On scaling in relation to singular spectra

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Abstract

This paper relates uniform α-Hölder continuity, or α-dimensionality, of spectral measures in an arbitrary interval to the Fourier transform of the measure. This is used to show that scaling exponents of exponential sums obtained from time series give local upper bounds on the degree of Hölder continuity of the power spectrum of the series. The results have applications to generalized random walk, numerical detection of singular continuous spectra and to the energy growth in driven oscillators.

1 Introduction

An interesting method to numerically detect singular continuous spectum has been proposed by Aubry, Goldrèche and Luck [2, 3]. In essence, their method amounts to the following. Let $\{a_k\}_{k=0}^{\infty} \subset \mathbb{C}$ and $S_n(q) := n^{-1}|\sum_{k=0}^{n-1} a_k e^{-2\pi ikq}|^2$. Suppose that $S_n(q)\,dq$ converges weakly as $n \to \infty$ to a positive Borel measure μ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Then μ should be singular at values of q for which $S_n(q)$ 'scales' like $n^{\beta(q)}$ for $0 < \beta(q) < 1$. In a previous paper [22] we proved, without justifying any scaling argument, that the particular model Aubry et al. considered does indeed have purely singular continuous spectrum for generic parameters (see Section 5.1 for details).

The method of Aubry et al. has been used to argue that sequences obtained from aperiodically driven quantum systems [29, 10] and a time series obtained from a dynamical system [30, 31] have singular continuous power spectrum. Moreover, the growth of sums of the form $S_N(q)$ has been shown to detemine the growth of the energy in driven classical and quantum-mechanical oscillators [6, 8]. Finally, Luck [28] has perturbatively related

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the exponents $\beta(q)$ to the speed with which gaps close in the spectrum of a discrete Schrödinger operator with potential λa_n as $\lambda \to 0$.

It is desirable, therefore, that the mathematical meaning of the exponents $\beta(q)$ and their relation to the singularity of the measure μ is clarified. That is the aim of this paper. It will show that: (i) if $\{a_n\}$ is a time series of an ergodic dynamical system, then $\beta(q)$ exist a.e. with respect to the ergodic measure when interpreted as critical exponent for a limsup, and (ii) $1-\beta(q)$ is an upper bound on the degree of Hölder continuity of μ in a neighborhood of q (see Corollary 4.3). However, scaling exponents $\beta(q) \in (0,1)$ need not imply that μ has a singular continuous part (see Remarks 2.5 and 2.6).

Hölder continuity here means uniform Hölder continuity. A positive Borel Measure on $\mathbb{R}$ is uniformly α-Hölder continuous — UαH, for short — in an interval I if there is a constant C such that $\mu(I_0) \leq C|I_0|^\alpha$ for all intervals $I_0 \subset I$ with $|I_0| < 1$, where $|I_0|$ denotes the length of I_0. It is globally uniformly α-Hölder continuous if it is UαH on its support. Uniform α-Hölder continuity is often referred to as uniform α-dimensionality (e.g., in [35]). A measure that is UαH gives zero weight to sets that have measure zero for α-dimensional Hausdorff measure (cf. Section 3.3 in [34]).

Section 2 of this paper relates a double integral over the Fourier transform $\hat{\mu}(t) := \int e^{-2\pi i t x}\, d\mu(x)$ of a finite positive Borel measure—the convolution of the measure with the Fejér kernel— to the degree of Hölder continuity in arbitrary intervals. This should be contrasted to results that relate global uniform Hölder continuity to the averaged decay of $\hat{\mu}$. Strichartz [35] has shown that if μ is globally UαH then there is a constant C such that $T^{-1}\int_0^T |\hat{\mu}(t)|^2\, dt < CT^{-\alpha}$. A partial, but optimal, converse has been obtained by Last [26]: if there exists a constant C such that $T^{-1}\int_0^T |\hat{\mu}(t)|^2\, dt < CT^{-\alpha}$ for all $T > 0$ then μ is globally U$\frac{\alpha}{2}$H. Of course a measure may very well have a degree of uniform Hölder continuity α that varies over its support. Then it is the most singular part (i.e., the part with the smallest α) that matters for the results just mentioned. Global uniform Hölder continuity of spectral measures of Schrödinger operators has received attention because it gives lower bounds on the spread of wave functions [18, 7], see also [19, 20, 26, 24, 15, 4].

The remaining sections apply the result of Section 2 to spectral measures of dynamical systems. Section 3 relates the degree of Hölder continuity of a spectral measure near zero to critical exponents of 'generalized random walks'. The generalization to 'twisted generalized random walks' in Section 4 explains the meaning of the scaling exponents $\beta(q)$. Finally, Section 5 discusses the application to the model considered by Aubry et al., and to the energy growth of driven oscillators.

2 Uniform Hölder continuity in an interval

This section explains how the Fourier transform of a positive bounded Borel measures can be related to uniform Hölder continuity in arbitrary intervals. In view of the applications that follow, the result will be stated and proved for spectral measures of a unitary operator. A series of remarks discusses some generalizations and the relation to the decomposition of the measure into its discrete, singular continuous and absolutely continuous parts.

