Estimates on invariant tori near an elliptic equilibrium point of a Hamiltonian system

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Abstract

We give a precise statement for KAM theorem in a neighbourhood of an elliptic equilibrium point of a Hamiltonian system. If the frequencies of the elliptic point are nonresonant up to a certain order $K \geq 4$, and a nondegeneracy condition is fulfilled, we get an estimate for the measure of the complement of the KAM tori in a neighbourhood of given radius. Moreover, if the frequencies satisfy a Diophantine condition, with exponent τ, we show that in a neighbourhood of radius r the measure of the complement is exponentially small in $(1/r)^{1/(\tau+1)}$. We also give a related result for quasi-Diophantine frequencies, which is more useful for practical purposes. The results are obtained by putting the system in Birkhoff normal form up to an appropiate order, and the key point relies on giving accurate bounds for its terms.

1 Introduction

We consider an analytic Hamiltonian system, with n degrees of freedom, having the origin as an elliptic equilibrium point. In suitable canonical coordinates, the Hamiltonian takes the form

$$H(q,p) = \sum_{s\geq 2} H_s(q,p), \tag{1}$$

where H_s is a homogeneous polynomial of degree s in (q,p) for every $s \geq 2$, and

$$H_2(q,p) = \frac{1}{2}\sum_{j=1}^{n} \lambda_j \left(q_j^2 + p_j^2\right). \tag{2}$$

We are concerned with the existence of *n-dimensional invariant tori* in a neighbourhood of the elliptic point.

We begin by showing, in section 2, that the Hamiltonian (1–2) is nearly-integrable by putting it in *Birkhoff normal form* up to an appropiate degree $K \geq 4$, provided the frequency vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ is nonresonant up to order K. Using results from the paper [6] by Giorgilli *et al.*, we state a quantitative version of Birkhoff theorem, which gives estimates for the homogeneous terms of the part in normal form and for the homogeneous terms of the remainder (proposition 1).

In section 3, like in Pöschel's paper [13], we consider action–angle variables in a neighbourhood of radius r. Assuming a suitable nondegeneracy condition (we deal with the isoenergetic case), we apply the known *KAM theorem* and show in theorem 3 that most trajectories in a neighbourhood of radius r lie in invariant tori: we get for the relative measure of their complement an estimate of the type $\mathcal{O}\left(r^{(K-3)/2}\right)$. In fact, an estimate like this was already obtained in [13] but, furthermore, we specify the smallness condition on r required for its validity.

The extra information provided in theorem 3 with respect to [13] becomes important in section 4, where we assume that λ satisfies a Diophantine condition: with given $\tau \geq n-1$ and $\gamma > 0$,

$$|k \cdot \lambda| \geq \frac{\gamma}{|k|_1^\tau} \qquad \forall k \in \mathbf{Z}^n \setminus \{0\}\,, \tag{3}$$

where we write $|k|_1 = \sum_{j=1}^n |k_j|$. We say λ to be τ, γ-*Diophantine.* Our main contribution, already announced in [5], is to show that in this case we can choose the degree K as a function of r, giving rise to an *exponentially small* estimate of the type

$$\exp\left\{-\left(\frac{1}{r}\right)^{1/(\tau+1)}\right\} \tag{4}$$

for the measure of the complement of the invariant set (theorem 4). To understand the fact that, in the Diophantine case, the measure of the complement of the invariant tori is exponentially small, we notice that the size of the perturbation in applying KAM theorem is very small near the elliptic point. Hence, we can ensure the preservation of the invariant tori under a Diophantine condition with a very small value of the parameter.

However, our estimate (4) is not very useful from a practical point of view. Indeed, if the frequency vector λ is not exactly known, it cannot be decided if it satisfies the Diophantine condition (3). For this reason, we have also included estimates for the *"quasi-Diophantine"* case, in section 4. We remark that, if the vector λ is known up to a precision $\delta > 0$, it has no sense to check the Diophantine condition beyond a certain finite order $N = N(\tau, \gamma, \delta)$. So we assume λ to be "Diophantine up to precision δ" (see a concrete definition in section 4). Then, we see in theorem 5 that exponentially small estimates of the type (4) hold except in a neighbourhood of radius $\mathcal{O}(\delta)$. So we can say that such estimates are still valid, for practical purposes, if δ is small. This suggests that, in studying the behaviour of the system around an elliptic fixed point, it does not really matter whether its frequencies are or are not exactly Diophantine, unless we look at a very small neighbourhood of the fixed point.

Since the measure of the region not covered by invariant tori, near the elliptic point, is neglectible from a practical point of view, we can consider theorems 4 and 5 as results of practical stability. This agrees with the known fact that, in order to detect unstable trajectories numerically, one cannot begin too close to the elliptic point.

As a technical remark, we point out that the estimates given in [6], based in the *Giorgilli–Galgani algorithm,* did not allow us to obtain the exponent $1/(\tau+1)$ of (4) directly, but a worse one. Nevertheless, we have carried out an improvement of the estimates of [6], without modifying the algorithm. In this way we obtain the exponent $1/(\tau+1)$, that seems to be optimal in the frame of our scheme.

It has to be recalled that exponentially small measure estimates for the complement of the invariant tori were first obtained by Neishtadt [12], for a system with two degrees of freedom in the case of degeneracy.

We also quote a result, related to our theorem 4, which has recently been established in [10]: for a fixed KAM torus of a nearly-integrable Hamiltonian, it is shown that in a neighbourhood of radius r there exist many n-dimensional invariant tori, and the measure of their complement is exponentially small.

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2 The Birkhoff normal form

Let us consider the Hamiltonian (1–2) and, given $K \geq 4$, assume that its frequency vector λ is nonresonant up to order K:

$$k \cdot \lambda \neq 0 \qquad \forall k \in \mathbf{Z}^n,\ 0 < |k|_1 \leq K. \tag{5}$$

The well-known *Birkhoff theorem* [1, 11] states that, in some neighbourhood of the origin, there exists a canonical transformation $\Psi^{(K)}$, near to the identity map, such that $\mathcal{H}^{(K)} = H \circ \Psi^{(K)}$ is in Birkhoff normal form up to degree K:

$$\mathcal{H}^{(K)}(q,p) = \lambda \cdot I + \mathcal{Z}^{(K)}(I) + \mathcal{R}^{(K)}(q,p), \tag{6}$$

with

$$\mathcal{Z}^{(K)}(I) = \sum_{\substack{s \text{ even} \\ 4 \leq s \leq K}} \mathcal{Z}_s(I), \qquad \mathcal{R}^{(K)}(q,p) = \sum_{s \geq K+1} \mathcal{R}_s^{(K)}(q,p), \tag{7}$$

where every $\mathcal{Z}_s(I)$ (uniquely determined) is a homogeneous polynomial of degree $s/2$ in the action variables

$$I_j = \frac{1}{2}\left(q_j^2 + p_j^2\right), \qquad j = 1, \ldots, n,$$

and every $\mathcal{R}_s^{(K)}(q,p)$ is a homogeneous polynomial of degree s in (q,p). Since the Hamiltonian

$$h^{(K)}(I) := \lambda \cdot I + \mathcal{Z}^{(K)}(I)$$

is integrable and in a neighbourhood of radius r one has $\mathcal{R}^{(K)} = \mathcal{O}\left(r^{K+1}\right)$, it turns out that $\mathcal{H}^{(K)}$ is a nearly-integrable Hamiltonian near the origin. Our aim is to apply KAM theorem to $\mathcal{H}^{(K)}$.

However, for our purposes we need to bound from below the radius of the neighbourhood where the transformation to Birkhoff normal form holds. Besides, we need bounds for the terms of the normal form (to satisfy the smallness condition for KAM theorem). Having these ideas in mind, we state below a quantitative version for Birkhoff theorem (proposition 1). Such a version comes from the results obtained by Giorgilli *et al.* [6], but we improve their estimates on the terms $\mathcal{Z}_s$, $\mathcal{R}_s^{(K)}$. This improvement is crucial in order to get the exponent $1/(\tau+1)$ appearing in the bound (4).

In [6], the canonical transformation bringing to normal form is constructed through the *Giorgilli–Galgani algorithm* (see also [7, 8, 15]), a variant of the Lie series method. In that scheme, the transformation is obtained as the flow of a unique nonautonomous Hamiltonian. We point out that the case concerned in [6] is more general than the one considered here, since it also involves resonant normal forms. We give in appendix A a description of the Giorgilli–Galgani algorithm.

In dealing with normal forms near a fixed point of a Hamiltonian system, it is usual to consider the complex canonical coordinates (x, y) defined by the linear change

$$x_j = \frac{1}{\sqrt{2}}(q_j - ip_j), \qquad y_j = -\frac{i}{\sqrt{2}}(q_j + ip_j), \qquad j = 1, \ldots, n$$

(these coordinates make simpler the resolution, in terms of coefficients, of the homological equations arising in the construction of normal forms). Making use of the notation $x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$, $y^{\nu'} = y_1^{\nu'_1} \cdots y_n^{\nu'_n}$, we write

$$H_s(x, y) = \sum_{\substack{\nu,\nu' \in \mathbf{N}^n \\ |\nu+\nu'|_1 = s}} h_{\nu,\nu'} x^\nu y^{\nu'}.$$

Note that q, p are real if $\overline{y} = ix$, and hence the Hamiltonian H is "real" whenever its coefficients satisfy the relation $h_{\nu,\nu'} = i^{|\nu+\nu'|_1} h_{\nu',\nu}$.

