec{P},\vec{Q},\vec{Q}';\vec{Q}''_1,\ldots,\vec{Q}''_{n-1}) \tag{44}$$

where the kernels Θ_n can be deduced from (36), (31), (32) by an easy (and tedius) calculation. This expression shows clearly that the support property of c leads to a support property for all h_n, and thus for h, w.r.t. $\vec{Q},\vec{Q}'$.

For $j=1$ or 2, the jth derivation of h_n w.r.t. $\vec{Q}$, $\vec{Q}'$ is bounded by the estimate (43) multiplied by $(\|\partial^j c\|_\infty/\|c\|_\infty)^j$, by a constant due to the derivations of the kernels $\sqrt{x^2+4\vec{Q}^2+4m^2}$, $\sqrt{y^2+4\vec{Q}'^2+4m^2}$ (for various variables x, y) w.r.t. $\vec{Q}$, $\vec{Q}'$ this time, and the factor $(n-1)^j$ for the number of terms caused by j derivations, according to the Leibniz formula. As before, these new factors have no consequence on the convergence, and then these functions converges to an element of $\mathcal{B}$, which is nothing else than the corresponding derivation of h.

Proof of 5). We have already seen in the proofs of 2) and 3) that the derivation of h at any order w.r.t. $\|\vec{P}\|$ and at order ≤ 2 w.r.t. $\vec{Q},\vec{Q}'$, belong to $\mathcal{B}$. The derivations w.r.t. $\vec{P}^2$ are bounded by the derivations w.r.t. $\|\vec{P}\|$ for large $\|\vec{P}\|$; for small $\|\vec{P}\|$ the boundedness is assured by the analyticity of h w.r.t. $\vec{P}^2$.

Proof of 6). Let us introduce the norm

$$\|f\|_* \;=\; m^{-3}\sup_{\vec{P}\in I\!R^3}\sqrt{\int_{I\!R^6} d\vec{Q}\ d\vec{Q}'\ |f(\vec{P},\vec{Q},\vec{Q}')|^2} \tag{45}$$

for all $f\in\mathcal{B}$. From an easy estimate involving the Cauchy-Schwartz inequality we get

$$\begin{aligned}\|b(f,f)\|_* &< \frac{\sqrt{3}}{8}\left(\|f\|_*^2+\|Df\|_*^2\right)\\ \|Db(f,f)\|_* &< \frac{\sqrt{3}}{16}\left(\|f\|_*^2+\|Df\|_*^2\right)\end{aligned} \tag{46}$$

for all $f\in\mathcal{B}$. Now recall that $h=\lim_{n\to\infty}A^n$. Let us suppose that for some $n\in I\!N$ we know that $\|A^n\|_*\leq 2/3$ and $\|DA^n\|_*\leq 1/20$ (because of (41) this is true for $n=0$). Then from (41) and (46) we obtain

$$\begin{aligned}\|A^{n+1}\|_* &= \|c+b(A^n,A^n)\|_* \;\leq\; \|c\|_*+\|b(A^n,A^n)\|_*\\ &\leq \frac{1}{2}+\frac{\sqrt{3}}{8}\,(\|A^n\|_*^2+\|DA^n\|_*^2) \;\leq\; \frac{1}{2}+\frac{\sqrt{3}}{8}\left(\frac{4}{9}+\frac{1}{20^2}\right) \;<\; \frac{2}{3}\\ \|DA^{n+1}\|_* &= \|Db(A^n,A^n)\|_* \;\leq\; \frac{\sqrt{3}}{16}\,(\|A^n\|_*^2+\|DA^n\|_*^2)\\ &\leq \frac{\sqrt{3}}{16}\left(\frac{4}{9}+\frac{1}{20^2}\right) \;<\; \frac{1}{20}.\end{aligned}$$

It follows that these bounds hold for all n. Because the uniform convergence of A^n and DA^n for $n\to\infty$, the estimates (42) follow now simply from the Riemann theorem. []

4. Unitarity representations of the Poincaré group

Until now, the representations (13) of the algebra $\mathcal{G}$ were only formal, in the sense that the domains of the operators and of the commutators were not precised. The proposition 4 brings the results which allow us to show that these algebra representations are well defined and can be integrated to give unitary continuous representations of $\mathcal{P}_+^\uparrow$.

Theorem 5 *Let $c \in \mathcal{V}$. Let $\mathcal{O}$ be the interaction operator* (22) *with the kernel h deduced from c in Proposition* 3. *Then $\{H, \vec{P}, \vec{J}, \vec{L}\}$, given by* (13)*, are the generators of a unitary continuous representation of $\mathcal{P}_+^\uparrow$.*

Proof. According to the Theorem 5 of [12], H, $\vec{P}$, $\vec{J}$, $\vec{L}$ are the generators of a unitary continuous representation of $\mathcal{P}_+^\uparrow$ if the following three conditions are satisfied: 1) they are self-adjoint, 2) they admit a common invariant dense domain $\mathcal{D}$ on which the commutation rules of the algebra hold, 3) $\mathcal{D}$ is a domain of essential self-adjointness for the operator $\Delta = m^{-2}(H^2 + \vec{P}^2) + \vec{J}^2 + \vec{L}^2$. This scheme gives the three natural steps of the proof.

1st step: proof that H, $\vec{P}$, $\vec{J}$, $\vec{L}$ are self-adjoint. It is clear for $\vec{P}$ and $\vec{J}$. To study H and $\vec{L}$ we need a general estimate of the norm of vectors like

$$\phi_\xi(\vec{P}, \vec{Q}) \;:=\; \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P}, \vec{Q}')}\; \phi(\vec{P}, \vec{Q}')\; \xi(\vec{P}, \vec{Q}, \vec{Q}')$$

for $\phi \in \mathcal{H}$ and a suitable kernel ξ. By using the Cauchy-Schwarz inequality and the Fubini theorem we obtain

$$\begin{aligned}
\|\phi_\xi\|^2 \;&=\; \int \frac{d\vec{P} d\sigma(\vec{Q})}{\Omega(\vec{P}, \vec{Q})} \left| \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P}, \vec{Q}')}\; \phi(\vec{P}, \vec{Q}')\; \xi(\vec{P}, \vec{Q}, \vec{Q}') \right|^2 \\
&\leq\; \int \frac{d\vec{P} d\sigma(\vec{Q})}{\Omega(\vec{P}, \vec{Q})} \left[\int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P}, \vec{Q}')} \left|\phi(\vec{P}, \vec{Q}')\right|^2 \right] \left[\int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P}, \vec{Q}'')} \left|\xi(\vec{P}, \vec{Q}, \vec{Q}'')\right|^2 \right] \\
&\leq\; \|\!|\xi|\!\|^2\; \|\phi\|^2
\end{aligned} \tag{47}$$

where

$$\|\!|\xi|\!\|^2 \;:=\; \sup_{\vec{P} \in I\!R^3} \int \frac{d\sigma(\vec{Q})}{\Omega(\vec{P}, \vec{Q})} \int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P}, \vec{Q}'')}\; \left|\xi(\vec{P}, \vec{Q}, \vec{Q}'')\right|^2 . \tag{48}$$

For H, given by (25), we find the bound on the operator norm $\|H - H_0\|_{op} \leq \|\!|h|\!\|$. ¿From (42) follows : $\|\!|h|\!\| \leq m/6$. Thus $H - H_0$ is a bounded operator, and H is self-adjoint on the domain of H_0.

For $\vec{L}$ we consider (26) without the last term (because $\vec{J}^{\vec{Q}} h = \vec{J}^{\vec{Q}'} h = 0$). The estimate (47) leads to

$$\|(L_j - L_{0,j})\phi\| \;\leq\; \|\!|h/\Omega|\!\|\; \|L_{0,j}\phi\| \;+\; (\|\!|h/\Omega|\!\| + \|\!|Dh/\Omega|\!\|)\; \|\phi\| \tag{49}$$

for all $j \in \{1,2,3\}$. From the Kato-Rellich theorem (section X.2 of [13]) follows that $\vec{L}$ is self-adjoint on the domain of $\vec{L}_0$ provided $\|h/\Omega\| < 1$. Using the previous estimates we get $\|h/\Omega\| \leq \|h\|/2m < 1/12 < 1$. (Note that the condition $\|h\|/2m < 1$ implies the operator norm bound $\|H - H_0\|_{op} < 2m$ and then the positivity of H).