Let U be a unitary operator on a separable Hilbert space $\mathcal{H}$. Every $\psi \in \mathcal{H}$ defines a spectral measure μ_ψ on the circle $\mathbb{T}$ by $\int_{\mathbb{T}} e^{2\pi i n x}\, d\mu_\psi = (U^n\psi, \psi)$. For $\lambda \in \mathbb{T}$ let $U_\lambda := e^{-2\pi i \lambda} U$ and $G_n(\lambda) := \frac{1}{n}\|\sum_{k=0}^{n-1} U_\lambda^k \psi\|^2$. Then $\lim_{n\to\infty} G_n(\lambda)\, d\lambda = \mu_\psi$ weakly (as can e.g. be seen by integrating $e^{-2\pi i m\lambda}$ with respect to $G_n(\lambda)\, d\lambda$). The following theorem says that $G_n(\lambda)$ scales like n^β as $n \to \infty$ if and only if μ_ψ is U$(1-\beta)$H in a neighborhood of λ.

Theorem 2.1 *Let Δ be an open interval and $0 \le \beta \le 1$. The following two statements are equivalent:*
i) There exists a constant $C > 0$ such that $\limsup_{n\to\infty} n^{-\beta} G_n(\lambda) \le C < \infty$ uniformly in $\lambda \in \Delta$;
ii) μ_ψ is $U(1-\beta)H$ in Δ.

Proof. Direct computation shows that $G_n(\lambda) = \int_{\mathbb{T}} K_{n-1}(\lambda - x)\, d\mu_\psi(x)$, where $K_{n-1}(y) := \frac{1}{n}(\frac{\sin n\pi y}{\sin \pi y})^2$ is the Fejér kernel. Note that $K_n(0) = n$ for all n and that $\lim_{n\to\infty} K_n(y) = 0$ uniformly on every closed interval not containing 0.

i) implies ii). By *i)* there exists for every $\epsilon > 0$ an $N > 0$ such that $G_n(\lambda) \le (C+\epsilon)n^\beta$ for all $n > N$ and all $\lambda \in \Delta$. Let $I \subset \Delta$ be an interval of length $|I| < 1/N$. Let $n > N$ be such that $\frac{1}{n} < |I| \le \frac{1}{n-1}$. Then I can be covered by two adjacent intervals $I_j := [\lambda_j - \frac{1}{2n}, \lambda_j + \frac{1}{2n}]$ of length $1/n$. Note that $\frac{\pi^2}{4n} K_{n-1}(\lambda_j - x) \ge 1$ if $x \in I_j$. Therefore,

$$\begin{aligned} \mu_\psi(I) &\le \sum_{j=1}^{2} \int_{I_j} \frac{\pi^2}{4n} K_{n-1}(\lambda_j - x)\, d\mu_\psi(x) \\ &\le \frac{\pi^2}{4n}\{G_n(\lambda_1) + G_n(\lambda_2)\} \end{aligned}$$

$$\leq \frac{\pi^2}{4}(C+\epsilon)|I|^{1-\beta}.$$

ii) implies i). Let $\lambda \in \Delta$ and choose $0 < a < 1/2$ such that $[\lambda - a, \lambda + a] \subset \Delta$. It suffices to show that

$$n^{-\beta} \int_{\lambda-a}^{\lambda+a} K_{n-1}(\lambda - x)\, d\mu_\psi(x) \leq C < \infty$$

for a constant C independent of λ and n. Since μ_ψ is U$(1-\beta)$H in Δ there is a constants D, N such that $\mu_\psi(I) \leq Dn^{\beta-1}$ for every interval I of length $1/n$ with $n > N$. Now $K_{n-1}(\lambda - x) \leq n$ if $|\lambda - x| < 1/n$ and $K_{n-1}(\lambda - x) \leq 4n/k^2\pi^2$ if $k/n \leq |\lambda - x| \leq (k+1)/n$ since $\sin \pi y \geq y/2$ for $0 \leq y \leq 1/2$. Therefore, denoting the integer part of an by $[an]$,

$$\int_{\lambda-a}^{\lambda+a} K_{n-1}(\lambda - x)\, d\mu_\psi(x) \leq 2Dn^\beta + 2\sum_{k=1}^{[na]} \frac{4}{k^2\pi^2} Dn^\beta \leq Cn^\beta,$$

for some constant C independent of λ and n. $\square$

Remarks 2.2 Theorem 2.1 holds in greater generality than stated. Let μ be a finite positive Borel measure on $\mathbb{R}$ with Fourier transform $\hat{\mu}(t) = \int e^{-2\pi i t x}\, d\mu(x)$. For $T > 0$, define $G_T(\lambda) := T^{-1}\Big|\int_0^T \int_0^T e^{2\pi i(t-s)\lambda}\hat{\mu}(t-s)\, ds\, dt\Big|^2$. Again, $G_T(\lambda) = \int K_T(\lambda - x)\, d\mu(x)$, where

$$K_T(y) = \int_{-T}^{T} (1 - \frac{|u|}{T}) e^{2\pi i y u}\, du = \frac{1}{T}\Big(\frac{sin \pi T y}{\pi y}\Big)^2$$

is the Fejér kernel on $\mathbb{R}$. Now Theorem 2.1 holds for μ and this function G_T. The proof is the same; that of '*i) implies ii)*' simplifies in that one can take $T = |I|^{-1}$.