We introduce some definitions. Given $r > 0$, we consider the real and complex polydisks of radius r centred at the origin:

$$\begin{aligned}
\mathcal{B}_r &:= \left\{(q,p) \in \mathbf{R}^{2n} \ : \ |(q,p)| \le r\right\} = \left\{(x,y) \in \mathbf{C}^{2n} \ : \ |(x,y)| \le r, \ \overline{y} = ix\right\}, \\
\widehat{\mathcal{B}}_r &:= \left\{(x,y) \in \mathbf{C}^{2n} \ : \ |(x,y)| \le r\right\},
\end{aligned}$$

where we define

$$|(q,p)| := \max_{j=1,\ldots,n} \sqrt{q_j^2 + p_j^2}, \qquad |(x,y)| := \max_{j=1,\ldots,n} \sqrt{|x_j|^2 + |y_j|^2}.$$

In order to give estimates, we introduce a norm for the polynomials involved. Like in [6], for a given homogeneous polynomial $f_s(x,y) = \sum_{|\nu+\nu'|_1 = s} f_{\nu,\nu'} x^\nu y^{\nu'}$, we define the norm

$$\|f_s\| := \sum_{|\nu+\nu'|_1 = s} |f_{\nu,\nu'}| \tag{8}$$

(for an alternative norm, see [15]). This definition also makes sense if f_s is a vector-valued function; each coefficient $f_{\nu,\nu'}$ is then a vector and $|f_{\nu,\nu'}|$ denotes its Euclidean norm (the same remark holds for a matrix-valued or tensor-valued function).

For a function $f(x,y)$, we denote $|f|_r$ its supremum norm on $\widehat{\mathcal{B}}_r$. Given $f = \sum_s f_s$, one has

$$|f|_r \leq \sum_s \|f_s\| \, r^s.$$

If there exist a, b such that $\|f_s\| \leq a^s b$ for every s, then f is analytic on $\widehat{\mathcal{B}}_r$ for $r < 1/a$.

We consider lower bounds for the *small divisors* in the nonresonance condition (5), up to successive orders. For $s \leq K$, let α_s be the decreasing sequence ($\alpha_s \leq \alpha_{s-1}$) defined by

$$\alpha_s := \min_{0<|k|_1 \leq s} |k \cdot \lambda| \,. \tag{9}$$

The improvement of the estimates given in [6] comes from the following remark: in the construction of the normal form described above, the obtainment of $\mathcal{Z}_s$ only involves small divisors up to order $s-1$. This allows us to get better estimates, with expressions of the type $\alpha_3 \cdots \alpha_{s-1}$ in the denominators instead of α_K^{s-3}, as shown in the next proposition.

Proposition 1 *Let $H(x,y) = \sum_{s\geq 2} H_s(x,y)$ be a Hamiltonian with $H_2 = \lambda \cdot I$, and assume that $\|H_s\| \leq c^{s-2} d$ for $s \geq 2$. Let $K \geq 4$ given and assume that λ is nonresonant up to order K. Let α_s, for $s \leq K$, be lower bounds for the small divisors as in (9). Then, there exists a canonical transformation $\Psi^{(K)}$, near to the identity map, such that $\mathcal{H}^{(K)} = H \circ \Psi^{(K)}$ is in the Birkhoff normal form (6–7) up to degree K, and one has:*

a) $\displaystyle \|\mathcal{Z}_s\| \leq \frac{\frac{1}{6}\,(6cd)^{s-2}\,(s-2)!}{\alpha_3 \cdots \alpha_{s-1}}$ *for s even, $4 \leq s \leq K$.*

b) $\displaystyle \left\|\mathcal{R}_s^{(K)}\right\| \leq \frac{20d^2\,(20cd)^{s-2}\,(K-3)!(K-2)^{s-K+2}}{\alpha_3 \cdots \alpha_{K-1}\alpha_K^{s-K+2}}$ *for $s \geq K+1$.*

c) *Defining*

$$r_K^* := \frac{\alpha_K}{548ncd\,K} \,, \tag{10}$$

the transformation $\Psi^{(K)}$ is analytic on $\widehat{\mathcal{B}}_{r_K^*}$ and, for any $r \leq r_K^*$, one has the inclusion $\Psi^{(K)}(\mathcal{B}_r) \supset \mathcal{B}_{r/2}$.

This proposition improves the results of [6, theorem 5.5]. The proof is deferred to appendix A.

3 KAM tori and estimates

3.1 Recalling KAM theorem

In this section we recall a statement of KAM theorem to be used later. Let us consider a nearly-integrable Hamiltonian written in action–angle variables

$$\mathcal{H}(\phi, I) = h(I) + f(\phi, I),$$

with $\phi \in \mathbf{T}^n$ and $I \in \mathcal{G} \subset \mathbf{R}^n$. The perturbation f is assumed to be of size ε. To show that most of the trajectories of $\mathcal{H}$ lie in n-dimensional invariant tori, one usually imposes one of the following nondegeneracy conditions on the frequency map $\omega = \nabla h$:

$$\det\left(\frac{\partial\omega}{\partial I}(I)\right) \neq 0 \qquad \text{or} \qquad \det\begin{pmatrix} \frac{\partial\omega}{\partial I}(I) & \omega(I) \\ \omega(I)^\top & 0 \end{pmatrix} \neq 0$$

for every $I \in \mathcal{G}$. We call these conditions *Kolmogorov nondegeneracy* and *isoenergetic nondegeneracy,* respectively.

We need a statement of KAM theorem expliciting the smallness condition on ε and an estimate for the complement of the invariant set. Several statements, for the Kolmogorov version or for the isoenergetic one, have been established in [12, 13, 2, 4, 9] (see also [3] for general reference about the subject). The statement reproduced below is taken from [4, 9], where the isoenergetic version is considered. Nevertheless, the ideas there contained also apply to the Kolmogorov version, which is simpler.

We begin with some definitions. Given a set $G \subset \mathbf{R}^n$, we consider analytic functions on complex neighbourhoods of $\mathbf{T}^n \times G$. Given $\rho = (\rho_1, \rho_2) \geq 0$ (i.e. $\rho_j \geq 0$, $j = 1, 2$), we introduce the sets:

$$\begin{aligned} \mathcal{W}_{\rho_1}(\mathbf{T}^n) &:= \{\phi \ : \ \operatorname{Re}\phi \in \mathbf{T}^n, \ |\operatorname{Im}\phi|_\infty \leq \rho_1\}, \\ \mathcal{V}_{\rho_2}(G) &:= \{I \in \mathbf{C}^n \ : \ |I - I'| \leq \rho_2 \text{ with } I' \in G\}, \end{aligned}$$

where $|\cdot|_\infty$ and $|\cdot| = |\cdot|_2$ denote, respectively, the supremum norm and the Euclidean norm for n-vectors. We then define:

$$\mathcal{D}_\rho(G) := \mathcal{W}_{\rho_1}(\mathbf{T}^n) \times \mathcal{V}_{\rho_2}(G).$$

For a given function $g(I)$ of the action variables, defined on $\mathcal{V}_\eta(G)$, $\eta \geq 0$, we consider the supremum norm:

$$|g|_{G,\eta} := \sup_{I \in \mathcal{V}_\eta(G)} |g(I)|, \qquad |g|_G := |g|_{G,0} \ . \tag{11}$$

Even if g is vector-valued (or matrix-valued), these definitions make sense by considering in $|g(I)|$ the Euclidean norm.

Given a function $f(\phi, I)$ of the action–angle variables, analytic on the domain $\mathcal{D}_\rho(G)$, $\rho = (\rho_1, \rho_2) \geq 0$, we consider its Fourier expansion $f(\phi, I) = \sum_{k \in \mathbf{Z}^n} f_k(I)\, e^{i\, k \cdot \phi}$ and define the following exponentially weighted norm (see also [14]):

$$\|f\|_{G,\rho} := \sum_{k \in \mathbf{Z}^n} |f_k|_{G,\rho_2} \cdot e^{|k|_1 \rho_1}. \tag{12}$$

We use this norm to express the smallness condition for KAM theorem.

We introduce a quantitative version for the isoenergetic condition. For a function $h(I)$ defined on $G \subset \mathbf{R}^n$, and given $\mu > 0$, we say that the associated frequency map $\omega = \nabla h$ is *μ-isoenergetically nondegenerate* if ω does not vanish on G and

$$\left|\frac{\partial\omega}{\partial I}(I)\, v + \xi\, \omega(I)\right| \geq \mu\, |v| \qquad \forall v \in \langle\omega(I)\rangle^\perp, \ \forall \xi \in \mathbf{R}, \ \ \forall I \in G. \tag{13}$$

It may be assumed without loss of generality that $\omega_n(I) \neq 0$ for $I \in G$. Under the isoenergetic nondegeneracy, and given a constant $a > 0$, the following map is a local diffeomorphism (see [4]):

$$\Omega_{\omega,h,a}(I) := \left(\frac{\overline{\omega}(I)}{\omega_n(I)} \,,\, a\, h(I) \right), \qquad I \in G, \tag{14}$$

where we use the notation $\overline{v} = (v_1, \ldots, v_{n-1})$ for $v = (v_1, \ldots, v_{n-1}, v_n)$. Our choice of the constant a in theorem 2 is related to the estimates given in the technical lemma 8 (see appendix B), which are better in this way.