2nd step: the invariant domain and the commutation rules. Let $\mathcal{D}$ be the following domain

$$\begin{aligned} \mathcal{D} \;=\; & \Big\{ \phi \in \mathcal{H} \;\Big|\; \Omega(\vec{P},\vec{Q})^{\ell}\; P_1^{n_1} P_2^{n_2} P_3^{n_3}\; L_{0,1}^{m_1} L_{0,2}^{m_2} L_{0,3}^{m_3}\; J_1^{j_1} J_2^{j_2} J_3^{j_3} \;\; \phi(\vec{P},\vec{Q}) \in \mathcal{H} \\ & \qquad \text{for all } \;\ell, n_1, n_2, n_3, m_1, m_2, m_3, j_1, j_2, j_3 \in I\!N \Big\}. \end{aligned} \tag{50}$$

$\mathcal{D}$ is dense (because it contains $C_0^\infty(I\!R^6)$) and is clearly left invariant by H_0, $\vec{P}$, $\vec{J}$ and $\vec{L}_0$. To show that it is also invariant under H and $\vec{L}$ it is sufficient to show that it is invariant under $\mathcal{O}$. Let $\phi \in \mathcal{D}$ and let us apply an operator $\Omega(\vec{P},\vec{Q})^{\ell} P_1^{n_1} P_2^{n_2} P_3^{n_3} L_{0,1}^{m_1} L_{0,2}^{m_2} L_{0,3}^{m_3} J_1^{j_1} J_2^{j_2} J_3^{j_3}$ to the vector $\mathcal{O}\phi$. ¿From 2), proposition 4, we get 0 unless $j_1 = j_2 = j_3 = 0$. In that case, from the Leibniz rule we find

$$\begin{aligned} & \Omega(\vec{P},\vec{Q})^{\ell}\; P_1^{n_1} P_2^{n_2} P_3^{n_3}\; L_{0,1}^{m_1} L_{0,2}^{m_2} L_{0,3}^{m_3}\; \mathcal{O}\phi(\vec{P},\vec{Q}) \;\; = \\ & \sum_{\vec{\alpha}+\vec{\beta}=\vec{m}} C_{\vec{\alpha},\vec{\beta}} \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')} \;\Big[\Omega(\vec{P},\vec{Q}')^{\ell+\beta_1+\beta_2+\beta_3+1} P_1^{n_1} P_2^{n_2} P_3^{n_3} L_{0,1}^{\alpha_1} L_{0,2}^{\alpha_2} L_{0,3}^{\alpha_3}\; \phi(\vec{P},\vec{Q}') \Big] \\ & \quad \times \; \left\{ \left(\frac{\Omega(\vec{P},\vec{Q})}{\Omega(\vec{P},\vec{Q}')} \right)^{\ell+\beta_1+\beta_2+\beta_3} \partial_{P_1}^{\beta_1} \partial_{P_2}^{\beta_2} \partial_{P_3}^{\beta_3} \frac{h(\vec{P},\vec{Q},\vec{Q}')}{} \right. \end{aligned}$$

$\Omega(\vec{P},\vec{Q}')(\Omega(\vec{P},\vec{Q}) + \Omega(\vec{P},\vec{Q}'))$(51)for some coefficients $C_{\vec{\alpha},\vec{\beta}}$. Because $\phi \in \mathcal{D}$ the expression in [ ] is a vector of $\mathcal{H}$. By using the estimate (47) we find that (51) belongs to $\mathcal{H}$ if the expression in { } has well defined $\|.\|$ norm. It is the case because of the points 3), 4) and 5) of proposition 4), because the derivations of $\Omega(\vec{P},\vec{Q}')^{-1}(\Omega(\vec{P},\vec{Q}) + \Omega(\vec{P},\vec{Q}'))^{-1}$ w.r.t. $\vec{P}$ give bounded functions, and because

$$\frac{\Omega(\vec{P},\vec{Q})}{\Omega(\vec{P},\vec{Q}')} \;\leq\; \frac{\|\vec{P}\|}{\Omega(\vec{P},\vec{Q}')} \;+\; \frac{2\omega(\vec{Q})}{\Omega(\vec{P},\vec{Q}')} \;\leq\; 1 \;+\; \frac{\omega(\vec{Q})}{m} \tag{52}$$

gives a bounded contribution, due to the compact support of $\vec{Q} \mapsto h(\vec{P},\vec{Q},\vec{Q}')$. Thus $\mathcal{D}$ is an invariant domain for all operators H, $\vec{P}$, $\vec{J}$ and $\vec{L}$.

By working on $\mathcal{D}$ the commutation rules for $\{H, \vec{P}, \vec{J}, \vec{L}\}$ can be deduced from those for $\{H_0, \vec{P}, \vec{J}, \vec{L}_0\}$ without having to take care of their domain. Thus we have only to do algebraic calculations, and thus to apply proposition 2.

3th step: proof that $\mathcal{D}$ is a domain of essential self-adjointness for the operator Δ. We show that Δ is self-adjoint, using theorems by Nelson and Kato-Rellich. Because $\{\vec{P}, H_0, \vec{J}, \vec{L}_0\}$ are the infinitesimal generators of a unitary continuous representation of a Lie group it follows from a theorem by Nelson (see [12], theorem 3) that all these operators and $\Delta_0 = m^{-2}(H_0^2 + \vec{P}^2) + \vec{J}^2 + \vec{L}_0^2$ are self-adjoint and that there exists a domain $\mathcal{D}_0$ which is a common invariant domain and a common core for all of them. However the largest common invariant domain is given by (50), thus $\mathcal{D}$ is also a common core. By the Kato-Rellich theorem (Theorem X.12 of [13]), Δ is essentially self-adjoint on $\mathcal{D}$ if

$$\|(\Delta - \Delta_0)\phi\| \;\; \leq \;\; k_1\|\Delta_0\phi\| \; + \; k_2\|\phi\| \tag{53}$$

for all $\phi \in \mathcal{D}$, for some $0 < k_1 < 1$ and $0 < k_2 < \infty$.

The bound (53) concerns the interaction terms of H and $\vec{L}$ because

$$\Delta - \Delta_0 \;=\; m^{-2}(H^2 \; - \; H_0^2) \; + \; \vec{L}^2 \; - \; \vec{L}_0^2. \tag{54}$$

Now, for all $\phi \in \mathcal{D}$

$$(H^2 \; - \; H_0^2)\; \phi(\vec{P},\vec{Q}) \;=\; \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\; \phi(\vec{P},\vec{Q}')\; \Xi(\vec{P},\vec{Q},\vec{Q}') \tag{55}$$

where

$$\begin{aligned} \Xi(\vec{P},\vec{Q},\vec{Q}') \;\; &:= \;\; \Big(\Omega(\vec{P},\vec{Q}) \; + \; \Omega(\vec{P},\vec{Q}')\Big)\; h(\vec{P},\vec{Q},\vec{Q}') \\ &+ \;\; \int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P},\vec{Q}'')}\; h(\vec{P},\vec{Q}',\vec{Q}'')\; h(\vec{P},\vec{Q}'',\vec{Q}') \end{aligned} \tag{56}$$

¿From (47) follows $\|(H^2 - H_0^2)\phi\| \leq |\!|\!|\Xi|\!|\!| \; \|\phi\|$, provided $|\!|\!|\Xi|\!|\!|$ is defined, which gives a contribution to the second term of (53) only. To see the existence of $|\!|\!|\Xi|\!|\!|$ we use first (52) to obtain

$$\frac{\Big(\Omega(\vec{P},\vec{Q}) + \Omega(\vec{P},\vec{Q}')\Big)^2}{2\omega(\vec{Q})\Omega(\vec{P},\vec{Q})\; 2\omega(\vec{Q}')\Omega(\vec{P},\vec{Q}')} \;\; < \;\; \frac{3}{2m^2} \tag{57}$$

which gives $|\!|\!|(\Omega + \Omega')h|\!|\!| < m^2\sqrt{3/2}|\!|\!|h|\!|\!|_*$ (which is bounded by $m^2\sqrt{2/3}$ from (42)) and then the Cauchy-Schwartz inequality to get $|\!|\!| \int d\sigma'' hh/\Omega'' |\!|\!| \;\leq\; |\!|\!|h|\!|\!|^2 \;<\; m^2/36$, from (42) again.