2.3 In particular, if $\mu = \mu_\psi$ is a spectral measure of a strongly continuous group $\{U_t\}_{t\in\mathbb{R}}$ of unitary operators on a separable Hilbert $\mathcal{H}$ (i.e. $\int_{\mathbb{R}} e^{2\pi i t x}\, d\mu_\psi(x) = (U_t\psi, \psi)$ for a $\psi \in \mathcal{H}$) then Theorem 2.1 holds for μ_ψ and $G_T(\lambda) := T^{-1}\Big\|\int_0^T U_{\lambda,t}\psi\, dt\Big\|^2$, where $U_{\lambda,t} := e^{-2\pi i \lambda t} U_t$.

2.4 Theorem 2.1 also generalizes to higher dimensions. This is of interest for actions of $\mathbb{Z}^d$ on a probability space (Ω, ν) that leave ν invariant.

Let U^k, $k \in \mathbb{Z}^d$, be the unitary action of $\mathbb{Z}^d$ on $\mathrm{L}^2(\Omega, \nu)$ defined by $U^k f := f \circ T^k$, where T^k is the action of $\mathbb{Z}^d$ on (Ω, ν). For $\lambda \in \mathbb{T}^d$, let $U_\lambda^k := e^{-2\pi i \langle \lambda, k\rangle} U^k$, where $\langle \cdot\,, \cdot \rangle$ denotes the Euclidian inner product in $\mathbb{R}^d$. Let $C_n := \{m \in \mathbb{Z}^d \mid 0 \leq m_i < n\}$ and $G_n(\lambda) := n^{-d}\|\sum_{k\in C_n} U_\lambda^k \psi\|^2$. The

Fejér kernel now is $K_n(y) = n^{-d} \prod_{i=1}^{d} (\sin n\pi y_i / \sin \pi y_i)^2$. The theorem holds as before with $0 \le \beta \le d$, Δ a cube, and $\mathrm{U}(d-\beta)\mathrm{H}$ instead of $\mathrm{U}(1-\beta)\mathrm{H}$ in statement *ii)*.

2.5 The values of $\lim_{n\to\infty} n^{-\beta} G_n(\lambda)$ for $\beta = 1$ and $\beta = 0$ determine the discrete part and the absolutely continuous part of μ_ψ, respectively. Since the functions $n^{-1}K_n$ tend to zero uniformly outside any neighborhood of zero, and $n^{-1}K_n(0) = 1$, one has $\mu_\psi(\{\lambda\}) = \lim_{n\to\infty} n^{-1}G_n(\lambda)$ for all $\lambda \in \mathbb{T}$ (e.g., p. 42 in [25]). Thus $\beta = 1$ determines the discrete part of μ_ψ. To see that $\beta = 0$ determines the absolutely continuous part of μ_ψ, write $\mu_\psi = f\sigma + \rho$, where σ denotes Lebesgue measure, $f \in \mathrm{L}^1(\sigma)$ and $\rho \perp \sigma$. Then $G_n(\lambda) \to f(\lambda)$ at σ-a.e. λ, see e.g. Theorem III.8.1 in Volume 1 of [37]. If f can be chosen to be continuous in a neighborhood of λ and ρ is zero on that neighborhood then $\lim_{n\to\infty} G_n(\lambda) = f(\lambda)$.

2.6 A measure μ that is UαH for some $0 < \alpha < 1$ need not have a singular continuous part. For instance, let $\{c_i\}$ be a positive sequence with $\sum c_i < \infty$, $x_i \in \mathbb{T}$, $0 < \beta < 1$ and let $\mu := \sum c_i |x - x_i|^{-\beta}$. Then μ is absolutely continuous and $\mathrm{U}(1-\beta)\mathrm{H}$. Still, the behavior of $n^{-\beta}G_n(\lambda)$ for $\beta = 0$ and 1 can always be used in principle to determine whether μ_ψ has a singular continuous part $\mu_{\psi,\mathrm{sc}}$, because

$$\mu_{\psi,\mathrm{sc}}(\mathbb{T}) = \|\psi\|^2 - \int f\, d\sigma - \sum \lim_{n\to\infty} n^{-1} G_n(\lambda),$$

where, as in the previous remark, $f = \lim_{n\to\infty} G_n$.

2.7 The critical Hölder exponent $\alpha_\mu(x)$ of μ at x is defined by

$$\limsup_{r\to 0} \mu([x-r, x+r]) r^{-\alpha} = \begin{cases} 0 & \text{if } \alpha < \alpha_\mu(x) \\ \infty & \text{if } \alpha > \alpha_\mu(x) \end{cases}$$

Note that $\alpha_\mu(x)$ may be larger then 1 if μ has very little mass near x. If $-1 \le \beta \le 1$ and $\limsup n^{-\beta} G_n(\lambda) < \infty$ then $1 - \beta \le \alpha_\mu(\lambda)$. This follows from $\mu([\lambda - \frac{1}{2n}, \lambda + \frac{1}{2n}]) \le \pi^2/4n \int_{|\lambda - x| \le 1/2n} K_{n-1}(\lambda - x)\, d\mu(x)$.

