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Ergodic properties of quantized toral automorphisms

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Abstract

We study the ergodic properties for a class of quantized toral automorphisms, namely the cat and Kronecker maps. The present work uses and extends the results of [10]. We show that quantized cat maps are strongly mixing, while Kronecker maps are ergodic and non-mixing. We also study the structure of these quantum maps and show that they are effected by unitary endomorphisms of a suitable vector bundle over a torus. The fiberwise parts of these endomorphisms form a family of finite dimensional quantizations, parameterized by the points of a torus, which includes the quantization proposed in [9].

I. Introduction

I.A. Quite distinct from its classical counterpart, there remains as yet no well-accepted concept of quantum ergodicity. Several inequivalent yet very natural approaches have been introduced. On the one hand, a system is deemed "quantum ergodic" if it has a well-defined classical limit which is itself ergodic [14],[15], [13], [4], [7]. On the other hand, the original notion of quantum ergodicity proposed by von Neumann defines, roughly speaking, a system as quantum ergodic if any observable is eventually distributed over the eigenstates according to the weight of each eigenstate.

Let us discuss this latter notion first. Let $\mathcal{H}$ be a Hilbert space, $\mathfrak{A}$ a *-algebra of operators on $\mathcal{H}$, and F a unitary quantum evolution operator (called also the propagator). Then the quantum system is "quantum ergodic" if for all observables $A \in \mathfrak{A}$, and any $\varphi \in \mathcal{H}$,

$$\lim_{M\to\infty} \frac{1}{M} \sum_{0\leq m\leq M-1} (\varphi, F^m A F^{-m} \varphi) = \sum_{n=0}^{\infty} |c_n|^2 (\varphi_n, A\varphi_n), \tag{I.1}$$

where c_n is the n-th Fourier coefficient of a vector φ with respect to the eigenstates of F spanning $\mathcal{H}, \varphi = \sum_n c_n \varphi_n$. Although the physical motivation behind this definition is indeed appealing, it leads unfortunately to quite unexpected and somewhat counter-intuitive results. First of all, it applies only to systems whose propagators have purely discrete spectra. Furthermore, it can be readily shown that any system whose propagator spectrum is simple (e.g. the one dimensional harmonic oscillator with generic frequency) is, as a consequence of this definition, quantum ergodic.

In [13], [4], [14], a system is defined as quantum ergodic if the time average (which is essentially the left hand side of (I.1)), smears the quantum mechanics onto a "classical limit state" plus a quantum mechanical correction which vanishes asymptotically in the classical limit. The existence of such a state is a highly non-trivial result and often a quantum system will not have such a "classical limit". In [14], quantum ergodicity of a class of quantum dynamical systems, called "Gelfand-Segal systems" are studied. By definition, a Gelfand-Segal systems has a propagator whose spectrum is discrete. This concept of quantum ergodicity seems to be particularly useful in systems which arise as quantizations of the geodesic flow on a compact manifold. For a discussion of quantized toral automorphisms within this framework, see [15] and [3].

I.B. In this paper we study the ergodic properties of a class of quantum dynamical systems whose spectra are continuous. The examples we discuss are the quantized cat and Kronecker maps. We work within the algebraic quantization scheme which emphasizes the role of observables in quantum kinematics and dynamics. In the context of toral automorphisms, such a scheme was discussed in [10]. That paper contains also an extensive list of references to the original literature concerning quantized toral automorphisms and algebraic quantization. A particularly natural and convenient choice of the algebra of observables turns out to be the $\mathbb{C}^*$-algebra $\mathfrak{A}_\hbar$ generated by Toeplitz operators $T_\hbar(f)$ on the Bargmann space with $\mathbb{Z}^2$-invariant symbols f. These Toeplitz operators are simply anti-Wick ordered quantizations of classical observables. The two properties which make Toeplitz quantization very natural are: (i) Toeplitz quantization is positivity preserving,

$$T_\hbar(f) \geq 0, \quad \text{if } f \geq 0, \tag{I.2}$$

and (ii) Toeplitz quantization is continuous in the symbol,

$$\|T_\hbar(f)\| \leq \|f\|_\infty, \tag{I.3}$$

where $\|\cdot\|$ is the operator norm, and where $\|\cdot\|_\infty$ is the sup norm. These properties have important consequences for the study of the semiclassical limit of the quantum system.

The results established in this paper have the form

$$\lim_{M\to\infty} \frac{1}{M} \sum_{0\leq m\leq M-1} F^m A F^{-m} = \tau_\hbar(A) I, \tag{I.4}$$

where A is an element of $\mathfrak{A}_\hbar$, and where $\tau_\hbar$ is a trace on $\mathfrak{A}_\hbar$. This trace is invariant under the quantum dynamics and reduces to the classical ensemble average in the limit as $\hbar \to 0$. It can be thus thought of as the quantum ensemble average. The limit in (I.4) is in the sense of weak topology.

I.C. The paper is organized as follows. In section II we present the classical maps, briefly review some of the results of [10] relevant to this paper, in particular the quantum time evolution operator in Bargmann space, and introduce a trace $\tau_\hbar$ on the algebra of observables. The quantization of the dynamics derived in [10] together with the concept of the trace enable us to show ergodicity and mixing of the cat map in Section III. That is, (i) the time average of an observable converges weakly in the large time limit to the trace of that observable, and (ii) for observables A and B, $\tau_\hbar(F^M A F^{-M} B)$ converges in the large time limit to the product $\tau_\hbar(A)\tau_\hbar(B)$. Related results, within a different quantization scheme, had previously been discussed in [2]. In section IV we show that the quantum Kronecker map is ergodic but not mixing. In section V, we study the structure of the quantized cat maps. For the values of Planck's constant satisfying the geometric quantization "integrality condition", we construct an isomorphism between the Bargmann space and the Hilbert space of sections of a vector bundle over the torus. Under this isomorphism, the algebra automorphism defining the quantum cat map becomes a unitary vector bundle endomorphism. This yields a family of finite dimensional quantizations of the cat dynamics parameterized by the points on the torus. A particular element of this family reduces to the quantization scheme proposed originally in [9]. Section VI contains similar results for the quantized Kronecker maps.

II. Quantized toral automorphisms

II.A. We begin here with a brief review of the systems we shall study. We restrict ourselves to two of the simplest and well known maps of the torus: Arnold's cat map and the Kronecker map. For a more thorough treatment we refer the reader to [1], [5].

The cat map is a linear automorphism of the torus, with one step classical evolution represented by an element $\gamma \in SL(2,\mathbb{Z})$,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{II.1}$$

It can be readily verified that $|\operatorname{tr}(\gamma)| > 2$ corresponds to uniformly hyperbolic dynamics, while $|\operatorname{tr}(\gamma)| < 2$ yields elliptic motion. Since we are interested in chaotic dynamics, we restrict ourselves to $|\operatorname{tr}(\gamma)| > 2$. In this case, the dynamics evolves locally along two linearly independent eigenvectors which are not orthogonal. Indeed, the slopes are irrationally related. The two eigenvalues of (II.1) μ_1 and μ_2 satisfy

$$\mu_1\mu_2 = 1, \tag{II.2}$$

with $|\mu_1| > 1$ and $|\mu_2| < 1$ corresponding to flow along unstable and stable axes, respectively.

