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Quantum Riemann Surfaces

for Arbitrary Planck's Constant

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Abstract

We continue our study of quantum Riemann surfaces initiated in [1, 2, 3]. We construct a one parameter family of deformations of compact Riemann surfaces of genus $g \geq 2$. Our construction does not require any discretness condition on the value of Planck's constant. It coincides with the construction of [2] in the case when Planck's constant assumes the discrete set of values dictated by geometric quantization.

I. Introduction

In a series of papers [1], [2], [3], we studied non-perturbative deformation quantization of Riemann surfaces. Our approach is based on the ideas of [4] (for related developments, see also [5], [6], [7], and references therein.) A satisfactory picture of uniformization of exceptional quantum Riemann surfaces emerged from these investigations. In the case of higher genus ($g \geq 2$) Riemann surfaces, the uniformization on the quantum level is a more complex issue. In fact, if M is a Riemann surface and N is a covering of M, then the quantization of N is a covering of the quantization of M in the sense of [3] only if the fundamental group of the covering $N \to M$ is abelian. Part of the problem is the presence of toplogical sectors (similar to the θ-vacua in gauge theory) in the quantum theory, which is related to the non-simple connectedness of the classical phase space. These sectors are classified by the characters of the fundamental group of the phase space. This does not reflect the non-commutativity of that group. On the other hand, whether the fundamental group is commutative or not seems to be important for the quantum uniformization in the sense of [3]. A similar phenomenon was discussed previously in [8].

Quantization of Riemann surfaces in the framework of geometric quantization (see e.g. [9]) requires a restriction on the allowed values of the deformation parameter r ("the quantization condition".) However, it is desirable to have a definition of quantum Riemann surfaces for all values of r. A definition consistent with geometric quantization was given in [2], for $r = n(2g-2)^{-1}$, where $n \in \mathbb{N}$. In principle, quantum uniformization allows for a construction of quantum Riemann surfaces for all values of r, using the universal covering, the Poincaré disk, as the point of departure. Since the fundamental groups of higher genus Riemann surfaces are non-abelian, it is likely, however, that the so defined algebra of quantized functions does not reduce to the algebra of [2], when $r = n(2g-2)^{-1}$.

In this paper, we construct quantization of compact Riemann surfaces for arbitrary values of $r > 0$ in a manner consistent with [2]. Our starting point is a non-compact covering space $\hat{M}$ such that the group of cover transformation is the abelian group $\mathbb{Z}$. Since $\hat{M}$ is a non-compact Riemann surface, all holomorphic line bundles are holomorphically trivial [10], and geometric quantization does not impose any restrictions on Planck's constant. It was explained in [2] that geometric quantization of Riemann surfaces leads to certain operator algebras on Hilbert spaces of automorphic forms. We define the quantization of M in terms of Toeplitz operators with invariant symbols on a space of automorphic forms on $\hat{M}$, and prove deformation estimates. The proof of the estimates is based on the methods developed in [2] (see [11] for a different approach.)

The paper is organized as follows. In Section II, we study the fundamental groups of M and $\hat{M}$. We define and study suitable spaces of automorphic forms on M and $\hat{M}$ in Section III. Section IV contains proofs of deformation estimates.

II. The fundamental groups

In this section, we fix our notation and describe certain group theoretic properties of the fundamental groups of Riemann surfaces which will be useful in later sections. Let M be a compact Riemann surface of genus $g \geq 2$. We pick a basis $\{a_i, b_i\}$, $i = 1, .., g$, of one-cycles on M. The fundamental group Γ of M is a finitely generated group with generators $\{a_i, b_i\}$, $i = 1, \ldots, g$, obeying the relation (see e.g [12])

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 \ .$$

We denote $a := a_g$ and $b := b_g$. An element $\gamma \in \Gamma$ can be represented as

$$\gamma = \prod_{j=1}^{\infty} \prod_{i=1}^{g} a_i^{n_{i,j}} b_i^{m_{i,j}},$$

where almost all integers $n_{i,j}, m_{i,j}$ are equal to zero. Consider now the homomorphism

$$\gamma \to \sum_j m_{g,j} \in \mathbb{Z} \ .$$

One can easily verify that the above map is well defined, i.e. it does not depend on the way we represent γ, and is indeed a homomorphism of groups. We denote the kernel of this homomorphism by Γ_0. The group Γ_0 is no longer finitely generated. In fact, it is not difficult to see that the following elements are its generators:

$$a, \ b^n a_i b^{-n}, \ b^n b_i b^{-n}, \ \text{where } n \in \mathbb{N}, \ i = 0, 1, \ldots, g-1 \ ,$$

and there are no relations among them, i.e. Γ_0 is the free product of infinite cyclic groups generated by the above generators.