Because in (53) the coefficient of $\|\phi\|$ has not to be bounded, we omit to estimate it explicitly henceforth. Now let us put $\vec{L}_h := \vec{L} - \vec{L}_0$. We have to control the action of $\vec{L}^2 - \vec{L}_0^2 = \vec{L}_0 \cdot \vec{L}_h + \vec{L}_h \cdot \vec{L}_0 + \vec{L}_h^2$ on vectors $\phi \in \mathcal{D}$. Replacing ϕ by $\vec{L}_0\phi$ in (26) leads, with (43), to the estimate

$$\|\vec{L}_h \cdot \vec{L}_0\phi\| \;\; < \;\; |\!|\!|h/\Omega|\!|\!| \; \|\vec{L}_0^2\phi\| \; + \; (|\!|\!|h/\Omega|\!|\!| + |\!|\!|Dh/\Omega|\!|\!|) \; \sum_{j=1}^{3} \|L_{0,j}\phi\| \tag{58}$$

On the other hand, applying $\vec{L}_0$ to (26) and neglecting the contribution of $\vec{J}^{\vec{Q}}$ gives

$$\vec{L}_0 \cdot \vec{L}_h\phi(\vec{P},\vec{Q}) \qquad = \qquad \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\; \frac{\Omega(\vec{P},\vec{Q})}{\Omega(\vec{P},\vec{Q}')}\; \left[\vec{L}_0^2\phi(\vec{P},\vec{Q}')\right]\; \frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}$$

$$
\begin{aligned}
- \quad & i \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\, \frac{\Omega(\vec{P},\vec{Q})}{\Omega(\vec{P},\vec{Q}')}\, \left[\vec{P}\cdot\vec{L}_0 \phi(\vec{P},\vec{Q}')\right] \times \\
& \left[-3\frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')^2} + \left(\frac{2}{\Omega(\vec{P},\vec{Q}')} + \frac{1}{\Omega(\vec{P},\vec{Q})}\right) \frac{Dh(\vec{P},\vec{Q},\vec{Q}')}{\|\vec{P}\|}\right] \\
- \Omega(\vec{P},\vec{Q}) \int & \; d\sigma(\vec{Q}') \;\; \phi(\vec{P},\vec{Q}') \;\left(2 + \|\vec{P}\|\; \partial_{\|\vec{P}\|}\right) \left[\frac{Dh(\vec{P},\vec{Q},\vec{Q}')}{\|\vec{P}\|\Omega(\vec{P},\vec{Q}')^2} - \frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(P,q')^3}\right]
\end{aligned}
$$

for all $\phi \in \mathcal{D}$. We remark in the last term the factor $\|\vec{P}\|\partial_{\|\vec{P}\|} Dh \|\vec{P}\|^{-1}$ which is well defined, in norm $\|.\|_*$, because h is an analytic function of $\vec{P}^2$ with all derivatives in $\mathcal{B}$ (points 3) and 5) of proposition 4). From the estimate (47) we get

$$
\begin{aligned}
\|\vec{L}_0\cdot\vec{L}_h\phi\| \;\; \leq \;\; & |\!|\!|\Omega h/\Omega'^2|\!|\!| \; \|\vec{L}_0^2\phi\| \; + \; 3(|\!|\!|\Omega h/\Omega'^2|\!|\!| + |\!|\!|\Omega Dh/\Omega'^2|\!|\!|) \; \sum_{j=1}^3 \|L_{0,j}\phi\| \\
& + \; k_3\|\phi\| \qquad\qquad (59)
\end{aligned}
$$

for some $k_3 \in (0,\infty)$. By applying $\vec{L}_h$ to (26) and by using (59) we get an estimate of the last term

$$
\begin{aligned}
\|\vec{L}_h^2\phi\| \;\; \leq \;\; & |\!|\!|h/\Omega|\!|\!| \; \|\vec{L}_0\cdot\vec{L}_h\phi\| \; + \; (|\!|\!|h/\Omega|\!|\!| + |\!|\!|Dh/\Omega|\!|\!|) \; \sum_{j=1}^3 \|L_{h,j}\phi\| \\
\leq \;\; & |\!|\!|\Omega h/\Omega'^2|\!|\!| \; \|\vec{L}_0^2\phi\| \; + \; \Big[3|\!|\!|h/\Omega|\!|\!| \; \left(|\!|\!|\Omega h/\Omega'^2|\!|\!| + |\!|\!|\Omega Dh/\Omega'^2|\!|\!| \;\right) \\
& + |\!|\!|h/\Omega|\!|\!| + |\!|\!|Dh/\Omega|\!|\!| \Big] \; \sum_{j=1}^3 \|L_{0,j}\phi\| \; + \; k_4\|\phi\| \qquad\qquad (60)
\end{aligned}
$$

for some $k_4 \in (0,\infty)$. Finally by collecting the estimates (58), (59), (60) we obtain

$$
\begin{aligned}
\|(\Delta-\Delta_0)\phi\| \;\; \leq \;\; & \left(\left|\!\left|\!\left| \frac{h}{\Omega} \right|\!\right|\!\right| + \left|\!\left|\!\left| \frac{Dh}{\Omega} \right|\!\right|\!\right| \right) \|\vec{L}_0^2\phi\| \; + \; \left[2 \left|\!\left|\!\left| \frac{h}{\Omega} \right|\!\right|\!\right| + 2 \left|\!\left|\!\left| \frac{Dh}{\Omega} \right|\!\right|\!\right| \right. \\
& \left. + \; 3\left(1 + \left|\!\left|\!\left| \frac{h}{\Omega} \right|\!\right|\!\right|\right) \left(\left|\!\left|\!\left| \frac{\Omega h}{\Omega'^2} \right|\!\right|\!\right| + \left|\!\left|\!\left| \frac{\Omega Dh}{\Omega'^2} \right|\!\right|\!\right| \right) \right] \;\; \sum_{j=1}^3 \|L_{0,j}\phi\| \\
& + \;\; k_5 \; \|\phi\| \qquad\qquad (61)
\end{aligned}
$$

for some $k_5 \in (0,\infty)$ and all $\phi \in \mathcal{D}$. It remains to find a bound on $\|L_{0,j}\phi\|$, $\|\vec{L}_0^2\phi\|$ of the type $k_6\|\Delta_0\phi\| + k_7\|\phi\|$ for constantsk_6 and k_7. Following Nelson [12, proof of Lemma 6.1] we find

$$
\begin{aligned}
\sum_{j=1}^3 \|L_{0,j}\phi\| \;\; \leq \;\; & \sqrt{3} \left[\sum_{j=1}^3 \left(\|L_{0,j}\phi\|^2 + m^{-2}\|P_j\phi\|^2 + \|J_j\phi\|^2 \right) + m^{-2} \, \|H_0\phi\|^2 \right]^{1/2} \\
= \;\; & \sqrt{3} \;\; (\phi, \Delta_0\phi)^{1/2} \;\; < \;\; \sqrt{3} \;\; \left(\phi, (\tfrac{1}{2}\Delta_0^2 + \Delta_0 + \tfrac{1}{2})\phi\right)^{1/2} \\
= \;\; & \sqrt{3/2} \; \|(\Delta_0+1)\phi\| \;\; \leq \;\; \sqrt{3/2} \;\; (\|\Delta_0\phi\| + \|\phi\|) \, . \qquad\qquad (62)
\end{aligned}
$$