As in [10], we write the dynamics

$$(x_1,\ x_2) \longrightarrow (ax_1 + bx_2,\ cx_1 + dx_2) \tag{II.3}$$

in terms of the complex variable $z = (x_1 + ix_2)/\sqrt{2}$ via

$$z \longrightarrow \overline{\alpha}z + \beta\overline{z}. \tag{II.4}$$

The factor of $\sqrt{2}$ in the denominator serves to make the transformation $(x_1,\ x_2) \to z$ a symplectomorphism. Note that α and β are simply the complex cat map parameters, with

$$\begin{aligned} \alpha &= \big(a + d + i(b - c)\big)/2, \\ \beta &= \big(a - d + i(b + c)\big)/2, \end{aligned} \tag{II.5}$$

and

$$|\alpha|^2 - |\beta|^2 = 1. \tag{II.6}$$

The classical Kronecker map is an even simpler automorphism of the torus defined by

$$(x_1, x_2) \longrightarrow (x_1 + \omega_1, x_2 + \omega_2),$$

(or equivalently $z \longrightarrow z + \omega$,) where the frequencies ω_1 and ω_2 are linearly independent over $\mathbb{Z}$, i.e. ω_1/ω_2 is irrational. It is a well known result, see e.g. [5], that the map is ergodic; however because of its simple uniform motion, it is not mixing. Furthermore, if we consider the same map with ω_1/ω_2 rational, then the classical dynamics is no longer ergodic. Rather, it is described by periodic orbits whose lengths are related to how closely ω_1/ω_2 approximates an irrational number.

II.B. We shall use the quantization presented in [10]. For a full account of the method the reader is referred to the original paper. Here we only summarize the results.

We work in a Bargmann representation with a Hilbert space $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$ consisting of entire functions on $\mathbb{C}$ which are square integrable with respect to the measure $d\mu_\hbar(z) = (\pi\hbar)^{-1}\exp(-|z|^2/\hbar)d^2z$. Quantizations of classical functions over phase space (which are "functions" over the quantized phase space) generate naturally a quantum mechanical algebra of observables. Of course, a particular choice of quantization must be given. We choose, as in [10], the anti-Wick quantization, and define the algebra of observables to be the $\mathbb{C}^*$-algebra $\mathfrak{A}_\hbar$ generated by Toeplitz operators. A Toeplitz operator with symbol f is given by

$$T_\hbar(f)\varphi(z) = \int_{\mathbb{C}} e^{z\overline{w}/\hbar} f(w)\varphi(w)d\mu_\hbar(w). \tag{II.7}$$

The Hilbert space $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$ carries a unitary projective representation of the group of translations of $\mathbb{C}$ given by

$$U(\zeta)\varphi(z) = e^{(\overline{\zeta}z - |\zeta|^2/2)/\hbar}\varphi(z - \zeta), \tag{II.8}$$

with the property

$$U(\zeta)U(\xi) = e^{i\mathrm{Im}(\overline{\zeta}\xi)/\hbar}U(\zeta + \xi). \tag{II.9}$$

Consider the following operators:

$$\begin{aligned} U &= U(-i\hbar\pi\sqrt{2}), \\ V &= U(\hbar\pi\sqrt{2}). \end{aligned} \tag{II.10}$$

These operators are generators of the algebra $\mathfrak{A}_\hbar$, and obey the commutation relation

$$UV = e^{i\lambda}VU, \tag{II.11}$$

where $\lambda = 4\pi^2\hbar$.

Quantum cat dynamics in the Bargmann representation is effected by the unitary operator

$$F\varphi(z) = |\alpha|^{-1/2} \exp\big(-\frac{\overline{\beta} z^2}{2\hbar\alpha}\big) \int_{\mathbb{C}} \exp\big(\frac{\overline{w}z}{\hbar\alpha} + \frac{\beta\overline{w}^2}{2\hbar\alpha}\big)\varphi(w)\, d\mu_\hbar(w). \tag{II.12}$$

Indeed, it was shown explicitly that for U and V defined in (II.10),

$$\begin{aligned} U' &= FUF^{-1} = e^{-i\lambda ab/2} U^a V^b, \\ V' &= FVF^{-1} = e^{-i\lambda cd/2} U^c V^d. \end{aligned} \tag{II.13}$$

Furthermore, this F has a well defined $\hbar \to 0$ limit yielding the desired classical dynamics.

For the Kronecker map, the unitary operator which effects the dynamics is readily shown to be $K = U(-\omega)$:

$$\begin{aligned} KUK^{-1} &= e^{2\pi i\omega_1} U, \\ KVK^{-1} &= e^{2\pi i\omega_2} V, \end{aligned} \tag{II.14}$$

where $\omega = (\omega_1 + i\omega_2)/\sqrt{2}$.

II.C. We also need a trace on the algebra $\mathfrak{A}_\hbar$. Let $\varphi \in \mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$ be an arbitrary vector of norm one. For $S \in \mathfrak{A}_\hbar$ we define

$$\tau_\hbar(S) = \int_{\mathbb{T}^2} (U(l)\varphi, SU(l)\varphi) d^2 l. \tag{II.15}$$

This functional has a number of remarkable properties which we summarize in the theorems below.

Theorem II.1. *The functional $\tau_\hbar$ has the following properties:*
1°. *It is a state on $\mathfrak{A}_\hbar$;*
2°. *Its value is given by*

$$\tau_\hbar(U^m V^n) = \delta_{m0}\delta_{n0}; \tag{II.16}$$

3°. *It is a trace on $\mathfrak{A}_\hbar$.*

In particular, $\tau_\hbar$ is independent of the choice of φ and from (II.16) coincides with the standard trace on the quantized torus.

Proof. 1°. Indeed, $\tau_\hbar$ is continuous,

$$|\tau_\hbar(S)| \le \int_{\mathbb{T}^2} \|S\|\,\|U(l)\varphi\|^2 d^2 l = \|S\|,$$

positive,

$$\tau_\hbar(S^\dagger S) = \int_{\mathbb{T}^2} \|SU(l)\varphi\|^2 d^2 l \ge 0,$$

and normalized,

$$\tau_\hbar(I) = \int_{\mathbb{T}^2} (U(l)\varphi, U(l)\varphi) d^2 l = \int_{\mathbb{T}^2} \|\varphi\|^2 d^2 l = 1.$$

2°. This is a direct calculation:

$$\begin{aligned}
\tau_\hbar(U^m V^n) &= \int_{\mathbb{T}^2} (\varphi, U(-l)U(-im\pi\hbar\sqrt{2})U(n\pi\hbar\sqrt{2})U(l)\varphi) d^2 l \\
&= \int_{\mathbb{T}^2} e^{i\pi\sqrt{2}(m\mathrm{Re}l + n\mathrm{Im}l)} (\varphi, U(-l - im\pi\hbar\sqrt{2})U(l + n\pi\hbar\sqrt{2})\varphi) d^2 l \\
&= \int_{\mathbb{T}^2} e^{2i\pi\sqrt{2}(m\mathrm{Re}l + n\mathrm{Im}l) + 2i\pi^2\hbar mn} (\varphi, U((n - im)\pi\hbar\sqrt{2})\varphi) d^2 l \\
&= \int_0^1 \int_0^1 e^{2i\pi(mx+ny) + 2i\pi^2\hbar mn} dxdy \ (\varphi, U((n - im)\pi\hbar\sqrt{2})\varphi) = \delta_{m0}\delta_{n0}.
\end{aligned}$$

3°. This follows from 2°. □

Remark. Notice that if $\varphi \in \mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$ is chosen to be the ground state $\varphi_0 = 1$, then

$$\varphi_l(z) := U(l)\varphi_0(z) = e^{\bar{l}z/\hbar - |l|^2/2\hbar} = e^{-|l|^2/2\hbar} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\bar{l}A^\dagger/\hbar\right)^n \varphi_0(z), \tag{II.17}$$

where $A^\dagger$ is the creation operator. Thus the trace of the operator S is the sum over the coherent states basis representation restricted to the fundamental domain of the expectation value of S.