Let $\hat{M}$ be a covering of M with no branching points and such that $\pi_1(\hat{M}) = \Gamma_0$. The group of cover transformations of the covering $\hat{M} \to M$ is then equal to $\mathbb{Z}$. Intuitively, $\hat{M}$ is obtained from M by cutting along the a cycle and then continuing M across the cut along the b cycle. One can visualize $\hat{M}$ as an infinite cylinder with infinite number of handle bodies attached, each handle body having $g - 1$ handles. $\hat{M}$ is a non-compact Riemann surface.

The universal covering space of both M and $\hat{M}$ is the unit disk $\mathcal{U}$. The groups Γ and Γ_0 can be think of as Fuchsian groups on $\mathcal{U}$. For

$$\gamma = \begin{bmatrix} a & b \\ \bar{a} & \bar{b} \end{bmatrix} \in \Gamma,$$

we denote

$$\gamma(z) = \frac{az+b}{\bar{b}x+\bar{a}} \ , \quad z \in \mathcal{U}. \tag{II.1}$$

Since the derivative of (II.1) is $\gamma'(z) = (\bar{b}z+\bar{a})^{-2}$, we can set:

$$\gamma'(z)^{1/2} := (\bar{b}z+\bar{a})^{-1}.$$

Let G be either Γ or Γ_0, and let $r > 0$. A multiplier of weight r for the group G is a map $v : G \to \mathbb{C}$ such that [12]

$$|v(\gamma)| = 1,$$

and

$$\gamma_1'(\gamma_2(z))^{-r/2}\gamma_2'(z)^{-r/2}v(\gamma_1)v(\gamma_2) = (\gamma_1\gamma_2)'(z)^{-r/2}v(\gamma_1\gamma_2) \ . \tag{II.2}$$

In (II.2), the standard branch of the logarithm is taken to define the r-th power. The existence of multipliers is a cohomological question, see [13] for the modern treatment. Equation (II.2) can be interpreted as the triviality of a 2-cocycle in the group cohomology of G. A multiplier is essentially a 1-cochain whose coboundary is that 2-cocycle. It is well-known that multipliers exist for Γ if and only if $r = n(2g-2)^{-1}$, $n = 1, 2, \ldots$. The situation is different for Γ_0.

Proposition II.1. *The second cohomology group $H^2(\Gamma_0, \mathbb{Z})$ of Γ_0 is trivial.*

Proof. Since $\Gamma_0 = \pi_1(\hat{M})$, and the universal covering space of $\hat{M}$ is contractible, it follows from Eilenberg-MacLane's theorem [14] that

$$H^*(\Gamma_0, \mathbb{Z}) \simeq H^*(\hat{M}, \mathbb{Z}) \ .$$

By Poincaré duality [15], $H^2(\hat{M}, \mathbb{Z})$ is isomorphic to the compactly supported cohomology group of $\hat{M}$ in dimension zero. The latter is trivial, as $\hat{M}$ is non-compact. □

Consequently, multipliers for Γ_0 exist for arbitrary r. Let us also remark that the ratio of two multipliers is a character of the group.

III. Automorphic forms

As before, let G be either Γ or Γ_0, and let v be a multiplier for the group G. Recall that a holomorphic function $\phi : \mathcal{U} \to \mathbb{C}$ is called an automorphic form for G of weight $r > 0$ with multiplier v, if [12]:

$$\phi(\gamma(z)) = v(\gamma)\gamma'(z)^{-r/2}\phi(z) \ , \tag{III.1}$$

for each $\gamma \in G$. Automorphic forms for infinitely generated Fuchsian groups like Γ_0 have been studied less extensively than those for finitely generated groups, but there is a fair amount of information available, see [16], [17], and references therein.