The bound on $\|\vec{L}_0^2\phi\|$ needs more development. Following [12, proof of Lemma 6.1] again we try to write $\Delta_0^2 - (\vec{L}_0^2)^2$ as a sum of positive and negative operators. For that we use the commutation rules (1) to (4), which imply

$$\begin{aligned}
\vec{P}^2\vec{L}^2 + \vec{L}^2\vec{P}^2 &= 2\sum_{j=1}^{3} P_j\vec{L}^2 P_j \; - \; 6H^2 \\
\vec{P}^2\vec{J}^2 + \vec{J}^2\vec{P}^2 &= 2\sum_{j=1}^{3} P_j\vec{J}^2 P_j \; - \; 6\vec{P}^2 \\
\vec{J}^2\vec{L}^2 + \vec{L}^2\vec{J}^2 &= 2\sum_{j=1}^{3} J_j\vec{L}^2 J_j \\
H^2\vec{L}^2 + \vec{L}^2H^2 &= 2H\vec{L}^2H \; - \; 2\vec{P}^2.
\end{aligned}$$

¿From these relations follows that $\Delta_0^2 - (\vec{L}_0^2)^2 + 2m^{-2}(2\vec{P}^2 + 3H_0^2)$ is a positive operator and thus

$$(\vec{L}_0^2)^2 \; < \; \Delta_0^2 + 2m^{-2}(2\vec{P}^2 + 3H_0^2) \; < \; \Delta_0^2 + 6\Delta_0 + 9 \; = \; (\Delta_0 + 3)^2$$

and finally $\|\vec{L}_0^2\phi\| < \|(\Delta_0 + 3)\phi\| \le \|\Delta_0\phi\| + 3\|\phi\|$. By collecting all these results the estimate (61) can be written as (53) with

$$\begin{aligned}
k_1 \; &= \; \left(\left\| \frac{h}{\Omega} \right\| + \left\| \frac{Dh}{\Omega} \right\| \right) \\
&+ \; \sqrt{\frac{3}{2}} \left[2\left\| \frac{h}{\Omega} \right\| + 2\left\| \frac{Dh}{\Omega} \right\| + 3\left(1 + \left\| \frac{h}{\Omega} \right\| \right) \left(\left\| \frac{\Omega h}{\Omega'^2} \right\| + \left\| \frac{\Omega Dh}{\Omega'^2} \right\| \right) \right] \qquad (63)
\end{aligned}$$

Now from (42) follows

$$\left\| \frac{h}{\Omega} \right\| \le \frac{\|h\|}{2m} \le \frac{\|h\|_*}{16} \le \frac{1}{24} \qquad \left\| \frac{Dh}{\Omega} \right\| \le \frac{\|Dh\|_*}{16} \le \frac{1}{320} \qquad (64)$$

where $\| \, \|_*$ is the norm (45), and, with $\Omega(\vec{P}, \vec{Q})/\Omega(\vec{P}, \vec{Q}') \le 2\omega(\vec{Q})\omega(\vec{Q}')/m^{-2}$ from (52), we get

$$\left\| \frac{\Omega h}{\Omega'^2} \right\| \le \frac{\|\omega\omega' h\|}{m^3} \le \frac{\|h\|_*}{8} \le \frac{1}{12} \qquad \left\| \frac{\Omega Dh}{\Omega'^2} \right\| \le \frac{\|Dh\|_*}{8} \le \frac{1}{160} \qquad (65)$$

The substitution of (64) and (65) in (63) gives $k_1 < 1/2 < 1$. []

5. Non-triviality

We have obtained a family of unitary representations of the Poincaré group by doing some perturbations of the two free boson model. It remains to show that the 'interaction terms' we have introduced have a real physical effect. This is an essential point, because such a situation never occurs in the non-quantum case, according to the 'non-interaction theorem' [2].

To that purpose we study the modification of the Casimir operators due to the introduction of the interaction. We consider first the operators W^2, given by (6), which becomes in the interaction representation (13)

$$W^2 \;=\; W_0^2 \;+\; \{\{\mathcal{O}, \vec{W}_0\}, \vec{W}_0\} \;+\; \{\mathcal{O}, \vec{W}_0\}^2$$

where W_0^2 is given by (21) and $\vec{W}_0 = -H_0\vec{J}+\vec{P}\wedge\vec{L}_0$. Now the condition (27) implies $\{\mathcal{O}, \vec{W}_0\} = 0$ and thus

$$W^2 \;=\; W_0^2. \tag{66}$$

The introduction of interaction has no effect on the Pauli-Lubanski operator W^2. On the other hand the operator M^2 takes the form

$$M^2\phi(\vec{P},\vec{Q}) \;=\; 4(\vec{Q}^2+m^2)\;\phi(\vec{P},\vec{Q}) \;+\; \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\;\phi(\vec{P},\vec{Q}')\;\Xi(\vec{P},\vec{Q},\vec{Q}') \tag{67}$$

for all $\phi \in \mathcal{D}$, where the kernel Ξ is given by (56). Note that the spectrum of M_0^2, given by (20), is the complete interval $[4m^2,\infty)$ and is absolutely continuous. The introduction of interaction has the important effect that an isolated eigenvalue may occur.

Proposition 6 *For all $c \in \mathcal{V}$ the operator M^2 is essentially self-adjoint on the domain $\mathcal{D}$ given by* (50). *Moreover, there exists $c \in \mathcal{V}$ (for some $m > 0$, $R > 0$) such that M^2 has an isolated eigenvalue $m_B^2 \in (0, 4m^2)$, and its spectrum is given by $\{m_B^2\} \cup [4m^2, \infty)$.*

Proof. M_0^2 is clearly essentially self-adjoint on $\mathcal{D}$. By the estimate (47) we have the bound on the operator norm $\|M^2 - M_0^2\|_{op} \le \|\!|\Xi|\!\|$, which is bounded by $\|\!|\Xi|\!\| \le m^2(\sqrt{2/3}+1/36)$ (see the estimates just after (56)). Thus M^2 is essentially self-adjoint on $\mathcal{D}$.

Because H and $\vec{P}$ commute they admit a simultaneous spectral measure $dE(\rho,\vec{P})$ where $\rho > 0$ is the spectral variable associated to H. In this representation M^2 becomes simply the multiplication operator by $\rho^2 - \vec{P}^2$. Because it commutes with the hole representation, the invariant subspaces are limited in the $(\rho,\vec{P})$ space by half hyperboloids $\left\{\rho = (\vec{P}^2+K^2)^{1/2}\right\}$ with $K > 0$, and so is the support of dE. Thus the spectrum of M^2 can be obtained from the restiction to a neighbourhood of $\vec{P} = 0$, that is from the restriction of M to the subspace

$$\mathcal{H}_\varepsilon \;:=\; \text{closure}_{\mathcal{H}}\;\{\phi \in \mathcal{H} \mid \phi(\vec{P},\vec{Q}) = 0 \;\text{ for all }\; \|\vec{P}\| > \varepsilon\} \tag{68}$$

for arbitrary small $\varepsilon > 0$.

The interest of $\mathcal{H}_\varepsilon$ is that the $\vec{P}$ variable belongs to a compact set, and thus the theory of 'decomposable operators' of [14, Section XIII.16] can be applied. It consists in working with fixed $\vec{P}$. We consider the operator $M^2_{\vec{P}}$ given by

$$M^2_{\vec{P}}\varphi(\vec{Q}) \;:=\; 4(\vec{Q}^2+m^2)\;\varphi(\vec{Q}) \;+\; \int d\sigma_{\vec{P}}(\vec{Q}')\;\varphi(\vec{Q}')\;k(\vec{P},\vec{Q},\vec{Q}') \tag{69}$$

for suitable φ in the space $\mathcal{H}_{\vec{P}} = L^2(I\!R, d\sigma_{\vec{P}})$ where $d\sigma_{\vec{P}}(\vec{Q}) = d\sigma(\vec{Q})\Omega(\vec{P},\vec{Q})^{-1}$. Here $\vec{P}\in I\!R^2$ is just a set of three parameters.