Another interesting fact about $\tau_\hbar$ is that its value on a Toeplitz operator is equal to the integral of the symbol of that operator. This is quite remarkable in that with our choice of quantization, the quantum expectation value yields exactly the classical value independent of Planck's constant.

Theorem II.2. *For any symbol $f \in C(\mathbb{T}^2)$,*

$$\tau_\hbar((T_\hbar(f)) = \tau(f). \tag{II.18}$$

where $\tau(f)$ is the phase-space integral of f over the torus.

Proof. It is sufficient to prove this for f a pure harmonic. The general case will follow by linearity and continuity. Let

$$\begin{aligned}
f_\zeta(z) &:= e^{2\pi^2\hbar|\zeta|^2} e^{2\pi(\bar{\zeta}z - \zeta\bar{z})} \\
&= e^{\pi^2\hbar(m^2+n^2)} e^{2\pi i(nx+my)},
\end{aligned} \tag{II.19}$$

where $\zeta = (m - in)/\sqrt{2}$, $m, n \in \mathbb{Z}$. Then, using (II.7), we find

$$T_\hbar(f_\zeta) = U^n V^m e^{2\pi^2 i\hbar mn} \qquad \text{(II.20).}$$

Using part 2° of Theorem II.1, we conclude that

$$\tau_\hbar(T_\hbar(f_\zeta)) = \delta_{m0}\delta_{n0} = \int_{\mathbb{T}^2} f_\zeta(z) d^2 z,$$

as claimed. □

III. Ergodic properties of the quantized cat map

III.A. In this section we study the ergodic properties of the quantized cat map. We prove that the dynamics generated by this map has a property which is a quantum mechanical analog of the strong mixing property. Furthermore, we show that the quantized cat dynamics is ergodic in the sense that the time average of an observable tends to its ensemble average given by the trace $\tau_\hbar$. The results established in this section had previously been proved in [2] within a different quantization scheme and by different methods.

Theorem III.1. (Strong mixing) *For any* $A, B \in \mathfrak{A}_\hbar$,

$$\lim_{M\to\infty} \tau_\hbar(F^M A F^{-M} B) = \tau_\hbar(A)\tau_\hbar(B). \tag{III.1}$$

Proof. We proceed in two steps.
Step 1. We assume first that $A = T_\hbar(f_\zeta)$, $B = T_\hbar(f_\eta)$, with f_ζ, f_η of the form (II.19). We have to show that the limit in (III.1) is 1, if $\zeta = \eta = 0$, and 0, otherwise. A direct calculation (see Section III of [10]) shows that

$$F T_\hbar(f_\zeta) F^{-1} = T_\hbar(f_{\gamma^{-1}\zeta}), \tag{III.2}$$

and consequently

$$F^m T_\hbar(f_\zeta) F^{-m} = T_\hbar(f_{\gamma^{-m}\zeta}). \tag{III.3}$$

Furthermore, as a consequence of (II.20),

$$T_\hbar(f_\zeta) T_\hbar(f_\eta) = \epsilon(\zeta, \eta) T_\hbar(f_{\zeta+\eta}), \tag{III.4}$$

where $\epsilon(\zeta, \eta)$ is such that

$$|\epsilon(\zeta, \eta)| = 1, \qquad \epsilon(0, \eta) = \epsilon(\zeta, 0) = 1. \tag{III.5}$$

As a consequence,

$$\begin{aligned}\tau_\hbar(F^M T_\hbar(f_\zeta) F^{-M} T_\hbar(f_\eta)) &= \epsilon(\gamma^{-M}\zeta, \eta)\tau_\hbar(T_\hbar(f_{\gamma^{-M}\zeta+\eta})) \\ &= \epsilon(\gamma^{-M}\zeta, \eta) \int_{\mathbb{T}^2} f_{\gamma^{-M}\zeta+\eta}(z) d^2 z.\end{aligned}$$

If $\zeta = \eta = 0$, then the above expression is equal to 1. For $\zeta = 0$, $\eta \neq 0$, $\int_{\mathbb{T}^2} f_\eta(z) d^2 z = 0$. Let $\zeta \neq 0$, $\eta \neq 0$. Since γ is hyperbolic, there is M_0 such that for all $M \geq M_0$, $\gamma^{-M}\zeta + \eta \neq 0$, and thus $\int_{\mathbb{T}^2} f_{\gamma^{-M}\zeta+\eta}(z) d^2 z = 0$, for all $M \geq M_0$.

Step 2. As a consequence of *Step 1*, (III.1) holds for any $A = A_0 := T_\hbar(f)$ and $B = B_0 := T_\hbar(g)$, where f and g are finite linear combinations of simple harmonics. Any element of $\mathfrak{A}_\hbar$ is a norm limit of such operators. Using the continuity of $\tau_\hbar$ and unitarity of F we obtain the inequality

$$\begin{aligned} |\tau_\hbar(F^M A F^{-M} B) - \tau_\hbar(A)\tau_\hbar(B)| & \\ \leq |\tau_\hbar(F^M A_0 F^{-M} B_0) - \tau_\hbar(A_0)\tau_\hbar(B_0)| & \\ + |\tau_\hbar(F^M A_0 F^{-M}(B - B_0))| + |\tau_\hbar(A_0)||\tau_\hbar(B - B_0)| & \\ + |\tau_\hbar(F^M (A - A_0) F^{-M} B_0)| + |\tau_\hbar(A - A_0)||\tau_\hbar(B_0)| & \\ + |\tau_\hbar(F^M (A - A_0) F^{-M}(B - B_0))| + |\tau_\hbar(A - A_0)||\tau_\hbar(B - B_0)| & \\ \leq |\tau_\hbar(F^M A_0 F^{-M} B_0) - \tau_\hbar(A_0)\tau_\hbar(B_0)| & \\ + 2(\|A_0\|\|B - B_0\| + \|A - A_0\|\|B_0\| + \|A - A_0\|\|B - B_0\|), & \end{aligned}$$

from which (III.1) follows. □

As a corollary, we obtain the following mixing property for observables which are Toeplitz operators.