Let R be a fundamental polygon for Γ. Then $R_0 := \bigcup_{n\in\mathbb{Z}} b^n R$ is a fundamental polygon for Γ_0. It has infinitely many sides. Here we use the same symbol b to denote the group element of Γ corresponding to the cycle $b = b_g$. For $r = n(2g-2)^{-1}$, we define $\mathcal{H}^r(\Gamma, v)$ to be the Hilbert space of automorphic forms for Γ equipped with the following scalar product:

$$(\phi, \psi) := \int_R \overline{\phi(z)}\psi(z)\, d\mu^r(z), \tag{III.2}$$

where the measure $d\mu^r(z)$ is

$$d\mu^r(z) := \frac{r-1}{\pi}(1-|z|^2)^{r-2}\, d^2 z \ .$$

For arbitrary $r > 0$, let $\mathcal{H}^r(\Gamma_0, v)$ be the space of automorphic forms for Γ_0 with the scalar product as in (III.2) but R_0 replacing R.

If v is a multiplier for Γ and $e^{i\theta} \in S^1$, we let v_θ denote a new multiplier for Γ defined by:

$$v_\theta(b) = v(b)e^{i\theta},$$

and $v_\theta = v$ on all other generators. In particular $v_0 = v$. All of the multipliers v_θ are equal when restricted to the subgroup Γ_0. The restricted multiplier will be again denoted by v.

Theorem III.1. *With the above definitions, there is a canonical isomorphism*

$$\mathcal{H}^r(\Gamma_0, v) \cong \int_{S^1}^{\oplus} \mathcal{H}^r(\Gamma, v_\theta)\, d\theta, \qquad r = n(2g-2)^{-1},\ n = 1, 2, \dots \ .$$

Proof. For an automorphic form ϕ for Γ_0, define

$$U\phi(z) = b'(z)^{r/2}\phi(bz) \ . \tag{III.3}$$

We claim that (III.3) is again an automorphic form, and in fact U is a unitary operator on $\mathcal{H}^r(\Gamma_0, v)$. Let us first verify (III.1):

$$\begin{aligned} U\phi(\gamma z) &= b'(\gamma z)^{r/2}\phi(b\gamma b^{-1}bz) \\ &= v(b\gamma b^{-1})b'(\gamma z)^{r/2}(b\gamma b^{-1})'(bz)^{-r/2}\phi(bz). \end{aligned}$$

Using the chain rule and (II.2), we obtain (III.1). Unitarity of U is a consequence of the following calculation:

$$\begin{aligned} (U\phi, U\psi) &= \int\limits_{R_0} \overline{b'(z)^{r/2}\phi(bz)}\, b'(z)^{r/2}\psi(bz)\, d\mu^r(z) \\ &= \int\limits_{R_0} \overline{\phi(bz)}\psi(bz)\, d\mu^r(bz) = (\phi, \psi), \end{aligned}$$

where we have used the transformation properties of $d\mu^r(z)$ and the fact that R_0 is invariant under b.

The isomorphism of Hilbert spaces that we want to establish is in essence the spectral decomposition of U. Explicitly, we define

$$P:\ \mathcal{H}^r(\Gamma_0, v) \to \int_{S^1}^{\oplus} \mathcal{H}^r(\Gamma, v_\theta)\, d\theta$$

by the following formula:

$$P\phi(\zeta, \theta) = \sum_{n\in\mathbb{Z}} v_\theta(b^{-n})U^n\phi(z)\ . \tag{III.4}$$

We need to verify that the right hand side of (III.4) is in $\mathcal{H}^r(\Gamma, v_\theta)$. If $\gamma \in \Gamma_0$, this follows from the fact that $U^n\phi(z)$ are automorphic forms for Γ_0, and the fact that $v_\theta|_{\Gamma_0} = v|_{\Gamma_0}$. If $\gamma = b$, we have

$$U\left(\sum_{n\in\mathbb{Z}} v_\theta(b^{-n})U^n\phi(z)\right) = v_\theta(b)\left(\sum_{n\in\mathbb{Z}} v_\theta(b^{-n})U^n\phi(z)\right)\ .$$

The general case follows easily.

We verify that P is an isometry:

$$\begin{aligned}
\|P\phi\|^2 &= \int_{-\pi}^{\pi} \Big(\int_R |P\phi(z,\theta)|^2 \, d\mu^r(z) \Big) \frac{d\theta}{2\pi} \\
&= \int_{-\pi}^{\pi} \Big(\int_R \sum_{n,m} \overline{U^n \phi(z)} U^m \phi(z) v_\theta(b)^{n-m} \, d\mu^r(z) \Big) \frac{d\theta}{2\pi} \\
&= \int_R \sum_n |U^n \phi(z)|^2 \, d\mu^r(z) \\
&= \sum_n \int_{b^n R} |\phi(z)|^2 \, d\mu^r(z) \\
&= \int_{R_0} |\phi(z)|^2 \, d\mu^r(z) \ .
\end{aligned}$$

Similar calculations show that the inverse of P is given by:

$$P^{-1}\psi(z) = \int_{-\pi}^{\pi} \psi(z,\theta) \, d\theta \ . \quad \square$$

This result implies that the quantization of Riemann surfaces proposed in this paper reduces, when $r = n(2g-2)^{-1}$, to the definition of [2].