Let $H_{\vec{P}}$ be the restriction of H for fixed $\vec{P}$, as operator on $\mathcal{H}_{\vec{P}}$. Because $M^2_{\vec{P}} = H^2_{\vec{P}} - \vec{P}^2$ the spectrum of $M^2_{\vec{P}}$ is easily deduced from the spectrum of $H_{\vec{P}}$. We write $H_{\vec{P}} = \Omega + H_{h,\vec{P}}$, where $H_{h,\vec{P}}\varphi(\vec{Q}) := \int d\sigma_{\vec{P}}(\vec{Q}')\varphi(\vec{Q}')h(\vec{P},\vec{Q},\vec{Q}')$ for all $\varphi\in\mathcal{H}_{\vec{P}}$. Because $\int |h(\vec{P},\vec{Q},\vec{Q}')|^2 d\sigma_{\vec{P}}(\vec{Q})d\sigma_{\vec{P}}(\vec{Q}')$ is well defined, $H_{h,\vec{P}}$ is compact. By the 'classical Weyl theorem' [14, section XIII.4] the essential spectrum of $H_{\vec{P}}$ is the same as of $\Omega(\vec{P},\vec{Q})$ (for fixed $\vec{P}$), that is the interval $[\Omega(\vec{P},0),\infty)$. It may exist a finite number of eigenvalues in $(0,\Omega(\vec{P},0))$, with finite-dimensional eigenspaces. Moreover $\vec{P}\to H_{h,\vec{P}}$ is an analytic family of compact operators in a $\mathbb{C}^2$-neighbourhood of the real axis (because by giving a small imaginary value to $\vec{P}$ the integral $\int |h(\vec{P},\vec{Q},\vec{Q}')|^2 d\sigma_{\vec{P}}(\vec{Q})d\sigma_{\vec{P}}(\vec{Q}')$ is still defined). Thus $H_{\vec{P}}$ is an 'analytic family of type (A)' and by theorem XII.13 of [14] the eigenvalues, if there are any, are continuous functions of $\vec{P}$. Returning to the operator $M^2_{\vec{P}} = H^2_{\vec{P}} - \vec{P}^2$ we conclude that its spectrum is made of the interval $[4m^2,\infty)$ and of a finite number of possible eigenvalues in the interval $(0,4m^2)$ which are continuous with respect to $\vec{P}$.

Now from the point (d) of theorem XIII.85 of [14] follows that the spectrum of M^2 restricted to the subspace $\mathcal{H}_\varepsilon$, and then on the hole $\mathcal{H}$, is made of the continuous intervalle $[4m^2,\infty)$ and of a possible structure in $(0,4m^2)$. The existence of something below $4m^2$ depends on the existence of solutions to the eigenvalue equation $M^2\phi(\vec{P},\vec{Q}) = m_B^2\phi(\vec{P},\vec{Q})$, for $0\neq\phi\in\mathcal{D}(M^2)$. By the point d) of the theorem just mentioned it suffices to consider the case $\vec{P}=0$ and the reduced equation

$$2\sqrt{\vec{Q}^2+m^2}\;\varphi(\vec{Q}) \;+\; \int \frac{d\vec{Q}'}{4(\vec{Q}'^2+m^2)}\;\varphi(\vec{Q}')\;c(\vec{Q},\vec{Q}') \;=\; m_B\;\varphi(\vec{Q}) \tag{70}$$

where we have replaced the operator M^2 by H (because for $\vec{P}=0$, M^2 is just H^2), for suitable φ. In the case where the function $c\in\mathcal{V}$ can be written as $c(\vec{Q},\vec{Q}') = -d(\|\vec{Q}\|)d(\|\vec{Q}'\|)$ for a function $d\in C^2([0,R])$ with $\|d\|^2_\infty \le m^3/(2|\mathcal{K}|)$, the equation (70) is equivalent to $\varphi(\vec{Q}) = d(\|\vec{Q}\|)(2\omega(\vec{Q})-m_B)^{-1}$, which belongs to $\mathcal{D}(H_{\vec{P}=0})$, provided the implicit equation

$$1 \;=\; \int \frac{d\vec{Q}}{4(\vec{Q}^2+m^2)}\;\frac{d(\|\vec{Q}\|)^2}{2\sqrt{\vec{Q}^2+m^2}-m_B}. \tag{71}$$

admits a solution $0 < m_B < 2m$. To study this question we consider the r.h.s. of

(71) as a function of the variable m_B. For $m_B = 0$ we get

$$\int \frac{d\vec{Q}}{4(\vec{Q}^2+m^2)} \frac{d(\|\vec{Q}\|)^2}{2\sqrt{\vec{Q}^2+m^2}} \le \|d\|_\infty^2 \int_{\mathcal{K}} \frac{d\vec{Q}}{8(\vec{Q}^2+m^2)^{3/2}}$$
$$\le \frac{m^3}{2|\mathcal{K}|} \frac{|\mathcal{K}|}{8m^3} = \frac{1}{16} < 1. \tag{72}$$

Let us suppose now that the function d^2 is bounded from below as follows

$$d(r)^2 \ge \begin{cases} \frac{m^3}{3|\mathcal{K}|} = \frac{m^3}{4\pi R^3} & \text{if } r < \frac{R}{2} \\ 0 \quad \text{if } r \ge \frac{R}{2} \end{cases} \tag{73}$$

In that case, for $m_B = 2m$ the r.h.s. of (71) becomes larger than

$$\int \frac{d\vec{Q}}{4(\vec{Q}^2+m^2)} \frac{d(\|\vec{Q}\|)^2}{2\sqrt{\vec{Q}^2+m^2}-2m} = \frac{\pi}{2} \int_0^R dr\ d(r)^2 \frac{\sqrt{r^2+m^2}+m}{r^2+m^2}$$
$$\ge \frac{m^3}{8R^3} \int_0^{R/2m} dr \frac{1+\sqrt{r^2+1}}{r^2+1} = \left(\frac{m}{2R}\right)^3 \left(\arctan\frac{R}{2m} + \arg\sinh\frac{R}{2m}\right) \tag{74}$$

which can be > 1 for m/R large enough (because $x^3(\arctan(1/x) + \arg\sinh(1/x)) \to \infty$ when $x \to \infty$). ¿From (72) and (74) it follows that (71) has a solution $0 < m_B < 2m$, provided d satisfies (73) and m/R is large enough. Note that this solution is unique, and thus (70) has a unique eigenvalue.

¿From the theory of 'decomposable operators' and from the geometrical form of the support of dE, mentioned above, it follows that the spectrum of M^2, for that particular case, has the form given in the conclusion of the proposition. []

Remark. The eigenvalue equation for M^2, or its reduced form (70), can be considered as a *relativistic Schrödinger equation* because it generates the discrete structure of the bound states (see the discussion of [10] on this topic).

6. Scattering states and conclusion

Let us sum up what we have found. We have constructed a family of unitary, continuous representations of the Poincaré group in four space-time dimensions, as perturbations of the two free boson model. The perturbations are introduced by interaction operators $\mathcal{O}$, the range of which are contained in the s-wave subspace. These perturbations are non trivial, in the sense that isolated eigenvalues of the Casimir operator M may appear (but the Pauli-Lubanski Casimir operator W^2 remains unchanged). To affirm that we have really obtained models for two bosons with interaction, it remains to construct the asymptotic states.

A vector $\psi \in \mathcal{H}$ is said to be a 'two-particle scattering state' if its time evolution, for very large time (or very small time), is not distinguishable from the two free particle evolution, that is, if there exist ψ_{out} (or ψ_{in}) in $\mathcal{H}$ such that $\psi = \Omega^- \psi_{out}$ (or $\psi = \Omega^+ \psi_{in}$), where the 'wave operators' Ω^+, Ω^- are given by

$$\Omega^{\pm} \;=\; s - \lim_{t \to \mp\infty} \, e^{itH} \, e^{-itH_0} \tag{75}$$

(do not confuse Ω^+, Ω^- with the function $\Omega(\vec{P}, \vec{Q})$). Here H_0 is the two free particle Hamiltonian (17). If the operators (75) exist on all $\mathcal{H}$, then their application to any vector of $\mathcal{H}$ give two-particle scattering states, and so a two particle scattering theory can be deduced.