Corollary III.2. *For $f, g \in C(\mathbb{T}^2)$,*

$$\lim_{m\to\infty} \tau_\hbar\big(F^m T_\hbar(f) F^{-m} T_\hbar(g)\big) = \tau(f)\tau(g). \tag{III.6}$$

Proof. This is a consequence of (II.18). □

III.B. Now we formulate the ergodic theorem for the quantized cat dynamics. For an operator S, define its time average over a period of time M:

$$\langle S\rangle_M := \frac{1}{M} \sum_{0\leq m\leq M-1} F^m S F^{-m}. \tag{III.7}$$

The theorem below asserts that for any $A \in \mathfrak{A}_\hbar$, the sequence $\langle A\rangle_M$ converges to $\tau_\hbar(A)I$ in the weak operator topology.

Theorem III.3. (Ergodicity of the quantized cat map) *For any $A \in \mathfrak{A}_\hbar$, and $\varphi, \psi \in \mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$,*

$$\lim_{M\to\infty} (\varphi, \langle A\rangle_M \psi) = \tau_\hbar(A)(\varphi, \psi). \tag{III.8}$$

Proof. We proceed in three steps.
Step 1. Let $A = T_\hbar(f_\zeta)$, where f_ζ is a simple harmonic. If $\zeta = 0$, then $\langle A\rangle_M = I$, and (III.8) holds trivially. Let $\zeta \neq 0$; we have to show that

$$\lim_{M\to\infty}(\varphi, \langle T_\hbar(f_\zeta)\rangle_M\psi) = 0. \tag{III.9}$$

Assume now that φ and ψ are normalized coherent states of the form (II.17), $\varphi = \varphi_\xi$, $\psi = \varphi_\eta$. Then

$$\begin{aligned}(\varphi_\xi, T_\hbar(f_\zeta)\varphi_\eta) &= \int_{\mathbb{C}} \overline{\varphi_\xi(z)} f_\zeta(z)\varphi_\eta(z)d\mu_\hbar(z)\\ &= e^{-(|\xi|^2-2\xi\overline{\eta}+|\eta|^2)/2\hbar-2\pi^2\hbar|\zeta|^2+2\pi(\overline{\zeta}\xi-\zeta\overline{\eta})},\end{aligned}$$

and so

$$|(\varphi_\xi, T_\hbar(f_\zeta)\varphi_\eta)| \leq e^{-2\pi^2\hbar|\zeta|^2+2\pi(|\xi|+|\eta|)|\zeta|}.$$

Consequently,

$$\begin{aligned}|(\varphi_\xi, \langle T_\hbar(f_\zeta)\rangle_M\varphi_\eta)| &\leq \frac{1}{M}\sum_{0\leq m\leq M-1} e^{-2\pi^2\hbar|\gamma^{-m}\zeta|^2+2\pi(|\xi|+|\eta|)|\gamma^{-m}\zeta|}\\ &\leq \frac{1}{M}\sum_{0\leq m\leq M-1} e^{-O(1)|\mu_1|^{2m}} \longrightarrow 0,\end{aligned}$$

where μ_1 is the eigenvalue of γ with $|\mu_1| > 1$.
Step 2. By means of *Step 1*, (III.9) holds for $\varphi = \varphi_0$ and $\psi = \psi_0$ which are finite linear combinations of coherent states. Any vector in $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$ is a norm limit of such elements. Observe also that the time average of an operator obeys the following inequality:

$$\|\langle S\rangle_M\| \leq \|S\|. \tag{III.10}$$

This leads to the inequality

$$\begin{aligned}|(\varphi, \langle T_\hbar(f_\zeta)\rangle_M\psi)| &\leq |(\varphi_0, \langle T_\hbar(f_\zeta)\rangle_M\psi_0)|\\ &\quad + \|T_\hbar(f_\zeta)\|(\|\varphi-\varphi_0\|\|\psi_0\| + \|\varphi_0\|\|\psi-\psi_0\| + \|\varphi-\varphi_0\|\|\psi-\psi_0\|),\end{aligned}$$

which yields (III.9).
Step 3. As a consequence of *Step 2*, (III.8) holds for any $A = A_0 := T_\hbar(f)$, where f is a finite linear combination of simple harmonics. Any element of $\mathfrak{A}_\hbar$ is a norm limit of such operators. Using the continuity of $\tau_\hbar$ and (III.10) we obtain the inequality:

$$|(\varphi, \langle A\rangle_M\psi) - \tau_\hbar(A)(\varphi,\psi)| \leq |(\varphi, \langle A_0\rangle_M\psi) - \tau_\hbar(A_0)(\varphi,\psi)| + 2\|A-A_0\|\|\varphi\|\|\psi\|,$$

and our claim follows. □

A simple corollary to the above theorem is the following result. It states that the time average of a quantum observable $T_\hbar(f)$ (namely the anti-Wick quantization of the classical observable f) converges weakly to the ensemble average of the classical observable f.

Corollary III.4. *For $f \in C(\mathbb{T}^2)$, and $\varphi, \psi \in \mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$,*

$$\lim_{M\to\infty} (\varphi, \langle T_\hbar(f)\rangle_M \psi) = \tau(f)(\varphi, \psi). \tag{III.11}$$

IV. Ergodic properties of the quantized Kronecker map

We turn now to ergodicity of the Kronecker map. In this section, we let $\langle\,\cdot\,\rangle_M$ denote the time average defined by (III.7), with F replaced by K. First, we prove the ergodic theorem for the quantized Kronecker dynamics. It states that the time averages of an observable converge in norm to its ensemble average (this is a somewhat stronger property than Theorem III.3 which involves weak convergence of time averages).

Theorem IV.1. (Ergodicity of the quantized Kronecker map) *For $A \in \mathfrak{A}_\hbar$,*

$$\lim_{M\to\infty} \|\langle A\rangle_M - \tau_\hbar(A)I\| = 0. \tag{IV.1}$$

In particular, for $f \in C(\mathbb{T}^2)$,

$$\lim_{m\to\infty} \|\langle T_\hbar(f)\rangle_M - \tau(f)I\| = 0. \tag{IV.2}$$

Proof. Let $f = f_\zeta$, where f_ζ is given by (II.19), with the corresponding Toeplitz operator $T_\hbar(f_\zeta)$. Then

$$K^m T_\hbar(f_\zeta) K^{-m} = e^{m\pi\sqrt{2}(\zeta\overline{\omega}-\overline{\zeta}\omega)} T_\hbar(f_\zeta), \tag{IV.3}$$

and so

$$\langle T_\hbar(f_\zeta)\rangle = \frac{1}{M} \sum_{0\le m\le M-1} e^{m\pi\sqrt{2}(\zeta\overline{\omega}-\overline{\zeta}\omega)} T_\hbar(f_\zeta).$$

This is equal to 1 if $\zeta = 0$, while for $\zeta \neq 0$,

$$\|\langle T_\hbar(f_\zeta)\rangle\| \le O(1)\|T(f_\zeta)\|/M = O(1)/M,$$

uniformly in ζ.

Simple continuity arguments similar to *Step 3* in the proof of Theorem III.3 conclude the proof of (IV.1). □

True to its classical origins, while the quantum Kronecker map is ergodic, it is not mixing.