IV. Toeplitz quantization

In this section, we construct a quantization of the Riemann surface M in terms of Toeplitz operators on $\mathcal{H}^r(\Gamma, v_\theta)$. We first recall the relevant definitions. The reproducing kernel $K^r_{\Gamma_0,v}(z,w)$ for $\mathcal{H}^r(\Gamma_0, v)$ is given by the following Poincaré series:

$$K^r_{\Gamma_0,v}(z,w) = \sum_{\gamma \in \Gamma_0} v(\gamma)^{-1} \gamma'(z)^{r/2} K^r(\gamma(z), w), \tag{IV.1}$$

where

$$K^r(z,w) = (1 - z\bar{w})^{-1} \ .$$

Let $C_\Gamma(\mathcal{U})$ be the $\mathbb{C}^*$-algebra of bounded continuous functions on $\mathcal{U}$ which are invariant under Γ, so that $C_\Gamma(\mathcal{U}) \cong C(M)$. For $f \in C_\Gamma(\mathcal{U})$, we define the Toeplitz operator $T^r_{\Gamma_0,v}(f)$ on $\mathcal{H}^r(\Gamma_0, v)$ with symbol f by:

$$(T^r_{\Gamma_0,v}(f)\phi)(z) = \int_{\Gamma_0} K^r_{\Gamma_0,v}(z,w) f(w) \phi(w) \, d\mu^r(w) \ . \tag{IV.2}$$

The goal of this section is to prove that the correspondence $f \mapsto T^r_{\Gamma_0,v}(f)$ is a quantization of M. This means that for $f \in C_\Gamma(\mathcal{U})$ we have the norm limit

$$\lim_{r\to\infty} \|T^r_{\Gamma_0,v}(f)\| = \|f\|_\infty, \tag{IV.3}$$

where $||\cdot||$ denotes the operator norm, and where $||\cdot||_\infty$ denotes the sup-norm. If, moreover, f, g are smooth, then

$$\lim_{r\to\infty} \|r\left[T^r_{\Gamma_0,v}(f), T^r_{\Gamma_0,v}(g)\right] + T^r_{\Gamma_0,v}(i\{f,g\})\| = 0, \tag{IV.4}$$

where $\{f,g\}$ is the usual Poisson bracket,

$$\{f,g\}(z) = i(1-|z|^2)^2[\partial f(z)\bar{\partial} g(z) - \partial g(z)\bar{\partial} f(z)] \ . \tag{IV.5}$$

Theorem IV.1. *With the above definitions, the correspondence*

$$f \mapsto T^r_{\Gamma_0,v}(f)$$

is a quantization of M.

Proof. The details of analogous estimates were explained in [1] and [2]. Here we follow [2], where the estimates were proved for the quantization based on automorphic forms of Γ. However, the compactness of the fundamental domain of Γ was used in an essential way in several places, so that the results cannot be applied to the case of Γ_0 (as R_0 is not compact in $\mathcal{U}$). The main difference is that "transfer of regularity" argument has to be done more carefully in the present case.

To prove (IV.3), we consider the vectors

$$\phi_w(z) := K^r_{\Gamma_0,v}(w,w)^{-1/2} K^r_{\Gamma_0,v}(z,w)$$

in $\mathcal{H}^r(\Gamma_0, v)$, and verify that:

$$\sup_{x\in R} |f(w) - (\phi_w, T^r_{\Gamma_0,v}(f)\phi_w)| \to 0, \quad \text{as } r\to\infty, \tag{IV.6}$$

in a way analogous to [2]. The proof there was based on Lemmas 4.1 and 4.2 which are also valid for $\Gamma_0 \subset \Gamma$. Since f is invariant under Γ, and not just Γ_0, and since the supremum in (IV.6) is taken over a compact set, (IV.6) follows exactly as in [2].