Proposition 7 *For all $c \in \mathcal{V}$ the operator $\Omega^{\pm}$ exist on $\mathcal{H}$.*

Proof. We apply Cook's method (theorem XI.4 of [15]), with $A = H$ and $B = H_0$. Note that $D(H) = D(H_0)$. Let us consider the function

$$F(t, \vec{P}, \vec{Q}) \;=\; \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P}, \vec{Q}')} \, \phi(\vec{P}, \vec{Q}') \, h(\vec{P}, \vec{Q}, \vec{Q}') \, e^{-it\Omega(\vec{P}, \vec{Q}')} \tag{76}$$

for $\phi \in C_0^\infty(I\!R^2)$ satisfying $\phi(\vec{P}, 0, Q_2) = \partial_{Q_1}\phi(\vec{P}, 0, Q_2) = 0$. Such vectors ϕ generate a dense subspace of $\mathcal{H}$. Performing two integrations by parts w.r.t. Q_1 leads to

$$F(t, \vec{P}, \vec{Q}) = -\frac{1}{16t^2} \int d\vec{Q}' e^{-it\Omega(\vec{P}, \vec{Q}')} \partial_{Q_1} \left(\frac{\Omega(\vec{P}, \vec{Q}')}{Q_1'} \, \partial_{Q_1} \frac{\phi(\vec{P}, \vec{Q}') h(\vec{P}, \vec{Q}, \vec{Q}')}{Q_1' \omega(\vec{Q}')} \right)$$

(recall that $Q_1 \to h(\vec{P}, \vec{Q}, \vec{Q}')$ is in $C_0^2(I\!R)$ with derivatives in $\mathcal{B}$, by the points 4) and 5) of proposition 4) from which it follows that the function

$$t \mapsto \left[\int \frac{d\vec{P} \, d\sigma(\vec{Q}')}{\Omega(\vec{P}, \vec{Q}')} \, \left| F(t, \vec{P}, \vec{Q}) \right|^2 \right]^{1/2} \;=\; \left\| (H - H_0) e^{-itH_0} \phi \right\| \tag{77}$$

is in $L^1([1, \infty))$, as required by Cook's method. []

Appendix A. One-particle representation

We show that the operators (7) are the infinitesimal generators of a unitary, continuous and irreducible representation of the universal covering of the Poincaré group $\mathcal{P}_+^\uparrow$. Such a representation is a good candidate for a theory describing a quantum relativistic spinless particle of mass $m > 0$. The Poincaré representation describing one particle of mass $m > 0$ without spin in given by the following result.

Lemma 8 *The operators* (7) *are the infinitesimal generators of a unitary, continuous and irreducible representation of the universal covering of the Poincaré group $\mathcal{P}_+^\uparrow$ in the space $L^2(I\!R^3, \sigma)$.*

We recall that $d\sigma(\vec{p}) = d\vec{p}\,[2\omega(\vec{p})]^{-1}$ where $\omega(\vec{p}) = \sqrt{\vec{p}^{\,2} + m^2}$.

Proof. Let us consider the following transformations of functions $\phi \in L^2(I\!R^3, \sigma)$:

$$(\tau, \vec{\xi}, \vec{\alpha}, \vec{\beta}) \cdot \ \phi(\vec{p}) \ := \ e^{i(\tau\omega(\vec{p}) + \vec{\xi} \cdot \vec{p})}\ \phi(R_{\vec{\alpha}}\lambda_{\vec{\beta}}\vec{p}) \tag{78}$$

for all $\tau \in I\!R$, $\vec{\xi}, \vec{\alpha}, \vec{\beta} \in I\!R^3$, $\|\vec{\beta}\| < 1$, and almost all $\vec{p} \in I\!R^2$, where $R_{\vec{\alpha}}$ is the matrix of the rotation of $I\!R^3$ of the Euler angles $\vec{\alpha}$ and $\lambda_{\vec{\beta}}\,\vec{p}$ is the 'spatial' component of the pure Lorentz transform of $(\omega(\vec{p}), \vec{p})$ of speed $\vec{\beta}$ (i.e. the 'spatial' component of (81) (see below) for such four-vector).

Let us show that the transformation (78) defines a unitary, continuous and irreducible representation in $L^2(I\!R^3, \sigma)$ of the Poincaré group $\mathcal{P}_+^\uparrow$, the infinitesimal generators of which are given by (7).

The unitarity and the group law are easily seen by introducing a fourth variable p^0 together with a Dirac-delta function in the measure, that is by replacing $L^2(I\!R^3, \sigma)$ by $L^2(I\!R^4, \Sigma)$ where $d\Sigma(p^0, \vec{p}) = dp^0 dp_1 dp_2 dp_3 \delta({p^0}^2 - \vec{p}^{\,2} - m^2)\theta(p^0)$, θ being the Heaviside function. Because of the Σ measure, a function $\phi(p^0, \vec{p})$ is not distinguishable from $\phi(\omega(\vec{p}), \vec{p})$. The transformation (78) becomes, after Fourier transformation

$$(\tau, \vec{\xi}, \vec{\alpha}, \vec{\beta}) \cdot \ \tilde{\phi}(t, \vec{x}) \ = \ \tilde{\phi}\left((\tau, \vec{\xi}, \vec{\alpha}, \vec{\beta})^{-1} \cdot (t, \vec{x})\right)$$

(where $\tilde{\phi}$ is the Fourier transform of ϕ) that is, given in terms of the ordinary Poincaré transformation on space-time. The group law is now obvious. The unitarity follows from the Poincaré invariance of the measure Σ.

The continuity of the representation is easily seen in $L^2(I\!R^3, \sigma)$, by the standard analysis methods. The irreducibility is obtained as follows. Let ϕ be a non zero function of $L^2(I\!R^3, \sigma)$. Let us show that the only vector orthogonal to $\{e^{i\vec{\xi} \cdot \vec{p}}\phi(\lambda_{\vec{\beta}}\,\vec{p}) \mid \vec{\xi}, \vec{\beta} \in I\!R^3\}$ is 0. By the Fourier theory

$$\int_{I\!R^3} d\sigma(\vec{p})\ \psi(\vec{p})^*\ e^{i\vec{\xi} \cdot \vec{p}}\phi(\lambda_{\vec{\beta}}\,\vec{p}) \ = \ 0 \quad \text{for all} \quad \vec{\xi} \in I\!R^3 \tag{79}$$

implies that $\psi(\vec{p}) = 0$ for almost all $\vec{p}$ such that $\phi(\lambda_{\vec{\beta}}\,\vec{p}) \neq 0$. Without loss of generality we may assume that $\phi(0) \neq 0$. For all $\vec{p} \in I\!R^3$ the vector $\vec{\beta} = -\,\vec{p}/\omega(\vec{p})$ satisfies $\lambda_{\vec{\beta}}\,\vec{p} = 0$ (see Appendix B). Thus if moreover we impose that (79) holds for all $\vec{\beta} \in I\!R^2$ we get $\psi = 0$.

After an elementary calculation the operators (7) turn out to be the infinitesimal generators of the representation (78). []

Appendix B. The relative momentum

We explain the change of variable $(\vec{p}_1, \vec{p}_2) \to (\vec{P}, \vec{Q})$ given in (16) and establish somes useful formulas.

The meaning of $\vec{P} = \vec{p}_1 + \vec{p}_2$, the total momentum, is clear. The notion of relative momentum is more subtle. Physically it must be related to the momentum of one of the particles in a centre-of-mass frame.

Let $p_1 = (\omega(\vec{p}_1), \vec{p}_1)$ and $p_2 = (\omega(\vec{p}_2), \vec{p}_2)$ be the energy-momenta of two free particles, of the same mass to simplify. We define first

$$P = \left(P^0, \vec{P}\right) = p_1 + p_2 \qquad q = \left(q^0, \vec{q}\right) = \frac{1}{2}(p_1 - p_2). \tag{80}$$

Note the useful relation $\vec{P}\cdot\vec{q} = P^0 q^0$. Let $\Lambda_{\vec{\beta}}$ be a Lorentz transformation such that $\Lambda_{\vec{\beta}} P = (M_0, \vec{0}\ )$, where $M_0 = [(P^0)^2 - \vec{P}^2]^{1/2}$. Physically $\Lambda_{\vec{\beta}}$ is associated with a change of referential frame, which put the two particles of momenta $\vec{p}_1, \vec{p}_2$ on one of their centre-of-mass frames. Then we define the relative momentum $\vec{Q}$ as the spatial projection of $\Lambda_{\vec{\beta}}\ q$. Note that the relative momentum just defined depends on the centre-of-mass frame, which is not unique.