3 Generalized random walk

Let Ω be a compact metric space with its Borel σ-algebra, $T\colon \Omega \to \Omega$ a measurable invertible map and ν a T-invariant ergodic probability measure on Ω. Each $\psi \in \mathrm{L}^2(\Omega, \nu)$ gives rise to a so-called generalized random walk (GRW)

$$S_n^\omega := \sum_{k=0}^{n-1} \psi(T^k \omega). \tag{1}$$

Simple random walk on $\mathbb{Z}^2$ is the GRW defined by $\Omega = \{0,1,2,3\}^{\mathbb{Z}}$, T the left shift on Ω, $\nu = \prod_{j\in\mathbb{Z}}[(\delta_{\omega_j,0} + \delta_{\omega_j,1} + \delta_{\omega_j,2} + \delta_{\omega_j,3})/4]$ and $\psi(\omega) = \psi(\omega_j) = 1$, i, -1, $-i$ if $\omega_j = 0$, 1, 2, and 3, respectively. Generalized random walks, and especially their recurrence properties, have been considered in e.g. [1, 5, 11, 36, 12, 13]. The aim of this section is to relate the Hölder continuity at 0 of the spectral measure μ_ψ of ψ to the speed with which the walk wanders off to infinity, as expressed by the critical exponents of the mean squared displacement and of $|S_n^\omega|^2$.

Recall that the critical exponent α_c of a sequence $N_n \in \mathbb{C}$ is defined by

$$\limsup_{n\to\infty} n^{-\alpha}|N_n| = \begin{cases} \infty & \alpha < \alpha_c \\ 0 & \alpha > \alpha_c \end{cases} \tag{2}$$

The critical exponent of $|S_n^\omega|^2$ is denoted by $\gamma(\omega)$, that of the mean squared displacement

$$\langle |S_n^\omega|^2\rangle := \int |S_n^\omega|^2 \, d\nu(\omega) = \Big\| \sum_{k=0}^{n-1} U^k\psi \Big\|^2_{\mathrm{L}^2}$$

by γ_{ms}; here $U\psi := \psi\circ T$ is the unitary operator implementing the dynamics in $\mathrm{L}^2(\Omega,\nu)$. Note that $0 \le \gamma_{\mathrm{ms}} \le 2$.

Lemma 3.1 *There exists a $\gamma \in [0,\gamma_{\mathrm{ms}}]$ such that $\gamma = \gamma(\omega)$ for ν-a.e. $\omega \in \Omega$.*

Proof. For $\alpha > 0$ and $I \subset [0,\infty)$ an interval, let $B_{\alpha,I} := \{\omega \in \Omega \mid \limsup_{n\to\infty} n^{-\alpha}|S_n^\omega|^2 \in I\}$. This set is measurable since $B_{\alpha,I} = \bigcap_{K=0}^\infty \bigcup_{L=K}^\infty \{\omega \in \Omega \mid n^{-\alpha}|S_n^\omega|^2 \in I \text{ for an } n \in [K,\ldots,L]\}$. It is also invariant. Hence the sets $B_{\alpha,0} := \bigcap_{m=1}^\infty B_{\alpha,[0,1/m]}$ and $B_{\alpha,\infty} := \bigcap_{m=1}^\infty B_{\alpha,[m,\infty)}$ are measurable and invariant. Thus Ω is the disjoint union of the measurable and invariant sets $B_{\alpha,0}$, $B_{\alpha,(0,\infty)}$ and $B_{\alpha,\infty}$. By ergodicity, one of these three sets has measure one. Since there is at most one value of α for which $\nu(B_{\alpha,(0,\infty)}) = 1$ there is a γ such that $\nu(B_{\alpha,\infty}) = 1$ if $\alpha < \gamma$ and $\nu(B_{\alpha,0}) = 1$ if $\alpha > \gamma$. This shows that $\gamma(\omega) = \gamma$ for ν-a.e. $\omega \in \Omega$.

If $\limsup_{n\to\infty} n^{-\beta}\langle |S_n^\omega|^2\rangle = 0$, then there is a subsequence n_k such that $\lim_{k\to\infty} n_k^{-\beta}|S_{n_k}^\omega|^2 = 0$ for ν-a.e. ω. This means that $\nu(B_{\beta,0}) = 1$. Hence $\beta > \gamma_{\mathrm{ms}}$ implies $\beta > \gamma$, and $\gamma \le \gamma_{\mathrm{ms}}$. □

It is now clear that the degree of Hölder continuity of μ_ψ at zero gives an upperbound on γ_{ms}, and that, conversely, γ_{ms} and γ provide upperbounds on the degree of Hölder continuity of μ_ψ near zero. This follows from the fact that, in the notation of Theorem 2.1, $G_n(0) = n^{-1}\langle |S_n^\omega|^2\rangle$. If μ_ψ is UβH on a neighborhood of 0, then $\gamma_{\rm ms} \le 2-\beta$. Conversely, if $\gamma_{\rm ms} > 1$ (or $\gamma > 1$), then μ_ψ is not U$(2-\beta)$H on any neighborhood of 0, for any $\beta \in (1, \gamma_{\rm ms})$ ($\beta \in (1,\gamma)$). Thus superdiffusive behavior of the GRW requires that μ_ψ is not U1H on any neighborhood of 0. On the other hand, the GRW can only be subdiffusive ($\gamma_{\rm ms} < 1$) if μ_ψ has very little weight near 0. Indeed, by Remark 2.7 this requires $\alpha_{\mu_\psi}(0) > 1$. Theorem 1 in [27] gives that $\langle |S_n^\omega|^2\rangle$ is bounded in n if and only if $\int (\sin \pi\lambda)^{-2}\, d\mu_\psi(\lambda) < \infty$.