Before giving the statement of the isoenergetic KAM theorem to be used later, we introduce some technical definitions. Given $G \subset \mathbf{R}^n$ and $b \geq 0$, we define the set

$$G - b := \{ I \in G \ : \ I + b \subset G \} \,,$$

where $I + b$ means the closed ball of radius b centred at I. Moreover, given $F \subset \mathbf{R}^n$ and $D > 0$, we say that F is a *D-set* if, for any $0 \leq b_1 < b_2$,

$$\text{mes}\,[(F - b_1) \setminus (F - b_2)] \leq D(b_2 - b_1).$$

Theorem 2 (Isoenergetic KAM Theorem) *Let $\mathcal{G} \subset \mathbf{R}^n$ a compact, and consider the Hamiltonian $\mathcal{H}(\phi, I) = h(I) + f(\phi, I)$, real analytic on $\mathcal{D}_\rho(\mathcal{G})$. Let $\omega = \nabla h$, and assume the bounds:*

$$\left| \frac{\partial^2 h}{\partial I^2} \right|_{\mathcal{G},\rho_2} \leq M, \qquad |\omega|_{\mathcal{G}} \leq L \qquad \text{and} \qquad |\omega_n(I)| \geq l \ \ \forall I \in \mathcal{G}.$$

Assume also that ω is μ-isoenergetically nondegenerate on $\mathcal{G}$. With $a = 16M/l^2$, assume that the map $\Omega = \Omega_{\omega,h,a}$ is one-to-one on $\mathcal{G}$, and that its range $F = \Omega(\mathcal{G})$ is a D-set; denote $P = \operatorname{diam} F$. Let $\tau > n - 1$ and $\gamma > 0$ given, and define the set

$$\widehat{G}_\gamma := \left\{ I \in \mathcal{G} - \frac{2\gamma}{\mu} \ : \ \omega(I) \ \textit{is}\ \tau,\gamma\textit{-Diophantine} \right\}.$$

For some constants C_1, C_2, C_3, C_4, C_5, depending only on n, τ, ρ_1, M, L, l, μ, if

$$\varepsilon := \|f\|_{\mathcal{G},\rho} \ \leq \ C_1 \gamma^2, \qquad \gamma \leq \min\left(C_2 \rho_2, C_3 \right), \tag{15}$$

then there exists a real continuous map $\mathcal{T} : \mathcal{W}_{\frac{\rho_1}{4}}(\mathbf{T}^n) \times \widehat{G}_\gamma \longrightarrow \mathcal{D}_\rho(\mathcal{G})$, analytic with respect to the angular variables, such that:

a) *For every $I \in \widehat{G}_\gamma$, the set $\mathcal{T}\left(\mathbf{T}^n \times \{I\}\right)$ is an invariant torus of $\mathcal{H}$, contained in $\mathbf{T}^n \times \mathcal{G}$, its frequency vector is colinear to $\omega(I)$ and its energy is $h(I)$.*

b) $\text{mes}\, \left[(\mathbf{T}^n \times \mathcal{G}) \setminus \mathcal{T}\left(\mathbf{T}^n \times \widehat{G}_\gamma \right) \right] \leq \left(C_4 D + C_5 P^{n-1} \right) \gamma.$

See the proof in [4, 9]. The statement (and a somewhat simpler proof) is also valid in the Kolmogorov case, with small changes. We also remark that, for a fixed ε, we may choose $\gamma \sim \sqrt{\varepsilon}$ and the measure of the complement in part (b) becomes $\mathcal{O}\left(\sqrt{\varepsilon}\right)$.

3.2 Applying KAM theorem

Now, our aim is to apply KAM theorem to the Hamiltonian $\mathcal{H}^{(K)} = h^{(K)} + \mathcal{R}^{(K)}$ introduced in (6–7). We put this Hamiltonian in action–angle variables through the known canonical change

$$q_j = \sqrt{2I_j} \cdot \cos\phi_j, \qquad p_j = \sqrt{2I_j} \cdot \sin\phi_j, \qquad j = 1, \ldots, n.$$

We have:

$$x_j = \sqrt{I_j} \cdot e^{-i\phi_j}, \qquad y_j = -i\sqrt{I_j} \cdot e^{i\phi_j}, \qquad j = 1, \ldots, n. \tag{16}$$

To obtain invariant tori in $\mathcal{B}_r$, we consider the set of actions corresponding to this neighbourhood:

$$\mathcal{G}_r := \left\{ I \in \mathbf{R}^n \ : \ I \geq 0, \ |I|_\infty \leq \frac{r^2}{2} \right\}.$$

We use the notation $I \geq a$, where $a \in \mathbf{R}$, to mean that $I_j \geq a$ for $j = 1, \ldots, n$. The change (16) maps $(\phi, I) \in \mathbf{T}^n \times \mathcal{G}_r \longmapsto (x, y) \in \mathcal{B}_r$. However, to fulfil the conditions of theorem 2 the Hamiltonian $\mathcal{H}^{(K)}$ should be analytic on some complex neighbourhood $\mathcal{D}_\rho(\mathcal{G}_r)$. This cannot be guaranteed because it is not possible to define $\sqrt{I_j}$ analytically around the coordinate hyperplanes $I_j = 0$. Hence we have to remove a suitable neighbourhood of these hyperplanes, but we shall show in the proof of theorem 3 that this does not affect essentially our measure estimates. Such an approach has already been carried out by Pöschel [13], also in applying KAM theorem to a Hamiltonian in Birkhoff normal form up to a certain order. For r, ρ_2 given, we shall take for theorem 2 the domain

$$\mathcal{G}_{r,\rho_2} := \left\{ I \in \mathbf{R}^n \ : \ I \geq 2\rho_2, \ |I|_\infty \leq \frac{r^2}{2} \right\},$$

which is nonempty if $\rho_2 < r^2/4$. Fixed r, we shall see in theorem 3 that the appropiate values for the main parameters of theorem 2 are $\varepsilon \sim r^{K+1}$ and $\gamma \sim \rho_2 \sim r^{(K+1)/2}$.

To apply theorem 2, we have to require the frequency map $\omega^{(K)} = \nabla h^{(K)}$ to be isoenergetically nondegenerate on the neighbourhood considered. In fact we only assume the nondegeneracy at the origin itself, since this suffices to ensure it in a small neighbourhood. Defining

$$A := \frac{\partial^2 \mathcal{Z}_4}{\partial I^2}\,, \tag{17}$$

a constant symmetric matrix, we have

$$\omega^{(K)}(I) = \lambda + AI + \sum_{\substack{s \text{ even} \\ 6 \leq s \leq K}} \nabla \mathcal{Z}_s(I),$$

and we require that

$$\det \begin{pmatrix} A & \lambda \\ \lambda^\top & 0 \end{pmatrix} \neq 0. \tag{18}$$

This implies that, for some $\mu > 0$ depending on A and λ, one has

$$|Av + \xi\,\lambda| \geq \mu\,|v| \qquad \forall v \in \langle\lambda\rangle^\top, \ \forall \xi \in \mathbf{R}. \tag{19}$$

This expression of the isoenergetic nondegeneracy will allow us to use the quantitative version (13), which is more useful for giving estimates.

If we were dealing with the Kolmogorov nondegeneracy, we should impose the condition

$$\det A \neq 0$$

instead of (18). In this case, the statement and the proof of our results would be analogous and somewhat simpler. Moreover, we point out that higher order conditions are also possible for both types of nondegeneracy: such conditions would be useful if the frequency map were degenerate at the origin and nondegenerate near it.

With the setup described above and some technical lemmas stated in appendix B, we are able to apply theorem 2 to our Birkhoff normal form.

Theorem 3 *Let $H(x,y) = \sum_{s\geq 2} H_s(x,y)$ be a real Hamiltonian with $H_2 = \lambda \cdot I$, and assume that $\|H_s\| \leq c^{s-2}d$ for $s \geq 2$. Let $K \geq 4$ given and assume that λ is nonresonant up to order K. Consider the transformed Hamiltonian $\mathcal{H}^{(K)}$ given by proposition 1, and denote $\omega^{(K)} = \nabla h^{(K)}$. Let r_K^* defined as in (10). Assume that the isoenergetic condition (18) holds, with A defined by (17). Let $\tau > n-1$ given. For some constants c_1, c_2 and c_3 depending only on n, τ, c, d, λ, A, given*

$$0 < r \leq c_1 r_K^* \tag{20}$$

and defining

$$\sigma_r^{(K)} = c_2 r^2 \left(\frac{7r}{r_K^*}\right)^{(K-3)/2}, \tag{21}$$

one has:

a) *There exists a subset $\widehat{G}_r^{(K)} \subset \mathcal{G}_r$ such that, for every $I \in \widehat{G}_r^{(K)}$, the vector $\omega^{(K)}(I)$ is $\tau, \sigma_r^{(K)}$-Diophantine, and there is an n-dimensional invariant torus of the Hamiltonian $\mathcal{H}^{(K)}$, contained in $\mathcal{B}_r$, having the frequency vector colinear to $\omega^{(K)}(I)$ and energy $h^{(K)}(I)$.*

b) *Denoting $\mathcal{T}_r^{(K)}$ the set filled with the invariant tori of part (a), the following bound holds:*

$$\operatorname{mes}\left[\mathcal{B}_r \setminus \mathcal{T}_r^{(K)}\right] \leq c_3 \left(\frac{7r}{r_K^*}\right)^{(K-3)/2} \cdot \operatorname{mes} \mathcal{B}_r \ . \tag{22}$$

Proof (Along this proof, the symbols $\preceq$ and $\sim$ express that the involved constants do not depend on r, ρ_2, K.)