Theorem IV.2. *The quantized Kronecker map is not mixing in the sense of (III.1).*

Proof. We construct a counter-example. We take A to be the Toeplitz operator for the pure harmonic, $A = T_\hbar(f_\zeta)$, where $\zeta \neq 0$, and $B = T_\hbar(f_{-\zeta})$. Then, by means of (IV.3), (II.18), and (III.4),

$$\begin{aligned} |\tau_\hbar(K^m A K^{-m} B) - \tau_\hbar(A)\tau_\hbar(B)| &= |\tau_\hbar(K^m T_\hbar(f_\zeta) K^{-m} T_\hbar(f_{-\zeta})) - \tau(f_\zeta)\tau(f_{-\zeta})| \\ &= |\tau_\hbar(e^{m\pi\sqrt{2}(\zeta\overline{w}-\overline{\zeta}w)} T_\hbar(f_\zeta) T_\hbar(f_{-\zeta}))| \\ &= 1, \end{aligned}$$

uniformly in m. Thus the Kronecker map does not satisfy the mixing condition. □

V. The structure of the quantized cat map

V.A. The quantization method used in [10] and in the present paper is convenient to study global properties of the quantized cat dynamics. For numerical analysis and detailed spectral properties of the dynamics, the previous quantization schemes ([9], [6]) seem more suitable. In this section we establish a connection between our quantization method of the cat dynamics to the previous methods. Indeed, we will see that our scheme yields a continuum of quantizations (similar to those of [9]) parameterized by a point θ on the two-dimensional torus. In the special case of $\theta = 0$ and γ of the structure

$$\begin{pmatrix} odd & even \\ even & odd \end{pmatrix} \text{ or } \begin{pmatrix} even & odd \\ odd & even \end{pmatrix}, \tag{V.1}$$

which was referred to as "quantizable" in [9], we reproduce exactly the result of [9]. In the general case, the angle variable θ parametrizes the different quantizations that result from having a classical phase space with a non-trivial topology. As in these earlier works, we restrict Planck's constant to satisfy the integrality condition

$$\hbar = 1/2\pi N. \tag{V.2}$$

We introduce the following notation:

$$X := U(-i/\sqrt{2}), \qquad Y := U(1/\sqrt{2}), \tag{V.3}$$

and observe that as a consequence of (II.11),

$$[X, Y] = 0. \tag{V.4}$$

The operators X and Y generate an action of the group $\mathbb{Z}^2$ on $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$. We also verify easily that,

$$\begin{aligned} [X,U] &= 0, \qquad [X,V] = 0, \\ [Y,U] &= 0, \qquad [Y,V] = 0, \end{aligned} \tag{V.5}$$

and so X and Y are in the commutant of $\mathfrak{A}_\hbar$. Finally, we note that

$$X = U^N, \qquad Y = V^N. \tag{V.6}$$

V.B. We shall call a holomorphic function ϕ on $\mathbb{C}$ a $\mathbb{Z}^2$-*automorphic form* if

$$\begin{aligned} X\phi(z) &= e^{2\pi i\theta_1}\phi(z), \\ Y\phi(z) &= e^{2\pi i\theta_2}\phi(z), \end{aligned} \tag{V.7}$$

where $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$. In other words, $\mathbb{Z}^2$-automorphic forms are simultaneous generalized eigenvectors of X and Y. Let $\mathcal{H}_\hbar(\theta)$ denote the space of all $\mathbb{Z}^2$-automorphic forms with fixed θ. Clearly, $\phi \in \mathcal{H}_\hbar(\theta)$ is uniquely determined once defined on the fundamental domain $D = [0,1] \times [0,1] \subset \mathbb{R}^2$. The space $\mathcal{H}_\hbar(\theta)$ has a natural inner product defined as an integral over this domain:

$$(\phi_1, \phi_2) = \int_D \overline{\phi_1(z)}\phi_2(z)d\mu_\hbar(z). \tag{V.8}$$

(Note a similar integral over the entire complex plane *does not converge*, hence the $\mathbb{Z}^2$-automorphic forms are not in $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$.) This inner product is a $\mathbb{Z}^2$ version of the familiar Petersson inner product. In the following lemma we construct a natural orthonormal basis for the space $\mathcal{H}_\hbar(\theta)$.

Lemma V.1.

(i) The following functions are elements of $\mathcal{H}_\hbar(\theta)$:

$$\phi_m^{(\theta)}(z) = C_m(\theta)e^{-N\pi z^2+2\sqrt{2}\pi(\theta_1+m)z}\sum_{k\in\mathbb{Z}} e^{-N\pi k^2-2\pi(\theta_1+i\theta_2+m)k+2\sqrt{2}N\pi kz}, \tag{V.9}$$

where

$$C_m(\theta) := (2/N)^{1/4}e^{-\pi(\theta_1+m)^2/N-2\pi i\theta_2 m/N}. \tag{V.10}$$

They are periodic in m,

$$\phi_{m+N}^{(\theta)} = \phi_m^{(\theta)}, \tag{V.11}$$

and furthermore,

$$\phi_0^{(\theta)}, \ldots, \phi_{N-1}^{(\theta)} \tag{V.12}$$

are orthonormal vectors in $\mathcal{H}_\hbar(\theta)$.

(ii) The space $\mathcal{H}_\hbar(\theta)$ has dimension N. Consequently, the functions (V.12) form an orthonormal basis for $\mathcal{H}_\hbar(\theta)$.

Remark. We observe that our basis functions (V.9) can be written in terms of the Jacobi ϑ functions (see for example [12]):

$$\phi_m^{(\theta)}(z) = C_m(\theta)e^{-N\pi z^2+2\sqrt{2}\pi(\theta_1+m)z}\vartheta(-i\sqrt{2}Nz + i(\theta_1+i\theta_2+m), iN),$$

where

$$\vartheta(\omega,\tau) = \sum_{k\in\mathbb{Z}} e^{i\pi k^2\tau+2\pi ik\omega}.$$

Expressions similar to (V.9) have been used before, see e.g. [11], and references therein.

Proof. (i) Usual arguments show that (V.9) converges on compact subsets of $\mathbb{C}$ and thus defines an entire function. It can be readily checked that $X\phi_m^{(\theta)}(z) = e^{2\pi i\theta_1}\phi_m^{(\theta)}(z)$, $Y\phi_m^{(\theta)}(z) = e^{2\pi i\theta_2}\phi_m^{(\theta)}(z)$, and so $\phi_m^{(\theta)} \in \mathcal{H}_\hbar(\theta)$. The periodicity condition (V.11) can be verified easily.