Remarks 3.2 Dekking [13] shows that $\gamma_{\rm ms} \le 1$ for the 'Rudin-Shapiro walk'. This is the GRW in which $\Omega \subset \{0,1,2,3\}^{\mathbb{Z}}$ is the substitution dynamical system (e.g., [32]) arising from the primitive substitution $0 \to 02$, $1 \to 32$, $2 \to 01$, $3 \to 31$ with $\psi(\omega) = \psi(\omega_0) = 1,\ -i,\ i,\ -1$ if $\omega_0 = 0,\ 1,\ 2,\ 3$, respectively. The spectral measure μ_ψ can be computed from Proposition VII.5 and Example VIII.2.2 in [32] as Lebesgue measure, so $\gamma_{\rm ms} = 1$, as conjectured by Dekking, and the diffusion coefficient is 1. (In Proposition VII.5 of [32] one should read $\overline{\tau(\beta)}$ for $\tau(\beta)$; this τ is our ψ. The symbols $\{0,1,2,3\}$ in [32] correspond to the symbols $\{a,d,b,c\}$ in [13].)
3.3 By Remark 2.5, if μ_ψ is absolutely continuous on a neighborhood 0, with a density that can be chosen to be a continuous function g, then $\gamma_{\rm ms} = 1$ and the diffusion coefficient of the GRW is $g(0)$.
3.4 Dumont and Thomas [16] have given an asymptotic expression for S_n^ω for the case that ω is a fixed point of a primitive substitution and $\psi(\omega)$ depends only on ω_0 (here $\Omega \subset \{0,\ldots,a\}^{\mathbb{Z}}$). They find, under certain assumptions on the eigenvalues of the substitution matrix, an upperbound $\beta < 1$ on $\gamma(\omega)$ if $\int \psi\, d\nu = 0$. This is a result for one particular sequence in Ω, a fixed point of the substitution, and therefore has no direct implication for the Hölder continuity of μ_ψ at 0.

4 Twisted generalized random walk

The exponential sums

$$S_n^\omega(\lambda) := \sum_{k=0}^{n-1} e^{-2\pi i\lambda k}\psi(T^k\omega) \tag{3}$$

define a walk in the complex plane for each $\omega \in \Omega$ and each $\lambda \in \mathbb{T}$. One can think of these as walks with a rotational bias: at step k the walker changes its direction by $e^{-2\pi i\lambda}$ and then makes a step of length $|\psi(T^k\omega)|$ in the direction $\arg(\psi(T^k\omega))$. When plotted in the complex plane, such walks often give rise to pretty pictures [14, 3]. The mean squared displacement $\int |S_n^\omega(\lambda)|^2\, d\nu(\omega)$ of $S_n^\omega(\lambda)$ is given by $nG_n(\lambda)$, and now depends on λ. By Theorem 2.1, its critical exponent $\gamma_{\mathrm{ms}}(\lambda)$ satifies $\gamma_{\mathrm{ms}}(\lambda) \le 2-\alpha$ on any interval where μ_ψ is UαH. Again, superdiffusive behavior of $\int |S_n^\omega(\lambda)|^2\, d\nu(\omega)$ requires singularity of μ_ψ at λ.

The aim of this section is to show that $S_n^\omega(\lambda)$ is itself a GRW (i.e., a GRW for some ergodic dynamical system), except possibly for a countable set of λ, and to relate the critical exponents $\gamma(\lambda)$ of $|S_n^\omega(\lambda)|^2$ to μ_ψ. This will explain the meaning of the method of Aubry et al. [2, 3], see the next section. It is of interest because it is often easier to deal numerically with one—hopefully typical—trajectory then with $\int |S_n^\omega(\lambda)|^2\, d\nu(\omega)$.

An eigenvalue of (Ω, T, ν) is a $\lambda \in [0,1] \simeq \mathbb{T}$ for which there is a $\phi \in \mathrm{L}^2(\Omega,\nu)$ such that $U\phi = e^{2\pi i\lambda}\phi$. The set of eigenvalues forms a countable group, which will be denoted by Λ. Let $\mathbb{Q}[\Lambda] := \{\mu \in \mathbb{T} \mid \mu = q\lambda, \lambda \in \Lambda, q \in \mathbb{Q}\}$.

Proposition 4.1 *For $\lambda \notin \mathbb{Q}[\Lambda]$ there exists a $\gamma(\lambda) \in [0, \gamma_{\mathrm{ms}}(\lambda)]$ such that $\gamma(\lambda,\omega) = \gamma(\lambda)$ for ν-a.e. ω.*

Proof. Let X be a compact metric space and ρ a Borel probability measure on X that is ergodic for an invertible, measurable transformation R on X. The direct product of (Ω, T, ν) and (X, R, ρ) is the dynamical system $(\tilde\Omega, \tilde T, \tilde\nu)$ defined by $\tilde\Omega := \Omega \times X$, $\tilde T(\omega, x) := (T\omega, Rx)$, $\tilde\nu := \mu \times \rho$. This direct product is ergodic if and only if (Ω, T, ν) and (X, R, ρ) share no eigenvalues other than 0 (see e.g. [9], Theorem 10.1.1).