The estimate (IV.4) is a consequence of

$$||r\left(T^r_{\Gamma_0,v}(f)T^r_{\Gamma_0,v}(g) - T^r_{\Gamma_0,v}(fg)\right) + T^r_{\Gamma_0,v}((1-|z|)^2\partial f\bar{\partial} g)|| \to 0, \quad \text{as } r\to\infty, \tag{IV.7}$$

for $f, g \in C^\infty_\Gamma$. To prove (IV.7), one expands $(\phi, T^r_{\Gamma_0,v}(f)T^r_{\Gamma_0,v}(g)\psi)$ in a Taylor series as in [2] formula (5.6). The first three terms in that formula combine to give the second and third terms in (IV.7). The analog of the fourth term of [2], formula (5.6), is $O(r^{-2})$ as in [1]. It remains to estimate the remainder.

The technique developed in [1] for estimating the remainder terms can be applied to our case with one modification. The integral $\int_{\mathcal{U}} |\psi(w)|^2 \, d\mu^r(w)$ is infinite if $\psi \in \mathcal{H}^r(\Gamma_0, v)$, and one needs to transfer a power of $1 - |w|^2$ to make it convergent. This "transfer of regularity" trick was explained in detail on the last two pages of [2] for Γ-automorphic forms. However, the compactness of the fundamental domain of Γ was used in an essential way. We show below that a modification of the argument used in [2] can be used to in our case.

Lemma IV.2. *Let $\psi \in \mathcal{H}^r(\Gamma_0, v)$ and $s > 1$. Then we have:*

$$\int_{\mathcal{U}} |\psi(w)|^2 (1 - |w|^2)^s \, d\mu^r(w) = O(1)||\psi||^2 \ .$$

Lemma IV.3. *Let $b \in SU(1,1)$ be hyperbolic, let K be a compact set in $\mathcal{U}$, and let $t > 0$. Then we have:*

$$\sup_{z\in K} \sum_{n\in\mathbb{Z}} |(b^n)'(z)|^t = O(1) \ .$$

We will prove these lemmas after we have completed the main line of the argument.

We now use the above lemmas to estimate the term in formula (5.16) of [2]. This will conclude the proof of Theorem IV.1. That term reads:

$$\int_{R_0\times\mathcal{U}} |\phi(z)|(1 - |z|^2)^{1-r/2} |\psi(\gamma_z(w))||\gamma_z'(w)|^{r/2} \frac{|w|^2}{(1 - |w|^2)^{11}} \, d\mu^r(z) \, d\mu^r(z), \tag{IV.8}$$

where $\gamma_z(w) = (w + z)/(\bar{z}w + 1)$. Let $0 < \epsilon < 1/2$. We multiply and divide the integrand by $(1 - |\gamma_z(w)|^2)^{1-\epsilon}$, and use the following elementary bound:

$$\frac{1}{(1 - |\gamma_z(w)|^2)^{1-\epsilon}} \le \frac{O(1)}{(1 - |z|^2)^{1-\epsilon}(1 - |w|^2)} \ .$$

The integral in (IV.8) is consequently less then

$$\begin{aligned} O(1) \int_{R_0\times\mathcal{U}} & |\phi(z)|(1 - |z|^2)^{\epsilon-r/2} |\psi(\gamma_z(w))|(1 - |\gamma_z(w)|^2)^{1-\epsilon} |\gamma_z'(w)|^{r/2} \\ & \times \frac{|w|^2}{(1 - |w|^2)^{12}} \, d\mu^r(z) \, d\mu^r(z) \ . \end{aligned} \tag{IV.9}$$

Using the Schwarz inequality and changing variables in the ψ term, we get the following bound:

$$
\begin{aligned}
O(1)\,||\phi||\,\left(\int_{R_0}(1-|z|^2)^{2\epsilon-r}\,d\mu^r(z)\right)^{1/2}\left(\int_{\mathcal{U}}|\psi(w)|^2(1-|w|^2)^{2-2\epsilon}\,d\mu^r(w)\right)^{1/2} \\
\times\left(\int_{\mathcal{U}}\frac{|w|^4}{(1-|w|^2)^{24}}\,d\mu^r(w)\right)^{1/2}\ .
\end{aligned}
\tag{IV.10}
$$