To show that $\phi_m^{(\theta)}$, $0 \le m \le N-1$, form an orthonormal set, we compute

$$\begin{aligned}(\phi_m^{(\theta)}, \phi_n^{(\theta)}) = &\overline{C_m(\theta)}C_n(\theta)\sum_{k,l\in\mathbb{Z}} e^{-N\pi(k^2+l^2)-2\pi(\theta_1-i\theta_2)k-2\pi(\theta_1+i\theta_2)l-2\pi(mk+nl)}\\
&\times N\int_D e^{-N\pi|z+\overline{z}|^2+2\sqrt{2}\pi(\theta_1+m+Nk)\overline{z}+2\sqrt{2}\pi(\theta_1+n+Nl)z}d^2z\\
=&\overline{C_m(\theta)}C_n(\theta)\sum_{k,l\in\mathbb{Z}} e^{-N\pi(k^2+l^2)-2\pi(\theta_1-i\theta_2)k-2\pi(\theta_1+i\theta_2)l-2\pi(mk+nl)}\\
&\times N\int_0^1 e^{-2N\pi x^2+2\pi(2\theta_1+m+n+N(k+l))x}dx\int_0^1 e^{-2\pi i(m-n+N(k-l))y}dy.\end{aligned}$$

Let $m \ne n$. Since both m and n are between 0 and $N-1$, the expression $m-n+N(k-l)$ does not vanish and so $(\phi_m^{(\theta)}, \phi_n^{(\theta)}) = 0$. Let $m = n$. Then

$$\begin{aligned}(\phi_m^{(\theta)}, \phi_m^{(\theta)}) &= N|C_m(\theta)|^2\sum_{k\in\mathbb{Z}} e^{-2N\pi k^2-4\pi(\theta_1+m)k}\int_0^1 e^{-2N\pi x^2+4\pi(\theta_1+m+Nk)}dx\\
&= N|C_m(\theta)|^2\sum_{k\in\mathbb{Z}}\int_0^1 e^{-2N\pi(x-k)^2+4\pi(\theta_1+m)(x-k)}\\
&= N|C_m(\theta)|^2\int_{\mathbb{R}} e^{-2N\pi x^2+4\pi(\theta_1+m)}dx\\
&= N|C_m(\theta)|^2(2N)^{-1/2}e^{2\pi(\theta_1+m)^2/N}\\
&= 1,\end{aligned}$$

and the claim is proved.

(ii) We proceed in steps.

Step 1. We shall first show that $\phi \in \mathcal{H}_\hbar(\theta)$, when considered as a function of z, has exactly N zeros inside any fundamental domain. Observe that $\phi \in \mathcal{H}_\hbar(\theta)$ satisfies

$$\begin{aligned} \phi(z - 1/\sqrt{2}) &= e^{-\sqrt{2}\pi N z + N\pi/2\pi + 2\pi i \theta_2} \phi(z), \\ \phi(z + i/\sqrt{2}) &= e^{-\sqrt{2}\pi N i z + N\pi/2 + 2\pi i \theta_1} \phi(z). \end{aligned} \tag{V.13}$$

Using the argument principle of elementary complex analysis and (V.13), we readily see that ϕ has precisely N zeros inside a fundamental domain.

Step 2. For a torus, the Riemann-Roch theorem [8] can be stated as follows. For any divisor D,

$$r(D^{-1}) = \deg D + i(D). \tag{V.14}$$

Recall that a divisor $D = P_1^{n_1} \ldots P_k^{n_k}$ is the collection of points $P_1, \ldots, P_k$ on the torus, with integers $n_1, \ldots, n_k$ assigned to each point. Note that implicitly we assign to all other points $n = 0$. The inverse D^{-1} of the divisor D is simply $D^{-1} = P_1^{-n_1} \ldots P_k^{-n_k}$. An example of a divisor is the set of zeros ($n_j > 0$) and poles ($n_j < 0$) of a meromorphic function f. We denote such a divisor by (f). Similarly, given a meromorphic 1-form ω we denote its divisor by (ω). An order relation among divisors can be defined as follows: for $D_1 = P_1^{n_1} \ldots P_k^{n_k}$, $D_2 = P_1^{m_1} \ldots P_k^{m_k}$, $D_1 \geq D_2$, if $n_j \geq m_j$, for all j. The degree of a divisor $\deg D$ is defined by $\deg D = \sum_j n_j$. We set $r(D)$ equal to the dimension of the vector space $L(D)$ of meromorphic functions f such that $(f) \geq D$. Likewise, $i(D)$ is the dimension of the space of meromorphic one-forms ω such that $(\omega) \geq D$.

We now take D to be the zero divisor of ϕ. As a consequence of *Step 1*, $\deg D = N$. Furthermore [8], for $\deg D > 0$, $i(D) = 0$. Thus we see that

$$r(D^{-1}) = N. \tag{V.15}$$

Step 3. Notice that for $\phi \in \mathcal{H}_\hbar(\theta)$ and $\phi_m^{(\theta)}$ given by (V.9), the quotients

$$\psi_m(z) := \phi_m^{(\theta)}(z)/\phi(z), \tag{V.16}$$

define meromorphic functions on the torus. By means of part *(i)* of the theorem, they are linearly independent. Since by construction $(\psi_m) \geq D^{-1}$, the result of *Step 2* implies that the set $\{\psi_0, \ldots, \psi_{N-1}\}$ spans $L(D^{-1})$. Suppose now that ϕ is linearly independent of $\phi_0, \ldots, \phi_{N-1}$. It follows that $1, \psi_0, \ldots, \psi_{N-1}$ are linearly independent, which contradicts (V.15). $\square$

Remark. The space $\int_{\mathbb{T}^2}^{\oplus} \mathcal{H}_\hbar(\theta) d\theta$ is thus the Hilbert space of square integrable sections of an N-dimensional vector bundle $Q\mathbb{T}_\hbar^2 \to \mathbb{T}^2$. The bundle $Q\mathbb{T}_\hbar^2$ does not admit a global continuous section and is thus topologically non-trivial.

V.C. In the next lemma, we construct an isomorphism between $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$ and the direct integral of the spaces $\mathcal{H}_\hbar(\theta)$. Under this isomorphism, the actions of U and V are block diagonal. Let $\phi_m \in \int_{\mathbb{T}^2}^{\oplus} \mathcal{H}_\hbar(\theta) d\theta$ be a (discontinuous in θ) element defined by $\phi_m(\theta, z) = \phi_m^{(\theta)}(z),\ 0 \leq \theta_j < 1$.

Lemma V.2. *There is an isomorphism*

$$\kappa :\ \mathcal{H}^2(\mathbb{C}, d\mu_\hbar) \longrightarrow \int_{\mathbb{T}^2}^{\oplus} \mathcal{H}_\hbar(\theta)\, d\theta, \tag{V.17}$$

such that

$$\begin{aligned} \kappa U \kappa^{-1} \phi_m(\theta, z) &= e^{2\pi i(\theta_1 + m)/N} \phi_m(\theta, z), \\ \kappa V \kappa^{-1} \phi_m(\theta, z) &= e^{2\pi i \theta_2/N} \phi_{m+1}(\theta, z). \end{aligned} \tag{V.18}$$

Proof. We set for $\varphi \in \mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$,

$$\kappa(\varphi)(\theta, z) := \sum_{m,n \in \mathbb{Z}^2} X^m Y^n \varphi(z) e^{-2\pi i m \theta_1 - 2\pi i n \theta_2}, \tag{V.19}$$

and verify easily that $\kappa(\varphi)(\theta,\ \cdot) \in \mathcal{H}_\hbar(\theta)$. For $\psi \in \int_{\mathbb{T}^2}^{\oplus} \mathcal{H}_\hbar(\theta) d\theta$ we set

$$\kappa^{-1}(\psi)(z) := \int_{\mathbb{T}^2} \psi(\theta, z) d\theta, \tag{V.20}$$

and note that $\kappa^{-1}(\psi) \in \mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$. We verify that κ and κ^{-1} are inverse to each other. Indeed, for $\varphi \in \mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$,

$$\begin{aligned} (\kappa^{-1}\kappa(\varphi))(z) &= \int_{\mathbb{T}^2} \kappa(\varphi)(\theta, z) d\theta \\ &= \int_{\mathbb{T}^2} \sum_{m,n \in \mathbb{Z}} X^m Y^n \varphi(z) e^{-2\pi i m \theta_1 - 2\pi i n \theta_2} d\theta \\ &= \varphi(z). \end{aligned}$$