If λ is irrational, take $X = \mathbb{T}$, ρ normalized Lebesgue measure and let $R = R_\lambda$ be defined by $R_\lambda x := x + \lambda \pmod 1$. If $\lambda = p/q$ is rational, take $X = \{kp/q\}_{k=0}^{q-1}$, let ρ be normalized counting measure on X and again let $R = R_\lambda$. In both cases (X, R_λ, ρ) is ergodic and has eigenvalues $e^{-2\pi i m\lambda}$, $m \in \mathbb{Z}$. Therefore the direct product $(\tilde\Omega, \tilde T, \tilde\nu)$ is ergodic if $\lambda \notin \mathbb{Q}[\Lambda]$. Take such a λ.

For $\psi \in \mathrm{L}^2(\Omega,\nu)$ let $\tilde\psi \in \mathrm{L}^2(\tilde\Omega, \tilde\nu)$ be defined by $\tilde\psi(\omega, x) := e^{-2\pi i x}\psi(\omega)$. Let $\tilde U$ act on $\mathrm{L}^2(\tilde\Omega,\tilde\nu)$ by $\tilde U\phi = \phi \circ \tilde T$, and define $\tilde S_n^{(\omega,x)} := \sum_{k=0}^{n-1} \tilde U^k \tilde\psi(\omega, x)$. (Note that $\tilde S_n^{(\omega,x)}$ implicitly depends on λ.) Then $S_n^\omega(\lambda) = \tilde S_n^{(\omega,0)}$. Since $|\tilde S_n^{(\omega,x)}|$ is independent of x, the critical exponent of $\int |\tilde S_n^{(\,\cdot\,)}|^2\, d\tilde\nu$ is the same as the critical exponent $\gamma_{\mathrm{ms}}(\lambda)$ of $\int |S_n^\omega(\lambda)|^2\, d\nu(\omega)$. By Lemma 3.1 there is a $\gamma(\lambda) \in [0, \gamma_{\mathrm{ms}}(\lambda)]$ such that the critical exponent $\tilde\gamma(\lambda, \omega, x)$ is equal to $\gamma(\lambda)$ for $\tilde\nu$-a.e. (ω, x). But $\tilde\gamma(\lambda,\omega,x) = \tilde\gamma(\lambda,\omega,0) = \gamma(\lambda,\omega)$ for all $x \in X$, so $\gamma(\lambda,\omega) = \gamma(\lambda)$ for ν-a.e. ω. □

Corollary 4.2 *If (Ω, T, ν) is weakly mixing then there exists for all $\lambda \in \mathbb{T}$ a $\gamma(\lambda) \in [0, \gamma_{\mathrm{ms}}(\lambda)]$ such that $\gamma(\lambda, \omega) = \gamma(\lambda)$ for ν-a.e. ω.*

Proof. Recall that, by definition, (Ω, T, ν) is weakly mixing if it has no eigenvalues other than 1. The case $\lambda = 0$ is covered by Lemma 3.1. □

Corollary 4.3 *Suppose $\lambda \notin \mathbb{Q}[\Lambda]$ and $\gamma(\lambda) > 1$. Let $0 < \epsilon < \gamma(\lambda) - 1$. Then μ_ψ is not $U(2 - \gamma(\lambda) + \epsilon)H$ on any neighborhood of λ.*

Proof. Let $\beta := \gamma(\lambda) - \epsilon$. Then $1 < \beta < \gamma_{\mathrm{ms}}(\lambda)$ and $\limsup_{n\to\infty} n^{-(\beta-1)} G_n(\lambda) = \infty$. Let Δ be any neighborhood of λ. By Theorem 2.1, μ_ψ is not $\mathrm{U}(2-\beta)\mathrm{H}$ on Δ. □

Note that the exceptional set in Proposition 4.1 depends on λ.

Corollary 4.3 shows that a single (but typical) trajectory provides upper bounds on the degree of Hölder continuity of μ_ψ. As explained in Remark 2.6, a degree of Hölder continuity strictly less than one does not prove that a measure is purely singular continuous. Thus the question arises whether one can determine from a single (typical) trajectory that the spectral measure has no absolutely continuous part. In general, this is not possible, as is explained below.

The following lemma is well known.