We assume $\lambda_n \neq 0$; otherwise a permutation of the variables may be done. We define

$$M = |A|, \qquad L = |\lambda|, \qquad l = |\lambda_n|, \tag{23}$$

and consider $\mu > 0$ (depending only on A and λ) such that (19) holds. In order to apply theorem 2 to the normal form $\mathcal{H}^{(K)} = h^{(K)} + \mathcal{R}^{(K)}$, we first see that for r small enough $\omega^{(K)}$ satisfies on $\mathcal{G}_{r,\rho_2}$ the conditions required for the frequency map, with the constants $2M$, $2L$, $l/2$, $\mu/2$ instead of M, L, l, μ respectively. Actually, in the first part of this proof we do not need to restrict ourselves to the set $\mathcal{G}_{r,\rho_2}$, since $\omega^{(K)}$ is a polynomial map. For technical reasons to be clarified later, we consider the set

$$\widetilde{\mathcal{G}}_r = \left\{ I \in \mathbf{R}^n \ :\ |I|_\infty \le \frac{3r^2}{4} \right\}$$

(without the restriction $I \ge 0$), which contains a neighbourhood of $\mathcal{G}_{r,\rho_2}$.

We are going to estimate the functions

$$\omega^{(K)}(I) - \lambda = \sum_{\substack{s \text{ even} \\ 4\le s\le K}} \nabla \mathcal{Z}_s(I), \qquad \frac{\partial \omega^{(K)}}{\partial I}(I) - A = \sum_{\substack{s \text{ even} \\ 6\le s\le K}} \frac{\partial^2 \mathcal{Z}_s}{\partial I^2}(I)$$

on $\widetilde{\mathcal{G}}_r$ and $\mathcal{V}_{\rho_2}\left(\widetilde{\mathcal{G}}_r\right)$, respectively, with $\rho_2 < r^2/4$ to be fixed appropiately. Taking into account that $I_j = i\, x_j y_j$, we can look at the derivatives of $\mathcal{Z}_s$ as homogeneous polynomials in x, y. So we can consider their norm as defined in (8). Let us check the following inequalities:

$$\left\| \nabla \mathcal{Z}_s \right\| \le \frac{s}{2} \left\| \mathcal{Z}_s \right\|, \qquad \left\| \frac{\partial^2 \mathcal{Z}_s}{\partial I^2} \right\| \le \frac{s^2}{4} \left\| \mathcal{Z}_s \right\|. \tag{24}$$

Indeed, let $\tilde{\mathcal{Z}}_{\nu,\nu} = \left(\tilde{\mathcal{Z}}_{\nu,\nu,j} \right)_{j=1,\dots,n}$ be the coefficient of I^ν in $\nabla \mathcal{Z}_s$. Then,

$$\left\| \nabla \mathcal{Z}_s \right\| = \sum_{2|\nu|_1 = s-2} \left| \tilde{\mathcal{Z}}_{\nu,\nu} \right| \le \sum_{2|\nu|_1 = s-2} \sum_{j=1}^{n} \left| \tilde{\mathcal{Z}}_{\nu,\nu,j} \right| = \sum_{j=1}^{n} \sum_{2|\nu|_1 = s} \nu_j \left| \mathcal{Z}_{\nu,\nu} \right| = \frac{s}{2} \left\| \mathcal{Z}_s \right\|,$$

namely the first inequality of (24). The second one follows in a similar way.

For s even, $4 \le s \le K$, we get from proposition 1 the inequality

$$\left\| \mathcal{Z}_s \right\| \le \frac{\frac{1}{6}(6cd)^{s-2}\, 2K^{s-4}}{\alpha_3\, \alpha_K^{s-4}} \le C \left(\frac{1}{r_K^*} \right)^{s-4}, \tag{25}$$

where C is a constant not depending on s (we need this exponent $s-4$ in order to obtain $1/(\tau+1)$ in (4); it cannot be reached from the original estimates of [6]). Then, making use of the notations (11), we get the estimates

$$\left| \omega^{(K)} - \lambda \right|_{\widetilde{\mathcal{G}}_r} \le \sum_{\substack{s \text{ even} \\ 4\le s\le K}} \frac{s}{2} \left\| \mathcal{Z}_s \right\| \left(\frac{3r^2}{4} \right)^{(s-2)/2} \preceq \sum_{\substack{s \text{ even} \\ 4\le s\le K}} s \left(\frac{r}{r_K^*} \right)^{s-4} r^2 \preceq r^2, \tag{26}$$

$$\left| \frac{\partial \omega^{(K)}}{\partial I} - A \right|_{\widetilde{\mathcal{G}}_r,\rho_2} \le \sum_{\substack{s \text{ even} \\ 6\le s\le K}} \frac{s^2}{4} \left\| \mathcal{Z}_s \right\| \left(r^2 \right)^{(s-4)/2} \preceq \sum_{\substack{s \text{ even} \\ 6\le s\le K}} s^2 \left(\frac{r}{r_K^*} \right)^{s-4} \preceq \frac{r^2}{\left(r_K^*\right)^2}, \tag{27}$$

where we have bounded the finite sums by the corresponding series and we have assumed, for instance, that $r \le r_K^*/2$. Then, with an appropiate value for c_1 in (20), we can obtain the inequalities

$$\left| \frac{\partial \omega^{(K)}}{\partial I} \right|_{\widetilde{\mathcal{G}}_r,\rho_2} \le 2M, \qquad \left| \omega^{(K)} \right|_{\widetilde{\mathcal{G}}_r} \le 2L \qquad \text{and} \qquad \left| \omega_n^{(K)}(I) \right| \ge \frac{l}{2} \quad \forall I \in \widetilde{\mathcal{G}}_r. \tag{28}$$

Moreover, using (19) and applying lema 9 with $\left|\omega^{(K)}-\lambda\right|_{\widetilde{\mathcal{G}}_r}$, $\left|\frac{\partial\omega^{(K)}}{\partial I}-A\right|_{\widetilde{\mathcal{G}}_r}$, $l/2$, $2M$ instead of ε, ε', l, M respectively, we can deduce that $\omega^{(K)}$ is $\frac{\mu}{2}$-isoenergetically nondegenerate on $\widetilde{\mathcal{G}}_r$.

Next we prove that the map $\Omega^{(K)}:=\Omega_{\omega^{(K)},h^{(K)},a}$, defined according to (14) and taking $a=2^7M/l^2$, is one-to-one on $\mathcal{G}_r\subset\widetilde{\mathcal{G}}_r$. First, we consider the case $K=4$:

$$\Omega^{(4)}(I)=\left(\frac{\overline{\lambda}+\overline{A}I}{\lambda_n+A_nI}\,,\,a\left(\lambda\cdot I+\frac{1}{2}I^\top AI\right)\right),$$

where $\overline{A}$ and A_n denote, respectively, the first $n-1$ rows and the last row of the matrix A. From the isoenergetic condition, and taking into account (14), we see that the map $\Omega^{(4)}$ is a local diffeomorphism, and thus there exists a constant $r_0>0$ (depending only on A and λ) such that $\Omega^{(4)}$ is one-to-one on $\widetilde{\mathcal{G}}_{r_0}$. To deduce that $\Omega^{(K)}$ is one-to-one on $\mathcal{G}_r$ for r small enough, we will use lemma 10. Proceeding like in (26–27), we can obtain the following bounds:

$$\left|\omega^{(K)}-\omega^{(4)}\right|_{\widetilde{\mathcal{G}}_r}\preceq\frac{r^4}{(r_K^*)^2}\,,\qquad\left|h^{(K)}-h^{(4)}\right|_{\widetilde{\mathcal{G}}_r}\preceq\frac{r^6}{(r_K^*)^2}\,.$$

After some calculations one gets, for any $I\in\widetilde{\mathcal{G}}_r$, the bounds:

$$\left|\overline{\Omega^{(K)}}(I)-\overline{\Omega^{(4)}}(I)\right|\le\frac{\left|\omega^{(K)}(I)-\omega^{(4)}(I)\right|\cdot\left|\omega^{(4)}(I)\right|}{\left|\omega_n^{(4)}(I)\right|\cdot\left|\omega_n^{(K)}(I)\right|}\preceq\frac{r^4}{(r_K^*)^2}\,,$$

$$\left|\Omega_n^{(K)}(I)-\Omega_n^{(4)}(I)\right|=a\left|h^{(K)}(I)-h^{(4)}(I)\right|\preceq\frac{r^6}{(r_K^*)^2}\,,$$

which lead to

$$\left|\Omega^{(K)}-\Omega^{(4)}\right|_{\widetilde{\mathcal{G}}_r}\preceq\frac{r^4}{(r_K^*)^2}\,.$$