Now, let $\phi \in \int_{\mathbb{T}^2}^{\oplus} \mathcal{H}_\hbar(\theta) d\theta$. Expanding in a Fourier series, we have

$$\phi(\theta, z) = \sum_{m,n \in \mathbb{Z}} \widehat{\phi}_{m,n}(z) e^{-2\pi i m \theta_1 - 2\pi i n \theta_2},$$

where $\widehat{\phi}_{m,n}(z) = \int_{\mathbb{T}^2} \phi(\theta, z) e^{2\pi i m \theta_1 + 2\pi i n \theta_2}\, d\theta$. Consequently,

$$\begin{aligned} (\kappa\kappa^{-1}(\phi))(\theta, z) &= \sum_{m,n \in \mathbb{Z}} X^m Y^n \kappa^{-1}(\phi)(z)\ e^{-2\pi i m \theta_1 - 2\pi i n \theta_2} \\ &= \sum_{m,n \in \mathbb{Z}} X^m Y^n \int_{\mathbb{T}^2} \phi(\eta, z) d\eta\ \ e^{-2\pi i m \theta_1 - 2\pi i n \theta_2} \\ &= \sum_{m,n \in \mathbb{Z}} \int_{\mathbb{T}^2} \phi(\eta, z) e^{2\pi i m \eta_1 + 2\pi i n \eta_2} d\eta\ \ e^{-2\pi i m \theta_1 - 2\pi i n \theta_2} \\ &= \phi(\theta, z), \end{aligned}$$

proving that κ is an isomorphism.

We will prove the first of the identities (V.18) only; the proof of the second one is identical. The calculation goes as follows:

$$\begin{aligned}
U\kappa^{-1}\phi_m(z) &= \int_{\mathbb{T}^2} C_m(\theta) e^{2N\pi(iz/N\sqrt{2}-1/4N^2)-N\pi(z+i/N\sqrt{2})^2+2\sqrt{2}\pi(\theta_1+m)(z+i/N\sqrt{2})} \\
&\qquad \times \sum_{k\in\mathbb{Z}} e^{-N\pi k^2-2\pi(\theta_1+i\theta_2+m)k+2\sqrt{2}N\pi k(z+i/N\sqrt{2})}\, d\theta \\
&= (2N)^{-1/4}\int_{\mathbb{T}^2} e^{-\pi(\theta_1+m)^2/N-2\pi i\theta_2 m/N-N\pi z^2+2\sqrt{2}\pi(\theta_1+m)z+2\pi i(\theta_1+m)/N} \\
&\qquad \times \sum_{k\in\mathbb{Z}} e^{-N\pi k^2-2\pi(\theta_1+i\theta_2+m)k+2\sqrt{2}N\pi kz}\, d\theta \\
&= \int_{\mathbb{T}^2} e^{2\pi i(\theta_1+m)/N}\phi_m(\theta,z)\, d\theta.
\end{aligned}$$

Hence $\kappa U\kappa^{-1}\phi_m(\theta,z) = e^{2\pi i(\theta_1+m)/N}\phi_m(\theta,z)$. $\square$

V.D. We now wish to reformulate the quantum cat dynamics on the direct integral space $\int_{\mathbb{T}^2}^{\oplus}\mathcal{H}_\hbar(\theta)d\theta$. We begin with the following lemma, which demonstrates that the quantum evolution operator F yields a smooth endomorphism of the bundle $Q\mathbb{T}^2_\hbar$. Recall that F is the integral operator given by (II.12).

Lemma V.3. *For $\phi\in\mathcal{H}_\hbar(\theta)$,*

$$\begin{aligned}
XF\phi(z) &= e^{2\pi i(\gamma^{-1}\theta+\Delta_{\gamma^{-1}})_1}F\phi(z) \\
YF\phi(z) &= e^{2\pi i(\gamma^{-1}\theta+\Delta_{\gamma^{-1}})_2}F\phi(z),
\end{aligned}$$

where

$$\Delta_\gamma = (Nab/2, Ncd/2). \tag{V.21}$$

Consequently, F maps unitarily $\mathcal{H}_\hbar(\theta)$ onto $\mathcal{H}_\hbar(\gamma^{-1}\theta+\Delta_{\gamma^{-1}})$.

Proof. Using (II.12), (V.3), and (II.9), we compute

$$\begin{aligned}
XF\phi(z) =& |\alpha|^{-1/2} e^{2\pi N(iz/\sqrt{2}-1/4)+\pi N\overline{\beta}(z+i/\sqrt{2})^2/\alpha} \\
&\times N\int_{\mathbb{C}} e^{2\pi N\overline{w}(z+i/\sqrt{2})/\alpha+\pi N\beta\overline{w}^2/\alpha-2\pi N|w|^2}\phi(w)\, d^2w.
\end{aligned}$$

Making the change of variables $w' = w-2^{-1/2}(b+id)$ and using $X\phi(z)=e^{2\pi i\theta_1}\phi(z)$ and $Y\phi(z)=e^{2\pi i\theta_2}\phi(z)$ we may reduce this, after some straightforward algebra, to

$$\begin{aligned}
XF\phi(z) =& e^{-i\pi Nbd+2\pi i(\theta_1 d-\theta_2 b)}F\phi(z) \\
=& e^{2\pi i(\gamma^{-1}\theta+\Delta_{\gamma^{-1}})_1}F\phi(z).
\end{aligned}$$

Similarly,

$$YF\phi(z) = e^{2\pi i(\gamma^{-1}\theta+\Delta_{\gamma^{-1}})_2} F\phi(z),$$

and the lemma is thus proved. □

An equivalent way of stating the above lemma is in the form of the following commutation relations:

$$\begin{aligned} F^{-1}XF &= e^{2\pi i(\Delta_{\gamma^{-1}})_1} X^d Y^{-b}, \\ F^{-1}YF &= e^{2\pi i(\Delta_{\gamma^{-1}})_2} X^{-c} Y^a. \end{aligned} \tag{V.22}$$

Theorem V.4.

(i) Under the isomorphism κ, the action of the evolution operator F is given by

$$\kappa F \kappa^{-1} \phi(\theta, z) = F\phi(\gamma\theta + \Delta_\gamma, z), \tag{V.23}$$

with the understanding that, on the right hand side of this equation, F acts on the z variable.