Lemma 4.4 $\lim_{n\to\infty} n^{-1} |S_n^\omega(\lambda)|^2\, d\lambda = \mu_\psi$ *weakly for ν-a.e. ω. If (Ω, T, ν) is uniquely ergodic and ψ is continuous then 'ν-a.e. ω' can be replaced by 'for all $\omega \in \Omega$'.*

Proof. For $p \in \mathbb{Z}$,

$$\int e^{-2\pi i \lambda p} S_n^\omega(\lambda)\, d\lambda = \frac{1}{n} \sum_{m=0}^{n-1} \psi(T^{m-p}\omega) \overline{\psi(T^m\omega)}, \tag{4}$$

which by the ergodic theorem converges to $(U^{-p}\psi, \psi)$ as $n \to \infty$ for ν-a.e. ω. Hence the Fourier coefficients of $S_n^\omega(\lambda)\, d\lambda$ converge pointwise to those of μ_ψ, for ν-a.e. ω. This shows that $\lim_{n\to\infty} n^{-1}|S_n^\omega(\lambda)|^2\, d\lambda = \mu_\psi$ weakly for ν-a.e. ω. The averages (4) converge for all $\omega \in \Omega$ as $n \to \infty$ if (Ω, T, ν) is uniquely ergodic and ψ is continuous (e.g. [9], Theorem 1.8.2). □

Thus both $\frac{1}{n}|S_n^\omega(\lambda)|^2\, d\lambda$ (for a.e.-ω) and its expectation $G_n(\lambda)\, d\lambda$ converge weakly to μ_ψ as $n \to \infty$. There is a difference, however. The sequence

$G_n(\lambda)$ converges for Lebesgue-a.e. λ and its limit is the value of the density of the absolutely continuous part of μ_ψ (see Remark 2.5). But $\lim \frac{1}{n}|S_n^\omega(\lambda)|^2$ in general does *not* exist for Lebesgue-a.e. λ. (E.g., if $\psi(T^k\omega)$, $k \in \mathbb{Z}$, are independent and uniformly distributed in $\mathbb{T}$ then for every λ and ν-a.e. ω the limit does not exist.)

Now assume that (Ω, T, ν) is uniquely ergodic and that ψ is continuous. Then, if $\lim_{n\to\infty} \frac{1}{n}|S_n^\omega(\lambda)|^2$ exists for Lebesgue-a.e. λ in some interval, the limit is equal to the density of μ_ψ with respect to the Lebesgue measure in that interval. In particular, if $\lim_{n\to\infty} \frac{1}{n}|S_n^\omega(\lambda)|^2 = 0$ a.e. on $\mathbb{T}$, then μ_ψ has no absolutely continuous part. In this special case examining a single trajectory allows to conclude that the spectral measure has no absolutely continous part.

Remarks 4.5 The results in Sections 3 and 4 generalize to flows, if, for $T > 0$ and $\lambda \in \mathbb{R}$, $S_T^\omega(\lambda)$ is defined by $S_T^\omega(\lambda) := \int_0^T e^{-2\pi i\lambda x}\psi(T_t\omega)\,dt$. An eigenvalue of the flow T_t is a $\lambda \in \mathbb{R}$ for which there is a $\phi \in \mathrm{L}^2(\Omega, \nu)$ such that $U_t\phi = e^{2\pi i\lambda t}\phi$.

4.6 If $\mu_\psi(\{\lambda\}) > 0$ then $S_n^\omega(\lambda)$ behaves ballistically, $\gamma_{\mathrm{ms}}(\lambda) = 2$, with diffusion coefficient $\mu_\psi(\{\lambda\})$, see Remark 2.5. If (Ω, T, ν) is uniquely ergodic and the eigenfunction ϕ for λ is continuous then $\gamma(\lambda, \omega) = 2$ for all $\omega \in \Omega$ [33].

5 Applications

5.1 The model of Aubry et al.

Aubry et al. [2, 3] have considered a model of atoms on the line defined by

$$x_n - x_{n-1} = 1 + \xi 1_{[0,\beta)}(n\alpha + \theta), \tag{5}$$

where $\{x_n\}$ is the set of atomic positions, α is irrational, and $x_0 \in \mathbb{R}$, $\beta, \theta \in \mathbb{T}$ and $\xi > 0$ are parameters. (Changing x_0 translates the structure; θ selects a structure within a 'local isomorphism class' determined by α, β and ξ.) They were interested in the 'structure factor' $S(\lambda)$, an unbounded measure defined by

$$dS(\lambda) = \lim_{T\to\infty} T^{-1}\Big|\sum_{x_n\in[0,T]} e^{-2\pi i\lambda x_n}\Big|^2\,d\lambda, \tag{6}$$

where the limit is taken in the vague topology (cf. [21, 23]). The structure factor does not depend on the choice of x_0 or θ. Aubry et al. argued that

S should be purely singular continuous if $\beta \neq n\alpha$ (mod 1). One reason for this assertion was that they found critical exponents $\eta(\lambda)$ of the sums

$$F_T(\lambda) := \sum_{x_n \in [0,T]} e^{-2\pi i \lambda x_n} \tag{7}$$

to be strictly between 1/2 and 1. For the case $\beta = 1/2$, $\alpha = \tau^{-1}$ ($\tau = (\sqrt{5}+1)/2$) they showed analytically [3] that $\eta(\lambda) = c$ for some $\frac{1}{2} < c < 1$ and λ in a countable dense set.

This section explains that $2 - 2\eta(\lambda)$ is an upperbound on the degree of Hölder continuity of S on neighborhoods of λ. The idea is to view S as a limit of spectral measures of a dynamical system, the flow under the function $f(x) = 1 + \xi 1_{[0,\beta)}(x)$ over the irrational rotation α on $\mathbb{T}$. This was explained in detail in [22]. The essence is the following. Let $\phi \geq 0$ be a C^∞-function with support in $[-1/2, 1/2]$, $\int \phi(x)\, dx = 1$, and for $1 > \epsilon > 0$ let $\phi_\epsilon(x) := \epsilon^{-1}\phi(x/\epsilon)$. Then the function $\rho_\epsilon := \phi_\epsilon * \sum_{n\in\mathbb{Z}} \delta_{x_n}$ is of the form $x \to \psi_\epsilon(T_x\omega)$ for an $\omega \in \Omega$ and a $\psi_\epsilon \in \mathrm{L}^2(\Omega, \nu)$, where $\Omega = \Omega_{\beta,\xi} = \{(x,y) \mid x \in \mathbb{T},\ 0 \leq y < f(x)\}$ and ν is the normalized Lebesgue measure on Ω. The spectral measure μ_{ψ_ϵ} of ψ_ϵ satisfies

$$\mu_{\psi_\epsilon} = S|\hat{\phi}_\epsilon|^2, \tag{8}$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ. Thus $\mu_{\psi_\epsilon} \to S$ vaguely as $\epsilon \to 0$. Since $\hat{\phi}_\epsilon$ is smooth is follows that for every interval I there is an ϵ such that the degrees of Hölder continuity of S and μ_{ψ_ϵ} are the same on I.

Now consider the integral $S_T^\omega(\lambda) := \int_0^T e^{-2\pi i\lambda x}\psi_\epsilon(T_x\omega)\, dx$. For each $\omega \in \Omega$ there is an A_ω, $|A_\omega| \leq 2$ such that

$$\begin{aligned} S_T^\omega(\lambda) &= \int_0^\infty e^{-2\pi i\lambda x}\Big(\sum_{x_n \in [0,T]} \phi_\epsilon * \delta_{x_n}\Big)(x)\, dx + A_\omega \\ &= \hat{\phi}_\epsilon(\lambda) F_T(\lambda) + A_\omega \end{aligned}$$

It follows that the critical exponent $\eta(\lambda)$ of $F_L(\lambda)$ is related to the critical exponent $\gamma(\lambda)$ of $|S_L^\omega(\lambda)|^2$ by $\gamma(\lambda) = 2\eta(\lambda)$. Hence corollary 4.3 (in 'flow form', cf. Remark 4.5) gives that S is not $\mathrm{U}(2 - 2\eta(\lambda) + \delta)\mathrm{H}$ on any neighborhood of λ, for all $0 < \delta < 2\eta(\lambda) - 1$.

Remark 5.1 The connection with the flow under the function f was used in [22] to determine the parameters α, β, ξ for which the unbounded measure S is continuous (apart from a delta function at zero, which is present for all parameters). This in turn was used to prove that for every irrational ξ and every β the measure S is purely singular continuous for generic α (i.e., for α in a dense G_δ). We recently found a paper by Goodson and Whitman [17] that proves by periodic approximation that for each ξ (irrational or not) and each irrational α the flow under f has singular spectrum (i.e., no spectral measure has an absolutely continuous part) for Lebesgue-a.e. β. Combined with the continuity result of [22] this gives that for every irrational ξ and every irrational α, the flow has purely singular continuous spectrum for almost every β. For these parameters S is purely singular continuous apart from the delta function a zero. For $\beta = 1/2$, $\alpha = \tau^{-1}$, the parameters considered in [3], S is continuous [22]. It has not been proved that it is singular continuous although the work of Aubry et al. does suggest that it is singular continuous.

5.2 Energy growth in driven oscillators

Bunimovitch et al. [6] consider a classical driven oscillator with natural frequency ω_0 described by the Hamiltonian $H = p^2/2 + \omega_0 q^2/2 - q\psi(T_t x)$, where T_t is an ergodic flow on a compact metric space with invariant probability measure ν. They are interested in the expectation of the time evolution $\langle E_t\rangle_\nu$ of the energy of the oscillator. Their equations (2.9) and (2.10) can be rewritten to give

$$\langle E_t\rangle_\nu = \omega_0^2 q_0^2/2 + p_0^2/2 + \frac{\omega_0^2}{2}\, tG_t(\omega_0)$$

where G_t is as in Remark 2.3. Bunimovich et al. express this in terms of the correlation function $\hat{\mu}_\psi$. In Proposition 3.1 they show that $\langle E_t\rangle_\nu$ grows linearly with t and has a well defined diffusion coefficient if the correlation functions decays sufficiently fast. Remark 3.3 generalizes this: it suffices to require that μ_ψ is absolutely continuous on a neighborhood of of ω_0 with a density that can be chosen to be continuous at ω_0. The correlation function need not decay at all in order to have diffusive behavior of $\langle E_t\rangle_\nu$.

Again, superdiffusive behavior of $\langle E_t\rangle$ requires μ_ψ to be singular at ω_0, in the sense that μ_ψ is not UαH on any neighborhood of ω_0, for some $\alpha < 1$.

Note that by Proposition 4.1 the critical exponent of E_t is independent of x (ν-a.e.), for all but possibly countably many ω_0.

The expection of the energy growth for driven quantum oscillators is also determined by tG_t [6, 8].

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