This bound is going to substitute ε in lemma 10. The parameters M, $\tilde{M}$, m, $\tilde{m}$, $\tilde{M}'$ appearing in that lemma are provided by lemma 8. Indeed, using (28), it is easy to check that we can take as M, $\tilde{M}$, m, $\tilde{m}$ some constants depending only on the current M, L, l, μ introduced in (23) and (19). For $\tilde{M}'$, we use the bound

$$\left|\frac{\partial^3h^{(K)}}{\partial I^3}\right|_{\widetilde{\mathcal{G}}_r}\le\sum_{\substack{s\text{ even}\\6\le s\le K}}\frac{s^3}{8}\left\|\mathcal{Z}_s\right\|\left(\frac{3r^2}{4}\right)^{(s-6)/2}\preceq\sum_{\substack{s\text{ even}\\6\le s\le K}}\frac{s^3r^{s-6}}{(r_K^*)^{s-4}}\preceq\frac{1}{(r_K^*)^2}\,,$$

which comes from the inequality

$$\left\|\frac{\partial^3\mathcal{Z}_s}{\partial I^3}\right\|\le\frac{s^3}{8}\left\|\mathcal{Z}_s\right\|,$$

obtained like (24). So we can take

$$\tilde{M}'=\left(\frac{1}{4M}\left|\frac{\partial^3h^{(K)}}{\partial I^3}\right|_{\widetilde{\mathcal{G}}_r}+\frac{12M}{l}\right)\frac{2^8ML}{l^2}\preceq\frac{1}{(r_K^*)^2}\,.$$

Now we are ready to apply lemma 10. With a convenient value for c_1, it is easy to check that (20) implies the smallness condition required in lemma 10. Then, we deduce that $\Omega^{(K)}$ is one-to-one on the set $\tilde{\mathcal{G}}_r - \frac{c' r^4}{\left(r_K^*\right)^2}$, where c' is a constant. This set contains $\mathcal{G}_r$ provided we assume the inequality

$$\frac{r^2}{2} + \frac{c' r^4}{\left(r_K^*\right)^2} \leq \frac{3r^2}{4} ,$$

which can also be included in condition (20).

From now onwards we restrict ourselves to $\mathcal{G}_{r,\rho_2}$; note that $\Omega^{(K)}\left(\mathcal{G}_{r,\rho_2}\right)$ is a D-set with $D \sim \left(r^2\right)^{n-1}$, and its diameter is $P \sim r^2$. It has to be noticed that if we had applied lemma 10 directly on $\mathcal{G}_r$ or $\mathcal{G}_{r,\rho_2}$, then we would have had to remove a relatively large neighbourhood of the coordenate hyperplanes $I_j = 0$, and this would affect the estimate for the measure of the complement given in part (b).

We have to check (15) in order to apply theorem 2. We consider the parameter $\sigma_r^{(K)}$ defined in (21) instead of γ, and the complex domain will be $\mathcal{D}_\rho\left(\mathcal{G}_{r,\rho_2}\right)$, with $\rho_1 = 1$ and $\rho_2 \sim \sigma_r^{(K)}$, in such a way that the choice of ρ_2 allows us to fulfil the second inequality of (15).

The remainder $\mathcal{R}^{(K)}(\phi, I)$ is analytic on the complex neighbourhood $\mathcal{D}_\rho\left(\mathcal{G}_{r,\rho_2}\right)$, with $\rho = (1, \rho_2)$. Indeed, since $\operatorname{Re} I_j > 0$ on this neighbourhood, the coordinate change (16) is analytic on it. To check the first inequality of (15), we have to consider the norm (12), defined in terms of Fourier coefficients. By a property of the norm (12) established in [14], one has

$$\left\|\mathcal{R}^{(K)}\right\|_{\mathcal{G}_{r,\rho_2},\rho} \leq \left(\coth^n \frac{1}{2}\right) \left|\mathcal{R}^{(K)}\right|_{\mathcal{D}_{(2,\rho_2)}\left(\mathcal{G}_{r,\rho_2}\right)} \tag{29}$$

where, in the right hand side of this inequality, we have considered the supremum norm on $\mathcal{D}_{(2,\rho_2)}\left(\mathcal{G}_{r,\rho_2}\right)$. To bound this norm, it will be better to consider the coordinates (x, y) because we can then use our estimates on the homogeneous terms. From proposition 1, and proceeding as in (25), we obtain for $s \geq K+1$ the estimate

$$\left\|\mathcal{R}_s^{(K)}\right\| \leq \tilde{C} \left(\frac{1}{r_K^*}\right)^{s-4} ,$$

where $\tilde{C}$ is a constant not depending on s. Using that $\rho_2 < r^2/4$ (otherwise the set $\mathcal{G}_{r,\rho_2}$ is empty), for $(\phi, I) \in \mathcal{D}_{(2,\rho_2)}\left(\mathcal{G}_{r,\rho_2}\right)$ we have

$$|x_j|\,, |y_j| \leq \sqrt{|I_j|} \cdot e^{|\operatorname{Im}\phi_j|} \leq \sqrt{\frac{r^2}{2} + \frac{r^2}{4}} \cdot e^2 \leq 7r.$$

Then, proceeding like in (26–27),

$$\left|\mathcal{R}^{(K)}\right|_{\mathcal{D}_{(2,\rho_2)}\left(\mathcal{G}_{r,\rho_2}\right)} \leq \sum_{s \geq K+1} \left\|\mathcal{R}_s^{(K)}\right\| (7r)^s \preceq \sum_{s \geq K+1} \frac{(7r)^s}{\left(r_K^*\right)^{s-4}} \preceq \frac{(7r)^{K+1}}{\left(r_K^*\right)^{K-3}} \tag{30}$$

provided we assume, for instance, the inequality $r \leq r_K^*/14$, which can be included in (20). Putting the bounds (29–30) together, we see from (21) that $\left\|\mathcal{R}^{(K)}\right\|_{\mathcal{G}_{r,\rho_2},\rho} \preceq \left(\sigma_r^{(K)}\right)^2$ and hence the first inequality of (15) is satisfied taking the constant c_2 appropiately.

Applying theorem 2, we obtain invariant tori parametrized by the set

$$\widehat{G}_r^{(K)} = \left\{ I \in \mathcal{G}_{r,\rho_2} - \frac{4\sigma_r^{(K)}}{\mu} \ : \ \omega(I) \text{ is } \tau, \sigma_r^{(K)}\text{-Diophantine} \right\},$$

and we have proved part (a). Let us denote $\mathcal{S}_r^{(K)}$ the set filled with these invariant tori in the action–angle coordinates, and $\mathcal{T}_r^{(K)}$ the same set in the original coordinates. By part (b) of theorem 2, and recalling that $D \sim r^{2n-2}$, $P \sim r^2$, we get the estimate

$$\operatorname{mes} \left[(\mathbf{T}^n \times \mathcal{G}_{r,\rho_2}) \setminus \mathcal{S}_r^{(K)} \right] \preceq r^{2n-2} \sigma_r^{(K)} \sim r^{2n} \left(\frac{7r}{r_K^*} \right)^{(K-3)/2} .$$

To bound the measure of the complement of the invariant set with respect the whole neighbourhood of radius r, we have to add to this estimate the measure of the part removed in considering $\mathcal{G}_{r,\rho_2}$ instead of $\mathcal{G}_r$. However, this does not affect our estimate, since the measure of the part removed has the same order. Indeed, we have:

$$\operatorname{mes} \left(\mathcal{G}_r \setminus \mathcal{G}_{r,\rho_2} \right) \leq n \left(r^2 \right)^{n-1} \cdot 2\rho_2 \sim r^{2n-2} \sigma_r^{(K)} \sim r^{2n} \left(\frac{7r}{r_K^*} \right)^{(K-3)/2} ,$$

and hence

$$\begin{aligned} \operatorname{mes} \left[(\mathbf{T}^n \times \mathcal{G}_r) \setminus \mathcal{S}_r^{(K)} \right] &\leq \operatorname{mes} \left[(\mathbf{T}^n \times \mathcal{G}_{r,\rho_2}) \setminus \mathcal{S}_r^{(K)} \right] + (2\pi)^n \cdot \operatorname{mes} \left(\mathcal{G}_r \setminus \mathcal{G}_{r,\rho_2} \right) \\ &\preceq r^{2n} \left(\frac{7r}{r_K^*} \right)^{(K-3)/2} . \end{aligned} \tag{31}$$

Finally, we have to move this bound to the neighbourhood $\mathcal{B}_r$, defined in terms of the old coordinates. The change to action–angle variables is measure-preserving, since it is canonical. This change relates $\mathbf{T}^n \times \mathcal{G}_r$ and $\mathcal{B}_r$; hence we get for the measure of $\mathcal{B}_r \setminus \mathcal{T}_r^{(K)}$ the same bound (31). Using that $\operatorname{mes} \mathcal{B}_r \sim r^{2n}$, we deduce the bound of part (b), concerning the relative measure inside $\mathcal{B}_r$. □

We remark that this result is a more elaborated version of Pöschel's result [13], which provides a measure estimate like (22), also with the exponent $(K-3)/2$. But we point out that the result given in [13] does not come from a quantitative version of Birkhoff theorem, and hence it is valid "for r small enough", without imposing any explicit condition like (20). We show in the next section that such a condition is crucial in order to obtain an exponentially small estimate for the measure of the complement.