To get more explicit expressions we need the formula of a pure Lorentz transform $\Lambda_{\vec{\beta}}$ of speed $\vec{\beta} \in I\!R^3$, $\|\vec{\beta}\| < 1$. It acts on a four-dimensional vector $(p^0, \vec{p}) \in I\!R \times I\!R^3$ as follows

$$\Lambda_{\vec{\beta}} \begin{pmatrix} p^0 \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \gamma(p^0 + \vec{\beta}\cdot\vec{p}) \\ \\ \vec{p} + \vec{\beta}\left[\gamma p^0 + \frac{\gamma - 1}{\beta^2}\vec{p}\cdot\vec{\beta}\right] \end{pmatrix} \tag{81}$$

where we have put $\beta = \|\vec{\beta}\|$ and $\gamma = (1 - \beta^2)^{-1/2}$. The condition that the spatial component of $\Lambda_{\vec{\beta}} P$ vanishes gives

$$\vec{P} \ + \ \vec{\beta}\left[\gamma P^0 \ + \ \frac{\gamma - 1}{\beta^2}\ \vec{P}\cdot\vec{\beta}\right] \ = \ 0. \tag{82}$$

If we choose the direction of $\vec{\beta}$ parallel to $\vec{P}$ (this condition leads to a particular centre-of-mass frame), we get the unique solution $\vec{\beta} = -\vec{P}/P^0$. After some calculation one gets $\Lambda_{-\vec{P}/P^0}\ q = (0, \vec{Q})$ where

$$\vec{Q} \ = \ \vec{q} \ - \ \frac{\vec{P}}{P^0}\left[\frac{P^0 q^0}{M_0} \ - \ \frac{P^0 - M_0}{M_0}\ \frac{(P^0)^2}{\vec{P}^2}\ \frac{\vec{q}\cdot\vec{P}}{P^0}\right] \tag{83}$$

which, after some easy development, gives finally the formula (16). Let us calculate the operators of the two free boson representation (8) in these variables, to establish (17). The norm $\|\vec{Q}\|$ is obtained by the properties of the Lorentz transform, which gives $\vec{Q}^2 = (q^0)^2 - \vec{q}^{\ 2}$. From the trivial identity $(P^0)^2 - \vec{P}^2 = 4m^2 + 4[(q^0)^2 - \vec{q}^{\ 2}]$ it follows

$$M_0 \ = \ 2\omega(\vec{Q}) \tag{84}$$

and then

$$P^0 \ = \ \omega(\vec{p}_1) + \omega(\vec{p}_1) \ = \ M_0^2 + \vec{P}^2 \ = \ \sqrt{\vec{P}^2 + 4\vec{Q}^2 + 4m^2} \ = \ \Omega(\vec{P}, \vec{Q}) \tag{85}$$

in agreement with (20) and (18). The angular momentum $\vec{J}$ is easily obtained because a rotation of the system $\vec{p}_1, \vec{p}_2$ leads to the same rotation of $\vec{P}, \vec{Q}$ (this is an obvious consequence of the formula (16)). The Lorentz generator $L_{0,1}$ is obtained as follows. Under an infinitesimal Lorentz transform of hyperbolic angle γ in the first direction (of unit vector $\vec{e}_1$) the fundamental quantities become

$$\begin{array}{ll} \vec{P} \to \vec{P} + \gamma P^0 \vec{e}_1 & \qquad \vec{q} \to \vec{q} + \gamma q^0 \vec{e}_1 \\ P^0 \to P^0 + \gamma P_1 & \qquad q^0 \to q^0 + \gamma q_1 \ . \end{array}$$

Under such a transformation $\vec{Q}$ becomes

$$\begin{aligned} \vec{Q} \ \ \to \ \ & \vec{q} + \gamma q^0 \vec{e}_1 - \frac{q^0 + \gamma q_1}{M_0 + P^0 + \gamma P_1} \ \left(\vec{P} + \gamma P^0 \vec{e}_1 \right) \\ \approx \ \ & \vec{Q} \ + \ \gamma \ \frac{1}{M_0 + P^0} \ \left((\vec{P} \cdot \vec{Q}) \vec{e}_1 - Q_1 \vec{P} \right) \end{aligned}$$

where we have performed a first order development and used the relation $q^0 M_0 = \vec{P} \cdot \vec{Q}$. The formula for $L_{0,1}$ given in (17) follows from the calculation

$$L_{0,1} \phi(\vec{P}, \vec{Q}) \ = \ -i \partial_\gamma \ \phi \left(\vec{P} + \gamma P^0 \vec{e}_1, \vec{Q} + \gamma \ \frac{(\vec{P} \cdot \vec{Q}) \vec{e}_1 - Q_1 \vec{P}}{M_0 + P^0} \right) \Bigg|_{\gamma=0} \tag{86}$$

$L_{0,2}$ and $L_{0,3}$ are obtained in the same way.

The Jacobian of the transformation $(\vec{p}_1, \vec{p}_2) \to (\vec{P}, \vec{Q})$ is calculated in two steps. First we perform the transformation $(\vec{p}_1, \vec{p}_2) \to (\vec{P}, \vec{q}\ )$, with Jacobian 1, and then the transformation $(\vec{P}, \vec{q}\ ) \to (\vec{P}, \vec{Q})$ with Jacobian (using the formula (16) for $\vec{Q}$)

$$\left| \det \left(\frac{\partial Q_i}{\partial q_j} \right) \right|^{-1} \ = \ \left[1 \ - \ \vec{P} \cdot \vec{\nabla}_{\vec{q}} \ \frac{q^0}{M_0 + P^0} \right]^{-1} \tag{87}$$

which, after a long and tedious calculation, gives $4\omega(\vec{p}_1)\omega(\vec{p}_2)[M_0 P^0]^{-1}$. Thus the measure σ_2 becomes $d\sigma_2(\vec{p}_1, \vec{p}_2) = d\mu(\vec{P}, \vec{Q}) = d\vec{P} d\sigma(\vec{Q}) \Omega(\vec{P}, \vec{Q})^{-1}$.

Appendix C. The fundamental equation

The fundamental equation (29) is the condition on a kernel $h(\vec{P}, \vec{Q}, \vec{Q}')$, depending only on the norms $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$, for the relation (14) and (15) to hold. This norm dependence leads to (27) and thus to $L_{0,j}\mathcal{O} = L^{\vec{P}}_{0,j}\mathcal{O}$ and $\mathcal{O}L_{0,j} = \mathcal{O}L^{\vec{P}}_{0,j}$ where $L^{\vec{P}}_{0,j} := -i\Omega(\vec{P}, \vec{Q})\partial_{P_j}$ for all $j \in \{1, 2, 3\}$.

Let us calculate the jth equation of (14). The linear part in $\mathcal{O}$ can be written

$$A_j \ := \ [\{H_0, \mathcal{O}\}, L_{0,j}] + [H_0, \{L_{0,j}, \mathcal{O}\}] \ = \ 2(H_0 \mathcal{O} L_{0,j} - L_{0,j} \mathcal{O} H_0 + i P_j \mathcal{O})$$

where we have used $[H_0, L_{0,j}] = iP_j$ and the fact that P_j and $\mathcal{O}$ commute. By applying this operator to a vector $\phi \in \mathcal{D}$ (the domain given by (50)) we obtain, in obvious symbolic notation

$$\begin{aligned}
A_j\phi(\vec{P},\vec{Q}) \quad &= \quad -2i\Omega \int d\sigma(\vec{Q}') \frac{\partial_{P_j}\phi h}{\Omega+\Omega'} \;+\; 2i\Omega\partial_{P_j}\int d\sigma(\vec{Q}')\frac{\phi h}{\Omega+\Omega'} \\
&+ \;\; 2iP_j \quad \int \frac{d\sigma(\vec{Q}')}{\Omega'}\frac{\phi h}{\Omega+\Omega'} \;=\; 2i\int d\sigma(\vec{Q}')\phi\left(\Omega\partial_{P_j}\frac{h}{\Omega+\Omega'} \;+\frac{P_j}{\Omega'}\frac{h}{\Omega+\Omega'}\right) \\
&= \quad 2i\int d\sigma(\vec{Q}')\;\phi(\vec{P},\vec{Q}')\;\frac{\Omega(\vec{P},\vec{Q})\;\partial_{P_j}h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}')}.
\end{aligned}$$