(ii) The matrix elements of the operator F,

$$(\phi_m^{(\tilde\theta)},\ F\phi_n^{(\theta)}) = \int_D \overline{\phi_m^{(\tilde\theta)}(z)}\, F\phi_n^{(\theta)}(z)\, d\mu_\hbar(z), \tag{V.24}$$

where

$$\tilde\theta = \gamma^{-1}\theta + \Delta_{\gamma^{-1}},$$

are explicitly given by

$$\begin{aligned} (\phi_m^{(\tilde\theta)},\ F\phi_n^{(\theta)}) &= (Nb)^{-1/2} \exp(\frac{i\nu}{2})\ \exp\frac{2\pi i}{N}(m\tilde\theta_2 - n\theta_2) \\ &\times \sum_{0\le r\le |b|-1} \exp(-2\pi i r\theta_2)\ \exp\frac{i\pi}{Nb}\Phi_\gamma(m+\tilde\theta_1,\ n+Nr+\theta_1), \end{aligned} \tag{V.25}$$

where $\exp(i\nu) = -i\alpha/|\alpha|$, *and where*

$$\Phi_\gamma(x,\ y) = ax^2 - 2xy + dy^2. \tag{V.26}$$

Proof. Part (i) of the theorem is a straightforward consequence of the previous lemma, and we leave the details to the reader.

To prove part (ii), we note first the following fact. If $A, B, C, D, E \in \mathbb{C}$ are such that $\mathrm{Re}A > 0$, and $(\mathrm{Re}A)^2 > (\mathrm{Re}B + \mathrm{Re}C)^2 + (\mathrm{Im}B - \mathrm{Im}C)^2$, then

$$\begin{aligned} &\int_{\mathbb{C}} \exp\big(-A|w|^2 + Bw^2 + C\overline{w}^2 + Dw + E\overline{w}\big) dw\, d\overline{w} \\ &= \frac{2\pi}{\sqrt{A^2-4BC}}\ \exp\frac{(D+E)^2(A^2-4BC) - (A(D-E)+2(EB-CD))^2}{4(A-B-C)(A^2-4BC)}. \end{aligned} \tag{V.27}$$

Let $e_m^{(\theta)}(z)$ be the following function:

$$e_m^{(\theta)}(z) = C_m(\theta) e^{-N\pi z^2 + 2\sqrt{2}\pi(\theta_1+m)z}. \tag{V.28}$$

Note that

$$X e_m^{(\theta)}(z) = e^{2\pi i\theta_1} e_m^{(\theta)}(z), \tag{V.29}$$

and

$$Y^k e_m^{(\theta)}(z) = e^{2\pi i k\theta_2} e_m^{(\theta)}(z) e^{-N\pi k^2 + 2\sqrt{2}\pi N k z - 2\pi(\theta_1+i\theta_2+m)k}. \tag{V.30}$$

As a consequence of (V.30),

$$\phi_m^{(\theta)} = \sum_k \left(e^{-2\pi i\theta_2} Y\right)^k e_m^{(\theta)}, \tag{V.31}$$

an identity which we will find useful later. As a consequence of (V.27), for any $r \in \mathbb{Z}$,

$$\begin{aligned}\int_{\mathbb{C}} \overline{e_m^{(\tilde{\theta})}(z)}\ F Y^r e_n^{(\theta)}(z)\ d\mu_\hbar(z) &= (Nb)^{-1/2} \exp(\frac{i\nu}{2})\ \exp\frac{2\pi i}{N}(m\tilde{\theta}_1 - n\theta_1) \\ &\times \exp\frac{i\pi}{Nb}\big(a(m+\tilde{\theta}_1)^2 - 2(m+\tilde{\theta}_1)(n+Nr+\theta_1) + d(n+Nr+\theta_1)^2\big).\end{aligned} \tag{V.32}$$

Consider now the sum:

$$\begin{aligned}&\sum_{0\le r\le |b|-1} e^{-2\pi i r\theta_2} \int_{\mathbb{C}} \overline{e_m^{(\tilde{\theta})}(z)}\ F Y^r e_n^{(\theta)}(z)\ d\mu_\hbar(z) = \\ &\qquad \sum_{0\le r\le |b|-1} e^{-2\pi i r\theta_2} \sum_{k,l} \int_D \overline{X^k Y^l e_m^{(\tilde{\theta})}(z)}\ X^k Y^l F Y^r e_n^{(\theta)}(z)\ d\mu_\hbar(z).\end{aligned}$$

Using the commutation relations (V.22) as well as (V.29) and (V.31), we can rewrite it as

$$\begin{aligned}&\sum_{0\le r\le |b|-1} e^{-2\pi i r\theta_2} \sum_{k,l} e^{2\pi i(-k\tilde{\theta}_1 + (dk-cl)\theta_1 + k(\Delta_{\gamma^{-1}})_1 + l(\Delta_{\gamma^{-1}})_2)} \\ &\quad\times \int_D \overline{Y^l e_m^{(\tilde{\theta})}(z)}\ F Y^{-bk+al+r} e_n^{(\theta)}(z)\ d\mu_\hbar(z) \\ &\quad= \sum_{0\le r\le |b|-1} \sum_{k,l} \int_D \overline{(e^{-2\pi i\tilde{\theta}_2} Y)^l e_m^{(\tilde{\theta})}(z)}\ F (e^{-2\pi i\theta_2} Y)^{-bk+al+r} e_n^{(\theta)}(z)\ d\mu_\hbar(z) \\ &\quad= \sum_{k,l} \int_D \overline{(e^{-2\pi i\tilde{\theta}_2} Y)^l e_m^{(\tilde{\theta})}(z)}\ F (e^{-2\pi i\theta_2} Y)^k e_n^{(\theta)}(z)\ d\mu_\hbar(z) \\ &\quad= \int_D \overline{\phi_m^{(\tilde{\theta})}(z)}\ F \phi_n^{(\theta)}(z)\ d\mu_\hbar(z),\end{aligned}$$

which, together with (V.32) proves the theorem. □

The sum in (V.25) is a generalized Gauß sum which, for the case of γ of the form (V.1) and $\theta = 0$, reduces to the Gauß sum studied in [9].

VI. The structure of the quantized Kronecker map

Using the isomorphism we introduced in the previous section, we construct here the quantized Kronecker dynamics on $\int_{\mathbb{T}^2}^{\oplus} \mathcal{H}_\hbar(\theta)\, d\theta$. In analogy with Theorem V.4, we have the following result.

Theorem VI.1.

(i) Under the isomorphism κ, the action of the evolution operator K is given by

$$\kappa K \kappa^{-1} \phi(\theta, z) = U(-\omega)\phi(\theta + N\omega, z), \tag{VI.1}$$

where $U(-\omega)$ acts on the z variable.

(ii) The matrix elements of the evolution operator K,

$$(\phi_m^{(\tilde{\theta})},\ K\phi_n^{(\theta)}) = \int_D \overline{\phi_m^{(\tilde{\theta})}(z)}\ K\phi_n^{(\theta)}(z)\ d\mu_\hbar(z),$$

where

$$\tilde{\theta} = \theta - N\omega,$$

are explicitly given by

$$(\phi_m^{(\tilde{\theta})},\ K\phi_n^{(\theta)}) = \exp\big(2\pi i\omega_2(\theta_1 - N\omega_1/2)\big)\ \delta_{mn}. \tag{VI.2}$$

Proof. The proof of (i) follows by a simple calculation involving Fourier series. To prove (ii), we verify by an explicit computation that

$$K\phi_m^{(\theta)} = e^{-iN\pi\omega_1\omega_2 + 2\pi i\omega_2\theta_1} \phi_m^{(\theta - N\omega)}. \quad \square \tag{VI.3}$$

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