4 The Diophantine and quasi-Diophantine cases

We now assume that the frequency vector λ satisfies a Diophantine condition, with some exponent τ. In this case, we prove that the parameter K may be chosen as a function of r. We then get, for the complement of the set filled with the invariant tori of the normal form $\mathcal{H}^{(K)}$ in a neighbourhood of radius r, an estimate which is exponentially small in $(1/r)^{1/(\tau+1)}$. The fact that the transformation $\Psi^{(K)}$ is canonical allows us to ensure that this estimate also holds for the complement of the invariant tori of the original Hamiltonian H.

Theorem 4 *Let* $H(x,y)=\sum_{s\geq 2} H_s(x,y)$ *be a real Hamiltonian with* $H_2=\lambda\cdot I$, *and assume* $\|H_s\|\leq c^{s-2}d$ *for* $s\geq 2$. *Assume that the vector* λ *is* τ,γ*-Diophantine, with* $\tau\geq n-1$ *and* $\gamma>0$. *Assume also that the isoenergetic nondegeneracy condition (18) holds, with* A *as in (17). For some constants* c_4 *and* c_5 *depending only on* n, c, d, λ, A, *if*

$$0<r\leq \frac{c_4\gamma}{4^{\tau+1}}\ , \tag{32}$$

then there exists a set $T_r\subset\mathcal{B}_r$ *such that every point of* T_r *belongs to an* n*-dimensional invariant torus of* H, *and one has the bound*

$$\mathrm{mes}\,[\mathcal{B}_r\setminus T_r]\leq c_5\,\exp\left\{-\frac{1}{16}\left(\frac{c_4\gamma}{r}\right)^{1/(\tau+1)}\right\}\cdot\mathrm{mes}\,\mathcal{B}_r\ . \tag{33}$$

Proof Since λ is τ,γ-Diophantine, one has $\alpha_s\geq\gamma/s^\tau$ for every $s>0$. Let $K\geq 4$ to be chosen. Applying proposition 1, we obtain a canonical transformation $\Psi^{(K)}$ such that $\mathcal{H}^{(K)}=H\circ\Psi^{(K)}$ is in Birkhoff normal form up to degree K. The transformation $\Psi^{(K)}$ is analytic on $\widehat{\mathcal{B}}_{r^*_K}$, with

$$r^*_K\geq\frac{\gamma}{548ncd\,K^{\tau+1}}\ . \tag{34}$$

By part (c) of proposition 1, we have $\Psi^{(K)}(\mathcal{B}_{2r})\supset\mathcal{B}_r$ if $r\leq r^*_K/2$. We shall apply theorem 3 with $2r$ instead of r in order to get invariant tori of $\mathcal{H}^{(K)}$ on $\mathcal{B}_{2r}$. Many of these tori will give, through the transformation $\Psi^{(K)}$, invariant tori of H on $\mathcal{B}_r$. Indeed, we assume

$$2r\leq c_1' r^*_K, \tag{35}$$

with $c_1'=\min(c_1,1/7e)$. Applying theorem 3 (taking some $\tau'>n-1$ instead of τ, for instance $\tau'=n$), we get a subset $\mathcal{T}^{(K)}_{2r}\subset\mathcal{B}_{2r}$ filled with invariant tori of $\mathcal{H}^{(K)}$, and satisfying the estimate:

$$\mathrm{mes}\,\left[\mathcal{B}_{2r}\setminus\mathcal{T}^{(K)}_{2r}\right]\leq c_3\,e^{-(K-3)/2}\cdot\mathrm{mes}\,\mathcal{B}_{2r}\leq c_3\,e^{-K/8}\cdot\mathrm{mes}\,\mathcal{B}_{2r}\ .$$

Taking $T_r:=\Psi^{(K)}\left(\mathcal{T}^{(K)}_{2r}\right)\cap\mathcal{B}_r$, we have $\Psi^{(K)}\left(\mathcal{B}_{2r}\setminus\mathcal{T}^{(K)}_{2r}\right)\supset\mathcal{B}_r\setminus T_r$. Using that $\Psi^{(K)}$ is measure-preserving, we obtain

$$\mathrm{mes}\,[\mathcal{B}_r\setminus T_r]\leq\mathrm{mes}\,\left[\mathcal{B}_{2r}\setminus\mathcal{T}^{(K)}_{2r}\right]\leq c_3\,e^{-K/8}\cdot\mathrm{mes}\,\mathcal{B}_{2r}\ . \tag{36}$$

Since the only restriction on K is the inequality (35), we choose $K=\left[(c_4\gamma/r)^{1/(\tau+1)}\right]$ with $c_4=c_1'/1096ncd$ (we use the notation $[\cdot]$ for the integer part of a real number). Note that condition (32) guarantees that $K\geq 4$. With our choice of K, we easily get from (36) the bound (33). $\square$

In the frame of theorem 4, another remarkable fact which can be deduced from theorem 3 is that the frequencies of the invariant tori in $\mathcal{B}_r$ are τ',σ^*_r-Diophantine, with some $\tau'>n-1$ (which can be different from τ), and

$$\sigma^*_r\preceq e^{-(K-3)/2}\sim\exp\left\{-\left(\frac{\gamma}{r}\right)^{1/(\tau+1)}\right\}.$$

This indicates that most of the invariant tori obtained for r small are very fragile, in the sense that a very small perturbation of the Hamiltonian would destroy them.

An important question from a practical point of view, which was proposed to us by M.V. Matveyev, is whether the exponentially small estimates are still valid if the frequency vector λ is not exactly Diophantine. In fact, if λ is known only approximately it cannot be decided if it satisfies a Diophantine condition. Nevertheless, if λ is "quasi-Diophantine", we can still expect good measure estimates for the complement of the invariant tori.

Note that, if we know λ up to a precision $\delta > 0$ (i.e. we know an approximation λ', with $|\lambda' - \lambda| \leq \delta$), then it has no sense to check the Diophantine condition (3) beyond a certain finite order. Given τ, γ and δ, we say λ to be τ, γ-*Diophantine up to precision* δ if

$$|k \cdot \lambda| \geq \frac{\gamma}{|k|_1^\tau} \qquad \forall k \in \mathbf{Z}^n,\ 0 < |k|_1 \leq N, \tag{37}$$

where

$$N = N(\tau, \gamma, \delta) := \left[\left(\frac{\gamma}{\delta} \right)^{1/(\tau+1)} \right].$$

For this definition, the restriction $\tau \geq n - 1$ is not necessary. Note that if λ' is an approximation to λ with $|\lambda' - \lambda| \leq \delta$ and

$$|k \cdot \lambda'| \geq \frac{2\gamma}{|k|_1^\tau} \qquad \forall k \in \mathbf{Z}^n,\ 0 < |k|_1 \leq N, \tag{38}$$

then we can deduce that λ is τ, γ-Diophantine up to precision δ.

The next theorem gives estimates for the quasi-Diophantine case. The only significative difference with respect to theorem 4 is that, for very small values of r (of the order of the precision δ), we cannot choose the parameter K larger than N.

Theorem 5 *Consider the same situation of theorem 4, but assuming only that λ is τ, γ-Diophantine up to precision δ. Define*

$$\tilde{r} := \max(r, c_4 \delta)$$

and assume

$$0 < \tilde{r} \leq \frac{c_4 \gamma}{4^{\tau+1}}\ .$$

Then, one has the bound

$$\operatorname{mes}\, [\mathcal{B}_r \setminus T_r] \leq c_5 \exp\left\{ -\frac{1}{16} \left(\frac{c_4 \gamma}{\tilde{r}} \right)^{1/(\tau+1)} \right\} \cdot \operatorname{mes} \mathcal{B}_r\ . \tag{39}$$

Proof We proceed as in theorem 4. By (37), one has $\alpha_s \geq \gamma / s^\tau$ for $0 < s \leq N$. Let K to be chosen, with $4 \leq K \leq N$. The inequalities (34–35) and the restriction $K \leq N$ lead us to take $K = \left[(c_4 \gamma / \tilde{r})^{1/(\tau+1)} \right]$, and we get the bound (39). □

It follows from this theorem that exponentially small estimates in $1/r$, like those of theorem 4, can be ensured in an "annulus" centred at the elliptic point. They hold for

$$c_4 \delta \leq r \leq \frac{c_4 \gamma}{4^{\tau+1}}$$

but not for $r < c_4\delta$, so they cannot be considered as asymptotic estimates. The relative width of this annulus is given by

$$\beta = \beta(\tau, \gamma, \delta) := 1 - \frac{4^{\tau+1}\delta}{\gamma} .$$

Note that, if $\delta \ll \gamma$, the neighbourhood where these estimates do not hold is not relevant, since its radius is of the order of the precision. Thus, in the quasi-Diophantine case we can still say that exponentially small estimates (in $1/r$) hold for practical purposes, since such estimates are not essentially modified by the fact that the frequency vector is not exactly Diophantine.

As an illustration, we consider in the tridimensional Restricted Three Body Problem a neighbourhood of the equilibrium point L_4 (see for instance [6, 15]). If μ denotes the mass parameter, the associated frequencies $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ satisfy the characteristic equation

$$\left(x^4 - x^2 + \frac{27}{16} - a^2\right)\left(x^2 - 1\right) = 0,$$

with

$$a = -\frac{(1-2\mu)3\sqrt{3}}{4} .$$

In the Sun–Jupiter case, we have $\mu \approx 1048.355^{-1} \approx 0.95387536 \cdot 10^{-3}$ and then

$$\lambda \approx \lambda' = (0.996757526, -0.080463876, 1).$$

Assuming μ given with precision 10^{-9}, we easily see that λ is known up to precision $\delta \sim 4 \cdot 10^{-8}$. After some computation, and taking into consideration (38), we have obtained values of γ, N and β for several values of τ.

τ	γ	N	β
0	$0.114445 \cdot 10^{-4}$	286	0.986019
0.5	$0.182052 \cdot 10^{-3}$	274	0.998242
1	$0.175564 \cdot 10^{-2}$	209	0.999635
1.5	$0.458555 \cdot 10^{-2}$	105	0.999721
2	$0.648495 \cdot 10^{-2}$	54	0.999605
2.5	$0.917110 \cdot 10^{-2}$	34	0.999442

We see that, for the values of τ considered, we get β quite near to 1. However, for a big value of δ one should be careful in the choice of τ in order to get β as large as possible.

Finally, we mention that quantitative estimates of the inner and outer radius of the annulus where our estimates are exponentially small in $1/r$ would require explicit knowledge of the constant c_4. This could be done reviewing thoroughly the bounds given in the present paper. However, these bounds have been carried out always considering the worst possible case. It is clear that, in concrete examples, the explicit computation of the Birkhoff normal form would give much better results.

A Estimates for the Giorgilli–Galgani algorithm

Let us recall the Giorgilli–Galgani algorithm as presented in [6]. For a given "generating" function $\chi = \sum_{s\geq 3} \chi_s$ (the subscripts denote the degrees of homogeneous polynomials), one defines a linear operator T_χ in the following way: if $f = \sum_{s\geq 1} f_s$, then

$$T_\chi f = \sum_{s\geq 1} F_s\,, \tag{40}$$

where

$$F_s = \sum_{l=1}^{s} f_{l,s-l} \tag{41}$$

and

$$f_{l,0} = f_l\,, \qquad f_{l,s} = \sum_{j=1}^{s} \frac{j}{s} L_{\chi_{2+j}} f_{l,s-j} \tag{42}$$

(the Poisson bracket is denoted $L_g f = \{g, f\}$). As pointed out in [6], the operator T_χ induces a canonical transformation, which can be written as $(X, Y) \mapsto (x, y)$, with

$$x_j = T_\chi X_j, \qquad y_j = T_\chi Y_j, \qquad j = 1, \ldots, n. \tag{43}$$

These equations are formal. However, if the χ_s satisfy suitable estimates then the corresponding series are convergent and the transformation (43) is analytic in a neighbourhood of the origin.

Given the Hamiltonian $H = \sum_{s\geq 2} H_s$ with $H_2 = \lambda \cdot I$, and assuming that λ is nonresonant up to order K, one can construct a generating function $\chi^{(K)} = \sum_{s=3}^{K} \chi_s$ such that $\mathcal{H}^{(K)} = T_{\chi^{(K)}} H$ is formally in Birkhoff normal form up to order K. Writing $\mathcal{H}^{(K)}$ as in (6), the following homological equations have to be satisfied:

$$L_{H_2} \chi_s + \mathcal{Z}_s = F_s\,, \qquad s \geq 3, \tag{44}$$

where

$$F_3 = H_3\,, \tag{45}$$

$$F_s = \sum_{l=1}^{s-3} \frac{l}{s-2} L_{\chi_{2+l}} \mathcal{Z}_{s-l} + \sum_{l=1}^{s-2} \frac{l}{s-2} H_{2+l,s-l-2}\,, \qquad s \geq 4. \tag{46}$$

We next give the quantitative lemmas required for the proof of proposition 1. The first lemma improves the results of lemma 3.2 and propositions 3.3 and 3.4 of [6].

Lemma 6 *Let* $\chi = \sum_{s\geq 3} \chi_s$, *with the hypothesis*

$$\|\chi_s\| \leq \frac{a^{s-3} b}{\beta_3 \cdots \beta_s} \qquad \forall s \geq 3,$$

where β_s *is a decreasing sequence of positive numbers. Let* $f = \sum_{s\geq 1} f_s$ *be given such that* $\|f_s\| \leq c^{s-1} d$ *for* $s \geq 1$. *Then, for the scheme described in (40–42) one has:*

a) $\|f_{l,s}\| \leq C_{l,s} \|f_l\|$ *for* $l, s \geq 1$, *where* $\displaystyle C_{l,s} = \frac{3b\binom{l+s-1}{s}\left(3b + \frac{8}{3}a\right)^{s-1}}{\beta_3 \cdots \beta_{s+2}}$.

b) $\|F_s\| \le \dfrac{\left(3b+\frac{8}{3}a+\beta_3 c\right)^{s-1} d}{\beta_3\cdots\beta_{s+1}}$ *for* $s\ge 1$.

c) *Assuming that for some* $K\ge 3$ *one has* $\beta_s=\beta_K$ *for every* $s\ge K+1$ *(for instance if* χ *is a polynomial of degree* K*) and writing*

$$r^* = \frac{\beta_K}{470nb+13a}\ ,$$

the canonical transformation Ψ *introduced in (43) is analytic on* $\widehat{\mathcal{B}}_{r^*}$ *and, for any* $r\le r^*$*, one has the inclusion* $\Psi(\mathcal{B}_r)\supset\mathcal{B}_{r/2}$.

Proof We point out that parts (a) and (b) run as in [6]; so we do not prove them here. To see (c), write the transformation formally defined in (43) as $\Psi=\left(\Psi^{(1)},\dots,\Psi^{(2n)}\right)$. If $Z^{(j)}$ denotes, for $j=1,\dots,2n$, the coordinates $X_1,\dots,X_n,Y_1,\dots,Y_n$, one can write

$$\Psi^{(j)}=T_\chi Z^{(j)}=Z^{(j)}+\sum_{s\ge 2}\Psi_s^{(j)},\qquad j=1,\dots,2n, \tag{47}$$

where every $\Psi_s^{(j)}$ is a homogeneous polynomial of degree s in (X,Y). One has:

$$\left\|\Psi_s^{(j)}\right\|\le C_{1,s-1}=\frac{3b\left(3b+\frac{8}{3}a\right)^{s-2}}{\beta_3\cdots\beta_{s+1}}\le\frac{3b}{\beta_3}\left(\frac{3b+\frac{8}{3}a}{\beta_K}\right)^{s-2}.$$

It follows that, for $r\le r_0=\dfrac{\beta_K}{6b+\frac{16}{3}a}$,

$$\left|\Psi^{(j)}-Z^{(j)}\right|_r\le\sum_{s\ge 2}\left\|\Psi_s^{(j)}\right\|r^s\le\frac{6b}{\beta_3}r^2. \tag{48}$$

We deduce that the series (47) are convergent on $\widehat{\mathcal{B}}_{r_0}$ and hence Ψ is analytic on this neighbourhood.

We are going to prove the inclusion $\Psi(\mathcal{B}_r)\supset\mathcal{B}_{r/2}$ from the fact that Ψ is near to the identity, applying lemma 10. Using the Cauchy inequalities and that $r\le r^*\le r_0/\left(1+\sqrt{2}\right)$, we get the bounds

$$\left|\frac{\partial\Psi^{(j)}}{\partial Z^{(\nu)}}-\delta_{j,\nu}\right|_r\le\frac{1}{r}\left|\Psi^{(j)}-Z^{(j)}\right|_{(1+\sqrt{2})r},\qquad\left|\frac{\partial^2\Psi^{(j)}}{\partial Z^{(\nu)}\,\partial Z^{(\nu')}}\right|_r\le\frac{2}{r^2}\left|\Psi^{(j)}-Z^{(j)}\right|_{(1+\sqrt{2})r}. \tag{49}$$

We obtain from (48–49) the following bounds for Ψ and its total derivatives:

$$\begin{aligned}
|\Psi-\mathrm{id}|_r &\le\sqrt{2n}\,\frac{6b}{\beta_3}\left(\left(1+\sqrt{2}\right)r\right)^2\le\frac{50\sqrt{n}\,b}{\beta_3}r^2,\\
|D\Psi-\mathrm{Id}|_r &\le\frac{2n}{r}\,\frac{6b}{\beta_3}\left(\left(1+\sqrt{2}\right)r\right)^2\le\frac{70nb}{\beta_3}r,\\
\left|D^2\Psi\right|_r &\le\frac{2(2n)^{3/2}}{r^2}\,\frac{6b}{\beta_3}\left(\left(1+\sqrt{2}\right)r\right)^2\le\frac{198n^{3/2}b}{\beta_3}.
\end{aligned}$$

We remark that we are using the Euclidean norm for vectors and matrices, because this is the norm in which lemma 10 has been stated. To apply lemma 10 on the domain $\mathcal{B}_r$, we can consider the following parameters:

$$\varepsilon = \frac{50\sqrt{n}\,b}{\beta_3}\,r^2\,, \quad M = m = 1\,, \quad \tilde{M} = 1 + \frac{70nb}{\beta_3}\,r\,, \quad \tilde{m} = 1 - \frac{70nb}{\beta_3}\,r\,, \quad \tilde{M}' = \frac{198n^{3/2}b}{\beta_3}\,.$$

With these parameters, the smallness condition of lemma 10 is easily verified. Then, we obtain for $r \le r^*$ the inclusion

$$\Psi\left(\mathcal{B}_r\right) \supset \mathcal{B}_r - \frac{200\sqrt{n}\,b\,r^2}{\beta_3\left(1 - \frac{70nb\,r}{\beta_3}\right)} \supset \mathcal{B}_{r/2}.$$

□

Next we give estimates for the procedure leading to the normal form, introduced in (44–46), improving the results contained in proposition 5.1 of [6].