To study the bilinear part, we rewrite (25) and (26) :

$$\begin{aligned}
\{H_0,\mathcal{O}\}\;\phi(\vec{P},\vec{Q}) \qquad &= \qquad \int\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\;\phi(\vec{P},\vec{Q}')\;h(\vec{P},\vec{Q},\vec{Q}') \\
\{L_{0,j},\mathcal{O}\}\;\phi(\vec{P},\vec{Q}) \qquad &= \qquad -i\int\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}(\partial_{P_j}\phi(\vec{P},\vec{Q}'))\;h(\vec{P},\vec{Q},\vec{Q}) \\
&- \quad i\Omega(P,q) \quad \int d\sigma(\vec{Q}')\phi(\vec{P},\vec{Q}')\;\partial_{P_j}\frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')(\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}'))}\;.
\end{aligned}$$

Thus the bilinear part in $\mathcal{O}$ gives, in symbolic notation

$$\begin{aligned}
&B_j\phi(\vec{P},\vec{Q}) \;:=\; [\{H_0,\mathcal{O}\},\{L_0,\mathcal{O}\}]\;\phi(\vec{P},\vec{Q}) \;= \\
&\int\frac{d\sigma''}{\Omega''}h^{Q,Q''}\left[-i\int\frac{d\sigma'}{\Omega'}(\partial_j\phi')h^{Q'',Q'} \;-\; i\Omega''\int d\sigma'\phi'\partial_j\frac{h^{Q'',Q'}}{\Omega'(\Omega''+\Omega')}\right. \\
&\qquad\left. +\; i\partial_j\int\frac{d\sigma'}{\Omega'}\phi' h^{Q'',Q'}\right] \;+\; i\Omega\int d\sigma''\left(\partial_j\frac{h^{Q,Q''}}{\Omega''(\Omega+\Omega'')}\right)\int\frac{dq'}{\Omega'}\phi' h^{Q'',Q'} \\
&= \; i\int d\sigma''\int d\sigma'\phi'\left(-\;h^{Q,Q''}\partial_j\frac{h^{Q'',Q'}}{\Omega'(\Omega''+\Omega')} \;+\; \frac{h^{Q,Q''}}{\Omega''}\partial_j\frac{h^{Q'',Q'}}{\Omega'}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\left. +\;\frac{\Omega}{\Omega'}\left(\partial_j\frac{h^{Q,Q''}}{\Omega''(\Omega+\Omega'')}\right)h^{Q'',Q'}\right) \\
&= \; i\int\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\;\phi(\vec{P},\vec{Q}')\int\frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P},\vec{Q}'')^2}\left\{-\;\frac{P_j}{\Omega(\vec{P},\vec{Q}'')}\;h(\vec{P},\vec{Q},\vec{Q}'')h(\vec{P},\vec{Q}'',\vec{Q}')\right. \\
&\qquad\qquad +\;\frac{\Omega(\vec{P},\vec{Q})\Omega(\vec{P},\vec{Q}'')\partial_{P_j}h(\vec{P},\vec{Q},\vec{Q}'')}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}'')}\;h(\vec{P},\vec{Q}'',\vec{Q}') \\
&\qquad\qquad \left. +\;h(\vec{P},\vec{Q},\vec{Q}'')\;\frac{\Omega(\vec{P},\vec{Q}'')\Omega(\vec{P},\vec{Q}')\partial_{P_j}h(\vec{P},\vec{Q}'',\vec{Q}')}{\Omega(\vec{P},\vec{Q}'')+\Omega(\vec{P},\vec{Q}')}\right\}.
\end{aligned}$$

Because h depends on $\vec{P}$ only through $\|\vec{P}\|$ we can replace everywhere ∂_{P_j} by $P_j\|\vec{P}\|^{-1}\partial_{\|\vec{P}\|}$. The condition $A_j + B_j = 0$, which must hold for all $\phi \in \mathcal{D}$, leads to the fundamental equation (29), for all $j \in \{1, 2, 3\}$.

The equations (15) are slightly more difficult. By using $\vec{J}^{\vec{Q}}\mathcal{O} = 0$ and by putting $\vec{J}^{\vec{P}} := \vec{J} - \vec{J}^{\vec{Q}}$ the linear part of the first equation of (15) can be written, in symbolic notations :

$$
\begin{aligned}
C\phi(\vec{P},\vec{Q}) \;&:=\; 2\left(L^{\vec{P}}_{0,1}\mathcal{O}L^{\vec{P}}_{0,2} - L^{\vec{P}}_{0,2}\mathcal{O}L^{\vec{P}}_{0,1} - i\mathcal{O}J^{\vec{P}}_{3}\right)\phi(\vec{P},\vec{Q}) \;=\\
&-2\Omega\int d\sigma'\left[(\partial_2\phi')\,\partial_1 - (\partial_1\phi')\,\partial_2\right]\frac{h}{\Omega+\Omega'} \;-\; 2\int\frac{d\sigma'}{\Omega'}\left[(P_1\partial_2 - P_2\partial_1)\,\phi'\right]\frac{h}{\Omega+\Omega'}\\
&= \;-\frac{2i}{\|\vec{P}\|}\int\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\left(J^{\vec{P}}_3\phi(\vec{P},\vec{Q}')\right)\; Dh(\vec{P},\vec{Q},\vec{Q}')
\end{aligned}
$$

The bilinear part of this equation gives

$$
\begin{aligned}
D\phi(\vec{P},\vec{Q}) \;&:=\; \left(\frac{1}{2}(C\mathcal{O}+\mathcal{O}C) + L^{\vec{P}}_{0,1}\mathcal{O}^2L^{\vec{P}}_{0,2} - L^{\vec{P}}_{0,2}\mathcal{O}^2L^{\vec{P}}_{0,1} + i\mathcal{O}^2J^{\vec{P}}_3\right)\phi(\vec{P},\vec{Q}) =\\
&= \;-\frac{i}{\|\vec{P}\|}\int\frac{d\sigma'}{\Omega'}J^{\vec{P}}_3\phi'\int\frac{d\sigma''}{\Omega''}\left(Dh^{\vec{Q},\vec{Q}''}\frac{h^{\vec{Q}'',\vec{Q}'}}{\Omega'+\Omega''} \;+\; \frac{h^{\vec{Q},\vec{Q}''}}{\Omega+\Omega''}Dh^{\vec{Q}'',\vec{Q}'}\right)\\
&(-i)^2\int\frac{d\sigma'}{\Omega'}\,\Omega\Omega'\left[(\partial_2\phi')\partial_1 - (\partial_1\phi')\partial_2\right]\int\frac{d\sigma''}{\Omega''}\,\frac{h^{\vec{Q},\vec{Q}''}}{\Omega+\Omega''}\,\frac{h^{\vec{Q}'',\vec{Q}'}}{\Omega'+\Omega''}\\
&+\int\frac{d\sigma'}{\Omega'}\,\left[(P_1\partial_2 - P_2\partial_1)\phi'\right]\int\frac{d\sigma''}{\Omega''}\,\frac{h^{\vec{Q},\vec{Q}''}}{\Omega+\Omega''}\,\frac{h^{\vec{Q}'',\vec{Q}'}}{\Omega'+\Omega''}\\
&= \;-\;\frac{i}{\|\vec{P}\|}\int\frac{d\sigma'}{\Omega'}\,J^{\vec{P}}_3\phi'\int\frac{d\sigma''}{\Omega''^2}\left(Dh^{\vec{Q},\vec{Q}''}h^{\vec{Q}'',\vec{Q}'} \;+\; h^{\vec{Q},\vec{Q}''}Dh^{\vec{Q}'',\vec{Q}'}\right)\\
&\quad + i\int\frac{d\sigma'}{\Omega'}\,J^{\vec{P}}_3\phi'\int\frac{d\sigma''}{\Omega''^3}\,h^{\vec{Q},\vec{Q}''}h^{\vec{Q}'',\vec{Q}'}
\end{aligned}
$$

The condition $C + D = 0$, which must hold for all $J^{\vec{P}}_3\phi$, leads to the fundamental equation (29) again. The two other equations of (15) lead to the same result.

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