With our choice $0<\epsilon<1/2$, the exponent $2-2\epsilon$ in the third factor is greater than 1, and so we can apply Lemma IV.2 to it, and conclude that it is $O(1)||\psi||^2$. The fourth factor is $O(r^{-1})$ by [2], formula (5.18). One can analyze the second factor in (IV.10) as follows:

$$
\begin{aligned}
\int_{R_0}(1-|z|^2)^{2\epsilon-r}\,d\mu^r(z) &= O(r)\int_{R_0}(1-|z|^2)^{2\epsilon}\,d\mu_P(z) \\
&= O(r)\sum_{n\in\mathbb{Z}}\int_{b^{-n}R}(1-|z|^2)^{2\epsilon}\,d\mu_P(z) \\
&= O(r)\sum_{n\in\mathbb{Z}}\int_{R}(1-|b^n z|^2)^{2\epsilon}\,d\mu_P(z) \\
&= O(r)\int_{R}\sum_{n\in\mathbb{Z}}\left|(b^n)'(z)\right|^{2\epsilon}(1-|z|^2)^{2\epsilon}\,d\mu_P(z) \\
&\le O(r)\sup_{z\in R}\sum_{n\in\mathbb{Z}}\left|(b^n)'(z)\right|^{2\epsilon}\ .
\end{aligned}
\tag{IV.11}
$$

In (IV.11), $d\mu_P(z)$ is the Poincaré measure on $\mathcal{U}$, and we have used the fact that R is compact. Since $2\epsilon>0$ it follows from Lemma IV.3 that (IV.11) is $O(r)$. This concludes the transfer of regularity argument.

Proof of Lemma IV.2 : We decompose $\mathcal{U}$ into the translates of R_0:

$$
\int_{\mathcal{U}}|\psi(w)|^2(1-|w|^2)^s\,d\mu^r(w) = \sum_{\gamma\in\Gamma_0}\int_{\gamma^{-1}R_0}|\psi(w)|^2(1-|w|^2)^s\,d\mu^r(w)
$$

$$
= \sum_{\gamma\in\Gamma_0}\int_{R_0}|\psi(w)|^2(1-|\gamma(w)|^2)^s\,d\mu^r(w) \le \left(\sup_{w\in\mathcal{U}}\sum_{\gamma\in\Gamma_0}(1-|\gamma(w)|^2)^s\right)||\psi||^2\ .
$$

To estimate the supremum factor, we proceed as in the proof of [2] lemma 4.2:

$$
\begin{aligned}
1 &= (1-|w|^2)^s \int_{\mathcal{U}} |K^s(w,z)|^2 \, d\mu^s(z) \\
&= (1-|w|^2)^s \sum_{\gamma\in\Gamma_0} \int_{\gamma^{-1}R_0} |K^s(w,z)|^2 \, d\mu^s(z) \\
&= (1-|w|^2)^s \sum_{\gamma\in\Gamma_0} |\gamma'(w)|^s \int_{R_0} |K^s(\gamma(w),z)|^2 \, d\mu^s(z) \\
&= \sum_{\gamma\in\Gamma_0} (1-|\gamma(w)|^2)^s \int_{R_0} |K^s(\gamma(w),z)|^2 \, d\mu^s(z) \\
&\geq \sum_{\gamma\in\Gamma_0} (1-|\gamma(w)|^2)^s \frac{1}{2^{2s}} \int_{R_0} d\mu^s(z) \ .
\end{aligned}
$$

Hence, $\sup_{w\in\mathcal{U}} \sum_{\gamma\in\Gamma_0} (1-|\gamma(w)|^2)^s = O(1)$, if $s>1$. This concludes the proof of Lemma IV.2.

Proof of Lemma IV.3 : Since b is a hyperbolic element of $SU(1,1)$, it has two real eigenvalues λ, $1/\lambda$ with $|\lambda|>1$. Letting

$$
b^n = \begin{bmatrix} \alpha_n & \beta_n \\ \bar{\beta}_n & \bar{\alpha}_n \end{bmatrix},
$$

we have $\alpha_n = O(|\lambda|^{|n|})$. Furthermore, we have the following bound:

$$
\begin{aligned}
|(b^n)'(z)| &= |\bar{\beta}_n z + \bar{\alpha}_n|^{-2} = |\alpha_n|^{-2} |1 + \frac{\bar{\beta}_n}{\bar{\alpha}_N} z|^{-2} \\
&\leq |\alpha_n|^{-2} (1-|z|)^{-2}.
\end{aligned}
$$

Since z varies over a compact set and $t>0$, it follows that the series $\sum_{n\in\mathbb{Z}} |(b^n)'(z)|^t$ is bounded, uniformly in z, by a convergent geometric series. This concludes the proof of Lemma IV.3 and Theorem IV.1 . □

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