Lemma 7 *Let $H = \sum_{s\ge 2} H_s$, with $H_2 = \lambda\cdot I$, and assume that $\|H_s\| \le c^{s-2}d$ for $s \ge 2$. Assume that λ is nonresonant up to order K and let α_s, for $s \le K$, be lower bounds for the small divisors as in (9). Then, for the scheme (44–46) one has, for $s = 3,\ldots,K$,*

$$\|F_s\| \le \frac{\frac{1}{6}(6cd)^{s-2}(s-2)!}{\alpha_3\cdots\alpha_{s-1}}\,. \tag{50}$$

Moreover,

$$\|\chi_s\| \le \frac{1}{\alpha_s}\,\|F_s\|\,, \qquad \|\mathcal{Z}_s\| \le \|F_s\|\,. \tag{51}$$

Proof It is enough to prove (50), since it implies the the inequalities (51) in view of the well-known resolution of the homological equation (44). We look for positive numbers $\theta_{l,s}$, η_s such that

$$\|H_{2+l,s}\| \le \frac{\theta_{l,s}cd}{\alpha_3\cdots\alpha_{s+2}}\,, \qquad l \ge 1,\ s \ge 0, \tag{52}$$

and

$$\|F_{2+s}\| \le \frac{\eta_s cd}{\alpha_3\cdots\alpha_{s+1}}\,, \qquad s \ge 1. \tag{53}$$

Like in [6, proposition 5.1] it is easy to see, by induction, that we can take $\theta_{l,0} = c^{l-1}$, $\eta_1 = 1$, and

$$\theta_{l,s} = \frac{cd}{s}\sum_{j=1}^{s} j(2+j)(2+l+s-j)\eta_j\theta_{l,s-j}\,, \qquad l \ge 1,\ s \ge 1,$$

$$\eta_s = \frac{cd}{s}\sum_{j=1}^{s-1} j(2+j)(2+s-j)\eta_j\eta_{s-j} + \frac{1}{s}\sum_{j=1}^{s} jd^{j-1}\theta_{j,s-j}\,, \qquad s \ge 2.$$

The main difference with respect to [6] is that the α_s have now been included directly in (52–53) as denominators and not inside the $\theta_{l,s}$, η_s. Proceeding like in [6], one sees that

$$\eta_s \le d^{s-1}c^{s-1}b_s,$$

where b_s denotes a sequence satisfying

$$b_s \le 6^{s-1} s! \qquad \forall s \ge 1 .$$

It then suffices to put these inequalities together. □

Using the two previous lemmas, we are able to give estimates for the Birkhoff normal form, including the terms of the remainder, as in theorem 5.5 of [6].

Proof of proposition 1 We recall that part (a) has already been stated in lemma 7. To get parts (b) and (c), we apply lemma 6 to the function H, taking $\chi^{(K)}$ as the generating function. We consider in that lemma the values c, d/c, $6cd$, cd instead of c, d, a, b, respectively, and

$$\beta_s = \frac{\alpha_s}{s-2} \qquad \text{for } \ 3 \le s \le K,$$
$$\beta_s = \beta_K \qquad \text{for } \ s \ge K+1,$$

as provided by lemma 7. In this way, we get

$$\left\| \mathcal{R}_s^{(K)} \right\| \le \frac{\left(3b + \frac{8}{3}a + \beta_3 c\right)^{s-1} \frac{d}{c}}{\beta_3 \cdots \beta_{s+1}}$$

for $s \ge K+1$. Using the identity

$$\beta_3 \cdots \beta_{s+1} = \beta_3 \cdots \beta_{K-1} \beta_K^{s-K+2} = \frac{\alpha_3 \cdots \alpha_{K-1} \alpha_K^{s-K+2}}{(K-3)!(K-2)^{s-K+2}}$$

and also the fact that $\beta_3 \le d$, we may arrange the bound on $\left\| \mathcal{R}_s^{(K)} \right\|$ and we get (b). Finally, the assertion of part (c) is deduced taking r_K^* somewhat smaller than the value given by lemma 6. □

B Isoenergetic nondegeneracy: technical results

We now include some lemmas concerning the isoenergetic nondegeneracy. The first one gives estimates for the local diffeomorphism introduced in (14). For its proof (and a thorough motivation to the constant a), see [4].

Lemma 8 *Let h be a real function of class $\mathcal{C}^3$ on $G \subset \mathbf{R}^n$, and $\omega = \nabla h$. Assume the bounds:*

$$\left| \frac{\partial^2 h}{\partial I^2} \right|_G \le M, \qquad \left| \frac{\partial^3 h}{\partial I^3} \right|_G \le M', \qquad |\omega|_G \le L \qquad \textit{and} \qquad |\omega_n(I)| \ge l \ \ \forall I \in G.$$

Assume also that ω is μ-isoenergetically nondegenerate on G. Let $a \ge 2M/l^2$ a fixed constant, and denote $\Omega = \Omega_{\omega,h,a}$. One has:

a) $\displaystyle \left| \frac{\partial \Omega}{\partial I} \right|_G \le 2La.$

b) $\left|\dfrac{\partial\Omega}{\partial I}(I)\,v\right| \geq \dfrac{\mu}{2L}\,|v| \quad \forall v\in\mathbf{R}^n,\ \forall I\in G.$

c) $\left|\dfrac{\partial^2\Omega}{\partial I^2}\right|_G \leq \left(\dfrac{M'}{2M}+\dfrac{3M}{l}\right) La.$

The next result establishes how a perturbation on the frequency map affects the constant μ of condition (13). See the proof in [4].

Lemma 9 *Let* $\lambda,\tilde\lambda\in\mathbf{R}^n$, *and let* $A,\tilde A$ *be* $(n\times n)$*-matrices. Assume* $\left|\tilde\lambda-\lambda\right|\leq\varepsilon$, $\left|\tilde A-A\right|\leq\varepsilon'$, *and* $l\leq\min\left(|\lambda|,\left|\tilde\lambda\right|\right)$, $M\geq\max\left(|A|,\left|\tilde A\right|\right)$. *For some* $\mu>0$, *assume that*

$$|Av+\xi\,\lambda|\geq\mu\,|v| \qquad \forall v\in\langle\lambda\rangle^{\perp},\ \forall\xi\in\mathbf{R}.$$

Then,

$$\left|\tilde Av+\xi\,\tilde\lambda\right|\geq\left(\mu-\frac{4M\varepsilon}{l}-\varepsilon'\right)|v| \qquad \forall v\in\left\langle\tilde\lambda\right\rangle^{\perp},\ \forall\xi\in\mathbf{R}.$$

The last lemma says that a small perturbation of a one-to-one map is also one-to-one provided its domain is slightly restricted. The proof is essentially given in [4] (see also [9]).

Lemma 10 *Let* $G\subset\mathbf{R}^n$ *a compact, and let* $\Omega\,,\tilde\Omega: G\longrightarrow\mathbf{R}^n$ *maps of class* $\mathcal{C}^2$, *with* $\left|\tilde\Omega-\Omega\right|_G\leq\varepsilon$. *Assume that* Ω *is one-to-one on* G, *and let* $F=\Omega(G)$. *Assume the bounds:*

$$\left|\frac{\partial\Omega}{\partial I}\right|_G\leq M, \qquad \left|\frac{\partial\tilde\Omega}{\partial I}\right|_G\leq\tilde M, \qquad \left|\frac{\partial^2\tilde\Omega}{\partial I^2}\right|_G\leq\tilde M',$$
$$\left|\frac{\partial\Omega}{\partial I}(I)\,v\right|\geq m\,|v|, \qquad \left|\frac{\partial\tilde\Omega}{\partial I}(I)\,v\right|\geq\tilde m\,|v| \quad \forall v\in\mathbf{R}^n,\quad \forall I\in G,$$

with $0<\tilde m<m$, $\tilde M>M$. *Assume also that*

$$\varepsilon\leq\frac{\tilde m^2}{4\tilde M'}\ .$$

Define $\tilde F=F-\dfrac{4M\varepsilon}{\tilde m}$, $\tilde G=\left(\tilde\Omega\right)^{-1}\left(\tilde F\right)$. *One has:*

a) $\tilde\Omega$ *is one-to-one on* $\tilde G$, *and* $\tilde\Omega\left(\tilde G\right)=\tilde F$.

b) $G-\dfrac{5M\varepsilon}{m\tilde m}\subset\tilde G\subset G-\dfrac{2\varepsilon}{\tilde m}\,.$

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