PERTURBATIVE OSCILLATION CRITERIA AND HARDY-TYPE INEQUALITIES

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Abstract. We prove a natural generalization of Kneser's oscillation criterion and Hardy's inequality for Sturm–Liouville differential expressions. In particular, assuming $-\frac{d}{dx}p_0(x)\frac{d}{dx}+q_0(x)$, $x\in(a,b)$, $-\infty\leq a<b\leq\infty$ to be nonoscillatory near a (or b), we determine conditions on $q(x)$ such that $-\frac{d}{dx}p_0(x)\frac{d}{dx}+q_0(x)+q(x)$ is nonoscillatory, respectively, oscillatory near a (or b).

1. Introduction

In this note we compare oscillation properties of solutions of Sturm–Liouville equations $\tau_0\psi_0=\lambda\psi_0$ and $\tau\psi=\lambda\psi$, where τ_0 is of the type $\tau_0=-\frac{d}{dx}p_0\frac{d}{dx}+q_0(x)$ and its perturbation τ is of the form $\tau=\tau_0+q(x)$. More precisely, assuming

$$0<p_0^{-1}\in L^1_{loc}(((a,b)),\ q_0\in L^1_{loc}((a,b))\text{ real–valued,}\tag{1.1}$$

consider the (quasi) differential expression

$$\tau_0=-\frac{d}{dx}p_0(x)\frac{d}{dx}+q_0(x),\quad x\in(a,b),\ -\infty\leq a<b\leq\infty\tag{1.2}$$

and its perturbation

$$\tau=\tau_0+q(x),\quad x\in(a,b),\tag{1.3}$$

where

$$q\in L^1_{loc}((a,b))\text{ is real–valued.}\tag{1.4}$$

Assuming $(\tau_0-\lambda_0)$ to be nonoscillatory near b, that is, assuming the existence of a real–valued solution $\psi_0(\lambda_0,x)$ of $\tau_0\psi=\lambda_0\psi$ which has a finite number of zeros in a neighborhood of b, we shall arrive at the following statements in Theorem 2.2:
Assume $|\int^\infty p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt|=\infty$.
(i) If

$$\liminf_{x\uparrow b}\ \{p_0(x)\psi_0(\lambda_0,x)^4\Big(\int^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^2q(x)\}>-1/4,\tag{1.5}$$

then $(\tau-\lambda_0)$ *is nonoscillatory near* b,
and
(ii) If

$$\limsup_{x\uparrow b}p_0(x)\{\psi_0(\lambda_0,x)^4\Big(\int^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}dt\Big)^2q(x)\}<-1/4,\tag{1.6}$$

then $(\tau-\lambda_0)$ *is oscillatory near* b.
(The case $|\int^\infty p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt|<\infty$ will be dealt with analogously.)

The special case $p_0=\psi_0=1$, $q_0=\lambda_0=0$ in (1.5) and (1.6) represents Kneser's [21] original (non)oscillation criterion dating back to 1893.

In fact, we shall prove in Theorem 2.3 a scale of (non)oscillation criteria generalizing the well–known (non)oscillation criteria by Weber [34], p. 53–62 (see also Hartman [12] and Hille [14]) which, in turn, extended Kneser's original criterion.

Before we continue the description of the content of this paper, a few hints concerning the literature on oscillation theory might be in order. Since an exhaustive list of references is impossible due to the incredible amount of papers devoted to this subject, we are forced to confine ourselves to monographs and reviews. Comprehensive treatments of oscillation theory can be found, for instance, in [1], [3], Ch. 1, [4], Sects. XIII.7–XIII.10, [13], Ch. XI, [15], