The Quasi-periodic Centre-saddle Bifurcation

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Abstract

Nearly integrable families of Hamiltonian systems are considered in the neighbourhood of normally parabolic invariant tori. In the integrable case such tori bifurcate into normally elliptic and normally hyperbolic invariant tori. With a KAM-theoretic approach it is shown that both the normally parabolic tori and the bifurcation scenario survive the perturbation, parametrised by pertinent large Cantor sets. These results are applied to rigid body dynamics.

§1 Introduction

Given a non-degenerate integrable Hamiltonian system, we know from KAM-theory that most maximal invariant tori survive a small (Hamiltonian) perturbation, cf. [Arnol'd;63a], [Pöschel;82], [Broer,Huitema,Sevryuk;96] or references therein. These maximal tori are the regular fibres of the *ramified torus bundle* defined by the integrable system. The singular fibres of the ramified torus bundle, *i.e.* the lower dimensional invariant tori together with their stable and unstable manifolds, determine how the maximal tori are distributed in phase space. Hence, it is instructive to understand the persistence of lower dimensional invariant tori under small perturbations.

The persistence of *normally hyperbolic* invariant tori, together with their stable and unstable manifolds, is addressed in [Graff;74] and [Zehnder;75,76], see also [Broer,Huitema,Takens;90]. It turns out that these survive a small Hamiltonian perturbation, provided that the (internal) frequencies satisfy certain *diophantine conditions*. In fact, these diophantine conditions are completely analogous to those that guarantee persistence of maximal tori. The set of *diophantine frequency vectors* is a Cantor set of large relative measure and may be used to parametrise the persisting invariant tori.

The persistence of *normally elliptic* invariant tori is more involved, cf. [Mel'nikov;65,68], [Moser;67], [Lieberman;72], [Eliasson;88], [Pöschel;89,94], [Broer,Huitema,Takens;90], [Kuksin;93], [Chierchia,Falcolini;96], [Jorba,Villanueva;96] and [Broer,Huitema,Sevryuk;96]. Next to the diophantine conditions on the internal frequencies there are also diophantine conditions involving the *normal frequencies* that have to be satisfied. Again this defines a Cantor set of large relative measure, however, the structure of this set differs from the above analogue, see Section 2 for the details. These results carry over to general lower dimensional tori, provided that corresponding diophantine conditions on the internal and normal frequencies hold.

Next in line are those lower dimensional tori that fail to fulfil the diophantine conditions involving normal frequencies, but in an as mild as possible way. Let us denote by $\omega_1, \ldots, \omega_n$ the (internal) frequencies of an n-torus and by Ω and Λ two of its normal frequencies, *i.e.* imaginary parts of purely imaginary eigenvalues of the linearization at the torus. Momentarily disregarding the necessary transversality conditions, there are three types of bifurcations, determined by equations

$$\Omega + \sum_{\nu=1}^{n} k_\nu \omega_\nu = 0 \tag{1}$$

$$2\,\Omega + \sum_{\nu=1}^{n} k_\nu \omega_\nu = 0 \tag{2}$$

$$\Omega - \Lambda + \sum_{\nu=1}^{n} k_\nu \omega_\nu = 0 \quad . \tag{3}$$

Choosing all $k_\nu = 0$ leads to the (quasi-periodic) centre-saddle bifurcation and to the resonance $\Omega = \Lambda$. For $k_\nu \neq 0$ it is instructive to consider the periodic case $n = 1$. Passing from Floquet exponents to Floquet multipliers and writing $\omega = \omega_1$, equation (1) reads $\mathrm{e}^{2\pi\mathrm{i}\frac{\Omega}{\omega}} = 1$, corresponding to a (periodic) centre-saddle bifurcation. For uneven k equation (2) becomes $\mathrm{e}^{2\pi\mathrm{i}\frac{\Omega}{\omega}} = -1$, leading to a Hamiltonian flip or period doubling bifurcation where, under variation of *e.g.* the energy, an elliptic periodic orbit turns into a hyperbolic periodic orbit and a periodic orbit of twice the period emerges. Equation (3) leads again to resonance, *i.e.* a Hamiltonian Hopf bifurcation, cf. [v.d.Meer;85], or the 1-1-resonance, cf. [Cotter;86].

In this paper we treat lower dimensional tori with a vanishing normal frequency $\Omega = 0$. Under appropriate transversality conditions a quasi-periodic centre-saddle bifurcation takes place. For simplicity we restrict to vector fields, a completely analogous theory may be formulated and proven for diffeomorphisms. Furthermore, we concentrate on bifurcating n-tori in $n+1$ degrees of freedom, and we remain in the real analytic category. Both restrictions are not essential, but they considerably simplify the proof and still cover the most important applications. We will shortly comment on how the occurrence of further normal frequencies can be incorporated, and why the results carry over to (in)finitely differentiable vector fields.

One of the transversality conditions we impose is that the linearized normal behaviour of the bifurcating tori be governed by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the tori are *normally parabolic*. Under further transversality conditions on the nonlinear part and dependence on the actions conjugate to the toral angles two families of normally hyperbolic and elliptic invariant tori meet at the family of normally parabolic tori and vanish. This is well-known in the integrable case, cf. [Meyer;70,75], [Broer,Chow,Kim,Vegter;93], and the aim of the present paper is to understand a small non-integrable perturbation of this bifurcation. The setting of the problem is inspired by similar studies of dissipative quasi-periodic bifurcations in [Braaksma,Broer;87], [Braaksma,Broer,Huitema;90]. In particular, we are not only interested in persistence of the normally parabolic invariant tori themselves, but we also want to show that the whole quasi-periodic centre-saddle bifurcation persists, with all lower dimensional invariant tori parametrised by pertinent Cantor sets.

This paper is organised as follows. In Section 2 we formulate the main results, both for systems that depend on external parameters and for isolated systems, where the actions conjugate to the toral angles act as internal or *distinguished* parameters. Furthermore we indicate possible generalizations. The presentation of Section 2 is based on a normal form theorem, the proof of which constitutes Section 4. Section 3 contains an application of these results to the dynamics of a rigid body. For weak external forces this is a perturbation of the Euler top, a properly degenerate or *superintegrable* system. In particular, the bifurcation scenario is generated by the perturbation itself and not present in the unperturbed system.

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§2 Formulation of the results

Motivated by the results for equilibria and periodic orbits, we formulate a normal form theorem which immediately establishes the persistence of normally parabolic invariant tori. For the normally hyperbolic and normally elliptic tori that bifurcate from these we apply the results of [Moser;67], [Pöschel;89] and [Broer,Huitema,Takens;90]. Then we replace the occurring parameters by the actions conjugate to the toral angles, and finally we discuss possible generalizations.

2.1 The normal form theorem

In a centre-saddle bifurcation a centre and a saddle meet and vanish under variation of a parameter λ , compare *Figure 2.1.* In the literature this is also called an extremal bifurcation or the Hamiltonian saddle-node bifurcation. This is a local phenomenon and takes place in two dimensions, *i.e.* in one degree of freedom. A standard Hamilton function for the centre-saddle bifurcation is

$$a\,p^2 \;+\; \frac{b}{3}\,q^3 \;-\; d\,\lambda\,q \qquad .$$

Let us attach an invariant n-torus T to every point of the (q,p)–plane, *i.e.* consider the Hamilton function

$$\mathcal{K}(x,y,q,p,\omega,\lambda) \;=\; (\omega\,|\,y) \;+\; a(\omega)\,p^2 \;+\; \frac{b(\omega)}{3}\,q^3 \;-\; d\,\lambda\,q \tag{4}$$

on $\mathsf{T} \times \mathbb{R}^n \times \mathbb{R}^2$ (where we have the symplectic structure $\sum_{\nu=1}^{n} \mathrm{d}x_\nu \wedge \mathrm{d}y_\nu + \mathrm{d}q \wedge \mathrm{d}p$). Next to the 'bifurcation parameter' $\lambda \in \mathbb{R}$ we also consider the frequency vector $\omega \in \Sigma \subseteq \mathbb{R}^n$ as parameter. While we think of y, q, p and λ as being small, the set Σ of frequencies has a global character.

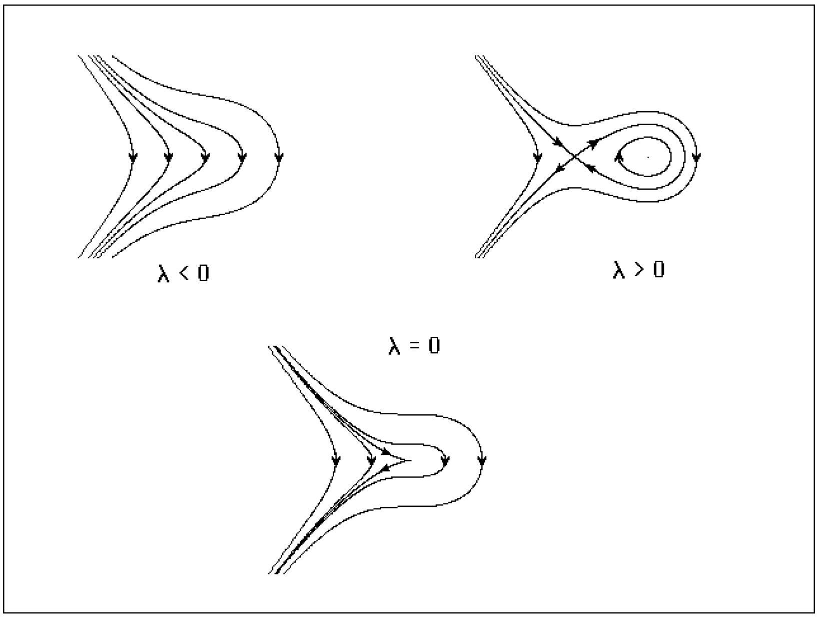

Figure 2.1: The centre-saddle bifurcation.

The aim of this paper is to understand the behaviour of a small perturbation $\mathcal{H}$ of $\mathcal{K}$ under variation of the parameter λ . Since $\mathcal{K}$ does not depend on x , this unperturbed

Hamilton function is integrable. Already for $n = 1$ the perturbed Hamilton function $\mathcal{H}$ will generally not be integrable. We are interested in the fate of the singular leaves of the ramified $(n+1)$-torus bundle defined by $\mathcal{K}$. On these n-tori the differential $\mathrm{d}\mathcal{K}$ depends linearly on the differentials of the other n integrals $y_1, \ldots, y_n$.

In the periodic case $n = 1$ this reads $\mathrm{d}\mathcal{K} = \omega \, \mathrm{d}y$. If the frequency ω is bounded away from zero, one can invert this relation and use the energy as the bifurcation parameter. Passing to iso-energetic Poincaré sections, the dynamics of the Poincaré map on the different energy shells is that depicted in *Figure 2.1.* Using the implicit mapping theorem, one can show that this scenario persists under small perturbations, cf. [Meyer;70,75], [Hanßmann;95]. In particular, all periodic orbits survive the perturbation.

For $n \geq 2$ we can only expect n-tori that have a diophantine frequency vector ω to persist. This is not an "open" condition (like $\omega \neq 0$ was), upon a small perturbation a single diophantine frequency vector may become resonant. Therefore, we consider a whole domain $\Sigma \subseteq \mathbb{R}^n$ of frequency vectors. For the diophantine condition we fix $\tau > n - 1$ and write

$$\Sigma_\gamma \;:=\; \left\{ \, \omega \in \Sigma \;\middle|\; |(\omega \,|\, k)| \geq \frac{\gamma}{|k|^\tau} \;\; \forall_{k \in \mathbb{Z}^n \setminus \{0\}} \, \right\} \quad .$$

We will also need $\Sigma'_\gamma \;:=\; \{ \omega \in \Sigma_\gamma \mid d(\omega, \partial\Sigma) \geq \gamma \}$. Our ultimate goal is to formulate conditions under which the internal parameter y can play the rôle of λ and ω . In the periodic case it is sufficient to require $\frac{\partial \lambda}{\partial y} \neq 0$ (next to $\frac{\partial \mathcal{K}}{\partial y} \neq 0$). We address this question in Theorem 2.3 below, here we decouple the frequency condition and treat λ and ω as externally given parameters.

When proving a centre-saddle bifurcation theorem for equilibria or periodic orbits, the difficult part is to find the parabolic equilibrium / periodic orbit in the perturbed system. Our first step is therefore to look for normally parabolic invariant n-tori of $X_{\mathcal{H}}$.

Theorem 2.1

Let T *be an* n*-torus,* Y *a neighbourhood of the origin in* $\mathbb{R}^n$ *and* S *a neighbourhood of the origin in* $\mathbb{R}^2$ *. Supply* $\mathsf{T} \times \mathsf{Y} \times \mathsf{S}$ *with the symplectic structure* $\sum_{\nu=1}^{n} \mathrm{d}x_\nu \wedge \mathrm{d}y_\nu + \mathrm{d}q \wedge \mathrm{d}p$ *, where* $x \in \mathsf{T}$ *,* $y \in \mathsf{Y}$ *and* $(q,p) \in \mathsf{S}$ *. Consider a real analytic Hamilton function* $\mathcal{K}$ *on* $\mathsf{T} \times \mathsf{Y} \times \mathsf{S}$ *, depending analytically on the parameters* $\lambda \in]-\epsilon, \epsilon[$ *and* $\omega \in \Sigma \subseteq \mathbb{R}^n$ *, where* Σ *is compact with non-empty interior. The Taylor series of* $\mathcal{K}$ *starts with the expression* (4)*, the derivatives* $\frac{\partial \mathcal{K}}{\partial q}$ *,* $\frac{\partial \mathcal{K}}{\partial p}$ *,* $\frac{\partial^2 \mathcal{K}}{\partial q^2}$ *and* $\frac{\partial^2 \mathcal{K}}{\partial q \partial p}$ *vanish for* $y = 0 = q = p = \lambda$ *. Furthermore,* $\mathcal{K}$ *does not depend on* x *. In* (4)*,* d *is a constant in* $]\,0,1]$ *, and the functions* $a, b : \Sigma \longrightarrow \mathbb{R}$ *are assumed to satisfy* $\frac{1}{\Gamma} d < a, b < \Gamma d$ *and* $\|Da\|, \|Db\| < \Gamma$ *for some constant* $\Gamma > 0$ *.*

Fixing $\tau > n - 1$ *,* $\gamma > 0$ *and* $N \in \mathbb{N}$ *, there exists a small positive constant* δ *, independent of* Σ *and* d *, with the following property. Given a real analytic perturbation*

$\mathcal{H}$ of $\mathcal{K}$ with $\frac{1}{d}\|\mathcal{H}-\mathcal{K}\|_{\mathsf{T}\times\mathsf{Y}\times\mathsf{S}\times\Sigma\times\,]-\epsilon,\epsilon[}\;<\;\delta$ *, there exists a C^∞-diffeomorphism Φ on* $\mathsf{T}\times\mathbb{R}^n\times\mathbb{R}^2\times\Sigma\times\mathbb{R}$ *such that*

1). *Φ is real analytic for fixed ω .*

2). *Φ is symplectic for fixed (ω,λ) .*

3). *Φ is C^∞-close to the identity.*

4). *On* $\mathsf{T}\times\mathbb{R}^n\times\mathbb{R}^2\times\Sigma'_\gamma\times\mathbb{R}\cap\Phi^{-1}(\mathsf{T}\times\mathsf{Y}\times\mathsf{S}\times\Sigma\times]-\epsilon,\epsilon[\,)$ *one can split* $\mathcal{H}\circ\Phi\;=\;\widetilde{\mathcal{K}}+H$ *into an integrable part $\widetilde{\mathcal{K}}$ and higher order terms H , both of class C^∞ , and real analytic for fixed ω . The 3-jet of $\widetilde{\mathcal{K}}$ reads*

$$\tilde{s}(\omega,\lambda)\;+\;(\omega\,|\,y)\;+\;(\tilde{c}\,|\,y)\,q\;+\;(\tilde{e}\,y\,|\,y)\;+\;\sum_{|j|=1}^{3}\;\sum_{k+l+m=0}^{3-|j|}\tilde{r}_{jklm}\,y^jq^kp^l\lambda^m$$
$$+\;\tilde{a}\,p^2\;+\;\frac{\tilde{b}}{3}q^3\;-\;d\,\lambda\,q\;+\;\tilde{r}_{0120}\,qp^2\;+\;\tilde{r}_{0030}\,p^3\;+\;\tilde{r}_{0102}\,q\lambda^2\;+\;\tilde{r}_{0021}\,p^2\lambda$$

where all coefficients but d depend on ω . The x-dependence is pushed into the higher order terms, i.e. $\dfrac{\partial^{|j|+k+l+m}H}{\partial y^j\partial q^k\partial p^l\partial\lambda^m}(x,0,0,0,\omega,0)\;=\;0$ *for all* $(x,\omega)\in\mathsf{T}\times\Sigma'_\gamma$ *and all j,k,l,m with $|j|+k+l+m\le N$.*

The proof of Theorem 2.1 shows that if Y and S are (small) ϵ-neighbourhoods of the origin, then the small positive constant δ may be chosen as $\delta=\epsilon^{3\frac{5}{6}}$. Furthermore, it turns out that for all but three terms in the Taylor series of $\mathcal{H}-\mathcal{K}$ a bound $\epsilon^{3\frac{1}{3}}$ is sufficient, and the y-dependent part only needs to be bounded by $\epsilon^{2\frac{5}{6}}$. We postpone this proof to Section 4 below. In this section we dwell on the implications. An immediate consequence of Theorem 2.1 is the existence of a Cantor family of normally parabolic invariant n-tori. In the new co-ordinates such tori lie for $\lambda=0$ at the "origin" $\mathsf{T}\times\{0\}\times\{(0,0)\}\subseteq\mathsf{T}\times\mathsf{Y}\times\mathsf{S}$. Note that these co-ordinates are valid only for diophantine frequency vectors $\omega\in\Sigma'_\gamma$, a Cantor set of large (n-dimensional) measure.

2.2 Persistence of the bifurcation scenario

Our next step is to show that the family of normally parabolic tori is part of an $(n+1)$-dimensional Cantor family of n-tori that are invariant under the flow of $X_{\mathcal{H}}$. For the system defined by the 3-jet of $\widetilde{\mathcal{K}}$ this family is given by $y=0=p$ and $b(\omega)q^2=d\,\lambda$. We treat the two half spaces $q > 0$ and $q < 0$ separately as the structure of the parametrising Cantor set depends on the normal behaviour of the invariant n-torus.

We start with the normally elliptic n-tori and apply to the new co-ordinates of Theorem 2.1 the non-symplectic co-ordinate transformation

$$(x, y, q, p) \mapsto (X, Y, Q, P) = (x, \lambda^{-\frac{1}{4}} y, q - \sqrt{\frac{d\lambda}{\tilde{b}(\omega)}}, \lambda^{-\frac{1}{4}} p) \tag{5}$$

together with the rescaling of time $T = \lambda^{\frac{1}{4}} t$. The equations of motion defined by the integrable part $\widetilde{\mathcal{K}}$ become

$$\begin{aligned}
\dot{X} &= \lambda^{-\frac{1}{4}} \left(\omega + \sqrt{\frac{d\lambda}{\tilde{b}(\omega)}}\, \tilde{c}(\omega) + Q\, \tilde{c}(\omega) \right) + \tilde{f}(Y, Q, P, \omega, \lambda) \\
\dot{Y} &= 0 \\
\dot{Q} &= 2\tilde{a}(\omega) P + \tilde{h}_1(Y, Q, P, \omega, \lambda) \\
\dot{P} &= -2\sqrt{d\tilde{b}(\omega)}\, Q - \lambda^{-\frac{1}{4}} (\tilde{c}(\omega) \,|\, Y) + \tilde{h}_2(Y, Q, P, \omega, \lambda) \quad .
\end{aligned}$$

In these co-ordinates the normally elliptic n-tori occur for every $\lambda > 0$ at the "origin" $Y = 0 = Q = P$. We investigate their persistence under the perturbation by the higher order terms H on the cone-like domain $D := \{\|Y\|, |Q|, |P| \leq \lambda^{\frac{3}{4}}\}$. Abbreviating $Z := (Q, P)$ and $\Omega := 2 \begin{pmatrix} 0 & \tilde{a}(\omega) \\ -\sqrt{d\,\tilde{b}(\omega)} & 0 \end{pmatrix}$ we write this vector field

$$\left(\frac{\omega}{\lambda^{\frac{1}{4}}} + \lambda^{\frac{1}{4}} \sqrt{\tfrac{d}{\tilde{b}}}\, \tilde{c} + \tfrac{Q}{\lambda^{\frac{1}{4}}} \tilde{c} + f \right) \frac{\partial}{\partial X} + g \frac{\partial}{\partial Y} + \left(\Omega Z - \tfrac{(\tilde{c} \,|\, Y)}{\lambda^{\frac{1}{4}}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + h \right) \frac{\partial}{\partial Z}$$

with

$$\begin{aligned}
f &= \tilde{f} + \tfrac{\partial H}{\partial Y} & \qquad g &= -\tfrac{\partial H}{\partial X} \\
h_1 &= \tilde{h}_1 + \tfrac{\partial H}{\partial P} & \qquad h_2 &= \tilde{h}_2 - \tfrac{\partial H}{\partial Q} \quad .
\end{aligned}$$

For our purpose it is sufficient to choose the order $N = 3$ in Theorem 2.1. On D the perturbative terms $f \frac{\partial}{\partial X} + g \frac{\partial}{\partial Y} + h \frac{\partial}{\partial Z}$ can be estimated by

$$\|f\|_D \leq C \lambda^{\frac{3}{4}} \quad , \quad \|g\|_D \leq C \lambda^{\frac{3}{2}} \quad \text{and} \quad \|h\|_D \leq C \lambda$$

with a constant C independent of λ . We want to apply Theorem 6.1 of [Broer,Huitema,Takens;90]. The relation between the size of the neighbourhood D of the n-tori and the smallness of the allowed perturbation is made precise in (the analogue of) Proposition 8.10 of [Broer,Huitema,Takens;90] : We have to make sure that $\|f\|_D < \lambda^{\frac{3}{4\mathsf{q}}}$, $\|g\|_D < \lambda^{\frac{9}{4\mathsf{q}}}$ and $\|h\|_D < \lambda^{\frac{3}{2\mathsf{q}}}$, where we may choose q from the interval $]1, 2[$. To fix thoughts we take $\mathsf{q} = \frac{5}{3}$. For every $\lambda < \frac{1}{C^{10}}$ we obtain normally elliptic invariant n-tori in D , parametrised by the Cantor set of those $\omega \in \Sigma'_\gamma$ that satisfy furthermore for all $k \in \mathbb{Z}^n \backslash \{0\}$ and all $l \in \{0, \pm 1, \pm 2\}$ the inequality

$$| \, (\omega \,|\, k) \; + \; \sqrt{\frac{d\lambda}{\tilde{b}(\omega)}} \, (\tilde{c}(\omega) \,|\, k) \; + \; 2 \, l \sqrt{\tilde{a}(\omega) \sqrt{d\lambda \tilde{b}(\omega)}} \; | \; \geq \; \frac{\gamma \lambda^{\frac{1}{4}}}{|k|^\tau} \quad . \tag{6}$$

Allowing λ to vary as well, we obtain the desired $(n+1)$-dimensional Cantor family of normally elliptic n-tori emerging from the normally parabolic $\mathsf{T} \times \{0\} \times \{(0,0)\} \times \Sigma'_\gamma \times \{0\}$.

For the normally hyperbolic n-tori near $\mathsf{T} \times \{0\} \times \{(-\sqrt{\frac{d\lambda}{\tilde{b}(\omega)}}, 0)\} \times \Sigma'_\gamma \times \{\lambda\}$ we consider a co-ordinate transformation similar to (5), with $Q = q + \sqrt{\frac{d\lambda}{\tilde{b}(\omega)}}$. Because of the hyperbolicity we can use a centre manifold and get by 'ordinary' KAM theory n-tori parametrised by Σ'_γ , see also [Graff;74] or [Broer,Huitema,Takens;90]. We have thus proven the following result.

Corollary 2.2

Under the assumptions of Theorem 2.1 the Hamiltonian vector field $X_{\mathcal{H}}$ has an $(n+1)$-dimensional Cantor family of invariant n-tori. In the new co-ordinates supplied by Theorem 2.1 the n-tori with $q > 0$ are normally elliptic and those with $q < 0$ are normally hyperbolic. They meet in an n-dimensional Cantor family of normally parabolic invariant n-tori.

We say that $\mathcal{H}$ undergoes a quasi-periodic centre-saddle bifurcation. We can get an impression on how the structure of the parametrising Cantor set changes for $n = 2$. To this end we further reduce the dimension by fixing *e.g.* the length of the frequency vector ω . In *Figure 2.2* the Cantor set is thus sketched projecting a slice of (ω, λ)-space. The upper (lower) part parametrises normally elliptic (hyperbolic) tori and the boundary between these is the Cantor set Σ'_γ that parametrises the normally parabolic tori.

To be precise, *Figure 2.2* depicts the Cantor set defined by (6) only if $\tilde{c}(\omega) \equiv 0$. Indeed, for $\tilde{c}(\omega) \neq 0$ the inequality (6) with $l = 0$ defines a Cantor set that slightly differs from Σ'_γ , and in particular the intersection of these would lead to *Cantor dust* also in the lower part of *Figure 2.2.* Let us discuss how to circumvent this problem.

There are two approaches to show persistence of lower dimensional tori. In [Broer,Huitema,Takens;90] the actions y conjugate to the toral angles x , and the co-ordinates $z = (q,p)$ in the normal symplectic direction are of equal size $\leq \varepsilon$, while the perturbing vector field $f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + h \frac{\partial}{\partial z}$ is estimated according to $\|f\| \leq \delta$, $\|g\| \leq \delta^3$, $\|h\| \leq \delta^2$

Figure 2.2: The structure of the parametrising Cantor set changes when the invariant torus passes from normally hyperbolic to normally elliptic

with $\delta^{\mathsf{q}} = \varepsilon$, $\mathsf{q} \in]1,2[$. In terms of the perturbing Hamilton function H this would lead to two kinds of estimates, splitting H into a z-dependent and a z-independent part. Fixing thoughts on the limiting case $\mathsf{q} = 2$, the former part would need an estimate $\leq \varepsilon^2$ and the latter only $\leq \varepsilon^{\frac{3}{2}}$.

In [Pöschel;89] one estimate $\|H\| \leq \varepsilon^2$ suffices, but the supremum norm $\|H\|$ is measured on a domain with $\|y\| \leq \varepsilon^2, \|z\| \leq \varepsilon$. This approach allows us to deal with the problematic term $(\tilde{c}(\omega) \,|\, y)\, q$ of $\widetilde{\mathcal{K}}$. Indeed, after the transformation (5) we may write $\lambda^{-\frac{1}{2}}\widetilde{\mathcal{K}}$ as

$$\tilde{s}(\omega,\lambda) \;+\; (\frac{\omega}{\lambda^{\frac{1}{4}}} \,|\, Y) \;+\; \tilde{a}(\omega)\, P^2 \;+\; \sqrt{d\tilde{b}(\omega)}\; Q^2 \;+\; \widetilde{H}$$

with $\|\widetilde{H}\|_{D} \leq C\lambda^{\frac{7}{4}}$ on the domain $D := \{ \|Y\| \leq \lambda^{\frac{3}{2}} , |Q|, |P| \leq \lambda^{\frac{3}{4}} \}$; the constant part $\tilde{s}(\omega,\lambda)$ is irrelevant for the dynamics. Note that we have to multiply the Hamilton function by $\lambda^{-\frac{1}{2}}$ to account for the time scaling by $\lambda^{\frac{1}{4}}$ and to compensate the factor $\lambda^{\frac{1}{4}}$ in the transformed symplectic 2-form. To ensure that also the X-dependent higher order terms satisfy $\|\lambda^{-\frac{1}{2}}H\|_{D} \leq C\lambda^{\frac{7}{4}}$ we have to choose the order $N = 4$ in Theorem 2.1. For every $\lambda < \frac{1}{C^4}$ the normally elliptic invariant n-tori in D are now parametrised by the Cantor set

$$\{ \; (\omega,\lambda) \in \Sigma'_{\gamma} \times]0,\epsilon[\;\;\Big|\;\; |\, (\omega \,|\, k) + 2l\sqrt{\tilde{a}(\omega)\sqrt{d\lambda\tilde{b}(\omega)}} \;| \geq \frac{\gamma\lambda^{\frac{1}{4}}}{|k|^{\tau}} \quad \forall_{k \in \mathbb{Z}^n \setminus \{0\}} \; \forall_{l \in \{\pm1,\pm2\}} \; \} \qquad ,$$

as desired. Similarly, $\Sigma'_{\gamma} \times]0,\epsilon[$ parametrises the normally hyperbolic invariant n-tori.

The persisting normally hyperbolic n-tori have stable and unstable manifolds. As is shown in [Graff;74] and [Zehnder;75,76], the stable (unstable) manifold of a persistent torus is a continuation of the stable (unstable) manifold in the unperturbed system. For the system $X_{\widetilde{\mathcal{K}}}$ defined by (4) the stable and unstable manifolds coincide in one arc, see *Figure 2.1.* But for $X_{\mathcal{H}}$ we cannot expect this any more. Already in the periodic case $n = 1$ it is a generic property that these manifolds intersect transversely, see [Robinson;70a,b] and [Broer,Tangerman;86].

For the unperturbed system $X_{\mathcal{K}}$ we have a family of invariant $(n+1)$-tori emerging from the normally elliptic n-tori that extends to the (un)stable manifolds of the normally hyperbolic n-tori. We expect the resonant $(n+1)$-tori to break up under the perturbation. It has been shown in [Sanders,Chow;86] that the KAM-frequency-conditions are fulfilled, ensuring the persistence of most $(n+1)$-tori, parametrised by a Cantor set.

2.3 Independence of external parameters

In the formulation of Theorem 2.1 the unperturbed Hamilton function $\mathcal{K}$ depends on the parameters ω and λ . We now want to account for these parameters by variables of the system. However, the parameter dependent version is also interesting in its own right. In applications, external parameters μ usually do not appear themselves as frequencies ω or 'bifurcation parameter' λ . It is sufficient that the map $\mu \mapsto (\omega(\mu), \lambda(\mu))$ is a submersion. Our aim is to use the action y conjugate to the torus angle x as internal or distinguished parameter. One difficulty is that Y is only n-dimensional, so $y \mapsto (\omega(y), \lambda(y))$ cannot be submersive.

This difficulty can be overcome as follows. Up to now we used a co-ordinate transformation ; on $\mathsf{T} \times \mathsf{Y} \times \mathsf{S} \times \Sigma'_\gamma \times \,]{-\epsilon}, \epsilon[$ the flows of $X_{\mathcal{H}}$ and $X_{(\widetilde{\mathcal{K}}+H)}$ were conjugate. Now we weaken this and require $X_{\mathcal{H}}$ and $X_{(\widetilde{\mathcal{K}}+H)}$ only to have equivalent flows. This corresponds to a reparametrisation of time and reduces the number of necessary parameters from $n+1$ to n . Already in the periodic case $n = 1$ one is interested in equivalences rather than conjugacies, allowing the use of a Poincaré section in the proof. It is not the frequency vector ω itself we want to control, but only the mutual ratios $[\omega_1, \omega_2, \ldots, \omega_n]$. See [Arnol'd;74] or [Broer,Huitema,Takens;90] for similar considerations.

Theorem 2.3

Let T *be an* n*-torus,* Y *a neighbourhood of the origin in* $\mathbb{R}^n$ *and* S *a neighbourhood of the origin in* $\mathbb{R}^2$ *. Supply* $\mathsf{T} \times \mathsf{Y} \times \mathsf{S}$ *with the symplectic structure* $\sum_{\nu=1}^{n} \mathrm{d}x_\nu \wedge \mathrm{d}y_\nu + \mathrm{d}q \wedge \mathrm{d}p$ *, where* $x \in \mathsf{T}$ *,* $y \in \mathsf{Y}$ *and* $(q,p) \in \mathsf{S}$ *. Consider a Hamilton function* $\mathcal{K}$ *on* $\mathsf{T} \times \mathsf{Y} \times \mathsf{S}$ *that* *does not depend on x . As a Taylor series in q and p it starts*

$$(\omega(y) \,|\, y) \;+\; a(y)\,p^2 \;+\; b(y)\,q^3 \;+\; c(y)\,q$$

with $c(0) = 0$, but $a(0) > 0$ and $b(0) > 0$. Furthermore the mapping

$$\begin{array}{rcl} \nu : \; \mathsf{Y} & \longrightarrow & \mathbb{R}\mathsf{P}^n \\ y & \mapsto & [c(y), \omega_1(y), \ldots, \omega_n(y)] \end{array}$$

is a submersion. Then a small perturbation $\mathcal{H}$ of $\mathcal{K}$ undergoes a quasi-periodic centre-saddle bifurcation.

Proof :

We add a parameter η and consider $\widehat{\mathcal{K}} := \eta\mathcal{K}$. Letting $\lambda := \hat{c}(y,\eta) := \eta c(y)$ and $\Sigma := \ker D\hat{c}(0,1)$ we choose new co-ordinates λ and ω on the (y,η)-space. Then we apply Corollary 2.2 ; **q.e.d.**

The smallness of the allowed perturbation depends on the diophantine condition on the persisting tori. We still consider $\tau > n-1$ as a constant. Thus we have to fix $\gamma > 0$ to get $\delta > 0$ such that $\|\mathcal{H} - \mathcal{K}\|_{\mathsf{T}\times\mathsf{Y}\times\mathsf{S}} < \delta$ quantifies small perturbations. Denoting $d := \min\{a(0), b(0), \|Dc(0)\|\}$ we even can find $\delta > 0$ independent of d such that $\frac{1}{d}\|\mathcal{H} - \mathcal{K}\|_{\mathsf{T}\times\mathsf{Y}\times\mathsf{S}} < \delta$ quantifies small perturbations. This becomes important in Section 3 where we consider the case that $\mathcal{H}$ is the perturbation of a superintegrable system and $\mathcal{K}$ its normal form of order one.

2.4 Generalizations

We chose to present our results in the real analytic category, though these are easily generalized to the context of infinite and finite differentiability. The corresponding proofs rely on a kind of diagonal procedure, cf. [Zehnder;75,76] or [Pöschel;82]. The key idea is to use the result in the real analytic category, approximating the system on every step of the KAM-iteration by a real analytic system. We use this technique in a simple case in Subsection 4.3 where we have to extend a result for real analytic Hamilton functions, allowing for C^∞-dependence on the frequency ω . Note that a smooth version of Theorem 2.1 immediately implies smooth versions of Corollary 2.2 and Theorem 2.3.

We also restricted to 'maximal' lower dimensional tori. It is straightforward to replace the integrable Hamilton function (4) by

$$(\omega \,|\, y) \;+\; (\Lambda w \,|\, w) \;+\; a\,p^2 \;+\; \frac{b}{3}\,q^3 \;-\; d\,\lambda\,q$$

with an invertible infinitesimally symplectic $m{\times}m$-matrix Λ . The hyperbolic part of Λ does not pose additional problems and can *e.g.* be dealt with by means of a centre manifold. Therefore we concentrate on the normally elliptic case

$$(\Lambda w \,|\, w) \;=\; \sum_{\mu=1}^{m} \Omega_\mu (u_\mu^2 + v_\mu^2)$$

where $w = (u_1, \ldots, u_m, v_1, \ldots, v_m)$. The normal frequencies Ω_μ are further parameters the coefficient functions a and b may depend upon. The Ω_μ also become involved in further diophantine conditions

$$|\,(\omega \,|\, k) + (\Omega \,|\, l)\,| \;\geq\; \frac{\gamma}{k^\tau} \qquad \forall_{k \in \mathbb{Z}^n \setminus \{0\}} \quad \forall_{l \in \mathbb{Z}^m, |l| \leq 2}$$

needed in the analogue of Theorem 2.1. This complicates the proof of the analogue of Theorem 2.3, and in fact one can only get a slightly weaker result. Already to allow for the actions y to play the rôle of the parameters ω and λ we had to reparametrise time, only the frequency ratio of the persistent tori could be controlled. For the mapping $y \mapsto (\omega, \Omega, \lambda)$ to have a large image within the Cantor set of diophantine frequencies, we resort to diophantine approximation of dependent quantities, cf. [Eliasson;88], [Pöschel;89], [Broer,Huitema,Sevryuk;96]. In this manner one can merely prove that the perturbed system possesses a Whitney-smooth family of invariant n-tori that undergoes a quasi-periodic centre-saddle bifurcation, it is no longer possible to connect the perturbed tori and the unperturbed ones. As shown in [Pöschel;89,94], [Wayne;90], [Kuksin;93] this kind of result can even be pushed to the case of infinitely many normal frequencies, thus proving persistence of finite-dimensional tori in infinite-dimensional Hamiltonian systems. This has applications to nonlinear partial differential equations.

In this paper we treat the quasi-periodic centre-saddle bifurcation in the Hamiltonian context. Indeed, under dissipative perturbations already the two-dimensional system

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= \lambda - q^2 \end{aligned} \tag{7}$$

is not structurally stable as a family in λ , but should be completed to the Bogdanov-Takens bifurcation

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= \lambda + \mu q - q^2 \pm pq \end{aligned} \quad .$$

The dissipative counterpart of the quasi-periodic centre-saddle bifurcation is the quasi-periodic saddle-node bifurcation, see [Braaksma,Broer,Huitema;90]. Things change if a perturbation of (7) has to preserve an additional structure, like a non-degenerate 2-form in the Hamiltonian context. In two dimensions such a 2-form is an area form, letting us conjecture that a persistence theorem for the quasi-periodic centre-saddle bifurcation also holds true in the volume-preserving context. For the unperturbed normal form one has to attach an invariant n-torus with equation $\dot{x} = 0$ to (7). At variance with the Hamiltonian case there are no internal parameters y one could use to replace (ω, λ) , for persistence results it is thus necessary to consider families of volume-preserving vector fields depending on external parameters.

The two-dimensional family (7) is also structurally stable in the reversible context. This leads us to the vector field

$$\omega(y)\frac{\partial}{\partial x} \;+\; a(y)p\frac{\partial}{\partial q} \;+\; [\lambda(y) - b(y)q^2]\frac{\partial}{\partial p}$$

on $\mathbb{T} \times \mathbb{R}^m \times \mathbb{R}^2$, and perturbations are assumed to preserve the involution

$$(x, y, q, p) \mapsto (-x, y, -q, p) \qquad .$$

In case $m \geq n$ we would require $y \mapsto [\lambda(y), \omega_1(y), \ldots, \omega_n(y)]$ to be a submersion, cf. Theorem 2.3. If this mapping is not submersive one could work with external parameters or resort to diophantine approximation of dependent quantities, see again [Broer,Huitema,Sevryuk;96].

The proof of persistence of the quasi-periodic centre-saddle bifurcation we give in Section 4 does not generalize to the volume preserving or reversible context since it is written on the level of Hamilton functions. However, as done in [Moser;67], [Broer,Huitema,Takens;90] and [Broer,Huitema;95] for persistent (non-bifurcating) lower dimensional tori, it should be possible to rewrite this proof in terms of structure preserving vector fields.

In the quasi-periodic centre-saddle bifurcation the non-degeneracy condition "no normal frequency vanishes" is violated in the mildest possible way. One can straightforwardly generalize to Hamilton functions in normal form

$$(\omega \,|\, y) \;+\; a\,p^2 \;+\; b\,q^d \;+\; \sum_{i=1}^{d-2} \lambda_i\, q^i \qquad . \tag{8}$$

Allowing also for the normal behaviour $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ leads to the elliptic and hyperbolic umbilics

$$p^2\, q \;\mp\; \frac{1}{3}q^3 \;+\; \lambda_1\,(p^2 \pm q^2) \;+\; \lambda_2\, p \;+\; \lambda_3\, q$$

or even to higher degenerate umbilics. Still more complicated are cases with several vanishing normal frequencies. In all these cases there are several unfolding-parameters $\lambda_1, \ldots, \lambda_m$ needed. Therefore, even after reparametrising time, there are not enough internal parameters $y_1, \ldots, y_n$ to replace (ω, λ) . One would thus resort to diophantine approximation of dependent quantities, with a necessary bound $m \leq n-1$. In general, it is not possible to avoid isolated n-tori in the unperturbed n-parameter family parametrised by the conjugate actions that undergo a bifurcation of co-dimension $m = n$. Though these are not persistent (in the absence of external parameters), a corresponding quasi-periodic bifurcation theory would yield information on other lower dimensional tori involved in the bifurcation, with the (non-persisting!) bifurcating torus playing the rôle of an organising centre.

When $d = 3$ the Hamilton function (8) describes the quasi-periodic centre-saddle bifurcation. In Section 4 we use bq^3 to get rid of terms q^2 during the KAM-step. For $d \geq 4$ it is not strictly necessary to get rid of terms q^{d-1} as these do not belong to the 2-jet. See [Broer,Hanßmann,You;97] for more details.

§3 Application to the rigid body

In this section we apply our results to perturbations of the Euler top, a dynamically symmetric free rigid body. The Euler top defines a superintegrable system, giving rise to a ramified torus bundle of invariant 2-tori. Such a 4-parameter family of 2-tori is a highly unstable situation, only 2-dimensional Cantor subfamilies can be expected to survive a perturbation. It is also the perturbation itself that has to lead to a quasi-periodic centre-saddle bifurcation. We consider a perturbation defined by a conservative force field, and concentrate on the case where the defining potential is a polynomial of degree two. The positional nature of the perturbing force leads to additional difficulties. In particular, we cannot directly apply Theorem 2.3, but have to resort to Theorem 2.1 and Corollary 2.2.

3.1 The Euler top

We perturb the free rigid body, fixed at one point. Therefore, we now recapitulate the main facts about it. The reader may find a comprehensive introduction in [Abraham,Marsden;78], [Arnol'd;74], [Cushman,Bates;96] or [Goldstein;50].

We choose a set of axes $\vec{e}_x, \vec{e}_y, \vec{e}_z$ fixed in space and a body set of axes $\vec{e}_1, \vec{e}_2, \vec{e}_3$ along the principal axes of inertia. The configuration space is the group $SO(3)$ of orientation preserving orthogonal three-by-three matrices. The elements of $SO(3)$ specify how to transform $\vec{e}_x, \vec{e}_y, \vec{e}_z$ into $\vec{e}_1, \vec{e}_2, \vec{e}_3$. The phase space is the cotangent bundle $T^*SO(3)$, the space of positions and (angular) momenta. The kinetic energy of the rigid body is defined by means of a left-invariant Riemannian metric $\langle\,..\,|\,..\,\rangle$, derived from the mass distribution. The relation to the bi-invariant Killing metric $(\,..\,|\,..\,)$ is given by the inertia tensor

$$I \;=\; \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \qquad ,$$

where $I_1 = I_2$ as we assume the body to be dynamically symmetric. An element $\alpha \in T^*SO(3)$ yields the components ℓ_1, ℓ_2, ℓ_3 of the angular momentum with respect to the body set of axes $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Then the kinetic energy is expressed as $T(\alpha) = \frac{1}{2}\langle\alpha\,|\,\alpha\rangle = \frac{\ell_1^2+\ell_2^2}{2I_1} + \frac{\ell_3^2}{2I_3}$.

Using the Killing metric we construct an S^1-action on $T^*SO(3)$, the *central action*. To this end define the function

$$|\mu| : \;\; T^*SO(3) \;\longrightarrow\; \mathbb{R}$$

by $|\mu|(\alpha) := \|\alpha\| := \sqrt{(\alpha\,|\,\alpha)}$, measuring the length of the angular momentum vector. On the complement $T^*SO(3)^{\setminus SO(3)}$ of the zero-section $SO(3) \subseteq T^*SO(3)$ we obtain a Hamiltonian vector field $X_{|\mu|}$, the trajectories of which are periodic with common period 2π . Thus the flow ψ_t of $X_{|\mu|}$ yields the central action

$$\begin{array}{rccc} \Gamma : & S^1 \times T^*SO(3)^{\setminus SO(3)} & \longrightarrow & T^*SO(3)^{\setminus SO(3)} \\ & (\xi, \alpha) & \mapsto & \psi_\xi(\alpha) \end{array} \qquad .$$

The orbits of the central action are related to a pure precession of the rigid body about the angular momentum. Therefore we call ξ the precession angle. The kinetic energy does not depend on the precession angle. Hence, we can use the central symmetry to reduce the phase space. Fixing the length $|\mu| \neq 0$ of the angular momentum vector, the orbit space $\{\alpha \in T^*SO(3) \;\big|\; \|\alpha\| = |\mu|\}\big/_{S^1}$ can be identified with

$$S^2_{|\mu|} \times S^2_{|\mu|} \;=\; \{\; (\mu, \ell) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\;\big|\;\; \|\mu\| = |\mu| = \|\ell\| \;\} \qquad ,$$

see [Hanßmann;95]. Here μ_1, μ_2, μ_3 are the components of the angular momentum with respect to the set of axes $\vec{e}_x, \vec{e}_y, \vec{e}_z$ fixed in space (and ℓ_1, ℓ_2, ℓ_3 those with respect to $\vec{e}_1, \vec{e}_2, \vec{e}_3$). The reduced Hamilton function is $H(\mu, \ell) = \frac{\ell_1^2+\ell_2^2}{2I_1} + \frac{\ell_3^2}{2I_3}$ and the symplectic structure on $T^*SO(3)$ reduces to $\dfrac{\sigma_{\scriptscriptstyle L}}{|\mu|} - \dfrac{\sigma_{\scriptscriptstyle R}}{|\mu|}$, where $\sigma_{\scriptscriptstyle L}$ ($\sigma_{\scriptscriptstyle R}$) denotes the area element on the left (right) hand sphere. Hence, the equations of motion are

$$\begin{aligned} \dot{\mu} \;&=\; \frac{\partial T}{\partial \mu} \times \mu \;=\; 0 \\ \dot{\ell} \;&=\; -\frac{\partial T}{\partial \ell} \times \ell \;=\; \ell \times I^{-1}(\ell) \end{aligned} \qquad .$$

In particular the equations decouple. On the right hand sphere we have Euler's equations ; since $I_1 = I_2$ these are

$$\dot{\ell}_1 \; = \; -\frac{I_3 - I_1}{I_1 I_3}\,\ell_2\ell_3 \; , \quad \dot{\ell}_2 \; = \; \frac{I_3 - I_1}{I_1 I_3}\,\ell_1\ell_3 \; , \quad \dot{\ell}_3 = 0 \qquad .$$

The vector field on the left hand sphere vanishes identically, the angular momentum is conserved.

From this we reconstruct the flow on $T^*SO(3)$. While the length $|\mu|$ of the angular momentum is conserved, the precession angle varies according to $\dot{\xi} = \frac{\partial T}{\partial|\mu|}$. To compute this we express T in generalized action-angle variables $(\xi, \rho, |\mu|, \Im, \mu_1, \mu_2, \mu_3)$. Here $\Im := \ell_3$ is the height on the right hand sphere, the component of the angular momentum along the *figure axis* $\vec{e}_3$, and ρ is the conjugate angle, measuring the rotation of the body about the figure axis. The kinetic energy now reads $T = \frac{|\mu|^2}{2I_1} - \frac{I_3 - I_1}{I_1 I_3} \cdot \frac{\Im^2}{2}$, and we conclude

$$\dot{\xi} = \frac{|\mu|}{I_1} \qquad \text{and} \qquad \dot{\rho} = -\frac{I_3 - I_1}{I_1 I_3}\Im \qquad .$$

The motion of the Euler top is a rotation with frequency $-\frac{I_3 - I_1}{I_1 I_3}\Im$ about the figure axis, superposed by a precession of the figure axis with frequency $\frac{|\mu|}{I_1}$ about the fixed angular momentum.

We perturb the free rigid body by a small conservative affine (constant+linear) force field. The set of axes $\vec{e}_x, \vec{e}_y, \vec{e}_z$ fixed in space may be chosen to consist of eigenvectors of the linear part of the force field. These axes give rise to co-ordinates x, y, z in space. The perturbing force field

$$\vec{G} \; = \; -\,\alpha\,\vec{e}_x \; - \; \beta\,\vec{e}_y \; - \; \gamma\,\vec{e}_z \; - \; 2ax\,\vec{e}_x \; - \; 2by\,\vec{e}_y \; - \; 2cz\,\vec{e}_z \tag{9}$$

leads to the potential energy

$$\begin{aligned} V_{\alpha,\beta,\gamma,a,b,c} \; &= \; \alpha s_1(\vec{e}_x \,|\, \vec{e}_1) \; + \; \beta s_1(\vec{e}_y \,|\, \vec{e}_1) \; + \; \ldots \; + \; \gamma s_3(\vec{e}_z \,|\, \vec{e}_3) \\ &\quad + \; a M_1(\vec{e}_x \,|\, \vec{e}_1)^2 \; + \; b M_1(\vec{e}_y \,|\, \vec{e}_1)^2 \; + \; \ldots \; + \; c M_3(\vec{e}_z \,|\, \vec{e}_3)^2 \end{aligned}$$

where $\vec{s} = s_1\vec{e}_1 + s_3\vec{e}_3$ is the centre of mass and $M_1 = M_2 = \frac{1}{2}I_3$ and $M_3 = I_1 - \frac{1}{2}I_3$ are the second moments of the mass distribution. It is straightforward (but lengthy) to express the matrix $g = \Big((\vec{e}_\zeta \,|\, \vec{e}_j)\Big)_{\substack{\zeta=x,y,z \\ j=1,2,3}}$ in the generalized action-angle variables $(\xi, \rho, |\mu|, \Im, \mu_1, \mu_2, \mu_3)$. In particular we see that the force field (9) is positional, *i.e.* the potential energy only depends on the configuration $g \in SO(3)$ of the system. As a result $V_{\alpha,\beta,\gamma,a,b,c}$ is homogeneous of degree zero in the momenta. Since an additive term $d \cdot (x^2 + y^2 + z^2)$ in the potential of the force field has no effect on the motion, we may assume $0 < a < b < c$.

3.2 The normal form

A 4-parameter family of invariant 2-tori is very degenerate, and we expect that most of these *Eulerian 2-tori* are destroyed under the perturbation by the affine force field, giving rise to a slow motion of the angular momentum. In this way the perturbation itself leads to a quasi-periodic centre-saddle bifurcation. The key idea is to use an intermediate (integrable) system 'between' the Euler top and the system with Hamilton function $H = T + V_\varepsilon$, where $\varepsilon = \sqrt{\alpha^2+\beta^2+\gamma^2+a^2+b^2+c^2}$. To obtain this intermediate system we normalise H with respect to the 2-torus symmetry invoked by the Euler top. Already the normal form of order one turns out to be sufficient for our purpose, this is exactly the average $\bar{H}$ of H along the Eulerian 2-tori.

Theorem 3.1

Let $H = T + V_\varepsilon$ be the Hamilton function of a rigid body, fixed at one point, with principal moments of inertia $I_1 = I_2 \neq I_3$ and moving in a conservative affine force field. Then the normal form of order one of H is

$$\bar{H}_{|\mu|,\Im}(\mu_1,\mu_2,\mu_3) \;=\; \frac{|\mu|^2}{2I_1} - \frac{I_3-I_1}{I_1I_3}\frac{\Im^2}{2} + \frac{a+b+c}{2}\,(I_1+(I_3-I_1)\frac{\Im^2}{|\mu|^2})$$
$$+\, s_3\frac{\Im}{|\mu|}\,(\alpha\frac{\mu_1}{|\mu|}+\beta\frac{\mu_2}{|\mu|}+\gamma\frac{\mu_3}{|\mu|}) - \frac{I_3-I_1}{2}\,\frac{3\Im^2-|\mu|^2}{|\mu|^2}\,(a\frac{\mu_1^2}{|\mu|^2}+b\frac{\mu_2^2}{|\mu|^2}+c\frac{\mu_3^2}{|\mu|^2}) \quad .$$

Let $\mathsf{T} = \{(\xi,\rho) \mid \xi \in S^1, \rho \in S^1\}$, denote by Y a set of $(|\mu|,\Im) \in \mathbb{R}^2$ with $\Im \in [-|\mu|,|\mu|]$ for which the frequency vector

$$\omega(|\mu|,\Im) = \begin{pmatrix} \dfrac{1}{I_1}\cdot|\mu| \\[2ex] -\dfrac{I_3-I_1}{I_1I_3}\cdot\Im \end{pmatrix}$$

is bounded away from resonances in $\{0,\pm1,\pm2\}^2$ and let $\mathsf{P} = \{(\mu_1,\mu_2,\mu_3) \in \mathbb{R}^3 \mid \mu_1^2 + \mu_2^2 \neq 0\}$. Then there is a Poisson co-ordinate transformation ψ on $\mathsf{T}\times\mathsf{Y}\times\mathsf{P}$ such that

$$||H\circ\psi - \bar{H}||_{\mathsf{T}\times\mathsf{Y}\times\mathsf{P}} \;\leq\; c_{\mathsf{Y}}\,\varepsilon^2$$

for some constant c_{Y} depending on the distance to the above resonances.

Proof :

See [Hanßmann;95] ; **q.e.d.**

The averaged Hamilton function $\bar{H}$ on $T^*SO(3)\backslash SO(3)$ is invariant under the central action Γ and under rotations about the figure axis. These two S^1-actions commute and define a 2-torus action. The corresponding momentum mapping is

$$(|\mu|, \Im): \quad T^*SO(3) \quad \longrightarrow \quad \mathbb{R}^2 \qquad .$$

The level sets of the momentum mapping are invariant manifolds of the Hamiltonian vector field $X_{\bar{H}}$. These level sets are also invariant under the 2-torus action. Dividing out this action leads to the reduced phase space $S^2_{|\mu|} \times \{\Im\}$. The reduced Hamilton function $\bar{H}_{|\mu|,\Im}$ defines a one-degree-of-freedom system on this reduced phase space. Recall that $S^2_{|\mu|} \times \{\Im\}$ comes equipped with the symplectic structure $\frac{\sigma_L}{|\mu|}$, where σ_L denotes the area element on the sphere. We consider the coefficients $\alpha, \beta, \gamma, a, b, c$ of the force field as fixed constants. The internal or *distinguished parameters* $|\mu|$ and $\Im$ on the other hand are allowed to vary. We want to show that $\bar{H}_{|\mu|,\Im}$ undergoes several centre-saddle bifurcations.

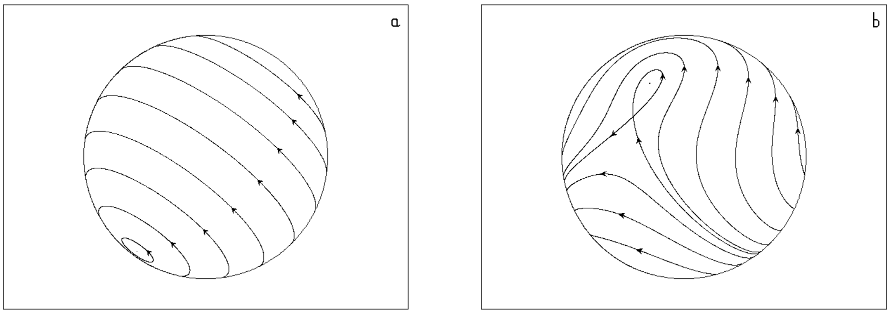

Figure 3.1: Possible flows on $S^2_{|\mu|} \times \{\Im\}$ to the reduced Hamilton function $\bar{H}_{|\mu|,\Im}$.

The orbits of the Hamiltonian vector field $X_{\bar{H}_{|\mu|,\Im}}$ are the intersections of the level sets $\{(\mu, \Im) \mid \bar{H}_{|\mu|,\Im}(\mu_1, \mu_2, \mu_3) = h\}$ with $S^2_{|\mu|} \times \{\Im\}$. Since $\bar{H}_{|\mu|,\Im}$ is a quadratic polynomial in μ_1, μ_2, μ_3 the level sets are quadrics. In this way the phase portraits are easily obtained. If $3\Im^2 = |\mu|^2$ the level sets are planes perpendicular to $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$. We get 2 centres, surrounded by periodic orbits, see *Figure 3.1a)*. When $3\Im^2 \neq |\mu|^2$ the level sets of $\bar{H}_{|\mu|,\Im}$ are tri-axial ellipsoids with centre

$$\frac{s_3 \, \Im \, |\mu|}{(I_3 - I_1)(3\Im^2 - |\mu|^2)} \begin{pmatrix} \alpha/a \\ \beta/b \\ \gamma/c \end{pmatrix} \qquad .$$

For values $(|\mu|, \Im)$ near $\{3\Im^2 = |\mu|^2\}$ this centre is close to infinity and the flow remains qualitatively as depicted in *Figure 3.1a)*. If the centre of the ellipsoid is closer to the sphere $S^2_{|\mu|} \times \{\Im\}$, the flow changes to the one of *Figure 3.1b)*. In between, a centre-saddle bifurcation takes place. There is a second centre-saddle bifurcation at the 'back side' of the sphere when the centre of the ellipsoid approaches the origin. For more details concerning the family $\bar{H}_{|\mu|,\Im}$ the reader is referred to [Hanßmann;95,97].

The vector field $X_{\bar{H}_{|\mu|,\Im}}$ on $S^2_{|\mu|} \times \{\Im\}$ depends only on the last two terms of $\bar{H}_{|\mu|,\Im}$. Furthermore, these terms are invariant under the transformation $\mu \mapsto \frac{\mu}{|\mu|}$ (which turns $\Im$ into $\frac{\Im}{|\mu|}$). Multiplying these last two terms of $\bar{H}_{|\mu|,\Im}$ by $\frac{1}{I_3 - I_1} \cdot \frac{|\mu|^2}{|\mu|^2 - 3\Im^2}$ we are led to

$$\mathcal{H}_{\mathcal{I}}(x,y,z) \;=\; \mathcal{I}\,(\alpha x + \beta y + \gamma z) \;+\; \frac{1}{2}(ax^2 + by^2 + cz^2) \tag{10}$$

with $x = \frac{\mu_1}{|\mu|}$, $y = \frac{\mu_2}{|\mu|}$, $z = \frac{\mu_3}{|\mu|}$ and $\mathcal{I} = \frac{s_3}{I_3 - I_1} \cdot \frac{\Im\,|\mu|}{|\mu|^2 - 3\Im^2}$. In fact we have again reduced the central S^1-action Γ : a (possibly time-reversing) scaling transforms the flow of $X_{\mathcal{H}_{\mathcal{I}}}$ on S^2 into the flow of $X_{\bar{H}_{|\mu|,\Im}}$ on $S^2_{|\mu|} \times \{\Im\}$. This shows that $\bar{H}_{|\mu|,\Im}$ is induced by a one-parameter family, the parameters $|\mu|$ and $\Im$ enter only as quotient $\frac{\Im}{|\mu|}$. We see that for (generic) positional forces only bifurcations of co-dimension one can occur.

Proposition 3.2

Let $(\alpha, \beta, \gamma, a, b, c)$ be in $\mathbb{R}^6$ with $\alpha, \beta, \gamma > 0$ and $0 < a < b < c$ and denote $\varepsilon = \sqrt{\alpha^2+\beta^2+\gamma^2+a^2+b^2+c^2}$. Then the one-parameter family $\mathcal{H}_{\mathcal{I}}$ of Hamiltonian systems defined by (10) has phase portraits as shown in Figure 3.1. Under variation of $\mathcal{I}$ the flow undergoes four centre-saddle bifurcations. Near such bifurcations, at a parameter value $\mathcal{I}_0$, one can choose local co-ordinates (q,p) on S^2 such that the 3-jet of $H_{\mathcal{I}}$ reads

$$H_{\mathcal{I}}^{\text{3-jet}}(q,p) \;=\; h(\mathcal{I}) \;+\; A(\mathcal{I})\,p^2 \;+\; \tfrac{1}{3}B(\mathcal{I})\,q^3 \;+\; C(\mathcal{I})(\mathcal{I}-\mathcal{I}_0)\,q \;+\; l(\mathcal{I})\,qp^2 \;+\; \tfrac{1}{3}m(\mathcal{I})\,p^3$$

where ε enters as a common factor and $\frac{\varepsilon}{\Gamma} \leq |A|, |B|, |C| \leq \Gamma\varepsilon$ for some constant $\Gamma > 0$.

Proof :

See [Hanßmann;95,97] ; **q.e.d.**

To return to the rigid body we have to translate this back to

$$\frac{\Im}{|\mu|} \;=\; \frac{1}{6\mathcal{I}}\Big(\frac{-s_3\beta}{I_3 - I_1} + \delta\sqrt{\frac{s_3^2\beta^2}{(I_3 - I_2)^2} + 12\mathcal{I}^2}\;\Big) \quad , \; \delta = \pm 1 \tag{11}$$

and attach the Eulerian 2-tori back to every point on the reduced phase spaces $S^2_{|\mu|} \times \{\Im\}$. The quasi-periodic centre-saddle bifurcation occurs when the distinguished parameters

$(|\mu|, \Im)$ cross a line defined by (11), with $\mathcal{I}$ taking one of the four values $\mathcal{I}_0$ guaranteed by Proposition 3.2.

The implications for the (averaged) dynamics of the rigid body are as follows. For $(|\mu|, \Im)$ near $\{3\Im^2 = |\mu|^2\}$ the motion can be derived from *Figure 3.1a)* : to the rotation of the rigid body about the figure axis and the precession of the figure axis about the angular momentum one has to superpose a slow rotational motion of the angular momentum nearly about the direction $\alpha\vec{e}_x + \beta\vec{e}_y + \gamma\vec{e}_z$ of the constant part of the affine force field. Near the quasi-periodic centre-saddle bifurcation the period of this slow rotational motion increases, and the arising normally parabolic 2-torus corresponds to a surviving Eulerian torus, a rotational-precessional motion with fixed angular momentum. After the bifurcation there are a normally hyperbolic and a normally elliptic surviving Eulerian torus, and for an open set of initial conditions the motion of the angular momentum is a libration rather than rotational.

3.3 Implications for the original system

In the previous subsection we have seen that the integrable normal form $\bar{H}$ undergoes several quasi-periodic centre-saddle bifurcations, and our present aim is to show that these persist when we return to the original system defined by $H = T + V_\varepsilon$.

In the case that the centre of mass lies on the figure axis the Hamilton function H is invariant under the right S^1-action. This allows to reduce to two degrees of freedom where it is considerably easier to show that the (reduced) periodic centre-saddle bifurcation persists, see [Hanßmann;95]. Here we concentrate on the case that the centre of mass is in general position.

One difficulty in applying the results of Section 2 is the positional character of the affine force field. Indeed, the same ratio $\frac{\Im}{|\mu|}$ that acts as the 'bifurcation parameter' λ in the quasi-periodic centre-saddle bifurcation also fixes the frequency ratio

$$\frac{\omega_1}{\omega_2} = -\frac{I_3 - I_1}{I_3} \cdot \frac{\Im}{|\mu|}$$

of the Eulerian 2-tori. Thus, the two distinguished parameters $|\mu|$ and $\Im$ are not sufficient to ensure persistence, the whole bifurcation scenario might fall into a resonance hole. Therefore we consider the inertia tensor as an external parameter. This weakens our result, we only get persistence of the quasi-periodic centre-saddle bifurcation for "most" rigid bodies.

Theorem 3.3

Let H_{I_1,I_3} be the family of Hamilton functions of dynamically symmetric rigid bodies, fixed at one point, with (fixed) centre of mass in general position and moving in a (fixed) generic small conservative affine force field. Then there is an open subset in (I_1, I_3)-space of large measure for which H_{I_1,I_3} undergoes several quasi-periodic centre-saddle bifurcations.

Proof :

The rôle of the 'bifurcation parameter' λ is played by the quotient $\frac{\Im}{|\mu|}$, and the values λ_0 where a quasi-periodic centre-saddle bifurcation takes place in the normal form $\bar{H}$ can be obtained from (11), inserting the corresponding values $\mathcal{I}_0$ given by Proposition 3.2. To apply Theorem 3.1 we have to bound $\frac{I_3-I_1}{I_3}\lambda_0$ away from $\{\pm\frac{1}{2}, \pm 1, \pm 2\}$, these are the first non-resonance conditions on (I_1, I_3) that have to be imposed. We want to apply Theorem 1.1 (and its Corollary) with $d := C(\mathcal{I}_0)$; from Proposition 3.2 we obtain $d = c_{\lambda_0}\varepsilon$ for some constant $c_{\lambda_0} \neq 0$. This allows to write the estimate given by Theorem 3.1 as

$$\frac{1}{d}\|H \circ \psi - \bar{H}\|_{\mathsf{T}\times\mathsf{Y}\times\mathsf{P}} \quad \leq \quad \frac{c_{\mathsf{Y}}}{c_{\lambda_0}}\,\varepsilon \qquad ,$$

so to ensure that the left hand side is smaller than the δ of Theorem 1.1 we simply fix $\varepsilon < \frac{c_{\lambda_0}}{c_{\mathsf{Y}}}\delta$. Since the ratio of the (internal) frequencies $\omega_1 = \frac{\partial\bar{H}}{\partial\Im}$ and $\omega_2 = \frac{\partial\bar{H}}{\partial|\mu|}$ of the normally parabolic torus is of the form

$$\frac{\omega_1}{\omega_2} \quad = \quad -\frac{I_3 - I_1}{I_3}\,\lambda_0 \; + \; O(\varepsilon) \qquad ,$$

the diophanticity of (ω_1, ω_2) for some $(|\mu|, \Im)$ is guaranteed by further non-resonance conditions on (I_1, I_3) . Varying $(|\mu|, \Im)$ along $\frac{\Im}{|\mu|} = \lambda_0$ makes sure that this does not lead to a Cantor set — only 'large' resonance zones have to be excluded **;** **q.e.d.**

For the persistence of the other main features of the averaged flow the reader is referred to [Hanßmann;95]. Note that the resonance strips of the Cantor set parametrising the normally hyperbolic tori are almost parallel to $\frac{\Im}{|\mu|} = \lambda_0$.

When dealing with perturbations of superintegrable systems like the Euler top, it is usually necessary to work with higher order normal forms, cf. [Arnol'd;63b], [Mazzocco;95]. The example we considered here is in this respect 'untypically simple' because almost all terms in the Fourier series of the perturbation V_ε of T vanish, allowing for the estimate of Theorem 3.1 to hold true on the relatively large domain $\mathsf{T}\times\mathsf{Y}\times\mathsf{P}$. In general one has to work in a small ϵ-neighbourhood of a fixed diophantine torus of the superintegrable system, and the careful estimates with respect to d and ϵ of Theorem 1.1 become more important. In [Lieberman;72] the theorem on persistence of normally elliptic invariant tori is refined in a similar way to take care of the special situations encountered in perturbations of superintegrable systems.

§4 The proof of Theorem 2.1

We want to find a co-ordinate transformation Φ that transforms away — to some extent — the small perturbation $\mathcal{H} - \mathcal{K}$ of the integrable quasi-periodic centre-saddle bifurcation $\mathcal{K}$. The idea is to construct Φ as a limit of transformations that approximately solve the equation $\mathcal{H} \circ \Phi = \mathcal{K}$. In the periodic case $n = 1$ such a sequence converges by means of a contraction argument and one can directly use the implicit mapping theorem. Here we encounter small denominators, so we have to argue differently. We use KAM-techniques as developed in [Kolmogorov;54], [Arnol'd;63a], [Moser;67], [Zehnder;75,76] and [Pöschel;82]. The quasi-periodic centre-saddle bifurcation can be thought of as a melange of the quasi-periodic saddle-node bifurcation as treated in §5 of [Braaksma,Broer,Huitema;90] with the structure preservation as dealt with in §8 of [Broer,Huitema,Takens;90]. Both proofs closely follow §5 of [Braaksma,Broer;87], and we also proceed along those lines. However, we work on the level of Hamilton functions rather than the level of vector fields.

4.1 Reduction to two propositions

The proof of Theorem 2.1 is rather involved, in particular as all details are included. The reader might prefer to skip the computational aspects at first (and probably also second) reading and concentrate on the line of argumentation. Let us centre the presentation around the following formulation of the result. On $\mathbb{R}^n$ we use the maximum norm $\|y\| = \max_\nu |y_\nu|$, and we abbreviate $\dfrac{\partial^{|j|}}{\partial y^j} = \dfrac{\partial^{j_{\nu_1} + \ldots + j_{\nu_n}}}{\partial y_{\nu_1}^{j_{\nu_1}} \ldots \partial y_{\nu_n}^{j_{\nu_n}}}$ for $|j| = 1$ to $\dfrac{\partial}{\partial y}$. Given $\Sigma \subseteq \mathbb{R}^n$, we still denote by Σ'_γ the (γ, τ)-diophantine elements of Σ with distance γ to $\partial\Sigma$.

Proposition 4.1

Let T *be an* n*-torus,* Y *a neighbourhood of the origin in* $\mathbb{R}^n$ *and* S *a neighbourhood of the origin in* $\mathbb{R}^2$ *. Supply* $\mathsf{T}\times\mathsf{Y}\times\mathsf{S}$ *with the symplectic structure* $\sum_{\nu=1}^{n} \mathrm{d}x_\nu \wedge \mathrm{d}y_\nu + \mathrm{d}q\wedge\mathrm{d}p$ *, where* $x\in\mathsf{T}$ *,* $y\in\mathsf{Y}$ *and* $(q,p)\in\mathsf{S}$ *. Let* $\mathcal{H}$ *be a real analytic perturbation of the Hamilton function*

$$\begin{aligned}\mathcal{K}(x,y,q,p,\omega,\lambda) \;=\; & (\omega\,|\,y) + a(\omega)\,p^2 + \frac{b(\omega)}{3}\,q^3 + (c(\omega)\,|\,y)\,q - d\,\lambda\,q \\ & + (e(\omega)\,y\,|\,y) + f(\omega)\lambda + g(\omega)\lambda^2 + h(\omega)\end{aligned} \tag{12}$$

which depends analytically on the parameters $\lambda\in\,]-\epsilon,\epsilon[$ *and* $\omega\in\Sigma\subseteq\mathbb{R}^n$ *,* Σ *open.*

To be precise, introduce a complex neighbourhood

$$\mathcal{D} \;:=\; U_\kappa(\mathsf{T})\times U_\epsilon(0)\times U_\epsilon(0,0)\times U_\kappa(\Sigma)\times U_\epsilon(0)$$

of $\mathsf{T}\times\mathsf{Y}\times\mathsf{S}\times\Sigma\times\,]-\epsilon,\epsilon[$ *; here* Y,S *and also* ϵ *may be chosen smaller. In* (12)*,* d *is a constant in* $]\,0,1]$ *, and the functions* a,b,c,e,f,g,h *on* $U_\kappa(\Sigma)$ *are assumed to satisfy* $\frac{1}{\Gamma}d\le a,b\le\Gamma d$ *,* $\|c\|,|g|\le\Gamma d$ *,* $\|e\|,|f|,|h|\le\Gamma$ *and* $|\frac{\partial a}{\partial\omega}|,|\frac{\partial b}{\partial\omega}|,\|\frac{\partial c}{\partial\omega}\|,\|\frac{\partial e}{\partial\omega}\|,|\frac{\partial f}{\partial\omega}|,|\frac{\partial g}{\partial\omega}|,|\frac{\partial h}{\partial\omega}|\le\Gamma$ *for some constant* $\Gamma>0$ *.*

Given $\tau>n-1$ *and* $\gamma>0$ *, there exist three positive constants* $\Delta\ge\delta\ge\partial>0$ *, independent of* Σ *and* d *, with the following property.*

Assume $\frac{1}{d}\|\mathcal{H}-\mathcal{K}\|_{\mathcal{D}}<\Delta$ *and also* $\frac{1}{d}\|\mathcal{H}-\mathcal{K}\|_{\mathcal{D}\cap\{y=0\}}<\delta$ *as well as*

$$\begin{aligned}&\tfrac{1}{2d}\Big\|\int_{\mathsf{T}}\tfrac{\partial^2}{\partial q^2}(\mathcal{H}-\mathcal{K})(x,0,0,0,\omega,0)\,q^2\,\mathrm{d}x\Big\|_{\mathcal{D}} < \partial\\ &\tfrac{1}{2d}\Big\|\int_{\mathsf{T}}\tfrac{\partial^3}{\partial q^2\partial\lambda}(\mathcal{H}-\mathcal{K})(x,0,0,0,\omega,0)\,q^2\lambda\,\mathrm{d}x\Big\|_{\mathcal{D}} < \partial\\ &\tfrac{1}{2d}\Big\|\int_{\mathsf{T}}\tfrac{\partial^3}{\partial q\partial p^2}(\mathcal{H}-\mathcal{K})(x,0,0,0,\omega,0)\,qp^2\,\mathrm{d}x\Big\|_{\mathcal{D}} < \partial \quad .\end{aligned}$$

Then there exists a C^∞*-diffeomorphism*

$$\begin{array}{rccc}\Phi: & \mathsf{T}\times\mathbb{R}^n\times\mathbb{R}^2\times\Sigma\times\mathbb{R} & \longrightarrow & \mathsf{T}\times\mathbb{R}^n\times\mathbb{R}^2\times\Sigma\times\mathbb{R}\\ & (x,y,q,p,\omega,\lambda) & \mapsto & (\phi_{\omega,\lambda}(x,y,q,p),\omega+\widetilde{\Omega}(\omega),\lambda+\widetilde{\Lambda}(\omega))\end{array}$$

such that

1). Φ *is real analytic in* x,y,q,p *and* λ *.*

2). $\phi_{\omega,\lambda}$ *is symplectic for every* $(\omega,\lambda)\in\Sigma\times\mathbb{R}$ *.*

3). Φ *is* C^∞*-close to the identity.*

4). *On* $\mathsf{T} \times \mathbb{R}^n \times \mathbb{R}^2 \times \Sigma_1' \times \mathbb{R} \ \cap \ \Phi^{-1}(\mathsf{T} \times \mathsf{Y} \times \mathsf{S} \times \Sigma \times]-\epsilon, \epsilon[\,)$ *one can write* $\mathcal{H} \circ \Phi = \mathcal{K}_\infty + H_\infty$ *. Here* $\mathcal{K}_\infty$ *is of the form (12) with* $d_\infty = d$ *, its coefficient functions* $a_\infty, \ldots, h_\infty$ *satisfy the above inequalities with* Γ *replaced by* 2Γ *, and* $\dfrac{\partial^{|j|+k+l+m} H_\infty}{\partial y^j \partial q^k \partial p^l \partial \lambda^m}(x,0,0,0,w,0) = 0$ *for all* $(x,\omega) \in \mathsf{T} \times \Sigma_1'$ *and all* j,k,l,m *with* $|j| + k + l + m \leq 2$ *.*

Before we prove this technical proposition (in the next subsection), we use it to derive Theorem 2.1. To this end we have to be able to push the x-dependence from 3rd order to order $N+1$. This is accomplished by the following proposition.

Proposition 4.2

Let T *be an* n*-torus,* Y *a neighbourhood of the origin in* $\mathbb{R}^n$ *and* S *a neighbourhood of the origin in* $\mathbb{R}^2$ *. Supply* $\mathsf{T} \times \mathsf{Y} \times \mathsf{S}$ *with the symplectic structure* $\sum_{\nu=1}^n \mathrm{d}x_\nu \wedge \mathrm{d}y_\nu + \mathrm{d}q \wedge \mathrm{d}p$ *, where* $x \in \mathsf{T}$ *,* $y \in \mathsf{Y}$ *and* $(q,p) \in \mathsf{S}$ *. Let* $\mathcal{H}$ *be a real analytic Hamilton function on* $\mathsf{T} \times \mathsf{Y} \times \mathsf{S}$ *, depending analytically on the parameter* $\lambda \in]-\epsilon, \epsilon[$ *and* C^∞ *on the parameter* $\omega \in \Sigma \subseteq \mathbb{R}^n$ *,* Σ *open. Let the 2-jet of* $\mathcal{H}$ *in* y,q,p,λ *be* x*-independent and of the form*

$$(\omega \,|\, y) \ + \ a(\omega)\, p^2 \ + \ (c(\omega) \,|\, y)\, q \ - \ d\, \lambda\, q \ + \ (e(\omega)\, y \,|\, y) \tag{13}$$

and let $\gamma > 0$ *,* $\tau > n-1$ *and* $N \in \mathbb{N}$ *. Then there exists a co-ordinate transformation*

$$\begin{array}{rccc} \Psi : & \mathsf{T} \times \mathbb{R}^n \times \mathbb{R}^2 \times \Sigma \times \mathbb{R} & \longrightarrow & \mathsf{T} \times \mathbb{R}^n \times \mathbb{R}^2 \times \Sigma \times \mathbb{R} \\ & (x,y,q,p,\omega,\lambda) & \mapsto & (\psi_{\omega,\lambda}(x,y,q,p),\omega,\lambda) \end{array}$$

such that

1). Ψ *is real analytic in* x,y,q,p *and* λ *.*

2). $\psi_{\omega,\lambda}$ *is symplectic for every* $(\omega,\lambda) \in \Sigma \times \mathbb{R}$ *.*

3). Ψ *is* C^∞*-close to the identity.*

4). *For* $\omega \in \Sigma_1$ *the* N*-jet of* $\mathcal{H} \circ \Psi$ *is* x*-independent.*

The proof, given in the last subsection, shows the inductive nature of the procedure. In particular, the 2-jet of $\mathcal{H} \circ \Psi$ is still given by (13). Furthermore, the 3-jet of $\mathcal{H} \circ \Psi$ equals the T-average of the 3-jet of $\mathcal{H}$.

Granted Proposition 4.1 and 4.2, we now prove Theorem 2.1 :

For some $\beta < \min\{\gamma, 1\}$ to be fixed later we consider the co-ordinate transformation

$$(x, y, q, p, \omega, \lambda) \;\mapsto\; (X, Y, Q, P, \Omega, \Lambda) \;=\; (x, \frac{y}{\beta^5}, \frac{q}{\beta^2}, \frac{p}{\beta^3}, \frac{\omega}{\beta}, \frac{\lambda}{\beta^4}) \qquad .$$

We have $\omega \in \Sigma'_\gamma$ if and only if $\Omega \in (\beta^{-1}\Sigma)'_{\frac{\gamma}{\beta}} \stackrel{!}{\subseteq} (\beta^{-1}\Sigma)'_1$; note that in the formulation of Proposition 4.1 and 4.2 the set of frequencies is not required to be bounded. In the new co-ordinates the truncated Hamilton function (4) becomes

$$\beta^6 \;\; [\; (\Omega \,|\, Y) \;+\; a(\beta\Omega)\, P^2 \;+\; \frac{1}{3} b(\beta\Omega)\, Q^3 \;-\; d\,\Lambda\, Q \;]$$

We apply Proposition 4.1 to the function in brackets and get a bound $\partial > 0$ for perturbations which is independent of β . Our candidate for the bound δ searched for in Theorem 2.1 is $\delta := \beta^6 \partial$. We consider the higher order terms of the unperturbed Hamilton function $\mathcal{K}$ as a perturbation of (4). In the new co-ordinates they are bounded by $C\,\beta^7$ for some constant C depending on $\mathcal{K}, \mathsf{Y}, \mathsf{S}$ and ϵ . Hence, we choose $\beta < \frac{\partial}{C}$. The 2-jet in the new co-ordinates already is in the form searched for, and we use Proposition 4.2 to make the N-jet independent of x . Note that the coefficients $\tilde{a}(\omega) = a_\infty(\omega)$ of p^2 and

$$\tilde{b}(\omega) \;\;=\;\; b_\infty(\omega) \;+\; \frac{1}{6} \int_{\mathsf{T}} \frac{\partial^3}{\partial q^3} H_\infty(x, 0, 0, 0, \omega, 0)\, \mathrm{d}x$$

of q^3 satisfy $\frac{1}{3\Gamma} d \leq \tilde{a}, \tilde{b} \leq 3\Gamma d$ since the co-ordinate transformation is close to the identity. To conclude the proof we have to get rid of the terms $q^2 p$, $p\lambda^2$, $qp\lambda$ and $q^2\lambda$. For $q^2 p$ this can be achieved with a transvection $q \mapsto q + \tau p$. As this further introduces terms yp and $p\lambda$ we combine this with a translation $p \mapsto p + u(y, \lambda)$. Then a transvection $p \mapsto p + \sigma(\lambda) q$ and a translation $q \mapsto q + v(y, \lambda)$ take care of the “mixed term” λqp , the λq^2-term and the λy-term we introduced during this process ; **q.e.d.(2.1)**

As we discussed in Subsection 2.2, there are two approaches to show persistence of lower dimensional tori. One of these is to use domains of different sizes in the various variables, and in fact the β-scaling in the above prove mimicked this idea so that terms $y^j q^k p^l \lambda^m$ with $5j + 2k + 3l + 4m > 6$ could be considered as higher order terms. However, for normally parabolic tori in a quasi-periodic centre-saddle bifurcation it is inevitable to consider different sizes of the perturbation. Therefore, we follow in Proposition 4.1 the approach of equal sizes $\leq \epsilon$, now leading to three different sizes of the perturbation.

4.2 Proof of Proposition 4.1

A proof of KAM-type usually follows the standard recipe laid out in *e.g.* [Pöschel;82], [Benettin,Galgani,Giorgilli,Strelcyn;84], [Broer,Huitema,Sevryuk;96], and the one presented here is no exception. Before coming to the technical details, let us explain the guiding ideas.

Our aim is 'to get rid of' the perturbation $H_0 := \mathcal{H} - \mathcal{K}$ of the integrable Hamilton function $\mathcal{K}_0 := \mathcal{K}$. To this end we use parameter shifts $\omega \mapsto \omega + \widetilde{\Omega}(\omega)$ and $\lambda \mapsto \lambda + \widetilde{\Lambda}(\omega)$ as well as a symplectic co-ordinate transformation $\phi_{\omega,\lambda}$ that together form a diffeomorphism Φ . Instead of solving directly $\mathcal{H} \circ \Phi = \mathcal{K}$ we work with a linearized version of this equation, yielding a transformation Φ_1 . We collect the differences to the "true" equation in a function H_1 and also allow changes in the coefficients of $\mathcal{K}$, only the constant d remains the same. The Hamilton function $\mathcal{H}_0 := \mathcal{H}$ we start with becomes in the new co-ordinates $\mathcal{H}_0 \circ \Phi_1 \stackrel{!}{=} \mathcal{K}_1 + H_1 =: \mathcal{H}_1$. If H_1 is "smaller" than H_0 we can repeat the procedure and inductively get a sequence $\mathcal{H}_i = \mathcal{K}_i + H_i$ of Hamilton functions as well as diffeomorphisms Φ_i with $\mathcal{H}_0 \circ \Phi_i = \mathcal{H}_i$.

A simple limit Φ_∞ would now solve our equation $\mathcal{H}_0 \circ \Phi_\infty = \mathcal{K}_\infty$ and reveal $\mathcal{H}_0$ to be integrable. We cannot expect this to happen. For instance, the linearization of $\mathcal{H} \circ \Phi = \mathcal{K}$ involves small denominators, the order of which we will increase in every step. As a result we have to exclude more and more resonance zones from $\mathcal{D}$. The pertinent factor of $\mathcal{D}$ will finally shrink to Σ_1' , compare the text of Proposition 4.1. In fact we let all factors of $\mathcal{D}$ shrink on every step and work with $\Phi_i : D_i \longrightarrow \mathcal{D}$. That allows us to use Cauchy's inequality for the estimates. Together with the mean value theorem this is the main ingredient to show $\lim_{i\to\infty} \|H_i\|_{D_i} = 0$, the "error term" $\|H_i\|_{D_i}$ will decrease exponentially fast.

Still, we cannot conclude $\mathcal{H}_0 \circ \Phi_\infty = \mathcal{K}_\infty$ on $\mathsf{T} \times \mathsf{Y} \times \mathsf{S} \times \Sigma_1' \times]-\epsilon, \epsilon[$ because the domains D_i will shrink to $\bigcap D_i = U_{\frac{1}{2}\kappa}(\mathsf{T}) \times \{0\} \times \{(0,0)\} \times \Sigma_1' \times \{0\}$. From the $\frac{\kappa}{2}$-piece in the complex direction we can conclude analyticity of Φ_∞ in x , and we want to apply the Inverse Approximation Lemma to show C^∞-(Whitney)-differentiability in ω . In order to define the limit Φ_∞ we use a polynomial truncation of the Φ_i . Accordingly, the limit Φ_∞ will be defined even on $\mathsf{T} \times \mathbb{R}^n \times \mathbb{R}^2 \times \Sigma_\gamma' \times \mathbb{R}$, but as a 'trade-in' we only get $\mathcal{H}_0 \circ \Phi_\infty = \mathcal{K}_\infty + H_\infty$, again compare the text of Proposition 4.1. Finally we will extend the domain of Φ_∞ in the ω-direction to Σ by means of Whitney's Extension Theorem.

Since we want to apply the Inverse Approximation Lemma, the D_i should only shrink in a geometric way in the ω-direction. We let D_i shrink geometrically in the x-direction as well. In the y- , (q,p)- and λ-direction and for the control of $\|H_i\|_{D_i}$ we use exponentially fast decreasing sequences.

We define $\sigma_i := (\frac{1}{4})^i \sigma_0$ and $\rho_i := \sigma_i^{2\tau+2}$ where we determine the value of σ_0 later. One assumption we already make is $\sigma_0 < \min\{\frac{\kappa}{2}, \frac{1}{2}\}$. For the exponentially fast decreasing sequences we first define $\delta_{i+1} := \delta_i \delta_i^{\mathsf{p}}$, leaving $\delta_0 < \frac{1}{2}$ undetermined. From this we derive the sequences $\Delta_i := \delta_i^{\mathsf{r}}$, $\partial_i := \delta_i^{\mathsf{t}}$ and $\varepsilon_i := \delta_i^{\mathsf{q}}$. (Since $\mathsf{p}, \mathsf{q}, \mathsf{r}$ and t only appear as exponents, the reader will not confound them with the co-ordinates (q, p) , or the time t .) The numbers $\Delta_0 = \delta_0^{\mathsf{r}}$, δ_0 and $\partial_0 = \delta_0^{\mathsf{t}}$ are the constants $\Delta \geq \delta \geq \partial$ sought for in Proposition 4.1.

Note that we do not aim for "best constants". For instance, we do not work with $\sigma_i := \varsigma^i \sigma_0$, trying to find an interval (probably $]0, \frac{1}{2}[$) from which we might choose ς . We also fix our thoughts on $\mathsf{t} = \mathsf{q} + \mathsf{r}$, *i.e.* $\partial = \varepsilon \Delta$. The shrinking domains D_i are now defined as

$$D_i \;:=\; U_{\frac{1}{2}\kappa + \sigma_i}(\mathsf{T}) \times U_{\varepsilon_i}(0) \times U_{\varepsilon_i}(0,0) \times U_{\rho_i}(\Sigma_1') \times U_{\varepsilon_i}(0) \qquad .$$

We do not assume ε_0 to be the ϵ in the text of Proposition 4.1, but merely require $\varepsilon_0 \leq \epsilon$ to ensure $D_0 \subseteq \mathcal{D}$. Since $\varepsilon_0 = \delta_0^{\mathsf{q}}$ this is an assumption on δ_0 . If one shrinks $\mathcal{D}$, the smallness assumptions on $\mathcal{H} - \mathcal{K}$ become weaker. Note that the case $D_0 = \mathcal{D}$ of "weakest assumptions" is included in our formulation of Proposition 4.1.

During the proof we will encounter functions f_i that are defined on a factor of D_i , *e.g.* do not depend on y, q, p and λ , or only depend on ω . We do not burden our exposition with different notations, but consider such f_i as defined on D_i . In particular we write $\|f_i\|_{D_i}$ for the supremum-norm ("the operator $f \mapsto f \circ \pi$ has norm 1"). The different domains only become important when we apply the Inverse Approximation Lemma.

We now concentrate on the ith iteration step. Our aim is to construct a diffeomorphism $\Psi_i : D_{i+1} \longrightarrow \Psi_i(D_{i+1}) \subseteq D_i$ with $\mathcal{H}_i \circ \Psi_i = \mathcal{H}_{i+1}$. By composition we can then define

$$\Phi_{i+1} \;:=\; \Psi_0 \circ \Psi_1 \circ \ldots \circ \Psi_i \;\;:\;\; D_{i+1} \;\longrightarrow\; \mathcal{D} \qquad .$$

We use the customary '+ notation' : we abbreviate $H_i, \varepsilon_i, \Psi_i, \ldots$ to $H, \varepsilon, \Psi, \ldots$ and write *e.g.* H_+ , ε_+ , Φ_+ for H_{i+1} , ε_{i+1} and Φ_{i+1} . This also emphasises that the constants in our estimates have to be independent of i . In fact they will only depend on n, τ and Γ .

We are given a Hamilton function $\mathcal{H} = \mathcal{K} + H$. We write H as a Taylor series

$$H(x, y, q, p, \omega, \lambda) \;\;=\;\; \sum_{|j|+k+l+m=0}^{\infty} H_{jklm}(x, \omega)\, y^j\, q^k\, p^l\, \lambda^m$$

and expand the coefficient functions as Fourier series

$$H_{jklm}(x, \omega) \;\;=\;\; \sum_{\mathsf{h} \in \mathbb{Z}^n} H_{jklm}^{\mathsf{h}}(\omega)\, \mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)} \qquad .$$

(The index h will only appear in relation to Fourier series, so the reader will not confound it with the constant part of $\mathcal{K}$.) We introduce the truncations

$$^T\!H \;=\; \sum_{|j|=1}^{2} \sum_{k+l+m=0}^{2-|j|} H_{jklm}\, y^j q^k p^l \lambda^m \;+\; \sum_{k+l+m=0}^{3} H_{0klm}\, q^k p^l \lambda^m$$

and

$$ {}_{\theta}H_{jklm} \;=\; \sum_{|\mathsf{h}|\le\theta} H^{\mathsf{h}}_{jklm} \mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)} \qquad , $$

here $\theta := [\rho^{\frac{-1}{\tau+1}}]$ is the maximal order $|\mathsf{h}| = \mathsf{h}_1 + \ldots + \mathsf{h}_n$ of the resonances we have to cope with at this stage. In particular we have

$$ {}^{T}_{\theta}H \;=\; \sum_{|j|+k+l+m=0}^{2} \sum_{|\mathsf{h}|\le\theta} H^{\mathsf{h}}_{jklm} \mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)} y^j q^k p^l \lambda^m \;+\; \sum_{k+l+m=3} \sum_{|\mathsf{h}|\le\theta} H^{\mathsf{h}}_{0klm} \mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)} q^k p^l \lambda^m \qquad . $$

We also define the functions

$$ H^{\delta} \;:=\; \sum_{k+l+m=0}^{\infty} \sum_{\mathsf{h}\in\mathbb{Z}^n} H^{\mathsf{h}}_{0klm}(\omega)\, \mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)}\, q^k p^l \lambda^m $$

and

$$ H^{\partial} \;:=\; H^{0}_{0200}(\omega)\, q^2 \;+\; H^{0}_{0201}(\omega)\, q^2\lambda \;+\; H^{0}_{0120}(\omega)\, qp^2 \qquad . $$

With these abbreviations our assumptions on H read $\frac{1}{d}\|H\|_{D} < \Delta$, $\frac{1}{d}\|H^{\delta}\|_{D} < \delta$ and $\frac{1}{d}\|H^{\partial}\|_{D} < \partial$. We want to construct $\Psi : D_+ \longrightarrow D$ such that these assumptions are also fulfilled on the next stage where we have $\mathcal{H} \circ \Psi = \mathcal{K}_+ + H_+$.

We use the assumptions from Proposition 4.1 on the coefficient functions $a, b, \ldots, h$ of $\mathcal{K}$ with Γ replaced by 2Γ and later derive an estimate for the coefficient functions $a_+, b_+, \ldots, h_+$ of $\mathcal{K}_+$ that inductively justifies this assumption. Note that there is no constant d_+ , we use the same d for $\mathcal{K}$ and $\mathcal{K}_+$. We will encounter various inequalities on the exponents, the most important are $3\mathsf{q} > \mathsf{r}$, $\mathsf{r}+\mathsf{t}-3\mathsf{q} > 1+\mathsf{p}$ and $4\mathsf{q} > 1$. The open set of positive $\mathsf{p}, \mathsf{q}, \mathsf{r}, \mathsf{t}$ satisfying all these inequalities is non-empty, in fact it contains the quadruple $\mathsf{t} = 1.15$, $\mathsf{r} = 0.85$, $\mathsf{q} = 0.3$ and $\mathsf{p} = 0.05$.

We are now able to formulate the key step in our proof of Proposition 4.1. Here and in the following lemmata and propositions we refrain from tediously repeating the assumptions we made so far.

Proposition 4.3

There are $\sigma_0 > 0$ and $\delta_0 > 0$ such that the following holds true. Given $\mathcal{H} = \mathcal{K} + H$ as above, there is a co-ordinate transformation

$$ \begin{array}{rccc} \Psi : & D_+ & \longrightarrow & D \\ & (x,y,q,p,\omega,\lambda) & \mapsto & (\psi_{\omega,\lambda}(x,y,q,p), \omega+\Omega(\omega), \lambda+\Lambda(\omega)) \end{array} $$

with symplectic $\psi_{\omega,\lambda}$ and a Hamilton function $\mathcal{K}_+$ of the form (12) such that $H_+ := \mathcal{H}\circ\Psi - \mathcal{K}_+$ satisfies $\frac{1}{d}\|H_+\|_{D_+} < \Delta_+$, $\frac{1}{d}\|H^{\delta}_+\|_{D_+} < \delta_+$ and $\frac{1}{d}\|H^{\partial}_+\|_{D_+} < \partial_+$.

Proof :

The proof is rather long and can be divided in the following parts :

$$\begin{array}{rl} \longrightarrow & \textit{construction of } \Psi \\ (ii) & \textit{estimates for } H_+ \textit{ (introduction)} \\ (iii) & \textit{estimates for } \Psi \\ (iv) & \textit{estimates for } H_+ \textit{ (conclusion)} \\ (v) & \textit{the assumptions on } \sigma_0 \textit{ and } \delta_0 \end{array}$$

The first and third part concern the co-ordinate transformation Ψ , part (iii) relies on several lemmata. In part (ii) and (iv) the individual terms of H_+ are estimated, the lengthy part (iv) consists of a series of inequalities and can be skipped at first reading. In the concluding part (v) the inequalities concerning σ_0 and δ_0 are assembled. The arrow indicates that we now start with part (i) .

In order to ensure the $\psi_{\omega,\lambda}$ to be symplectic we define them as time-1-flows of Hamiltonian vector fields $X_{K_{\omega,\lambda}}$. We work with

$$K \;=\; \sum_{|j|+k+l+m=0}^{2} \sum_{|\mathsf{h}|\leq\theta} K_{jklm}^{\mathsf{h}} \mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)} y^j q^k p^l \lambda^m \;+\; \sum_{k+l+m=3} \sum_{|\mathsf{h}|\leq\theta} K_{0klm}^{\mathsf{h}} \mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)} q^k p^l \lambda^m$$

and require K , Ω and Λ to solve the partial differential equation

$$\begin{aligned} (\omega\,|\,\tfrac{\partial K}{\partial x}) \;=\;& \,T_\theta H \;+\; (\Omega\,|\,y) \;-\; d\,\Lambda\,q \;+\; (a-a_+)\,p^2 \;+\; \tfrac{1}{3}(b-b_+)\,q^3 \\ &+\; (c-c_+\,|\,y)\,q \;+\; ((e-e_+)y\,|\,y) \\ &+\; (f\circ(\mathrm{id}+\Omega)-f_+)\,\lambda \;+\; (g-g_+)\,\lambda^2 \\ &+\; h\circ(\mathrm{id}+\Omega)-h_+ \;+\; f\,\Lambda \;+\; 2g\,\Lambda\,\lambda \\ &-\; 2a\,\tfrac{\partial K}{\partial q}\,p \;-\; d\,\tfrac{\partial K}{\partial p}\,\lambda \;+\; b\,\tfrac{\partial \widetilde{K}}{\partial p}\,q^2 \;+\; (c\,|\,y)\,\tfrac{\partial \widetilde{K}}{\partial p} \qquad . \end{aligned} \tag{14}$$

In the last two expressions we had to restrict to $\widetilde{K} \;=\; \sum_{|j|+k+l+m=0}^{2} K_{jklm}\, y^j q^k p^l \lambda^m$, indeed $T_\theta(b\frac{\partial K}{\partial p}q^2 \;+\; (c\,|\,y)\frac{\partial K}{\partial p}) \;=\; b\frac{\partial \widetilde{K}}{\partial p}q^2 \;+\; (c\,|\,y)\frac{\partial \widetilde{K}}{\partial p}$.

Equation (14) can be viewed as a linear version of $\mathcal{H}\circ\Psi = \mathcal{K}_+$. In fact (14) is derived from this equation writing

$$(\mathcal{H}_{\omega,\lambda}\circ\psi_{\omega,\lambda})(z) \;=\; \mathcal{H}_{\omega,\lambda}(z+X_{K_{\omega,\lambda}}(z)+\ldots) \;=\; \mathcal{H}_{\omega,\lambda}(z) \;+\; \mathrm{d}\mathcal{H}_{\omega,\lambda}(z)(X_{K_{\omega,\lambda}}(z)) \;+\; \ldots$$

and neglecting, next to the "higher order terms", also the expressions involving $\frac{\partial K}{\partial x}$ — except for $(\omega\,|\,\frac{\partial K}{\partial x})$. We drop the parameter transformation Λ whenever it leads to 'quadratic terms' ($i.e.$ we drop $g\Lambda^2$ and $-d\frac{\partial K}{\partial p}\Lambda$) and we almost completely drop Ω (replacing $a(\omega+\Omega(\omega))$ by $a(\omega)$ etc.), only keeping $(\omega+\Omega(\omega)\,|\,y)$, $f(\omega+\Omega(\omega))\lambda$ and $h(\omega+\Omega(\omega))$.

Equation (14) is equivalent to the following system of equations on the coefficient functions. For $u, v \in \mathbb{R}^n$ we denote the symmetrised tensor product by $u \bullet v = (\frac{1}{2}(u_\nu v_\mu + u_\mu v_\nu))_{\nu,\mu=1,\ldots,n}$.

$$\begin{aligned}
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0000}(x,\omega)) &= {}_\theta H_{0000}(x,\omega) \;+\; f(\omega)\,\Lambda(\omega) \;+\; h(\omega+\Omega(\omega)) \;-\; h_+(\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0100}(x,\omega)) &= {}_\theta H_{0100}(x,\omega) \;-\; d\,\Lambda(\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0010}(x,\omega)) &= {}_\theta H_{0010}(x,\omega) \;-\; 2a(\omega) K_{0100}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0001}(x,\omega)) &= {}_\theta H_{0001}(x,\omega) - d\,K_{0010}(x,\omega) + 2g(\omega)\,\Lambda(\omega) + f(\omega+\Omega(\omega)) - f_+(\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{1000}(x,\omega)) &= {}_\theta H_{1000}(x,\omega) \;+\; \Omega(\omega) \;+\; K_{0010}(x,\omega)\,c(\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0200}(x,\omega)) &= {}_\theta H_{0200}(x,\omega) \;+\; b(\omega)\,K_{0010}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0110}(x,\omega)) &= {}_\theta H_{0110}(x,\omega) \;-\; 4a(\omega)\,K_{0200}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0101}(x,\omega)) &= {}_\theta H_{0101}(x,\omega) \;-\; d\,K_{0110}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0020}(x,\omega)) &= {}_\theta H_{0020}(x,\omega) \;-\; 2a(\omega)\,K_{0110}(x,\omega) \;+\; a(\omega) \;-\; a_+(\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0011}(x,\omega)) &= {}_\theta H_{0011}(x,\omega) \;-\; 2a(\omega)\,K_{0101}(x,\omega) \;-\; 2d\,K_{0020}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0002}(x,\omega)) &= {}_\theta H_{0002}(x,\omega) \;-\; d\,K_{0011}(x,\omega) \;+\; g(\omega) \;-\; g_+(\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{1100}(x,\omega)) &= {}_\theta H_{1100}(x,\omega) \;+\; K_{0110}(x,\omega)\,c(\omega) \;+\; c(\omega) \;-\; c_+(\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{1010}(x,\omega)) &= {}_\theta H_{1010}(x,\omega) \;+\; 2K_{0020}(x,\omega)\,c(\omega) \;-\; 2a(\omega)\,K_{1100}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{1001}(x,\omega)) &= {}_\theta H_{1001}(x,\omega) \;+\; K_{0011}(x,\omega)\,c(\omega) \;-\; d\,K_{1010}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{2000}(x,\omega)) &= {}_\theta H_{2000}(x,\omega) \;+\; c(\omega) \bullet K_{1010}(x,\omega) \;+\; e(\omega) \;-\; e_+(\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0300}(x,\omega)) &= {}_\theta H_{0300}(x,\omega) \;+\; b(\omega)\,K_{0110}(x,\omega) \;+\; \tfrac{1}{3}(\,b(\omega) \;-\; b_+(\omega)\,) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0210}(x,\omega)) &= {}_\theta H_{0210}(x,\omega) \;-\; 6a(\omega)\,K_{0300}(x,\omega) \;+\; 2b(\omega)\,K_{0020}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0201}(x,\omega)) &= {}_\theta H_{0201}(x,\omega) \;-\; d\,K_{0210}(x,\omega) \;+\; b(\omega)\,K_{0011}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0120}(x,\omega)) &= {}_\theta H_{0120}(x,\omega) \;-\; 4a(\omega)\,K_{0210}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0111}(x,\omega)) &= {}_\theta H_{0111}(x,\omega) \;-\; 4a(\omega)\,K_{0201}(x,\omega) \;-\; 2d\,K_{0120}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0102}(x,\omega)) &= {}_\theta H_{0102}(x,\omega) \;-\; d\,K_{0111}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0030}(x,\omega)) &= {}_\theta H_{0030}(x,\omega) \;-\; 2a(\omega)\,K_{0120}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0021}(x,\omega)) &= {}_\theta H_{0021}(x,\omega) \;-\; 2a(\omega)\,K_{0111}(x,\omega) \;-\; 3d\,K_{0030}(x,\omega) \\
(\omega \,|\, \tfrac{\partial}{\partial x} K_{0012}(x,\omega)) &= {}_\theta H_{0012}(x,\omega) \;-\; 2a(\omega)\,K_{0102}(x,\omega) \;-\; 2d\,K_{0021}(x,\omega)
\end{aligned}$$

$$(\omega \,|\, \tfrac{\partial}{\partial x} K_{0003}(x,\omega)) \;=\; {}_{\theta}H_{0003}(x,\omega) \;-\; d\,K_{0012}(x,\omega)$$

We have to solve two different problems when dealing with these equations. On the one hand (14) has to be true when we average along T, *i.e.* for the zeroth Fourier coefficients. Then the left hand side vanishes, and we have to ensure that the right hand side is also zero. These are called the *integrability conditions*. On the other hand we have to solve the x-dependent part of the above equations and find the K^{h}_{jklm} with non-zero Fourier index. It is here that the small denominators $\frac{1}{\mathrm{i}(\mathsf{h}\,|\,\omega)}$ come into play.

We first deal with the integrability conditions. Those coefficient functions K^0_{jklm} that do not appear on the right hand side are not important, we set them zero. We are also free to choose $K^0_{0020} = 0$ and $K^0_{0102} = 0$ and define $K^0_{0111} = \frac{1}{d} H^0_{0102}$. Proceeding from bottom to top we now get a unique solution. Here we first omit equations involving the functions $a_+, \ldots, h_+$ and solve them only after all K^0_{jklm} are determined. For instance, we get

$$b_+(\omega) \;=\; b(\omega) \;+\; 3H^0_{0300}(\omega) \;+\; 3b(\omega)\,K^0_{0110}(\omega) \;\stackrel{!}{=}\; b \;+\; 3H^0_{0300} \;+\; \frac{3b}{d}\,H^0_{0101} \qquad .$$

(We will frequently suppress the parameter ω.) The particular form (12) of the integrable Hamilton function $\mathcal{K}$ was actually to a large extent dictated by the integrability conditions. The coefficient functions

$$K^0_{0010} \;:=\; -\frac{H^0_{0200}}{b} \quad , \quad K^0_{0011} \;:=\; -\frac{H^0_{0201}}{b} \;+\; \frac{d}{4ab} H^0_{0120}$$
$$\text{and} \qquad K^0_{1010} \;:=\; \frac{1}{d} H^0_{1001} \;-\; H^0_{0201}\frac{c}{db} \;+\; H^0_{0120}\frac{c}{4ab}$$

are peculiar in that they depend on coefficient functions of H of a higher order. The same applies to

$$\Omega(\omega) \;:=\; -\,H^0_{1000}(\omega) \;+\; \frac{H^0_{0200}(\omega)}{b(\omega)} c(\omega) \qquad .$$

We will use our assumption $\partial = \varepsilon\Delta$ to overcome the resulting difficulties — ∂ was introduced for this reason.

We now turn to the non-zero Fourier indices h, recalling that $|\mathsf{h}| \leq \theta$. The resonant frequencies ω with $(\mathsf{h}\,|\,\omega) = 0$ are excluded from our domain D, so we can solve the equations straightforwardly. We start at the top with

$$K^{\mathsf{h}}_{0000}(\omega) \;=\; \frac{H^{\mathsf{h}}_{0000}(\omega)}{\mathrm{i}(\mathsf{h}\,|\,\omega)} \qquad \text{and} \qquad K^{\mathsf{h}}_{0100}(\omega) \;=\; \frac{H^{\mathsf{h}}_{0100}(\omega)}{\mathrm{i}(\mathsf{h}\,|\,\omega)} \qquad ,$$

continue with

$$K^{\mathsf{h}}_{0010}(\omega) \;=\; \frac{H^{\mathsf{h}}_{0010}(\omega)}{\mathrm{i}(\mathsf{h}\,|\,\omega)} \;-\; 2a(\omega)\frac{K^{\mathsf{h}}_{0100}(\omega)}{\mathrm{i}(\mathsf{h}\,|\,\omega)} \;\stackrel{!}{=}\; \frac{H^{\mathsf{h}}_{0010}}{\mathrm{i}(\mathsf{h}\,|\,\omega)} \;+\; 2a\frac{H^{\mathsf{h}}_{0100}}{(\mathsf{h}\,|\,\omega)^2}$$

and finally reach at the bottom

$$K^{\mathsf{h}}_{0003} \;=\; \frac{H^{\mathsf{h}}_{0003}}{\mathsf{i}(\mathsf{h}\,|\,\omega)} \;+\;\ldots\;-\; 10752\, a^5 b^2 d^3\, \frac{H^{\mathsf{h}}_{0100}}{\mathsf{i}(\mathsf{h}\,|\,\omega)^{11}} \qquad .$$

In particular we see that no K^{h}_{0klm} depends on a H^{h}_{jklm} with $j \geq 1$. This completes the construction of K .

We obtain a parameter dependent time-1-flow ψ of X_K and define $\Psi := (\psi, \mathrm{id}{+}\Omega, \mathrm{id}{+}\Lambda)$. We will sometimes write

$$\psi(x,y,q,p,\omega,\lambda) \;=\; (x + V^1(x,y,q,p,\omega,\lambda), \ldots, p + V^4(x,y,q,p,\omega,\lambda))$$

and also $\mathrm{id} + X_K = \mathrm{id} + U$ so to avoid minus signs.

(i)	*construction of* Ψ
$\longrightarrow$	*estimates for* H_+ *(introduction)*
(iii)	*estimates for* Ψ
(iv)	*estimates for* H_+ *(conclusion)*
(v)	*the assumptions on* σ_0 *and* δ_0

In replacing $\mathcal{H} \circ \Psi = \mathcal{K}_+$ by our linearization (14) we introduced several kinds of "error terms" : we dropped the ω-transformation in the arguments, we replaced V by U , we neglected 'quadratic terms', we replaced $H \circ \Psi$ by H and truncated H . All these "errors" are collected in

$$\begin{aligned}
H_+ \;&=\; \mathcal{H} \circ \Psi \;-\; \mathcal{K}_+ \\
&=\; H \circ \Psi \;-\; H \;+\; H \;-\; {}^{T}\!H \;+\; {}^{T}\!H \;-\; {}^{T}_{\theta}\!H \;+\; (\omega\,|\,V^2 - U^2) \\
&\;+\; (a(\omega+\Omega) - a(\omega))\, p^2 \;+\; \tfrac{1}{3}(b(\omega+\Omega) - b(\omega))\, q^3 \\
&\;+\; (c(\omega+\Omega) - c(\omega)\,|\,y)\, q \;+\; ((e(\omega+\Omega) - e(\omega))\, y\,|\,y) \\
&\;+\; (g(\omega+\Omega) - g(\omega))\, \lambda^2 \;+\; (f(\omega+\Omega) - f(\omega))\, \Lambda \\
&\;+\; 2(g(\omega+\Omega) - g(\omega))\, \Lambda\, \lambda \;+\; g(\omega+\Omega)\, \Lambda\, \Lambda \\
&\;+\; 2(a(\omega+\Omega) - a(\omega))\, U^4\, p \;+\; (b(\omega+\Omega) - b(\omega))\, U^3\, q^2 \\
&\;+\; (c(\omega+\Omega) - c(\omega)\,|\,y)\, U^3 \;+\; (c(\omega+\Omega)\,|\,U^2)\, q \\
&\;+\; 2(e(\omega+\Omega)\, y\,|\,V^2) \;+\; (\Omega\,|\,V^2) \;-\; d\,\Lambda\, V^3 \\
&\;+\; b(\omega+\Omega)\, V^3\, V^3\, q \;+\; \tfrac{1}{3} b(\omega+\Omega)\, V^3\, V^3\, V^3 \\
&\;+\; (c(\omega+\Omega)\,|\,V^2)\, V^3 \;+\; (e(\omega+\Omega)\, V^2\,|\,V^2) \\
&\;+\; a(\omega+\Omega)\, U^4\, U^4 \;+\; a(\omega+\Omega)\, U^4\, (V^4 - U^4) \\
&\;+\; a(\omega+\Omega)\, V^4\, (V^4 - U^4) \;+\; 2a(\omega+\Omega)\, (V^4 - U^4)\, p \\
&\;+\; b(\omega+\Omega)\, (V^3 - U^3)\, q^2 \;-\; d\, (V^3 - U^3)\, \lambda \\
&\;+\; (c(\omega+\Omega)\,|\,y)\, (V^3 - U^3) \;+\; (c(\omega+\Omega)\,|\,V^2 - U^2)\, q \\
&\;+\; b(\omega) \sum_{k+l+m=3} l\, K_{0klm}\, q^{k+2} p^{l-1} \lambda^m \;+\; (c(\omega)\,|\,y) \sum_{k+l+m=3} l\, K_{0klm}\, q^k p^{l-1} \lambda^m
\end{aligned}$$

Our aim is to estimate H_+ . We have to make sure that every term of the above expression can be estimated by Δ_+ . Terms belonging to H_+^δ or H_+^∂ even have to be estimated by δ_+ and ∂_+ , respectively. The necessary computations rely on Cauchy's estimate and the mean value theorem. We defer most of them to part (iv).

Here we attack those terms of H_+ that do not involve K, U or V . These calculations illustrate how we obtain the estimates. For instance, we have

$$\frac{1}{d}\|H - {}^T\!H\|_{D_+} \;\leq\; \frac{1}{d}\|\sum_{|j|+k+l+m\geq 3} H_{jklm}\cdot(y,q,p,\lambda)^{(j,k,l,m)-3}\|_D\, \varepsilon_+^3 \;\overset{\text{Lemma 4.4}}{\leq}\; const\cdot\varepsilon^{-3}\,\Delta\,\varepsilon_+^3$$

where $(y,q,p,\lambda)^{(j,k,l,m)-3}$ denotes

$$y_1^{j_1-\min(j_1,3)}\cdot y_2^{j_2-\min(j_2,\max(0,3-j_1))}\cdot\ldots\cdot q^{k-\min(k,\max(0,3-|j|))}\cdot\ldots\cdot\lambda^{m-\max(0,3-|j|-k-l))} \qquad .$$

Granted $3\mathsf{q} > \mathsf{r}$ this can be estimated by Δ_+ . The constant has to be taken care of by ε_0 . This is an assumption on $\delta_0 = {\varepsilon_0}^{\frac{1}{\mathsf{q}}}$. Similar assumptions will accompany every inequality imposed upon the exponents, so let us be explicit here : we require $\delta_0 < (\frac{1}{const})^{\frac{1}{3\mathsf{qp}-\mathsf{rp}}}$. We will use the common '$\lesssim$ notation' for estimates involving a constant we do not need to remember. Like above, these constants only depend on n, τ and Γ .

We will estimate the terms of H_+ by δ_+ , or even ∂_+ , whenever this is possible. The weakening of δ to Δ was introduced for the above term. But $H - {}^T\!H$ also contains terms that belong to H_+^δ , we have to add the estimate

$$\frac{1}{d}\|H^\delta - {}^T\!H^\delta\|_{D_+} \;\leq\; \frac{1}{d}\|\sum_{k+l+m\geq 4} H_{0klm}\cdot(q,p,\lambda)^{(k,l,m)-4}\|_D\,\varepsilon_+^4 \;\overset{\text{Lemma 4.4}}{\lesssim}\; \varepsilon^{-4}\,\delta\,\varepsilon_+^4 \;\overset{!}{<}\; \delta_+$$

granted $4\mathsf{q} > 1$ and another assumption on δ_0 . Here and above we use the following lemma.

Lemma 4.4

There is a constant c_1 , depending only on n, τ and Γ , such that for all ι, j, k, l, m, ν with $|\iota| + |j| + k + l + m + |\nu| \leq 5$ the following inequalities hold true

$$\sigma^{|\iota|}\,\rho^{|\nu|}\,\varepsilon^{|j|+k+l+m}\;\|\frac{\partial^{|\iota|+|j|+k+l+|\nu|+m}H}{\partial x^\iota\partial y^j\partial q^k\partial p^l\partial\omega^\nu\partial\lambda^m}\|_{D_+} \;\leq\; c_1\,\|H\|_D$$

$$\sigma^{|\iota|}\,\rho^{|\nu|}\,\varepsilon^{k+l+m}\;\|\frac{\partial^{|\iota|+k+l+|\nu|+m}H^\delta}{\partial x^\iota\partial q^k\partial p^l\partial\omega^\nu\partial\lambda^m}\|_{D_+} \;\leq\; c_1\,\|H^\delta\|_D \qquad .$$

Proof :

The application of Cauchy's estimate yields the factors $(\sigma-\sigma_+)^{|\iota|}$, $(\rho-\rho_+)^{|\nu|}$ and $(\varepsilon-\varepsilon_+)^{|j|+k+l+m}$. Now use $\sigma_+=\frac{\sigma}{4}$, $\rho_+\le\frac{\rho}{4}$ and $\varepsilon_+\le\frac{\varepsilon}{4}$. Since $\varepsilon_+=\varepsilon\varepsilon^{\mathsf{p}}\le\varepsilon_0^{\mathsf{p}}\varepsilon$ the latter is an assumption on $\delta_0=\varepsilon_0{}^{\frac{1}{\mathsf{q}}}$. We can choose c_1 somewhat larger so that the inequalities also hold true when we replace D_+ by an 'intermediate' subset $D_\otimes$ we encounter in Lemma 4.5 ;
q.e.d.(4.4)

From Lemma 4.4 we conclude

$$\|\Omega\|_{D_+}\ \le\ c_1\varepsilon^{-1}\,\|H\|_{D}\ +\ 4\Gamma^2\varepsilon^{-2}\,\|H^\partial\|_{D}\ <\ d(c_1+4\Gamma^2)\,\varepsilon^{-2}\,\partial$$

where we used that $\mathsf{t}=\mathsf{r}+\mathsf{q}$. We use the mean value theorem to estimate those terms of H_+ that purely come from the ω-transformation in the argument. They all lead to inequalities on the exponents t,q and p , which are accompanied by assumptions on δ_0 . We additionally require $\mathsf{t}>2\mathsf{q}$ and

$$\delta_0^{\mathsf{t}-2\mathsf{q}}\ <\ \min\left\{\frac{\rho_0-\frac{1}{4}\rho_0}{c_1+4\Gamma^2}\,,\,(\frac{1}{4})^{\frac{2\tau+2}{\mathsf{p}}}\right\}\tag{15}$$

to ensure $(\mathrm{id},\mathrm{id}+\Omega,\mathrm{id})(D_+)\subseteq D$. This is an assumption relating δ_0 and $\sigma_0=\rho_0^{\frac{1}{2\tau+2}}$. To save space we use sloppy notations like $f(\omega+\Omega)$ for the function $f\circ(\mathrm{id}+\Omega)$. We denote in brackets the necessary inequalities on the exponents t,q and p that ensure, together with the accompanying assumptions on δ_0 , that the inequalities $\overset{!}{<}$ hold true.

$$\begin{array}{rcll}
\frac{1}{d}\|(a(\omega+\Omega)-a(\omega))\,p^2\|_{D_+} & \le & \frac{1}{d}\,\|\frac{\partial a}{\partial\omega}\|_{D}\ \|\Omega\|_{D_+}\,\varepsilon_+^2 & \\
 & \le. & \varepsilon^{-2}\partial\varepsilon_+^2\overset{!}{<}\delta_+ & (\mathsf{t}+2\mathsf{pq}>1+\mathsf{p})\\
\frac{1}{d}\|\frac{1}{3}(b(\omega+\Omega)-b(\omega))\,q^3\|_{D_+} & \le. & \varepsilon^{-2}\partial\varepsilon_+^3\overset{!}{<}\delta_+ & (\mathsf{t}+\mathsf{q}+3\mathsf{pq}>1+\mathsf{p})\\
\frac{1}{d}\|(c(\omega+\Omega)-c(\omega)\,|\,y)\,q\|_{D_+} & \le. & \varepsilon^{-2}\partial\varepsilon_+^2\overset{!}{<}\delta_+ & (\mathsf{t}+2\mathsf{pq}>1+\mathsf{p})\\
\frac{1}{d}\|((e(\omega+\Omega)-e(\omega))\,y\,|\,y)\|_{D_+} & \le. & \varepsilon^{-2}\partial\varepsilon_+^2\overset{!}{<}\delta_+ & (\mathsf{t}+2\mathsf{pq}>1+\mathsf{p})\\
\frac{1}{d}\|(g(\omega+\Omega)-g(\omega))\,\lambda^2\|_{D_+} & \le. & \varepsilon^{-2}\partial\varepsilon_+^2\overset{!}{<}\delta_+ & (\mathsf{t}+2\mathsf{pq}>1+\mathsf{p})
\end{array}$$

Obviously it would be sufficient to estimate the third and fourth term by Δ_+ . On the other hand the first term cannot be estimated by ∂_+ . We necessarily have to work with three different orders of smallness $\partial<\delta<\Delta$.

We also get $\|\Lambda\|_{D_+}=\frac{1}{d}\|H^0_{0100}\|_{D_+}<c_1\,\varepsilon^{-1}\,\delta$ and obtain

$$\begin{array}{rcll}
\frac{1}{d}\|(f(\omega+\Omega)-f(\omega))\,\Lambda\|_{D_+} & \le. & \varepsilon^{-2}\,\partial\,\varepsilon^{-1}\,\delta\overset{!}{<}\delta_+ & (\mathsf{t}-3\mathsf{q}>\mathsf{p})\\
\frac{2}{d}\|(g(\omega+\Omega)-g(\omega))\,\Lambda\,\lambda\|_{D_+} & \le. & \varepsilon^{-2}\,\partial\,\varepsilon^{-1}\,\delta\,\varepsilon_+\overset{!}{<}\delta_+ & (\mathsf{t}-2\mathsf{q}+\mathsf{pq}>\mathsf{p})\\
\|g(\omega+\Omega)\,\Lambda\,\Lambda\|_{D_+} & \le. & \varepsilon^{-2}\,\delta^2\overset{!}{<}\delta_+\quad . & (1-2\mathsf{q}>\mathsf{p})
\end{array}$$

$$\begin{array}{rl} (i) & \textit{construction of } \Psi \\ (ii) & \textit{estimates for } H_+ \textit{ (introduction)} \\ \longrightarrow & \textit{estimates for } \Psi \\ (iv) & \textit{estimates for } H_+ \textit{ (conclusion)} \\ (v) & \textit{the assumptions on } \sigma_0 \textit{ and } \delta_0 \end{array}$$

The other terms of H_+ necessite estimates on K , on $U = X_K$ and on V . We define an intermediate domain $D_+ \subseteq D_\otimes \subseteq D$ by means of $\sigma_\otimes = \frac{1}{3}(2\sigma + \sigma_+)$, $\rho_\otimes = \frac{1}{3}(2\rho + \rho_+)$ and $\varepsilon_\otimes = \frac{1}{3}(2\varepsilon + \varepsilon_+)$. In Lemma 4.4 we take c_1 large enough (multiplying it by three), such that the estimates hold true with D_+ replaced by $D_\otimes$.

Lemma 4.5

There is a constant c_2 (depending on c_1) such that

$$\begin{array}{rcl} \|\frac{\partial K}{\partial y}\|_{D_\otimes} & \leq & c_2\,\sigma^{-8\tau}\,\varepsilon^{-1}\,(\|\frac{H}{d}\|_D + \varepsilon^{-1}\|\frac{H^\partial}{d}\|_D) \\ \|\frac{\partial K}{\partial x}\|_{D_\otimes} & \leq & c_2\,\sigma^{-12\tau-1}\,\|H\|_D \\ \|\frac{\partial K}{\partial p}\|_{D_\otimes} & \leq & c_2\,\sigma^{-11\tau}\,\varepsilon^{-1}\,(\|\frac{H}{d}\|_D + \varepsilon^{-1}\|\frac{H^\partial}{d}\|_D) \\ \|\frac{\partial K}{\partial q}\|_{D_\otimes} & \leq & c_2\,\sigma^{-10\tau}\,\varepsilon^{-1}\,\frac{1}{d}\|H\|_D \qquad . \end{array}$$

Moreover,

$$\|\frac{\partial^{|j|+k+l+|\nu|+m}K}{\partial y^j\partial q^k\partial p^l\partial\omega^\nu\partial\lambda^m}\|_{D_\otimes} \;\leq\!\cdot\; \sigma^{-(11+2|\nu|)\tau-2|\nu|}\;\varepsilon^{-|j|-k-l-m}\left(\|\frac{H}{d}\|_D + \varepsilon^{-1}\,\|\frac{H^\partial}{d}\|_D\right)$$

and for $|\iota| \geq 1$

$$\|\frac{\partial^{|\iota|+|j|+k+l+|\nu|+m}K}{\partial x^\iota\partial y^j\partial q^k\partial p^l\partial\omega^\nu\partial\lambda^m}\|_{D_\otimes} \;\leq\!\cdot\; \sigma^{-(12+2|\nu|)\tau-2|\nu|-|\iota|}\;\varepsilon^{-|j|-k-l-m}\;\|H\|_D \qquad .$$

Proof :

We only derive the first estimate, the others are similarly obtained.
For $(x,y,q,p,\omega,\lambda) \in D_\otimes$ we have

$$\begin{aligned} |\frac{\partial K}{\partial y}(x,y,q,p,\omega,\lambda)| \;&\leq\; \sum_{|j|+k+l+m=1}^{2}\;\sum_{|\mathsf{h}|\leq\theta} |j|\,|K^{\mathsf{h}}_{jklm}(\omega)|\,|y|^{|j|-1}|q|^k|p|^l|\lambda|^m\,\mathrm{e}^{|\mathrm{Im}\,x||\mathsf{h}|} \\ &\leq\!\cdot\; \sum_{|j|+k+l+m=1}^{2} \|K^0_{jklm}\|_{D_\otimes}\,\varepsilon^{|j|+k+l+m-1} \\ &+\; \sum_{|j|+k+l+m=1}^{2}\;\sum_{|\mathsf{h}|\leq\theta}\;\sum_{\mu=1}^{7} |(\mathsf{h}\,|\,\omega)|^{-\mu}\,|H^{\mathsf{h}}_{jklm}(\omega)|\,\varepsilon^{|j|+k+l+m-1}\,\mathrm{e}^{(\frac{1}{2}\kappa+\sigma_\otimes)|\mathsf{h}|} \end{aligned}$$

$$\leq. \ \varepsilon^{-1} \| \frac{H}{d} \|_{D} \ + \ \varepsilon^{-2} \| \frac{H^{\partial}}{d} \|_{D} \ + \ \varepsilon^{-1} \| H \|_{D} \sum_{|\mathsf{h}| \leq \theta} \sum_{\mu=1}^{7} |(\mathsf{h} \,|\, \omega)|^{-\mu} \ \mathrm{e}^{-|\mathsf{h}|(\frac{1}{2}\kappa + \sigma - \frac{1}{2}\kappa - \sigma_{\otimes})} \qquad .$$

In the last step we used $|H^{\mathsf{h}}_{jklm}(\omega)| \leq \|H^{\mathsf{h}}_{jklm}\|_{D_{\otimes}} \mathrm{e}^{-|\mathsf{h}|(\frac{1}{2}\kappa+\sigma)}$ which expresses the familiar decay of the Fourier coefficients of a real analytic function (Payley-Wiener), cf. [Zehnder;75] or [Benettin,Galgani,Giorgilli,Strelcyn;84].

It remains to estimate the sum in the last term by $\sigma^{-8\tau}$. Since $\sigma - \sigma_{\otimes} = \frac{1}{4}\sigma$ this sum is $\sum_{|\mathsf{h}| \leq \theta} \sum_{\mu=1}^{7} |(\mathsf{h}\,|\,\omega)|^{-\mu} \mathrm{e}^{-\frac{1}{4}|\mathsf{h}|\sigma}$. To $\omega \in U_{\rho_{\otimes}}(\Sigma'_1)$ exists $w \in \Sigma'_1$ with $|\omega - w| < \rho_{\otimes} \leq \frac{3}{4}\rho$. Also recall $\theta = [\rho^{\frac{-1}{\tau+1}}]$, *i.e.* $-\rho \geq -\theta^{-\tau-1}$. We obtain

$$|(\mathsf{h}\,|\,\omega)| \ \geq \ |(\mathsf{h}\,|\,w)| \ - \ |\omega - w||\mathsf{h}| \ \stackrel{\text{Def. } \Sigma'_1}{\geq} \ |\mathsf{h}|^{-\tau} \ - \ \frac{3}{4}\rho\theta \ \geq \ |\mathsf{h}|^{-\tau} \ - \ \frac{3}{4}\theta^{-\tau} \ \geq \ \frac{1}{4}|\mathsf{h}|^{-\tau} \qquad .$$

Lemma 4.6 $\quad \bigwedge_{\ell \in \mathbb{N}} \ \bigwedge_{s \in]0,1]} \ \sum_{\mathsf{h} \in \mathbb{Z}^n} |\mathsf{h}|^{\ell\tau} \, \mathrm{e}^{-|\mathsf{h}|s} \ \leq \ C(\ell, \tau, n) \, s^{-(\ell+1)\tau}$

Proof : See Lemma 5.5 of [Braaksma,Broer;87]
and the considerations following it ; **q.e.d.(4.6)**

Now we can conclude (with $s = \frac{1}{4}\sigma$)

$$\sum_{|\mathsf{h}| \leq \theta} |\mathsf{h}|^{7\tau} \mathrm{e}^{-\frac{1}{4}|\mathsf{h}|\sigma} \ \leq. \ \sigma^{-8\tau} \qquad ; \qquad \textbf{q.e.d.(4.5)}$$

The terms K_{0klm} (with "$j = 0$") allow for the estimate $\|\frac{\partial^{|\iota|}}{\partial x^{\iota}} K_{0klm}\|_{D_{\otimes}} \leq. \sigma^{-12\tau-|\iota|} \cdot \varepsilon^{-k-l-m} \|H^{\delta}\|_{D}$ for $|\iota| \geq 1$. If they are different from K_{0010} and K_{0011} we also have $\|K_{0klm}\|_{D_{\otimes}} \leq. \sigma^{-12\tau} \varepsilon^{-k-l-m} \|\frac{H^{\delta}}{d}\|_{D}$.

For V we need another intermediate domain D_* . Let $\sigma_* = \frac{1}{2}(\sigma + \sigma_+)$, $\rho_* = \frac{1}{2}(\rho + \rho_+)$ and $\varepsilon_* = \frac{1}{2}(\varepsilon + \varepsilon_+)$, then $D_+ \subseteq D_* \subseteq D_{\otimes} \subseteq D$. We have $\varepsilon_{\otimes} - \varepsilon_* \geq \frac{1}{8}\varepsilon$ and $\|X_K\|_{D_{\otimes}} \leq c_2 \, \sigma^{-12\tau-1} \, \varepsilon^{-2} \, \partial$. We require $\delta^{\mathsf{t}-3\mathsf{q}} < \frac{\sigma^{12\tau+1}}{8c_2}$, *i.e.* $\mathsf{t} > 3\mathsf{q}$ and $\delta_0^{\mathsf{t}-3\mathsf{q}} < \min\left\{ \frac{\sigma_0^{12\tau+1}}{8c_2}, (\frac{1}{4})^{\frac{12\tau+1}{\mathsf{p}}} \right\}$, whereupon $\|X_K\|_{D_{\otimes}} < \varepsilon_{\otimes} - \varepsilon_*$. This assumption relating δ_0 and σ_0 intensifies the similar inequality (15) — note $c_1 + 4\Gamma \leq c_2$ — and we have $(\mathrm{id} + (X_K, \Omega, \Lambda))(D_*) \subseteq D_{\otimes}$. We want this inclusion to hold with X_K replaced by V .

Lemma 4.7

The time-1-flow $\psi = \mathrm{id} + V$ *can be estimated by the vector field* $U = X_K$ *— varying the domain :*

$$\|V^{\mu}\|_{D_*} \ \leq \ \|U^{\mu}\|_{D_{\otimes}} \qquad \textit{for} \quad \mu = 1, \ldots, 4 \qquad .$$

In particular

$$\|V\|_{D_*} \;\leq\; c_2\,\sigma^{-12\tau-1}\,\varepsilon^{-1}\Big(\|\frac{H}{d}\|_D + \varepsilon^{-1}\|\frac{H^\partial}{d}\|_D\Big)$$

and $\Psi(D_*) \subseteq D_\otimes$.

Proof :

We denote the flow of X_K by ψ_t , get

$$\|\psi_t - \mathrm{id}\|_{D_*} \;\leq\; t\,\|X_K \circ \psi_t\|_{D_*} \;\leq\; \|X_K\|_{\psi_t(D_*)}$$

and use the connectivity of $[0,1]$ to conclude ; **q.e.d.(4.7)**

Using Cauchy's estimate we get, since $\sigma_* - \sigma_+ = \frac{1}{8}\sigma$, $\rho_* - \rho_+ \geq \frac{1}{8}\rho$ and $\varepsilon_* - \varepsilon_+ \geq \varepsilon$,

$$\sigma^{2|\nu|\tau+2|\nu|+|\iota|}\,\varepsilon^{|j|+k+l+m}\,\|\frac{\partial^{|\iota|+|j|+k+l+|\nu|+m}V^\mu}{\partial x^\iota \partial y^j \partial q^k \partial p^l \partial\omega^\nu \partial\lambda^m}\|_{D_+} \;\leq\!\cdot\; \|U^\mu\|_{D_\otimes} \qquad .$$

In H_+ we also have terms that have components of $V-U$ as a factor. If we denote $V_t := \psi_t - \mathrm{id}$ we get $\frac{\mathrm{d}}{\mathrm{d}t}(V_t - tU) \;=\; X_K \circ \psi_t - X_K$ whence $\|V-U\| \;\leq\; \int_0^1 \|U\circ\psi_t - U\|\mathrm{d}t$. Using the mean value theorem and Cauchy's estimate we conclude

$$\begin{aligned}\|V^\mu - U^\mu\|_{D_*} \;\leq\; & \max\Big\{\; \|\frac{\partial U^\mu}{\partial x}\|_{D_\otimes}\cdot\int_0^1 \|V_t^1\|_{D_*}\,\mathrm{d}t \;,\; \|\frac{\partial U^\mu}{\partial y}\|_{D_\otimes}\cdot\int_0^1 \|V_t^2\|_{D_*}\,\mathrm{d}t \\ & \qquad \|\frac{\partial U^\mu}{\partial q}\|_{D_\otimes}\cdot\int_0^1 \|V_t^3\|_{D_*}\,\mathrm{d}t \;,\; \|\frac{\partial U^\mu}{\partial p}\|_{D_\otimes}\cdot\int_0^1 \|V_t^4\|_{D_*}\,\mathrm{d}t \;\Big\} \\ \leq\!\cdot\; & \quad \sigma^{-23\tau-2}\,\varepsilon^{-3}\,(\|\frac{H}{d}\|_D + \varepsilon^{-1}\|\frac{H^\partial}{d}\|_D)^2\end{aligned}$$

for $\mu = 3,4$. For $\mu = 2$ we have the better estimate

$$\|V^2 - U^2\|_{D_*} \;\leq\!\cdot\; \sigma^{-24\tau-3}\,\varepsilon^{-2}\,\|H\|_D\,(\|\frac{H}{d}\|_D + \varepsilon^{-1}\|\frac{H^\partial}{d}\|_D) \qquad .$$

Since we have no bounds on ω we finally need the following estimate.

Lemma 4.8

$$\|(\omega\,|\,\frac{\partial K}{\partial x})\|_{D_\otimes} \;\leq\!\cdot\; \sigma^{-11}\,\|H\|_D$$

Proof :

From (14) we have

$$(\omega \,|\, \frac{\partial K}{\partial x}) \;=\; \sum_{0<|\mathsf{h}|\leq\theta} ({}^{T}\!H^{\mathsf{h}} \;-\; 2a\,\frac{\partial K^{\mathsf{h}}}{\partial q}\,p \;-\; d\,\frac{\partial K^{\mathsf{h}}}{\partial p}\,\lambda \;+\; b\,\frac{\partial \widetilde{K}^{\mathsf{h}}}{\partial p}\,q^2 \;+\; (c\,|\,y)\,\frac{\partial \widetilde{K}^{\mathsf{h}}}{\partial p})\;\mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,\omega)} \qquad .$$

Like in the proof of Lemma 4.5 we use the Payley-Wiener estimate to conclude

$$|(\omega \,|\, \frac{\partial K}{\partial x})| \;\leq.\; \|H\|_{D} \sum_{0<|\mathsf{h}|\leq\theta} (1 \;+\; 2a|\mathsf{h}|^{9} \;+\; d|\mathsf{h}|^{10} \;+\; b|\mathsf{h}|^{7} \;+\; \|c\||\mathsf{h}|^{7})\;\mathrm{e}^{-|\mathsf{h}|(\sigma-\sigma_{\otimes})}$$

and again apply Lemma 4.6 ; **q.e.d.(4.8)**

(i) *construction of* Ψ
(ii) *estimates for* H_+ *(introduction)*
(iii) *estimates for* Ψ
$\longrightarrow$ *estimates for* H_+ *(conclusion)*
(v) *the assumptions on* σ_0 *and* δ_0

For the sake of completeness we now estimate one by one all the terms of H_+ that involve K, U or V . The strategy is always the same : if we cannot immediately estimate by ∂_+ , we isolate the terms that enter H_+^{∂} and show that they only depend on H^{δ} . Like in (ii) we denote in brackets the necessary inequalities on the exponents $\mathsf{r}, \mathsf{t}, \mathsf{q}$ and p , but we do not explicitly mention the accompanying assumptions on σ_0 and δ_0 . The point $(x, y, q, p, \omega, \lambda)$ is varying in D_+ .

$$\tfrac{1}{d}|b(\omega) \sum_{k+l+m=3} l\,K_{0klm}\,q^{k+2}p^{l-1}\lambda^{m}| \leq.\; \sigma^{-12\tau}\,\varepsilon^{-3}\,\|\tfrac{H^{\delta}}{d}\|_{D}\,\varepsilon_+^{4}$$
$$\leq\; \sigma^{-12\tau}\,\delta^{1+\mathsf{q}+4\mathsf{pq}} \;\overset{!}{<}\; \partial_+ \qquad (1{+}\mathsf{q}{>}\mathsf{t} \;\wedge\; 4\mathsf{q}{>}1)$$

$$\tfrac{1}{d}|(c(\omega)\,|\,y) \sum_{k+l+m=3} l\,K_{0klm}\,q^{k}p^{l-1}\lambda^{m}| \leq.\; \sigma^{-12\tau}\,\varepsilon^{-3}\,\delta\,\varepsilon_+^{3} \;\overset{!}{<}\; \Delta_+ \qquad (1{>}\mathsf{r} \;\wedge\; 3\mathsf{q}{>}\mathsf{r})$$

This is the other term where we need the (weaker) estimate Δ .
Note the similarity to the problematic terms in $H - {}^{T}\!H$.

$$\tfrac{1}{d}|2(a(\omega+\Omega) - a(\omega))\,U^{4}\,p| \;\leq\; \tfrac{2}{d}\,\|\tfrac{\partial a}{\partial\omega}\|_{D}\,\|\Omega\|_{D_+}\,\|U^{4}\|_{D_+}\,\varepsilon_+$$
$$\leq.\; \varepsilon^{-2}\,\partial\,\sigma^{-10\tau}\,\varepsilon^{-1}\,\Delta\,\varepsilon_+ \;\overset{!}{<}\; \partial_+ \qquad (\mathsf{r}{-}2\mathsf{q}{+}\mathsf{pq}{>}\mathsf{tp})$$

$$\tfrac{1}{d}|(b(\omega+\Omega) - b(\omega))\,U^{3}\,q^{2}| \leq.\; \varepsilon^{-2}\,\partial\,\sigma^{-11\tau}\,\varepsilon^{-2}\,\partial\,\varepsilon_+^{2} \;\overset{!}{<}\; \partial_+ \qquad (\mathsf{t}{-}2\mathsf{q}{+}2\mathsf{pq}{>}\mathsf{tp})$$

$$\tfrac{1}{d}|(c(\omega+\Omega) - c(\omega)\,|\,y)\,U^{3}| \leq.\; \varepsilon^{-2}\,\partial\,\sigma^{-11\tau}\,\varepsilon^{-2}\,\partial\,\varepsilon_+ \;\overset{!}{<}\; \partial_+ \qquad (\mathsf{t}{-}3\mathsf{q}{+}\mathsf{pq}{>}\mathsf{tp})$$

Note that it would be sufficient to estimate this term by Δ .

$$\tfrac{1}{d}|(c(\omega+\Omega)\,|\,U^{2})\,q| \;\leq\; \tfrac{1}{d}\|c\|_{D}\,c_2\,\sigma^{-12\tau-1}\,\|H\|_{D}\,\varepsilon_+$$
$$\leq.\; \sigma^{-12\tau-1}\,\Delta\,\varepsilon_+ \;\overset{!}{<}\; \delta_+ \qquad (\mathsf{r}{+}\mathsf{q}{+}\mathsf{pq}{>}1{+}\mathsf{p})$$

The average of this term vanishes, so it does not contribute to H_+ .

$\frac{2}{d}|(e(\omega+\Omega)\, y \,|\, V^2)| \leq\!\cdot\;\; \sigma^{-12\tau-1}\,\Delta\,\varepsilon_+ \;\overset{!}{<}\; \delta_+$ (r+q+pq>1+p)

Obviously "$\overset{!}{<} \Delta_+$" would be enough.

$\frac{1}{d}|(\Omega \,|\, V^2)| \leq\!\cdot\;\; \varepsilon^{-2}\,\partial\,\sigma^{-12\tau-1}\,\Delta \;\overset{!}{<}\; \partial_+$ (r−2q>tp)

$\frac{1}{d}|d\,\Lambda\, V^3)| \leq\!\cdot\;\; \varepsilon^{-1}\,\delta\,\sigma^{-11}\,\varepsilon^{-2}\,\partial \;\overset{!}{<}\; \partial_+$ (1−3q>tp)

$\frac{1}{d}|b(\omega+\Omega)\, V^3\, V^3\, q| \leq\!\cdot\;\; \sigma^{-22\tau}\,\varepsilon^{-4}\,\partial^2\,\varepsilon_+ \;\overset{!}{<}\; \partial_+$ (t−3q+pq>tp)

$\frac{1}{3d}|b(\omega+\Omega)\, V^3\, V^3\, V^3| \leq\!\cdot\;\; \sigma^{-33\tau}\,\varepsilon^{-6}\,\partial^3 \;\overset{!}{<}\; \partial_+$ (2t−6q>tp)

$\frac{1}{d}|(c(\omega+\Omega) \,|\, V^2)\, V^3| \leq\!\cdot\;\; \sigma^{-23\tau-1}\,\Delta\,\varepsilon^{-2}\,\partial \;\overset{!}{<}\; \partial_+$ (r−2q>tp)

$\frac{1}{d}|(e(\omega+\Omega)\, V^2 \,|\, V^2)| \leq\!\cdot\;\; \sigma^{-24\tau-2}\,\Delta^2 \;\overset{!}{<}\; \partial_+$ (2r>t+tp)

$\frac{1}{d}|a(\omega+\Omega)\, U^4\, U^4| \;\leq\; \frac{1}{d}\|a\|_D\; \|U^4\|_{D_+}\, \|U^4\|_{D_+} \leq\!\cdot\; \sigma^{-10\tau}\,\varepsilon^{-1}\,\|\frac{H}{d}\|_D\,\sigma^{-10\tau}\,\varepsilon^{-1}\,\|\frac{H}{d}\|_D$

$\leq\; \sigma^{-20\tau}\,\varepsilon^{-2}\,\Delta^2 \;\overset{!}{<}\; \delta_+$ (2r−2q>1+p)

Those terms of aU^4U^4 that enter H_+^∂ have to come from H^δ , for them we continue

$\leq\; \sigma^{-20\tau}\,\varepsilon^{-2}\,\delta^2 \;\overset{!}{<}\; \partial_+$ (2−2q>t+tp)

$\frac{1}{d}|a(\omega+\Omega)\, U^4\,(V^4-U^4)| \leq\!\cdot\;\; \|U^4\|_{D_+}\, \|V^4-U^4\|_{D_+}$

$\leq\!\cdot\; \sigma^{-10\tau}\,\varepsilon^{-1}\,\|\frac{H}{d}\|_D\,\sigma^{-23\tau-2}\,\varepsilon^{-3}\,(\|\frac{H}{d}\|_D+\varepsilon^{-1}\|\frac{H^\partial}{d}\|_D)^2$

$\leq\; \sigma^{-33\tau-2}\,\varepsilon^{-6}\,\Delta\,\partial^2 \;\overset{!}{<}\; \partial_+$ (t+r−6q>tp)

$\frac{1}{d}|a(\omega+\Omega)\, V^4\,(V^4-U^4)| \leq\!\cdot\;\; \sigma^{-33\tau-2}\,\varepsilon^{-6}\,\Delta\,\partial^2 \;\overset{!}{<}\; \partial_+$ (t+r−6q>tp)

$\frac{2}{d}|a(\omega+\Omega)\,(V^4-U^4)\, p| \leq\!\cdot\;\; \sigma^{-23\tau-2}\,\varepsilon^{-3}\,(\|\frac{H}{d}\|_D+\varepsilon^{-1}\|\frac{H^\partial}{d}\|_D)^2\,\varepsilon_+$

$\leq\!\cdot\; \sigma^{-23\tau-2}\,\varepsilon^{-5}\,\partial^2\,\varepsilon_+ \;\overset{!}{<}\; \delta_+$ (2t−4q+pq>1+p)

The Taylor series of $(V^4-U^4)p$ contains the monomial qp^2 which belongs to H_+^∂ . Here we estimate more carefully. We are interested in $\frac{\partial^2}{\partial q \partial p}(V^4-U^4)(x,0,0,0,\omega,0)$, so we differentiate $V^4-U^4 = \int_0^1 U^4 \circ \psi_t - U^4 \mathrm{d}t$ under the integral sign ($\psi_t = \mathrm{id}+V_t$ denotes the flow of X_K). Using the chain rule we write

$$\frac{\partial^2}{\partial q \partial p}(U^4\circ\psi_t) \;=\; \frac{\partial^2 U^4}{\partial q \partial p}\circ\psi_t \;+\; A \;+\; B \;+\; C \;+\; (\frac{\partial^2 U^4}{\partial x^2}\circ\psi_t)\,\frac{\partial V_t^1}{\partial q}\,\frac{\partial V_t^1}{\partial p} \qquad .$$

Now $U^4 = -\frac{\partial K}{\partial q}$ is a quadratic polynomial in q and p and only depends linearly on y . Therefore ψ_t alters $\frac{\partial^2 U^4}{\partial q \partial p} = -K_{0210}$ only in the x-direction and the mean value theorem yields

$\frac{2}{d}|a(\omega+\Omega)\, q\, p^2 \int_0^1 \frac{\partial^2 U^4}{\partial q \partial p}\circ\psi_t - \frac{\partial^2 U^4}{\partial q \partial p}\,\mathrm{d}t| \leq\!\cdot\;\; \varepsilon_+^3 \int_0^1 \|\frac{\partial}{\partial x}K_{0210}\|_{D_\otimes}\, \|V_t^1\|_{D_+}\,\mathrm{d}t$

$\leq\!\cdot\; \varepsilon_+^3\,\sigma^{-12\tau-1}\,\varepsilon^{-3}\,\|H^\delta\|_D\,\sigma^{-8\tau}\,\varepsilon^{-1}\,(\|\frac{H}{d}\|_D+\varepsilon^{-1}\|\frac{H^\partial}{d}\|_D)$

$\leq\!\cdot\; \sigma^{-20\tau-1}\,\varepsilon^{-5}\,\delta\,\partial\,\varepsilon_+^3 \;\overset{!}{<}\; \partial_+$ (1−2q+3pq>tp)

In A we collect the terms $k!\,l!\,K_{0kl0}(x+V_t^1,\omega)$, $k+l=3$, multiplied by one or two factors $\frac{\partial V_t^\mu}{\partial z}$, $\mu=3,4$, $z=q,p$. We have

$$\tfrac{2}{d}|a(\omega+\Omega)\,q\,p^2\int_0^1 A\,\mathrm{d}t| \le.\ \varepsilon_+^3\,\sigma^{-12\tau}\,\varepsilon^{-3}\,\|\tfrac{H^\delta}{d}\|_D\,\sigma^{-11\tau}\,\varepsilon^{-2}\,(\|\tfrac{H}{d}\|_D+\varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)$$
$$\le.\ \sigma^{-23\tau}\,\varepsilon^{-6}\,\delta\,\partial\,\varepsilon_+^3 \overset{!}{<} \partial_+ \qquad (1-3q+3pq>tp)$$

The expression B contains the terms $(\frac{\partial U^4}{\partial z}\circ\psi_t)\,\frac{\partial^2 V_t^\mu}{\partial q\partial p}$, $(z,\mu)=(x,1),\ldots,(p,4)$. While

$$|\int_0^1 (\frac{\partial U^4}{\partial x}\circ\psi_t)\,\frac{\partial^2 V_t^1}{\partial q\partial p} + (\frac{\partial U^4}{\partial y}\circ\psi_t)\,\frac{\partial^2 V_t^2}{\partial q\partial p}\,\mathrm{d}t\,|$$
$$\le.\ \sigma^{-20\tau-1}\,\varepsilon^{-4}\,\|H\|_D\,(\|\frac{H}{d}\|_D+\varepsilon^{-1}\|\frac{H^\partial}{d}\|_D)$$

the other two terms are more involved. We estimate for $z=q,p$

$$|\,\frac{\partial U^4}{\partial z}(\psi_t(x,0,0,0,\omega,0))\,| \le |\,\frac{\partial U^4}{\partial z}(x,0,0,0,\omega,0))\,| + \max\Big\{\|\frac{\partial^2 U^4}{\partial x\partial z}\|_{D_\otimes}\cdot\|V_t^1\|_{D_*}\,,$$
$$\|\frac{\partial^2 U^4}{\partial y\partial z}\|_{D_\otimes}\cdot\|V_t^2\|_{D_*}\,,\ \|\frac{\partial^2 U^4}{\partial q\partial z}\|_{D_\otimes}\cdot\|V_t^3\|_{D_*}\,,\ \|\frac{\partial^2 U^4}{\partial p\partial z}\|_{D_\otimes}\cdot\|V_t^4\|_{D_*}\Big\}$$
$$\le.\ \sigma^{-10\tau}\,\varepsilon^{-2}\,\|\frac{H^\delta}{d}\|_D + \sigma^{-22\tau-2}\,\varepsilon^{-4}\,\|\frac{H}{d}\|_D\,(\|\frac{H}{d}\|_D+\varepsilon^{-1}\|\frac{H^\partial}{d}\|_D)$$

and multiply by $|\frac{\partial^2 V_t^\mu}{\partial q\partial p}|\le.\ \sigma^{-11\tau}\,\varepsilon^{-4}\,\partial$, $(\mu=3,4)$. Summing up we get

$$\tfrac{2}{d}|a(\omega+\Omega)\,q\,p^2\int_0^1 B\,\mathrm{d}t| \qquad (r-2q+3pq>tp\ \wedge\ 1-3q+3pq>tp\ \wedge\ t+r-6q+3pq>tp)$$
$$\le.\ \sigma^{-20\tau-1}\,\varepsilon^{-5}\,\Delta\,\partial\,\varepsilon_+^3 + \sigma^{-21\tau}\,\varepsilon^{-6}\,\delta\,\partial\,\varepsilon_+^3 + \sigma^{-33\tau-2}\,\varepsilon^{-9}\,\Delta\,\partial^2\,\varepsilon_+^3 \overset{!}{<} \partial_+$$

The remainder C consists of terms $\frac{\partial^{|j|+k+l}U^2}{\partial y^j\partial q^k\partial p^l}\circ\psi_t$ with $|j|+k+l=2$, multiplied by a factor $\frac{\partial V_t^1}{\partial z}$, $z=q,p$, and eventually also a factor $\frac{\partial V_t^\mu}{\partial z}$, $\mu=2,3,4$, $z=q,p$. We conclude

$$\tfrac{2}{d}|a(\omega+\Omega)\,q\,p^2\int_0^1 C + (\tfrac{\partial^2 U^4}{\partial x^2}\circ\psi_t)\,\tfrac{\partial V_t^1}{\partial q}\,\tfrac{\partial V_t^1}{\partial p}\,\mathrm{d}t|$$
$$\le.\ \sigma^{-20\tau-1}\,\varepsilon^{-2}\,\Delta\,\varepsilon^{-3}\,\partial\,\varepsilon_+^3 \overset{!}{<} \partial_+ \qquad (r-2q+3pq>tp)$$

$$\tfrac{1}{d}|b(\omega+\Omega)\,(V^3-U^3)\,q^2| \le.\ \sigma^{-23\tau-2}\,\varepsilon^{-5}\,\partial^2\,\varepsilon_+^2 \overset{!}{<} \partial_+ \qquad (t-3q+2pq>tp)$$

$$\tfrac{1}{d}|d\,(V^3-U^3)\,\lambda| \le.\ \sigma^{-23\tau-2}\,\varepsilon^{-5}\,\partial^2\,\varepsilon_+ \overset{!}{<} \delta_+ \qquad (2t-4q+pq>1+p)$$

Here the term $\frac{\partial^2}{\partial q^2}(V^3-U^3)(x,0,0,0,\omega,0)\,q^2\,\lambda$ belongs to H_+^∂ . We treat this similar to $\frac{\partial^2}{\partial q\partial p}(V^4-U^4)(x,0,0,0,\omega,0)\,q\,p^2$. Again differentiating under the integral sign we write

$$\frac{\partial^2}{\partial q^2}(U^3\circ\psi_t) = \frac{\partial^2 U^3}{\partial q^2}\circ\psi_t + A + B + C + (\frac{\partial^2 U^3}{\partial x^2}\circ\psi_t)\,\frac{\partial V_t^1}{\partial q}\,\frac{\partial V_t^1}{\partial q}$$

and get

$$|q^2\lambda\int_0^1 \tfrac{\partial^2 U^3}{\partial q^2}\circ\psi_t - \tfrac{\partial^2 U^3}{\partial q^2}\,\mathrm{d}t| \leq\!\cdot\;\; \varepsilon_+^3\int_0^1 \|\tfrac{\partial}{\partial x}K_{0210}\|_{D_\otimes}\,\|V_t^1\|_{D_+}\,\mathrm{d}t$$

$$\leq\!\cdot\; \sigma^{-20\tau-1}\,\varepsilon^{-5}\,\delta\,\partial\,\varepsilon_+^3 \;\overset{!}{<}\; \partial_+ \qquad \text{(1−2q+3pq>tp)}$$

$$|q^2\lambda\int_0^1 A\,\mathrm{d}t| \leq\!\cdot\;\; \sigma^{-23\tau}\,\varepsilon^{-6}\,\delta\,\partial\,\varepsilon_+^3 \;\overset{!}{<}\; \partial_+ \qquad \text{(1−3q+3pq>tp)}$$

$$|q^2\lambda\int_0^1 B\,\mathrm{d}t| \qquad \text{(r−2q+3pq>tp} \;\wedge\; \text{1−3q+3pq>tp} \;\wedge\; \text{t+r−6q+3pq>tp)}$$

$$\leq\!\cdot\; \sigma^{-20\tau-1}\,\varepsilon^{-5}\,\Delta\,\partial\,\varepsilon_+^3 \;+\; \sigma^{-21\tau}\,\varepsilon^{-6}\,\delta\,\partial\,\varepsilon_+^3 \;+\; \sigma^{-33\tau-2}\,\varepsilon^{-9}\,\Delta\,\partial^2\,\varepsilon_+^3 \;\overset{!}{<}\; \partial_+$$

$$|q^2\lambda\int_0^1 C \;+\; (\tfrac{\partial^2 U^3}{\partial x^2}\circ\psi_t)\,\tfrac{\partial V_t^1}{\partial q}\,\tfrac{\partial V_t^1}{\partial q}\,\mathrm{d}t| \leq\!\cdot\;\; \sigma^{-20\tau-1}\,\varepsilon^{-5}\,\Delta\,\partial\,\varepsilon_+^3 \;\overset{!}{<}\; \partial_+ \qquad \text{(r−2q+3pq>tp)}$$

$$\tfrac{1}{d}|(c(\omega+\Omega)\,|\,y)\,(V^3-U^3)| \leq\!\cdot\;\; \sigma^{-23\tau-2}\,\varepsilon^{-5}\,\partial^2\,\varepsilon_+ \;\overset{!}{<}\; \delta_+ \qquad \text{(2t−4q+pq>1+p)}$$

It would be sufficient to estimate by Δ_+ .

$$\tfrac{1}{d}|(c(\omega+\Omega)\,|\,V^2-U^2)\,q| \leq\!\cdot\;\; \sigma^{-24\tau-3}\,\varepsilon^{-2}\,\|H\|_D\,(\|\tfrac{H}{d}\|_D + \varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)\,\varepsilon_+$$

$$\leq\; \sigma^{-24\tau-3}\,\varepsilon^{-3}\,\Delta\,\partial\,\varepsilon_+ \;\overset{!}{<}\; \partial_+ \qquad \text{(r−2q+pq>tp)}$$

$$\tfrac{1}{d}|(\omega\,|\,V^2-U^2)| \;=\; \tfrac{1}{d}|\int_0^1 (\omega\,|\,U^2\circ\psi_t - U^2)\,\mathrm{d}t\,|$$

$$\leq\; \tfrac{1}{d}\,\max\Big\{\,\|\tfrac{\partial}{\partial x}(\omega\,|\,\tfrac{\partial K}{\partial x})\|_{D_\otimes}\cdot\int_0^1\|V_t^1\|_{D_*}\,\mathrm{d}t\,,\;\|\tfrac{\partial}{\partial y}(\omega\,|\,\tfrac{\partial K}{\partial x})\|_{D_\otimes}\cdot\int_0^1\|V_t^2\|_{D_*}\,\mathrm{d}t\,,$$

$$\|\tfrac{\partial}{\partial q}(\omega\,|\,\tfrac{\partial K}{\partial x})\|_{D_\otimes}\cdot\int_0^1\|V_t^3\|_{D_*}\,\mathrm{d}t\,,\;\|\tfrac{\partial}{\partial p}(\omega\,|\,\tfrac{\partial K}{\partial x})\|_{D_\otimes}\cdot\int_0^1\|V_t^4\|_{D_*}\,\mathrm{d}t\,\Big\}$$

$$\leq\!\cdot\; \tfrac{1}{d}\,\sigma^{-11\tau-1}\,\varepsilon^{-1}\,\|H\|_D\,\sigma^{-12\tau-1}\,\varepsilon^{-1}\,(\|\tfrac{H}{d}\|_D + \varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)$$

$$\leq\; \sigma^{-23\tau-2}\,\varepsilon^{-3}\,\Delta\,\partial \;\overset{!}{<}\; \delta_+ \qquad \text{(t+r−3q>1+p)}$$

This time the Taylor series of $(\omega\,|\,V^2-U^2)$ contains all three monomials q^2 , $q^2\lambda$ and qp^2 that constitute H_+^∂ . For $\frac{\partial^2}{\partial q^2}(\omega\,|\,V^2-U^2)(x,0,0,0,\omega,0)\,q^2$ we write

$$\frac{\partial^2}{\partial q^2}(U^2\circ\psi_t) \;=\; \frac{\partial^2 U^2}{\partial q^2}\circ\psi_t \;+\; 2(\frac{\partial^2 U^2}{\partial y\partial q}\circ\psi_t)\,\frac{\partial V_t^2}{\partial q} \;+\; 2(\frac{\partial^2 U^2}{\partial q^2}\circ\psi_t)\,\frac{\partial V_t^3}{\partial q}$$
$$+\; 2(\frac{\partial^2 U^2}{\partial q\partial p}\circ\psi_t)\,\frac{\partial V_t^4}{\partial q} \;+\; A \;+\; B \;+\; C$$

and obtain, since $\frac{\partial^2 U^2}{\partial q^2}$ and $\frac{\partial^2 U^2}{\partial q\partial p}$ only depend on terms of H^δ ,

$$\tfrac{1}{d}|q^2\int_0^1(\omega\,|\,\tfrac{\partial^2 U^2}{\partial q^2}\circ\psi_t - \tfrac{\partial^2 U^2}{\partial q^2})\,\mathrm{d}t\,| \;\leq\; \tfrac{1}{d}\,\varepsilon_+^2\int_0^1 \max\Big\{\,\|\tfrac{\partial}{\partial x}(\omega\,|\,\tfrac{\partial^2 U^2}{\partial q^2})\|_{D_\otimes}\cdot\|V_t^1\|_{D_*}\,,$$

$$\|\tfrac{\partial}{\partial q}(\omega\,|\,\tfrac{\partial^2 U^2}{\partial q^2})\|_{D_\otimes}\cdot\|V_t^3\|_{D_*}\,,\;\|\tfrac{\partial}{\partial p}(\omega\,|\,\tfrac{\partial^2 U^2}{\partial q^2})\|_{D_\otimes}\cdot\|V_t^4\|_{D_*}\,\Big\}\;\mathrm{d}t$$

$$\leq\!\cdot\; \tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-11\tau-1}\,\varepsilon^{-3}\,\|H^\delta\|_D\,\sigma^{-12\tau-1}\,\varepsilon^{-1}\,(\|\tfrac{H}{d}\|_D + \varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)$$

$$\leq\; \sigma^{-23\tau-2}\,\varepsilon^{-5}\,\delta\,\partial\,\varepsilon_+^2 \;\overset{!}{<}\; \partial_+ \qquad \text{(1−3q+2pq>tp)}$$

$$\tfrac{2}{d}|q^2\int_0^1(\omega\,|\,(\tfrac{\partial^2 U^2}{\partial y\partial q}\circ\psi_t)\,\tfrac{\partial V_t^2}{\partial q})\,\mathrm{d}t\,| \;\leq\; \tfrac{2}{d}\,\varepsilon_+^2\,\|(\omega\,|\,\tfrac{\partial}{\partial x}K_{1100})\|_{D_\otimes}\cdot\int_0^1\|V_t^2\|_{D_*}\,\mathrm{d}t$$

$$\leq\!\cdot\; \tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-11\tau-1}\,\varepsilon^{-2}\,\|H\|_D\,\sigma^{-12\tau-1}\,\varepsilon^{-1}\,\|H\|_D$$

$$\leq\; \sigma^{-23\tau-2}\,\varepsilon^{-3}\,\Delta^2\,\varepsilon_+^2 \;\overset{!}{<}\; \partial_+ \qquad \text{(2r−q+2pq>t+tp)}$$

$$\tfrac{2}{d}|q^2\int_0^1 (\omega\,|\,(\tfrac{\partial^2 U^2}{\partial q^2}\circ\psi_t)\,\tfrac{\partial V_t^3}{\partial q})\,\mathrm{d}t\,| \;\le\; \tfrac{2}{d}\,\varepsilon_+^2\,\|(\omega\,|\,\tfrac{\partial}{\partial x}\tfrac{\partial^2 U^2}{\partial q^2})\|_{D_\otimes}\cdot\int_0^1\|V_t^3\|_{D_*}\,\mathrm{d}t$$

$$\le\cdot\; \tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-11\tau}\,\varepsilon^{-2}\,\|H^\delta\|_D\,\sigma^{-11\tau}\,\varepsilon^{-2}\,(\|\tfrac{H}{d}\|_D+\varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)$$

$$\le\; \sigma^{-22\tau}\,\varepsilon^{-5}\,\delta\,\partial\,\varepsilon_+^2 \;\overset{!}{<}\; \partial_+ \qquad (1-3\mathsf{q}+2\mathsf{pq}>\mathsf{tp})$$

$$\tfrac{2}{d}|q^2\int_0^1 (\omega\,|\,(\tfrac{\partial^2 U^2}{\partial q\partial p}\circ\psi_t)\,\tfrac{\partial V_t^4}{\partial q})\,\mathrm{d}t\,| \;\le\cdot\; \tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-11\tau}\,\varepsilon^{-2}\,\|H^\delta\|_D\,\sigma^{-10\tau}\,\varepsilon^{-2}\,\|\tfrac{H}{d}\|_D$$

$$\le\; \sigma^{-21\tau}\,\varepsilon^{-4}\,\delta\,\Delta\,\varepsilon_+^2 \;\overset{!}{<}\; \partial_+ \qquad (1+\mathsf{r}-2\mathsf{q}+2\mathsf{pq}>\mathsf{t}+\mathsf{tp})$$

In A we collect the terms $\frac{\partial^{|j|+k+l}U^2}{\partial y^j\partial q^k\partial p^l}\circ\psi_t$ with $|j|+k+l=2$, multiplied by two factors $\frac{\partial V_t^\mu}{\partial q}$, $\mu=2,3,4$. We have

$$\tfrac{1}{d}|q^2\int_0^1(\omega\,|\,A)\mathrm{d}t| \;\le\; \tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-11\tau}\,\varepsilon^{-2}\|H\|_D\,\sigma^{-24\tau-2}\,\varepsilon^{-4}\,(\|\tfrac{H}{d}\|_D+\varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)^2$$

$$\le\; \sigma^{-35\tau-2}\,\varepsilon^{-8}\,\Delta\,\partial^2\,\varepsilon_+^2 \;\overset{!}{<}\; \partial_+ \qquad (\mathsf{t}+\mathsf{r}-6\mathsf{q}+2\mathsf{pq}>\mathsf{tp})$$

The expression B contains the terms $(\frac{\partial U^2}{\partial z}\circ\psi_t)\,\frac{\partial^2 V_t^\mu}{\partial q^2}$, $(z,\mu)=(x,1),\dots,(p,4)$ and allows the estimate (similar to the one in $\frac{1}{d}|a(\omega+\Omega)\,(V^4-U^4)\,p|$)

$$\tfrac{1}{d}|q^2\int_0^1(\omega\,|\,B)\mathrm{d}t| \le\cdot\; \tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-23\tau-1}\,\varepsilon^{-3}\,\|H\|_D\,(\|\tfrac{H}{d}\|_D+\varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)$$

$$+\;\tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-11\tau}\,\varepsilon^{-1}\,\|H^\delta\|_D\sigma^{-11\tau}\,\varepsilon^{-4}\,\partial$$

$$+\;\tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-11\tau-1}\,\varepsilon^{-2}\,\|H\|_D\sigma^{-12\tau-1}\,\varepsilon^{-1}\,(\|\tfrac{H}{d}\|_D+\varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)\sigma^{-11\tau}\,\varepsilon^{-4}\,\partial$$

$$\le\; \sigma^{-23\tau-1}\,\varepsilon^{-4}\,\Delta\,\partial\,\varepsilon_+^2\;+\;\sigma^{-22\tau}\,\varepsilon^{-5}\,\delta\,\partial\,\varepsilon_+^2\;+\;\sigma^{-34\tau-2}\,\varepsilon^{-8}\,\Delta\,\partial^2\,\varepsilon_+^2 \;\overset{!}{<}\; \partial_+$$

$$(\mathsf{r}-2\mathsf{q}+2\mathsf{pq}>\mathsf{tp}\;\;\wedge\;\;1-3\mathsf{q}+2\mathsf{pq}>\mathsf{tp}\;\;\wedge\;\;\mathsf{t}+\mathsf{r}-6\mathsf{q}+2\mathsf{pq}>\mathsf{tp})$$

The remainder C consists of terms $\frac{\partial^{|\iota|+|j|+k+l}U^2}{\partial x^\iota\partial y^j\partial q^k\partial p^l}\circ\psi_t$ with $|\iota|+|j|+k+l=2$, $|\iota|\ge 1$, multiplied by one or two factors $\frac{\partial V_t^\mu}{\partial q}$, $\mu=1,2,3,4$. We conclude

$$\tfrac{1}{d}|q^2\int_0^1(\omega\,|\,c)\mathrm{d}t| \le\cdot\; \tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-11\tau-2}\,\varepsilon^{-1}\|H\|_D\,\sigma^{-12\tau-1}\,\varepsilon^{-2}\,(\|\tfrac{H}{d}\|_D+\varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)$$

$$\le\; \sigma^{-23\tau-3}\,\varepsilon^{-4}\,\Delta\,\partial\,\varepsilon_+^2 \;\overset{!}{<}\; \partial_+ \qquad (\mathsf{r}-2\mathsf{q}+2\mathsf{pq}>\mathsf{tp})$$

Remain the third order monomials. To fix thoughts we only treat $q^2\lambda$, qp^2 being similar. Using Cauchy's estimate we convert the extra derivative $\frac{\partial}{\partial\lambda}$ into ε^{-1} in all the above expressions. Since this is compensated by $|\lambda|<\varepsilon_+$ we merely have to replace 2pq by 3pq in all the inequalities of the exponents.

$$\tfrac{1}{d}|H\circ\Psi\,-\,H| \;=\; \tfrac{1}{d}\;\max\Big\{\,\|\tfrac{\partial H}{\partial x}\|_{D_\otimes}\cdot\|V^1\|_{D_+}\,,\;\|\tfrac{\partial H}{\partial y}\|_{D_\otimes}\cdot\|V^2\|_{D_+}\,,\;\|\tfrac{\partial H}{\partial q}\|_{D_\otimes}\cdot\|V^3\|_{D_+}\,,$$

$$\|\tfrac{\partial H}{\partial p}\|_{D_\otimes}\cdot\|V^4\|_{D_+}\,,\;\|\tfrac{\partial H}{\partial\omega}\|_{D_\otimes}\cdot\|\Omega\|_{D_+}\,,\;\|\tfrac{\partial H}{\partial\lambda}\|_{D_\otimes}\cdot\|\Lambda\|_{D_+}\,\Big\}$$

$$\le\cdot\; \tfrac{1}{d}\,\sigma^{-2\tau-2}\,\varepsilon^{-1}\,\|H\|_D\,\sigma^{-12\tau-1}\,\varepsilon^{-1}\,(\|\tfrac{H}{d}\|_D+\varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)$$

$$\le\; \sigma^{-14\tau-3}\,\varepsilon^{-3}\,\Delta\,\partial \;\overset{!}{<}\; \delta_+ \qquad (\mathsf{t}+\mathsf{r}-3\mathsf{q}>1+\mathsf{p})$$

Some terms have to be estimated by ∂_+ . First, we have

$$\tfrac{1}{d}|H^\delta\circ\Psi\;-\;H^\delta|\;\le\!\cdot\;\tfrac{1}{d}\,\sigma^{-2\tau-2}\,\varepsilon^{-1}\,\|H^\delta\|_D\,\sigma^{-12\tau-1}\,\varepsilon^{-1}\,(\|\tfrac{H}{d}\|_D+\varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)$$
$$\le\;\sigma^{-14\tau-3}\,\varepsilon^{-3}\,\delta\,\partial\;\overset{!}{<}\;\partial_+ \qquad \text{(1−3q>tp)}$$

Thus we are only concerned with the y-dependent part $H^\Delta:=H-H^\delta$.
We have $\frac{\partial^2}{\partial q^2}H^\Delta(x,0,0,0,\omega,0)\;=\;0$ and

$$\frac{\partial^2}{\partial q^2}(H^\Delta\circ\Psi)\;\;=\;\;\frac{\partial^2 H^\Delta}{\partial q^2}\circ\Psi\;+\;A\;+\;B\;+\;C\qquad .$$

From the mean value theorem we conclude that in the point $(x,0,0,0,\omega,0)$

$$|\,\frac{\partial^2}{\partial q^2}(H^\Delta\circ\Psi)\,|\;\;\le\;\;\max\Big\{\;\|\tfrac{\partial^3H^\Delta}{\partial x\partial q^2}\|_{[\Psi]}\cdot|V^1|\;,\;\|\tfrac{\partial^3H^\Delta}{\partial y\partial q^2}\|_{[\Psi]}\cdot|V^2|\;,\;\|\tfrac{\partial^3H^\Delta}{\partial q^3}\|_{[\Psi]}\cdot|V^3|\;,$$
$$\|\tfrac{\partial^3H^\Delta}{\partial q^2\partial p}\|_{[\Psi]}\cdot|V^4|\;,\;\|\tfrac{\partial^3H^\Delta}{\partial q^2\partial\omega}\|_{[\Psi]}\cdot|\Omega|\;,\;\|\tfrac{\partial^3H^\Delta}{\partial q^2\partial\lambda}\|_{[\Psi]}\cdot|\Lambda|\;\Big\}$$

Here $[\Psi]$ denotes the segment $\{t\cdot\Psi(x,0,0,0,\omega,0)\;\big|\;t\in[0,1]\}$. For the first two and the last two terms it is sufficient to note $[\Psi]\subseteq D_\otimes$ as we have the estimates

$$\tfrac{1}{d}\,|q^2|\,\|\tfrac{\partial^3H^\Delta}{\partial x\partial q^2}\|_{D_\otimes}\,\|V^1\|_{D_+}\;\le\!\cdot\;\tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-1}\,\varepsilon^{-2}\|H\|_D\,\sigma^{-8\tau}\,\varepsilon^{-1}\,(\|\tfrac{H}{d}\|_D+\varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)$$
$$\le\;\sigma^{-8\tau-1}\,\varepsilon^{-4}\,\Delta\,\partial\,\varepsilon_+^2\;\overset{!}{<}\;\partial_+ \qquad \text{(r−2q+2pq>tp)}$$
$$\tfrac{1}{d}\,|q^2|\,\|\tfrac{\partial^3H^\Delta}{\partial y\partial q^2}\|_{D_\otimes}\,\|V^2\|_{D_+}\;\le\!\cdot\;\tfrac{1}{d}\,\varepsilon_+^2\,\varepsilon^{-3}\,\|H\|_D\,\sigma^{-12\tau-1}\,\|H\|_D$$
$$\le\;\sigma^{-12\tau-1}\,\varepsilon^{-3}\,\Delta^2\,\varepsilon_+^2\;\overset{!}{<}\;\partial_+ \qquad \text{(2r−q+2pq>t+tp)}$$
$$\tfrac{1}{d}\,|q^2|\,\|\tfrac{\partial^3H^\Delta}{\partial q^2\partial\omega}\|_{D_\otimes}\,\|\Omega\|_{D_+}\;\le\!\cdot\;\tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-2\tau-2}\,\varepsilon^{-2}\|H\|_D\,\varepsilon^{-1}\,(\|\tfrac{H}{d}\|_D+\varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)$$
$$\le\;\sigma^{-2\tau-2}\,\varepsilon^{-4}\,\Delta\,\partial\,\varepsilon_+^2\;\overset{!}{<}\;\partial_+ \qquad \text{(r−2q+2pq>tp)}$$
$$\tfrac{1}{d}\,|q^2|\,\|\tfrac{\partial^3H^\Delta}{\partial q^2\partial\lambda}\|_{D_\otimes}\,\|\Lambda\|_{D_+}\;\le\!\cdot\;\tfrac{1}{d}\,\varepsilon_+^2\,\varepsilon^{-3}\|H\|_D\,\varepsilon^{-1}\,\|\tfrac{H^\delta}{d}\|_D$$
$$\le\;\varepsilon^{-4}\,\Delta\,\delta\,\varepsilon_+^2\;\overset{!}{<}\;\partial_+ \qquad \text{(1+r−2q+2pq>t+tp)}$$

For the third term we apply the mean value theorem a second time and obtain

$$\|\frac{\partial^3}{\partial q^3}(H^\Delta\circ\Psi)\|_{[\Psi]}\;\;\le\;\;\max\Big\{\;\|\tfrac{\partial^4H^\Delta}{\partial x\partial q^3}\|_{D_\otimes}\cdot\|V^1\|_{D_+}\;,\;\|\tfrac{\partial^4H^\Delta}{\partial y\partial q^3}\|_{D_\otimes}\cdot\|V^2\|_{D_+}\;,$$
$$\|\tfrac{\partial^4H^\Delta}{\partial q^4}\|_{D_\otimes}\cdot\|V^3\|_{D_+}\;,\;\|\tfrac{\partial^4H^\Delta}{\partial q^3\partial p}\|_{D_\otimes}\cdot\|V^4\|_{D_+}\;,\;\|\tfrac{\partial^4H^\Delta}{\partial q^3\partial\omega}\|_{D_\otimes}\cdot\|\Omega\|_{D_+}\;,\;\|\tfrac{\partial^4H^\Delta}{\partial q^3\partial\lambda}\|_{D_\otimes}\cdot\|\Lambda\|_{D_+}\;\Big\}$$

This yields

$$\tfrac{1}{d}\,|q^2|\,\|\tfrac{\partial^3H^\Delta}{\partial q^3}\|_{[\Psi]}\cdot\|V^3\|_{D_+}\;\;\le\!\cdot\;\tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-2\tau-2}\,\varepsilon^{-4}\|H\|_D\,\sigma^{-23\tau-1}\,\varepsilon^{-2}\,(\|\tfrac{H}{d}\|_D+\varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)^2$$
$$\le\;\sigma^{-25\tau-3}\,\varepsilon^{-8}\,\Delta\,\partial^2\,\varepsilon_+^2\;\overset{!}{<}\;\partial_+ \qquad \text{(t+r−6q+2pq>tp)}$$

The same estimate holds true for $\frac{1}{d}\,|q^2|\,\|\frac{\partial^3H^\Delta}{\partial q^2\partial p}\|_{[\Psi]}\cdot\|V^4\|_{D_+}$.
The expression A contains the terms $(\frac{\partial^2H^\Delta}{\partial q^2}\circ\Psi)\frac{\partial V^3}{\partial q}$, $(\frac{\partial H^\Delta}{\partial q}\circ\Psi)\frac{\partial^2V^3}{\partial q^2}$, $(\frac{\partial^2H^\Delta}{\partial q\partial p}\circ\Psi)\frac{\partial V^4}{\partial q}$

and $(\frac{\partial H^\Delta}{\partial p}\circ\Psi)\frac{\partial^2 V^4}{\partial q^2}$. We apply the mean value theorem and obtain the same estimate as above.
In B we assemble those terms where in comparison with the last two terms of A either $\frac{\partial}{\partial p}$ is replaced by $\frac{\partial}{\partial x},\frac{\partial}{\partial \omega}$, or V^4 is replaced by V^1,Λ . This gives the same four estimates as for the first two and the last two terms we encountered in $\frac{\partial^2 H^\Delta}{\partial q^2}\circ\Psi$.
The remainder C consists of second derivatives of H^Δ , multiplied by two terms of the form $\frac{\partial V^1}{\partial q},\ldots,\frac{\partial V^4}{\partial q},\frac{\partial \Omega}{\partial q},\frac{\partial \Lambda}{\partial q}$. We conclude

$$\begin{aligned}\tfrac{1}{d}\,|q^2\,C| &\leq\!\cdot\; \tfrac{1}{d}\,\varepsilon_+^2\,\sigma^{-4\tau-4}\,\varepsilon^{-2}\|H\|_D\,\sigma^{-24\tau-2}\,\varepsilon^{-4}\,(\|\tfrac{H}{d}\|_D+\varepsilon^{-1}\|\tfrac{H^\partial}{d}\|_D)^2\\ &\leq\; \sigma^{-28\tau-6}\,\varepsilon^{-8}\,\Delta\,\partial^2\,\varepsilon_+^2 \;\overset{!}{<}\; \partial_+ \qquad\qquad \text{(t+r}-\text{6q+2pq>tp)}\end{aligned}$$

The monomial $q^2\lambda$ leads to similar estimates. Using Cauchy's estimate we convert the extra derivative $\frac{\partial}{\partial\lambda}$ into ε^{-1} , which is compensated by $|\lambda|<\varepsilon_+$. We merely have to replace 2pq by 3pq in the inequalities of the exponents. The same estimates also hold true for $\frac{1}{d}\,|\,qp^2\,\frac{\partial^3}{\partial q\partial p^2}(H^\Delta\circ\Psi-H^\Delta)(x,0,0,0,\omega,0)\,|$.

Finally we have to estimate $\frac{1}{d}\|{}^T\!H-{}^T_\theta H\|_{D_+}$.

Lemma 4.9

If $\theta\sigma>2n$ then there is a constant c_3 , only depending on n and τ , such that

$$\|{}^T\!H-{}^T_\theta H\|_{D_+}\;\leq\; c_3\,\|H\|_D\,\theta^n\,\mathrm{e}^{-\frac{\theta\sigma}{2}}\qquad .$$

Proof :

See III of Proposition 5.7 in [Braaksma,Broer;87] ; **q.e.d.(4.9)**

Since $\theta=[\rho^{\frac{-1}{\tau+1}}]=[\sigma^{-2}]$ the condition $\theta\sigma>2n$ resorts to the assumption $\sigma_0<\frac{1}{2n+1}$. We compute

$$\begin{aligned}\frac{1}{d}\|{}^T\!H-{}^T_\theta H\|_{D_+} &\leq\; \frac{c_3}{d}\,\|H\|_D\,\rho^{\frac{-n}{\tau+1}}\,\mathrm{e}^{\frac{\sigma}{2}}\,\mathrm{e}^{-\frac{\sigma}{2}\rho^{\frac{-1}{\tau+1}}}\\ &\leq\; \mathrm{e}\,c_3\,\Delta\,\sigma^{-2n}\,\mathrm{e}^{-\frac{1}{2\sigma}}\qquad .\end{aligned}$$

The definition $\rho=\sigma^{2\tau+2}$ aimed for this estimate. For H^δ we obtain

$$\frac{1}{d}\|{}^T\!H^\delta-{}^T_\theta H^\delta\|_{D_+}\;\leq\;\mathrm{e}\,c_3\,\delta\,\sigma^{-2n}\,\mathrm{e}^{-\frac{1}{2\sigma}}$$

while ${}^{T}\!H^{\partial} - {}^{T}_{\theta}\!H^{\partial} = 0$ by definition. We want to estimate these expressions by, respectively, Δ_+ and δ_+ . It is sufficient to show

$$\mathrm{e}\, c_3\, \sigma^{-2n}\, \mathrm{e}^{-\frac{1}{2\sigma}} \;\overset{!}{<}\; \frac{\delta_+}{\delta} \;=\; \delta^{\mathsf{p}} \qquad .$$

Lemma 4.10

If $s > 0$, $A > 0$, $\eta > 0$ *and* $m \in \mathbb{N}$ *satisfy* $s \;\geq\; \frac{2m}{A}\, \ln(\frac{m}{A}\, \eta^{\frac{-1}{m}})$, *then* $s^m \mathrm{e}^{-As} \;\leq\; \eta$.

Proof :

See Lemma 5.10 of [Braaksma,Broer;87] ; **q.e.d.(4.10)**

We apply this lemma with $s = \frac{1}{\sigma}$, $A = \frac{1}{2}$, $\eta = \frac{\delta^{\mathsf{p}}}{\mathrm{e}c_3}$ and $m = 2n$. The condition reads $\dfrac{1}{\sigma} \geq 8n \ln(4n(\dfrac{\mathrm{e}c_3}{\delta^{\mathsf{p}}})^{\frac{1}{2n}})$ and holds true granted $1 + \mathsf{p} < 4$ and the assumption

$$\sigma_0 \;<\; \frac{1}{4\mathsf{p} \ln(\frac{1}{\delta_0}) \;+\; 4 \ln(c_3) \;+\; 4 \;+\; 8n \ln(4n)} \qquad .$$

(i)	*construction of* Ψ
(ii)	*estimates for* H_+ *(introduction)*
(iii)	*estimates for* Ψ
(iv)	*estimates for* H_+ *(conclusion)*
$\longrightarrow$	*the assumptions on* σ_0 *and* δ_0

During our estimation of H_+ we met a series of necessary inequalities on the exponents $\mathsf{r}, \mathsf{t}, \mathsf{q}$ and p which can all be fulfilled, *e.g.* by $\mathsf{t} = 1.15$, $\mathsf{r} = 0.85$, $\mathsf{q} = 0.3$ and $\mathsf{p} = 0.05$. We also made finitely many assumptions on σ_0 and δ_0 . These were all but one of the form

$$\sigma_0 \;\leq\; \frac{1}{c_4} \quad , \qquad \delta_0^{\mathsf{s}} \;\leq\; \frac{1}{c_5} \quad , \qquad \delta_0^{\mathsf{s}} \;\leq\; \frac{1}{c_6}\, \sigma_0$$

with $c_4, c_5, c_6 > 0$ denoting constants that only depend on n, τ and Γ (possibly through c_1 and c_2) and $\mathsf{s} > 0$ standing for constants defined by $\mathsf{r}, \mathsf{t}, \mathsf{q}, \mathsf{p}$ and τ . We are free to use more assumptions of these types in the remainder of the proof of Proposition 4.1 below. At the end we also made the assumption

$$\sigma_0 \;<\; \frac{1}{4\mathsf{p} \ln(\frac{1}{\delta_0}) \;+\; c_7} \qquad .$$

The first two inequalities can be satisfied choosing σ_0 and δ_0 small, but for the last two this has to be done in an interdependent way. Since $\delta_0^{\mathsf{s}} \ln(\frac{1}{\delta_0}) \overset{\delta_0 \to 0}{\longrightarrow} 0$ for all $\mathsf{s} > 0$ this can be achieved **;** **q.e.d.(4.3)**

To close the induction argument we have to make sure that the coefficient functions of $\mathcal{K}_+ = \mathcal{K}_{i+1}$ remain bounded by 2Γ . At the same time we show that the series $(\mathcal{K}_i)_{i\in\mathbf{N}}$ generated by the induction process converges to a limit $\mathcal{K}_\infty$.

Proposition 4.11

For all $i \in \mathbf{N}$ the coefficient functions of $\mathcal{K}_i$ satisfy

$$a_i(\omega), b_i(\omega) \ \geq \ \frac{d}{2\Gamma} \quad , \qquad a_i(\omega), b_i(\omega), \|c_i(\omega)\|, |g_i(\omega)| \ \leq \ 2\,\Gamma\, d$$

$$\|e_i(\omega)\|, |f_i(\omega)|, |h_i(\omega)| \ \leq \ 2\Gamma \quad , \qquad |\frac{\partial a_i}{\partial \omega}|, \ldots |\frac{\partial h_i}{\partial \omega}| \ \leq \ 2\Gamma$$

for all $\omega \in U_{\rho_i}(\Sigma_1')$. The sequences $(a_i), \ldots, (h_i)$ converge on Σ_1' .

Proof :

From the equations defining $\mathcal{K}_{i+1}$ out of $\mathcal{K}_i$ and H_i we obtain the estimates

$$\begin{aligned}
&\tfrac{1}{d}\|a_{i+1} - a_i\|_{D_{i+1}} \leq\cdot\ \varepsilon_i^{-2}\,\tfrac{1}{d}\|H_i^\delta\|_{D_i} \ \leq\ c_8\,\delta_i^{1-2\mathsf{q}} \\
&\tfrac{1}{d}\|b_{i+1} - b_i\|_{D_{i+1}} \leq\cdot\ \varepsilon_i^{-3}\,\tfrac{1}{d}\|H_i^\delta\|_{D_i} \ \leq\ c_8\,\delta_i^{1-3\mathsf{q}} \\
&\tfrac{1}{d}\|c_{i+1} - c_i\|_{D_{i+1}} \leq\cdot\ \varepsilon_i^{-2}\,\tfrac{1}{d}\|H_i\|_{D_i} \ \leq\ c_8\,\delta_i^{\mathsf{r}-2\mathsf{q}} \\
&\|e_{i+1} - e_i\|_{D_{i+1}} \leq\cdot\ \varepsilon_i^{-2}\,(\|H_i\|_{D_i} + \varepsilon^{-1}\|H_i^\partial\|_{D_i}) \ \leq\ c_8\,\delta_i^{\mathsf{t}-3\mathsf{q}} \\
&\|f_{i+1} - f_i\|_{D_{i+1}} \leq\cdot\ \|\tfrac{\partial f_i}{\partial \omega}\|_{D_i}\,\|\Omega\|_{D_{i+1}} \ +\ \varepsilon_i^{-1}\,(\|H_i^\delta\|_{D_i} + \varepsilon^{-1}\|H_i^\partial\|_{D_i}) \\
&\qquad \leq\cdot\ \varepsilon_i^{-1}\,(\|H_i\|_{D_i} + \varepsilon^{-1}\|H_i^\partial\|_{D_i}) \ \leq\ c_8\,\delta_i^{\mathsf{t}-2\mathsf{q}} \\
&\tfrac{1}{d}\|g_{i+1} - g_i\|_{D_{i+1}} \leq\cdot\ \varepsilon_i^{-2}\,\tfrac{1}{d}(\|H_i^\delta\|_{D_i} + \varepsilon^{-1}\|H_i^\partial\|_{D_i}) \ \leq\ c_8\,\delta_i^{\mathsf{t}-3\mathsf{q}} \\
&\|h_{i+1} - h_i\|_{D_{i+1}} \leq\cdot\ \|\tfrac{\partial h_i}{\partial \omega}\|_{D_i}\,\|\Omega\|_{D_{i+1}} \ +\ \|H_i^\delta\|_{D_i} + \varepsilon^{-1}\|\tfrac{H_i^\delta}{d}\|_{D_i} \\
&\qquad \leq\cdot\ \varepsilon_i^{-1}\,(\tfrac{1}{d}\|H_i\|_{D_i} + \varepsilon^{-1}\|H_i^\partial\|_{D_i}) \ \leq\ c_8\,\delta_i^{\mathsf{t}-2\mathsf{q}}
\end{aligned}$$

We have $1 - 3\mathsf{q} > 0$, so granted the assumption $\delta_0^{1-3\mathsf{q}} \ < \ \min\{\frac{1}{4\Gamma c_8}, (\frac{1}{2})^{\frac{1}{\mathsf{p}}}\}$ the series given by summation of the left hand sides remain bounded by $\frac{1}{2\Gamma}$. In particular the sequences of coefficient functions converge. Furthermore we have for $i \in \mathbf{N}$ and $\omega \in U_{\rho_i}(\Sigma_1')$

$$
\begin{aligned}
a_i(\omega) \;&\geq\; a_0(\omega) \;-\; |a_i(\omega) - a_0(\omega)| \;\geq\; \tfrac{d}{\Gamma} \;-\; \tfrac{d}{2\Gamma} \;=\; \tfrac{d}{2\Gamma} \\
b_i(\omega) \;&\geq\; b_0(\omega) \;-\; |b_i(\omega) - b_0(\omega)| \;\geq\; \tfrac{d}{\Gamma} \;-\; \tfrac{d}{2\Gamma} \;=\; \tfrac{d}{2\Gamma} \\
a_i(\omega) \;&\leq\; a_0(\omega) \;+\; |a_i(\omega) - a_0(\omega)| \;\leq\; \Gamma d \;+\; \tfrac{d}{2\Gamma} \;\leq\; 2\,\Gamma\, d
\end{aligned}
$$

and similarly $b_i(\omega), \|c_i(\omega)\|, |g_i(\omega)| \;\leq\; 2\,\Gamma\, d$ and $\|e_i(\omega)\|, |f_i(\omega)|, |h_i(\omega)| \;\leq\; 2\Gamma$. Using Cauchy's estimate and choosing c_8 appropriately we get the inequalities

$$
\|\frac{\partial}{\partial\omega}(a_{i+1} - a_i)\|_{D_{i+1}} \;, \ldots ,\; \|\frac{\partial}{\partial\omega}(h_{i+1} - h_i)\|_{D_{i+1}} \;\;\leq\;\; c_8\, \sigma^{-2\tau-2}\, \delta^{1-3\mathsf{q}} \qquad .
$$

Therefore we replace the above assumption by

$$
\delta_0^{1-3\mathsf{q}} \;\;<\;\; \min\left\{ \frac{\sigma_0^{2\tau+2}}{4\Gamma c_8} \;,\; (\frac{1}{2}\,(\frac{1}{4})^{2\tau+2}\,)^{\frac{1}{\mathsf{p}}} \right\}
$$

to obtain for $i \in \mathbb{N}$ and $\omega \in U_{\rho_i}(\Sigma_1')$

$$
|\frac{\partial a_i}{\partial\omega}| \;\;\leq\;\; |\frac{\partial a_0}{\partial\omega}| \;+\; |\frac{\partial}{\partial\omega}(a_i - a_0)| \;\;\leq\;\; \Gamma \;+\; \frac{1}{2\Gamma} \;\;\leq\;\; 2\Gamma
$$

and similar estimates for the other coefficient functions **;** **q.e.d.(4.11)**

This completes the ith iteration step. We have seen how to construct a diffeomorphism $\Psi_i : D_{i+1} \longrightarrow \Psi_i(D_{i+1}) \subseteq D_i$ that transforms $\mathcal{K}_i + H_i$ into $\mathcal{K}_{i+1} + H_{i+1}$. Since the perturbation H_{i+1} of $\mathcal{K}_{i+1}$ is smaller than the perturbation H_i of $\mathcal{K}_i$ we can repeat the procedure and inductively get $\mathcal{H}_i = \mathcal{K}_i + H_i$ for all $i \in \mathbb{N}$.

By composition $\Phi_{i+1} := \Psi_0 \circ \Psi_1 \circ \ldots \circ \Psi_i$ we obtain a co-ordinate transformation that turns the given $\mathcal{H}_0 = \mathcal{K}_0 + H_0$ into $\mathcal{K}_{i+1} + H_{i+1}$. Our aim is to find a "limit" Φ_∞ with

$$
\mathcal{H}_0 \circ \Phi_\infty \;\;=\;\; \mathcal{K}_\infty \;+\; H_\infty \qquad .
$$

The occurrence of H_∞ reflects that $\lim\limits_{i\to\infty} \Phi_i$ is only defined on $\bigcap D_i \;=\; U_{\frac{1}{2}\kappa}(\mathsf{T}) \times \{0\} \times \{(0,0)\} \times \Sigma_1' \times \{0\}$. We will define Φ_∞ as the limit of polynomial truncations of $\Phi_i = (\varphi_i, \mathrm{id} + \widetilde{\Omega}_i, \mathrm{id} + \widetilde{\Lambda}_i)$ in y, q, p and λ . Here we have to be careful not to destroy the symplectic structure. Instead of simply truncating φ_i , we truncate a generating function to define ϕ_i . This procedure leads to $\Phi_\infty \;=\; \lim\limits_{i\to\infty} (\phi_i, \mathrm{id} + \widetilde{\Omega}_i, \mathrm{id} + \widetilde{\Lambda}_i)$.

First we show that $(\widetilde{\Omega}_i)_i$ and $(\widetilde{\Lambda}_i)_i$ converge. We write $\Phi_i \;=\; \mathrm{id} + (W_i, \widetilde{\Omega}_i, \widetilde{\Lambda}_i)$ and use the C^2-norm $\|\Phi_i\|_{C^2(D_i)} \;=\; \max\{\|\Phi_i\|_{D_i}, \|D\Phi_i\|_{D_i}, \|D^2\Phi_i\|_{D_i}\}$.

Proposition 4.12

There is a constant c_9 , depending only on n, τ and Γ , such that

$\|\Phi_i\|_{C^2(D_i)} \le c_9$ *for every* $i \in \mathbf{N}$.

Proof :

From $\Phi_{i+1} = \Phi_i \circ \Psi_i$ follows $D\Phi_{i+1} = (D\Phi_i \circ \Psi_i) \bullet D\Psi_i$ and

$D^2\Phi_{i+1} = (D^2\Phi_i \circ \Psi_i) \bullet (D\Psi_i, D\Psi_i) + (D\Phi_i \circ \Psi_i) \bullet D^2\Psi_i$.

Since $\|(DV_i, D\Omega_i, D\Lambda_i)\|_{D_{i+1}} \le\cdot\ \sigma_i^{-14\tau-3}\,\varepsilon_i^{-2}\,(\|\frac{H_i}{d}\|_{D_i} + \varepsilon_i^{-1}\|\frac{H_i^\partial}{d}\|_{D_i})$

and $\|(D^2V_i, D^2\Omega_i, D^2\Lambda_i)\|_{D_{i+1}} \le \|D^2(V_i - U_i)\|_{D_{i+1}} + \|(D^2U_i, D^2\Omega_i, D^2\Lambda_i)\|_{D_{i+1}}$
$\le\cdot\ \sigma_i^{-28\tau-7}\,\varepsilon_i^{-5}\,(\|\frac{H_i}{d}\|_{D_i} + \varepsilon_i^{-1}\|\frac{H_i^\partial}{d}\|_{D_i})^2 + \sigma_i^{-16\tau-5}\,\varepsilon_i^{-2}\,(\|\frac{H_i}{d}\|_{D_i} + \varepsilon_i^{-1}\|\frac{H_i^\delta}{d}\|_{D_i})$

we have

$$\|\Phi_{i+1}\|_{C^2(D_{i+1})} \le \|\Phi_i\|_{C^2(D_i)} \cdot (1 + \mathit{const} \cdot \sigma_i^{-28\tau-7}\,\varepsilon_i^{-3}\,\delta_i) \quad .$$

Hence, it is sufficient that $\sum_{i=0}^{\infty} \sigma_i^{-28\tau-7}\,\delta^{1-3\mathsf{q}}$ has a universal bound. The assumption $\delta_0^{1-3\mathsf{q}} < \min\{\frac{1}{2}\sigma_0^{28\tau+7}, (\frac{1}{2}(\frac{1}{4})^{28\tau+7})^{\frac{1}{\mathsf{p}}}\}$ ensures this ; **q.e.d.(4.12)**

The inequalities

$$\begin{aligned}
\|\widetilde\Omega_{i+1} - \widetilde\Omega_i\|_{D_{i+1}} &\le \|\mathrm{id} + \widetilde\Omega_i\|_{C^2(D_i)}\,\|\Omega_i\|_{D_i} \le c_2\,c_9\,\varepsilon_i^{-2}\,\partial_i \\
\|\widetilde\Lambda_{i+1} - \widetilde\Lambda_i\|_{D_{i+1}} &\le \|\Lambda_i\|_{D_{i+1}} + \|D\widetilde\Lambda_i\|_{D_i}\,\|\Omega_i\|_{D_i} \\
&\le c_1\,\varepsilon_i^{-1}\,\delta_i + \|\Phi_i\|_{C^2(D_i)}\,c_2\,\varepsilon_i^{-2}\,\partial_i \le c_2\,c_9\,\varepsilon_i^{-2}\,\partial_i
\end{aligned}$$

imply the convergence of $(\widetilde\Omega_i)_i$ and $(\widetilde\Lambda_i)_i$.

To get a bound on DW_i we write $\Phi_i = \Psi_0 \circ \Upsilon_i$. Like in the proof of Proposition 4.12 we obtain $\|\Upsilon_i\|_{C^2(D_i)} \le c_9$ for every $i \in \mathbf{N}$. We add the assumption $\delta_0 < \left(\frac{\sigma_0^{12\tau+1}}{2c_2c_9}\right)^{\frac{1}{\mathsf{t}-2\mathsf{q}}}$ and conclude $\|DW_i\|_{D_i} \le \|DV_0\|_{\Upsilon_i(D_i)} \cdot \|D\Upsilon_i\|_{D_i} \le \frac{1}{2}$.

Since $\varphi_i : (x, y, q, p, \omega, \lambda) \mapsto (X, Y, Q, P)$ is for fixed (ω, λ) a symplectomorphism, the 1-form

$$\sum_{\nu=1}^{n} (y_\nu - Y_\nu)\,\mathrm{d}x_\nu + (X_\nu - x_\nu)\,\mathrm{d}Y_\nu + (p - P)\,\mathrm{d}q + (Q - q)\,\mathrm{d}P$$

is closed and can therefore be written as $\mathrm{d}S_i$. Note that the coefficient functions are 2π-periodic in the toral co-ordinates $x_1, \ldots, x_n$. The function $S_i = S_i(x, Y, q, P)$ is a generating function for φ_i . We use a third order truncation $\widetilde S_i$ of S_i to define the replacement ϕ_i of φ_i : the truncation $\widetilde{S}_i$ of S_i includes all terms up to order three in (Y,q,P,λ) , with the exception of $S_i^{0003}(x,\omega)\lambda^3$. Note that this truncation differs from the truncation ${}^T\!H$ of H we used in Proposition 4.3 in that we here also include third order terms in Y (but drop λ^3).

Proposition 4.13

The sequence $(\widetilde{S}_i)_{i\in\mathbb{N}}$ of truncations converges.

Proof :

Let $F_i : D_i \longrightarrow D_0$ denote the transformation of (x,y,q,p,ω,λ) into

$$(x \, , \, y + W_i^2(x,y,q,p,\omega,\lambda) \, , \, q \, , \, p + W_i^4(x,y,q,p,\omega,\lambda) \, , \, \omega \, , \, \lambda) \;\overset{!}{=}\; (x \, , \, Y \, , \, q \, , \, P \, , \, \omega \, , \, \lambda)$$

and $G_i := F_i^{-1}$. The truncations $\widetilde{S}_i$ are polynomials in Y,q,P and λ , the coefficients of which are holomorphic functions in x and ω . To truncate we write S_i as a Taylor series at $F_i(x,0,0,0,\omega,0) =: (x,Y_i,0,P_i,\omega,0)$. Therefore,

$$S_i^{jklm}(x,\omega) \;=\; \frac{\partial^{|j|+k+l+m} S_i}{\partial Y^j \partial q^k \partial P^l \partial \lambda^m}(x,Y_i,0,P_i,\omega,0) \qquad ,$$

and we have to show convergence as $i \to \infty$ for $|j|+k+l+m \le 3$, $m \neq 3$.

For $|j| \ge 1$ we can use $\dfrac{\partial S_i}{\partial Y} \;=\; W_i^1 \circ G_i$. We get immediately

$$\begin{aligned}
| \, S_{i+1}^{1000}(x,\omega) - S_i^{1000}(x,\omega) \, | \;&=\; | \, W_{i+1}^1(x,0,0,0,\omega,0) - W_i^1(x,0,0,0,\omega,0) \, | \\
&\le\; \|\Phi_i\|_{C^2(D_i)} \, \|V_i(x,0,0,0,\omega,0)\| \\
&\le\; c_2\, c_9\, \sigma_i^{-12\tau-1}\, \varepsilon_i^{-1} \, (\|\tfrac{H_i}{d}\|_{D_i} + \varepsilon_i^{-1} \|\tfrac{H_i^\partial}{d}\|_{D_i}) \;\le\; c_2\, c_9\, \sigma_i^{-12\tau-1}\, \delta^{\mathsf{t}-2\mathsf{q}} \qquad .
\end{aligned}$$

From the chain rule we have $\dfrac{\partial^2 S_i}{\partial Y^2} \;=\; (\dfrac{\partial W_i^1}{\partial y} \circ G_i)\dfrac{\partial G_i^2}{\partial Y} \;+\; (\dfrac{\partial W_i^1}{\partial p} \circ G_i)\dfrac{\partial G_i^4}{\partial Y}$ and hence

$$\begin{aligned}
| \, S_{i+1}^{2000}(x,\omega) - S_i^{2000}(x,\omega) \, | \;\le\; & | \, \tfrac{\partial W_{i+1}^1}{\partial y}(x,0,0,0,\omega,0) \tfrac{\partial G_{i+1}^2}{\partial Y}(x,Y_{i+1},0,P_{i+1},\omega,0) \\
& \qquad - \tfrac{\partial W_i^1}{\partial y}(x,0,0,0,\omega,0) \tfrac{\partial G_i^2}{\partial Y}(x,Y_i,0,P_i,\omega,0) \, | \\
+ \; & | \tfrac{\partial W_{i+1}^1}{\partial p}(x,0,0,0,\omega,0) | \; | \, \tfrac{\partial G_{i+1}^4}{\partial Y}(x,Y_{i+1},0,P_{i+1},\omega,0) \;-\; \tfrac{\partial G_i^4}{\partial Y}(x,Y_{i+1},0,P_{i+1},\omega,0) \, | \\
+ \; & | \tfrac{\partial W_{i+1}^1}{\partial p}(x,0,0,0,\omega,0) | \; \|DG_i\|_{D_0} \; | \, F_{i+1}(x,0,0,0,\omega,0) \;-\; F_i(x,0,0,0,\omega,0) \, | \\
+ \; & \|\Phi_i\|_{C^2(D_i)} \, \|V_i(x,0,0,0,\omega,0)\| \; \|DG_i\|_{D_0} \\
\lessdot \; & \sigma_i^{-12\tau-1}\, \varepsilon_i^{-1} \, (\|\tfrac{H_i}{d}\|_{D_i} + \varepsilon_i^{-1} \|\tfrac{H_i^\partial}{d}\|_{D_i}) \;\lessdot\; \sigma_i^{-12\tau-1}\, \delta_i^{\mathsf{t}-2\mathsf{q}}
\end{aligned}$$

since $DG_{i+1} - DG_i = DG_i(DF_i - DF_{i+1})DG_{i+1}$ and $\|DG\| \leq 2$. The same estimate holds true for the other S^{jklm} with $|j| \geq 1$ and $|j| + k + l + m = 2$.

We have no bound on the C^3-norm of the Φ_i , so we argue differently for $|\, S^{3000}_{i+1}(x,\omega) - S^{3000}_i(x,\omega)\,|$. We have

$$\frac{\partial^3 S_i}{\partial Y^3} \;=\; (\frac{\partial^2 W_i^1}{\partial y^2} \circ G_i)\,\frac{\partial G_i^2}{\partial Y}\,\frac{\partial G_i^2}{\partial Y} \;+\; 2(\frac{\partial^2 W_i^1}{\partial y \partial p} \circ G_i)\,\frac{\partial G_i^2}{\partial Y}\,\frac{\partial G_i^4}{\partial Y}$$

$$+\; (\frac{\partial^2 W_i^1}{\partial p^2} \circ G_i)\,\frac{\partial G_i^4}{\partial Y}\,\frac{\partial G_i^4}{\partial Y} \;+\; (\frac{\partial W_i^1}{\partial y} \circ G_i)\,\frac{\partial^2 G_i^2}{\partial Y^2} \;+\; (\frac{\partial W_i^1}{\partial p} \circ G_i)\,\frac{\partial^2 G_i^4}{\partial Y^2} \qquad .$$

Since $D^2G_{i+1} - D^2G_i = DG_i \bullet (D^2F_i \circ G_i) \bullet (DG_i, DG_i) - DG_{i+1} \bullet (D^2F_{i+1} \circ G_{i+1}) \bullet (DG_{i+1}, DG_{i+1})$ the only difficulty are the second derivatives of W_i . To fix thoughts we consider $\frac{\partial^2}{\partial y^2}(W_{i+1} - W_i)(x,0,0,0,\omega,0)$. From

$$\frac{\partial}{\partial y} W_{i+1} \;=\; (\frac{\partial W_i}{\partial x} \circ \Psi_i)\frac{\partial V_i^1}{\partial y} \;+\; (\frac{\partial W_i}{\partial y} \circ \Psi_i)(1 + \frac{\partial V_i^2}{\partial y})$$
$$+\; (\frac{\partial W_i}{\partial q} \circ \Psi_i)\frac{\partial V_i^3}{\partial y} \;+\; (\frac{\partial W_i}{\partial p} \circ \Psi_i)\frac{\partial V_i^4}{\partial y}$$

we conclude

$$\frac{\partial^2}{\partial y^2} W_{i+1} \;=\; \frac{\partial^2 W_i}{\partial y^2} \circ \Psi_i \;+\; 2(\frac{\partial^2 W_i}{\partial y^2} \circ \Psi_i)\frac{\partial V_i^2}{\partial y} \;+\; A \;+\; B$$

where A consists of the $(\frac{\partial W_i}{\partial z} \circ \Psi_i)\frac{\partial^2 V_i^\mu}{\partial y^2}$, $(z,\mu) = (x,1), \ldots, (p,4)$ and B is the sum of terms $(\frac{\partial^2 W_i}{\partial z \partial \zeta} \circ \Psi_i)\frac{\partial V_i^\mu}{\partial y}\frac{\partial V_i^\nu}{\partial y}$, $(z,\mu), (\zeta,\nu) = (x,1), \ldots, (p,4)$.

Let us proceed term by term. Recalling the intermediate domains $D_{i+1} \subseteq D_{i*} \subseteq D_{i\otimes} \subseteq D_i$ of part (iii) of the proof of Proposition 4.3 we first have

$$\begin{aligned}
\|\tfrac{\partial^2 W_i}{\partial y^2} \circ \Psi - \tfrac{\partial^2 W_i}{\partial y^2}\|_{D_{i+1}} \;\leq\; \max\Big\{ &\|\tfrac{\partial^3 W_i}{\partial x \partial y^2}\|_{D_{i\otimes}} \cdot \|V^1\|_{D_{i+1}} ,\ \|\tfrac{\partial^3 W_i}{\partial y^3}\|_{D_{i\otimes}} \cdot \|V^2\|_{D_{i+1}} , \\
&\|\tfrac{\partial^3 W_i}{\partial y^2 \partial q}\|_{D_{i\otimes}} \cdot \|V^3\|_{D_{i+1}} ,\ \|\tfrac{\partial^3 W_i}{\partial y^2 \partial p}\|_{D_{i\otimes}} \cdot \|V^4\|_{D_{i+1}} ,\ \|\tfrac{\partial^3 W_i}{\partial y^2 \partial \omega}\|_{D_{i\otimes}} \cdot \|\Omega\|_{D_{i+1}} , \\
&\|\tfrac{\partial^3 W_i}{\partial y^2 \partial \lambda}\|_{D_{i\otimes}} \cdot \|\Lambda\|_{D_{i+1}} \Big\} \;\leq\!\cdot\; \sigma_i^{-2\tau-2}\,\varepsilon_i^{-1}\,\|D^2 W_i\|_{D_i}\|V_i\|_{D_{i+1}} \\
&\leq\!\cdot\; \sigma_i^{-14\tau-3}\,\varepsilon_i^{-2}\,(\|\tfrac{H_i}{d}\|_{D_i} + \varepsilon_i^{-1}\|\tfrac{H_i^\partial}{d}\|_{D_i}) \;\leq\!\cdot\; \sigma_i^{-14\tau-3}\,\delta_i^{\mathsf{t}-3\mathsf{q}} \ .
\end{aligned}$$

For the other terms the W-factor is bounded by $\|\Phi_i\|_{C^2(D_i)}$ (or by $\frac{1}{2}$ for A). First derivatives of V_i can be estimated by

$$const \cdot \sigma_i^{-12\tau-1}\,\varepsilon_i^{-2}\,(\|\frac{H_i}{d}\|_{D_i} + \varepsilon_i^{-1}\|\frac{H_i^\partial}{d}\|_{D_i}) \;\leq\!\cdot\; \sigma_i^{-12\tau-1}\,\delta_i^{\mathsf{t}-3\mathsf{q}} \qquad ,$$

so we are left with $\frac{\partial^2 V_i^\mu}{\partial y^2} \;=\; \frac{\partial^2}{\partial y^2}(V_i^\mu - U_i^\mu) \;+\; \frac{\partial^2 U_i^\mu}{\partial y^2}$, evaluated at the point $(x,0,0,0,\omega,0)$. While

$$|\,\tfrac{\partial^2}{\partial y^2}(V_i^\mu - U_i^\mu)\,| \;\leq\!\cdot\; \sigma_i^{-23\tau-2}\,\varepsilon_i^{-5}\,(\|\tfrac{H_i}{d}\|_{D_i} + \varepsilon_i^{-1}\|\tfrac{H_i^\partial}{d}\|_{D_i})^2 \;\leq\!\cdot\; \sigma_i^{-23\tau-2}\,\delta_i^{2\mathsf{t}-7\mathsf{q}}$$

$$|\,\tfrac{\partial^2 U_i^2}{\partial y^2}\,| \;\leq\!\cdot\; \varepsilon_i^{-2}\,\|\tfrac{\partial K_i}{\partial x}\|_{D_{i*}} \;\leq\!\cdot\; \sigma_i^{-12\tau-1}\,\varepsilon_i^{-2}\,\|H_i\|_{D_i} \;\leq\!\cdot\; \sigma_i^{-12\tau-1}\,\delta_i^{\mathsf{r}-2\mathsf{q}} \qquad ,$$

we have $\frac{\partial^2 U_i^1}{\partial y^2}(x,0,0,0,\omega,0) = K_{i,3000}(x,\omega) \equiv 0$, $\frac{\partial^2 U_i^3}{\partial y^2}(x,0,0,0,\omega,0) = K_{i,2100}(x,\omega) \equiv 0$ and $\frac{\partial^2 U_i^4}{\partial y^2}(x,0,0,0,\omega,0) = -K_{i,2010}(x,\omega) \equiv 0$.

For second derivatives of $W_{i+1} - W_i$ in $(x,0,0,0,\omega,0)$ with respect to q and p we are led to the non-vanishing $K_{i,0kl0}(x,\omega)$ with $k+l=3$. However, these only involve terms $H_{i,0klm}$ (compare the definition of K_i in part (i) of the proof of Proposition 4.3). Hence, we get for them the estimate

$$const\cdot\sigma_i^{-11\tau}\,\varepsilon_i^{-3}\,\|H_i^\delta\|_{D_i} \;\leq\!\cdot\; \sigma_i^{-11\tau}\,\delta_i^{1-3\mathsf{q}} \qquad .$$

To sum up, we have shown that $|\,S_{i+1}^{3000}(x,\omega) - S_i^{3000}(x,\omega)\,|$ decays exponentially (granted another assumption on σ_0 and δ_0). The same estimates hold true for the other S^{jklm} with $|j| \geq 1$ and $|j|+k+l+m=3$.

Using also $\dfrac{\partial S_i}{\partial P} = W_i^3 \circ G_i$ and $\dfrac{\partial S_i}{\partial q} = -W_i^4 \circ G_i$ we similarly obtain an exponential decay of all differences $|\,S_{i+1}^{jklm}(x,\omega) - S_i^{jklm}(x,\omega)\,|$ with $m < |j|+k+l+m \leq 3$.

We are left with the terms S^{000m} , $m = 0,1,2$. Their average vanishes and they are determined by $\dfrac{\partial S_i}{\partial x} = -W_i^2 \circ G_i$. We conclude

$$\begin{aligned} |\,S_{i+1}^{000m}(x,\omega) - S_i^{000m}(x,\omega)\,| \;&\leq\; \max_{\nu=1,\ldots,n}\,|\int_0^{x_\nu} \frac{\partial^m}{\partial\lambda^m}(W_i^2 - W_i^2\circ G_i)\,\mathrm{d}x_\nu\,| \\ &\leq\; \|\tfrac{\partial^m}{\partial\lambda^m}(W_i^2 - W_i^2\circ G_i)\|_{D_i^0} \end{aligned}$$

with $D_i^0 := D_i \cap \{y = q = p = \lambda = 0\}$. Like above we get an exponential decay **;**

q.e.d.(4.13)

From our construction we know that the sequence of polynomial truncations $({}^T\!H_i)_{i\in\mathbb{N}}$ of the perturbative terms converges exponentially to zero. This does not remain true when we replace the $\varphi_i = \mathrm{id} + W_i$ by $\phi_i = \mathrm{id} + \widetilde{W}_i$, but the more restrictive 2-jet truncation of the difference

$$\mathcal{H}_0 \circ (\mathrm{id} + (\widetilde{W}_i, \tilde{\Omega}_i, \tilde{\Lambda}_i)) \quad - \quad \mathcal{H}_0 \circ (\mathrm{id} + (W_i, \tilde{\Omega}_i, \tilde{\Lambda}_i))$$

vanishes. This is the reason why we restrict in Proposition 4.1 to an x-independent 2-jet.

We finally have computed all the necessary estimates. We conclude the proof of Proposition 4.1 using the Inverse Approximation Theorem and Whitney's Extension Lemma. For the convenience of the reader we include explicit formulations of these classical results.

Inverse Approximation Lemma

Take a geometric sequence $r_i := \varrho^i r_0$. *Let* $A \subseteq \mathbb{C}^m$ *be an open or closed subset and consider the sequence of* r_i*-neighbourhoods* $U_i := U_{r_i}(A)$. *For each* $i \in \mathbb{N}$ *let* F_i *be a real analytic function on* U_i *and put* $F_0 = 0$. *Assume there are a constant* $M > 0$ *and a positive real number* $\beta \notin \mathbb{N}$ *such that*

$$\bigwedge_{i\in\mathbb{N}} \quad \|F_i - F_{i-1}\|_{U_i} \ \leq \ M\, r_i^\beta \qquad .$$

Then there exists a unique function $F \in C^\beta(A)$ *and a constant* C *(that only depends on* β , ϱ *and the dimension* m *) such that*

$$\|F\|_{C^\beta(A)} \ \leq \ C\,M \qquad \text{and} \qquad \|F - F_i\|_{C^\beta(A)} \stackrel{i\to\infty}{\longrightarrow} 0$$

For a proof see [Zehnder;75]. Let us explain "$C^\beta(A)$". If A is open, "β times differentiable" just means $\ell := [\beta]$ times differentiable in the usual sense and a Hölder condition with exponent $\beta - \ell$ on the ℓth derivative. If A is closed $C^\beta(A)$ consists of tuples $F := (F^0, F^1, \ldots, F^\ell)$. One should think of F^k as the kth derivative, so

$$P_k(z,\zeta) \ := \ \sum_{|j|=0}^{\ell-k} \frac{1}{j!}\, F^{k+|j|}(\zeta)\,(z-\zeta)^j$$

is the analogue of the Taylor polynomial of F^k of order $\ell - k$. For F to be in $C^\beta(A)$ there has to be a constant γ satisfying

$$\bigwedge_{k=0,\ldots,\ell} \ \bigwedge_{z,\zeta\in A} \quad |F^k(z)| \ < \ \gamma \quad \wedge \quad |\, F^k(z) - P_k(z,\zeta)\,| \ < \ \gamma\, \|z-\zeta\|^{\beta-k} \qquad .$$

The minimum of these γ is denoted by $\|F\|_{C^\beta(A)}$. A C^∞-(Whitney)-differentiable function on A is a sequence $F = (F^k)_{k\in\mathbb{N}}$ with $(F^k)_{k<\beta} \in C^\beta(A)$ for all $\beta \notin \mathbb{N}$.

Extension Theorem (Whitney)

For $A \subseteq \mathbb{C}^m$ closed there is a linear extension operator

$$\mathcal{E} \; : \; C^\infty(A) \longrightarrow C^\infty(\mathbb{C}^m)$$

such that $D^k(\mathcal{E}F)_{|A} = F^k \quad \forall_{k=0,\ldots,[\beta]}$.

A proof can be found in [Whitney;34]. For a C^β version see [Stein;70] or [Hörmander;58]. Thus, the elements of $C^\infty(A)$ can be characterised as restrictions of functions in $C^\infty(\mathbb{C}^m)$. In particular the F^k play indeed the rôle of derivatives.

We now can show differentiability of the limit functions $\mathcal{K}_\infty$ and Φ_∞ . We first apply the Inverse Approximation Lemma to $U_{\rho_i}(\Sigma_1') \subseteq \mathbb{C}$. From the proof of Proposition 4.11 we know that the coefficient functions of the $\mathcal{K}_i$ satisfy the assumptions of the Inverse Approximation Lemma for all $\beta \notin \mathbb{N}$. Therefore, the sequences of coefficient functions have C^∞-(Whitney)-differentiable limits $a_\infty, \ldots, h_\infty$ and define a C^∞-(Whitney)-differentiable Hamilton function $\mathcal{K}_\infty$ which is analytic (since polynomial) in y, q, p and λ .

The two inequalities after the proof of Proposition 4.12 show that also $(\widetilde{\Omega}_i)_{i\in\mathbb{N}}$ and $(\widetilde{\Lambda}_i)_{i\in\mathbb{N}}$ have C^∞-(Whitney)-differentiable limits $\widetilde{\Omega}_\infty$ and $\widetilde{\Lambda}_\infty$. Instead of investigating $\phi_\infty = \lim_{i\to\infty} \phi_i$ directly, we use the (truncated) generating functions $\widetilde{S}_i$. Their coefficients S_i^{jklm} depend also on x , so we now work on $U_{\rho_i}(\overline{U_{\frac{1}{2}\kappa}(\mathsf{T})} \times \Sigma_1')$. As the proof of Proposition 4.13 shows we get C^∞-(Whitney)-smooth limits S_∞^{jklm} . They constitute the coefficients of a generating function

$$\widetilde{S}_\infty \; : \; U_{\frac{1}{2}\kappa}(\mathsf{T}) \times \mathbb{C}^n \times \mathbb{C}^2 \times \Sigma_\gamma' \times \mathbb{C} \longrightarrow \mathbb{C}$$

which is analytic in x, Y, q, P and λ . With Whitney's Extension Theorem we get $\widetilde{S}_\infty(x, Y, q, P, \omega, \lambda)$ for all $\omega \in \mathbb{R}^n$. This defines for every (ω, λ) a symplectomorphism ϕ_∞ on $\mathsf{T} \times \mathbb{R}^n \times \mathbb{R}^2$. For $\omega \in \Sigma_1'$ we have $\phi_\infty = \lim_{i\to\infty} \phi_i$.

$$(x, y, q, p, \omega, \lambda) \mapsto (\, \phi_\infty(x, y, q, p, \omega, \lambda) \, , \, \omega + \widetilde{\Omega}(\omega) \, , \, \lambda + \widetilde{\Lambda}(\omega) \,)$$

is the diffeomorphism Φ_∞ sought for in Proposition 4.1. Letting $H_\infty := \mathcal{H}_0 \circ \Phi_\infty - \mathcal{K}_\infty$ we have $H_{\infty,\, jklm} = \lim_{i\to\infty} H_{i,\, jklm}$ for all coefficients of the Taylor series in y, q, p and λ . Hence, we can conclude from Lemma 4.4 that they vanish for $|j| + k + l + m \leq 2$ **;**

q.e.d.(4.1)

4.3 Proof of Proposition 4.2

We are given a perturbation of the integrable or T-symmetric 2-jet (13) by x-dependent higher order terms, and our aim is to simplify the perturbing Hamilton function by means of a series of co-ordinate transformations. We cannot expect to gain the full T-symmetry from the approximating 2-jet, but we can push the T-symmetry through the Taylor series. In each step the effect of the co-ordinate transformation amounts to averaging along the n-torus T .

To this end it is necessary that the dynamics on T is not resonant, in fact we need the frequency vector ω to be diophantine. Note that the transformation sought for in Proposition 4.2 leaves ω (and λ) invariant. Our proof follows §2 of [Braaksma,Broer,Huitema;90] and proceeds in three steps. First we fix $\omega \in \Sigma_\gamma$ and let only x, y, q, p and λ vary. Then we let also ω vary, but only in an analytic way. The third and final step consists in generalizing this result to the C^∞-category.

$\longrightarrow$ *A single diophantine* ω
(*ii*) $\mathcal{H}$ *analytic also in* ω
(*iii*) *The general case*

We construct our symplectic co-ordinate transformation inductively, and in each step as the time one flow ψ_1 of a Hamiltonian vector field X_K . The case $N = 2$ holds true by the assumptions of Proposition 4.2, so we may assume the conclusion being true for some $N - 1 \geq 2$. We write

$$\mathcal{H}^{N\text{-jet}}(x, y, q, p, \lambda) \;=\; (\omega \,|\, y) \,+\, a\,p^2 \,+\, (c \,|\, y)\,q \,-\, d\,\lambda\,q \,+\, (e\,y \,|\, y)$$

$$+ \sum_{|j|+k+l+m=3}^{N} H^0_{jklm}\, y^j q^k p^l \lambda^m \;+ \sum_{|j|+k+l+m=N} \;\sum_{\mathsf{h}\in\mathbb{Z}^n\setminus\{0\}} H^{\mathsf{h}}_{jklm}\, \mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)}\, y^j q^k p^l \lambda^m$$

and search for

$$K \;=\; \sum_{|j|+k+l+m=N} \;\sum_{\mathsf{h}\in\mathbb{Z}^n\setminus\{0\}} K^{\mathsf{h}}_{jklm}\, \mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)}\, y^j q^k p^l \lambda^m$$

to solve the equation

$$(\omega \,|\, \frac{\partial K}{\partial x}) \;=\; \sum_{|j|+k+l+m=N} \;\sum_{\mathsf{h}\in\mathbb{Z}^n\setminus\{0\}} H^{\mathsf{h}}_{jklm}\, \mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)}\, y^j q^k p^l \lambda^m \;-\; 2a\,p\,\frac{\partial K}{\partial q} \;-\; d\,\lambda\,\frac{\partial K}{\partial p} \tag{16}$$

which splits into a system of equations on the K^{h}_{jklm} . This type of linear equations, called 1-bite problem in [Boer,Huitema,Sevryuk;96], also defines the co-ordinate transformation needed for the KAM iteration procedure in the proof of Proposition 4.3. Since the unknowns K^{h}_{jklm} appear on the right hand side, it is important to solve (16) in the right order.

Note that blocks with different order j of y never interact. A (partial) lexicographical order on (k,l) puts the system into triangular form. Hence, we start with

$$K^{\mathsf{h}}_{0N00} \;=\; \frac{H^{\mathsf{h}}_{0N00}}{\mathrm{i}(\mathsf{h}\,|\,\omega)} \qquad , \qquad \mathsf{h} \in \mathbb{Z}^n\backslash\{0\}$$

and proceed to lower order exponents k of q . Within a block of constant k we first solve

$$\mathrm{i}(\mathsf{h}\,|\,\omega)\, K^{\mathsf{h}}_{0\,k\,N-k\,0}(\omega) \;=\; H^{\mathsf{h}}_{0\,k\,N-k\,0} \;-\; 2a\, K^{\mathsf{h}}_{0\,k+1\,N-k-1\,0}$$

and likewise proceed to lower orders l of p . The convergence of the Fourier series defining K is guaranteed by the diophanticity of ω , cf. [Zehnder;75] or [Benettin,Galgani,Giorgilli, Strelcyn;84]. We conclude that

$$\left(\mathcal{H}\circ\psi_1\right)^{N\text{-jet}} \;=\; \int_{\mathsf{T}} \mathcal{H}^{N\text{-jet}}\,\mathrm{d}x \qquad .$$

(i) *A single diophantine* ω
$\longrightarrow$ $\mathcal{H}$ *analytic also in* ω
(iii) *The general case*

Let now $\mathcal{H}$ be holomorphic on some complex neighbourhood

$$U_\kappa(\mathsf{T}) \times U_\epsilon(0) \times U_\epsilon(0,0) \times U_\kappa(\Sigma) \times U_\epsilon(0)$$

of $\mathsf{T}\times\mathsf{Y}\times\mathsf{S}\times\Sigma\times\,]-\epsilon,\epsilon[$ and consider the geometric sequence $\rho_i := \frac{1}{2}\gamma\,2^{-(\tau+1)i}$, starting at sufficiently high i to ensure $\rho_i \leq \frac{1}{3}\kappa$. We want to show that the co-ordinate transformation(s) ψ_1 , defined above for every $\omega\in\Sigma_\gamma$, depend on ω in a C^∞-(Whitney)-smooth way. The Fourier truncation

$$K_i \;=\; \sum_{|j|+k+l+m=N} \underbrace{\sum_{0<|\mathsf{h}|\leq 2^i} K^{\mathsf{h}}_{jklm}(\omega)\,\mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)}}_{=:\,K_{i,jklm}(x,\omega)}\; y^j q^k p^l \lambda^m$$

defines a holomorphic function. Indeed, the $K_{i,jklm}$ are holomorphic on $U_\kappa(\mathsf{T})\times U_{\rho_i}(\Sigma_\gamma)$ as for $\omega\in U_{\rho_i}(\Sigma_\gamma)$ and $\mathsf{h}\in\mathbb{Z}^n$ with $0<|\mathsf{h}|\leq 2^i$ the inequality $|(\mathsf{h}\,|\,\omega)| \geq \frac{\gamma}{2|\mathsf{h}|^\tau}$ holds true, cf. the proof of Lemma 4.5. Like at the end of the previous subsection we now use the Inverse Approximation Lemma to show that K is C^∞-(Whitney)-differentiable in ω . On $U_{\rho_i}(\overline{U_{\frac{1}{3}\kappa}(\mathsf{T})}\times\Sigma_\gamma)$ the differences $K_{i,jklm}-K_{i-1,jklm}$ can be estimated as in the proof of Lemma 4.5 by

$$\Big|\sum_{2^{i-1}<|\mathsf{h}|\leq 2^i} K^{\mathsf{h}}_{jklm}(\omega)\,\mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)}\Big| \;\leq\; \sum_{2^{i-1}<|\mathsf{h}|\leq 2^i} |K^{\mathsf{h}}_{jklm}(\omega)|\,\mathrm{e}^{|\mathrm{Im}\,x||\mathsf{h}|}$$

$$
\begin{aligned}
&\le \sum_{2^{i-1}<|\mathsf{h}|\le 2^i} \sum_{\mu=1}^{2m+l+1} |(\mathsf{h}\,|\,\omega)|^{-\mu}\, |H^{\mathsf{h}}_{jklm}(\omega)|\, \mathrm{e}^{(\frac{1}{3}\kappa+\rho_i)|\mathsf{h}|} \\
&\le \|H_{jklm}\| \sum_{2^{i-1}<|\mathsf{h}|\le 2^i} \sum_{\mu=1}^{2N+1} |(\mathsf{h}\,|\,\omega)|^{-\mu}\, \mathrm{e}^{(\rho_i-\frac{2}{3}\kappa)|\mathsf{h}|} \\
&\le \|H_{jklm}\| \sum_{2^{i-1}<|\mathsf{h}|\le 2^i} \tfrac{2}{\gamma}\, |\mathsf{h}|^{2N+1}\, \mathrm{e}^{-\frac{1}{3}\kappa|\mathsf{h}|} \qquad (17)\\
&\le \|H_{jklm}\|\, \tfrac{2}{\gamma}\, \mathrm{e}^{-\frac{\kappa}{6}2^{i-1}} \sum_{\mathsf{h}\in\mathbb{Z}^n} |\mathsf{h}|^{2N+1}\, \mathrm{e}^{-\frac{\kappa}{6}|\mathsf{h}|} \\
&\le \mathit{const}\cdot \exp(-\tfrac{\kappa}{12}2^i) \qquad \text{(by Lemma 4.6)}
\end{aligned}
$$

and thus decrease faster than any ρ_i^β , $\beta > 0$, as desired. This also proves the K_{jklm} to be analytic in x . With Whitney's Extension Lemma we obtain a Hamilton function that is defined on $\mathsf{T}\times\mathbb{R}^n\times\mathbb{R}^2\times\Sigma\times\mathbb{R}$ and generates the time one flow ψ_1 searched for.

(i) A single diophantine ω
(ii) $\mathcal{H}$ analytic also in ω
$\longrightarrow$ The general case

Finally let $\mathcal{H}$ only be of class C^∞ , though analytic in x, y, q, p and λ . To use the previous result, we approximate $\mathcal{H}$ by analytic functions $\mathcal{H}_i$ and replace (16) by

$$
(\omega\,|\,\frac{\partial K_i}{\partial x}) \;=\; \sum_{|j|+k+l+m=N}\;\sum_{0<|\mathsf{h}|\le 2^i} H^{\mathsf{h}}_{i,jklm}(\omega)\mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)} y^j q^k p^l \lambda^m \;-\; 2a(\omega)p\,\frac{\partial K}{\partial q} \;-\; d\lambda\,\frac{\partial K}{\partial p} \qquad (18)
$$

which is again solved by analytic functions K_i . Following [Zehnder;75] we choose the approximating sequence $(\mathcal{H}_i)_i$ to satisfy

$$
\|H^{\mathsf{h}}_{i,jklm} - H^{\mathsf{h}}_{i-1,jklm}\|_{U_{\rho_i}(\Sigma_\gamma)} \;\le\; \mathit{const}\cdot\rho_i^\beta
$$

for some fixed $\beta\notin\mathbb{N}$. Let L_i denote the solution of (18) with H_i replaced by H_{i-1} (but summation over h still from $|\mathsf{h}| = 1$ to $|\mathsf{h}| = 2^i$). Writing

$$
\begin{aligned}
&K_{i,jklm}(x,\omega) \;-\; K_{i-1,jklm}(x,\omega) \;= \\
&\quad K_{i,jklm}(x,\omega) \;-\; L_{i,jklm}(x,\omega) \;+ \sum_{2^{i-1}<|\mathsf{h}|\le 2^i} L^{\mathsf{h}}_{jklm}(\omega)\,\mathrm{e}^{\mathrm{i}(\mathsf{h}\,|\,x)}
\end{aligned}
$$

we complete the estimate (17) of the third term by

$$
\begin{aligned}
&\sum_{0<|\mathsf{h}|\le 2^i}\sum_{\mu=1}^{2m+l+1} |(\mathsf{h}\,|\,\omega)|^{-\mu}\,|\,H^{\mathsf{h}}_{i,jklm}(\omega) - H^{\mathsf{h}}_{i-1,jklm}(\omega)\,|\,\mathrm{e}^{(\frac{1}{3}\kappa+\rho_i)|\mathsf{h}|} \\
&\quad\le \max_{\mathsf{h}} \|H^{\mathsf{h}}_{i,jklm} - H^{\mathsf{h}}_{i-1,jklm}\|_{U_{\rho_i}(\Sigma_\gamma)} \sum_{\mathsf{h}\in\mathbb{Z}^n} |\mathsf{h}|^{2N+1}\,\mathrm{e}^{-\frac{1}{3}\kappa|\mathsf{h}|} \\
&\quad\le \mathit{const}\cdot\rho_i^\beta
\end{aligned}
$$

and derive from the Inverse Approximation Lemma the limit K of $(K_i)_i$ to be in C^β . Since this holds true for every $\beta > 0$ we have shown that K is C^∞-(Whitney)-smooth and thus can be extended to the desired Hamilton function on $\mathsf{T} \times \mathbb{R}^n \times \mathbb{R}^2 \times \Sigma \times \mathbb{R}$ **;**

q.e.d.(4.2)

The main reason we need Proposition 4.2 is that we had to truncate the co-ordinate transformations φ_i during the proof of Proposition 4.1, see the discussion before Proposition 4.13. Unlike in the case of *e.g.* normally elliptic tori, where the transformations one works with form a group, cf. [Pöschel;89], the φ_i are polynomials of increasing degree in y, q, p, λ . The same problem arises in [Braaksma,Broer;87] and [Braaksma,Broer,Huitema;90] where dissipative quasi-periodic bifurcations are treated. Since there is no structure to be preserved, one can truncate directly the co-ordinate transformations and thus obtain the desired limit, (Whitney)-smooth in ω and analytic in the other variables. This is in fact implicitly done in these papers, though not explicitly mentioned and justified. Like in Proposition 4.1 one loses one order in the normal form, but applying the analogue of Proposition 4.2, see §2 of [Braaksma,Broer,Huitema;90], allows to make terms of that order x-independent again and indeed push the x-dependence to any order desired.

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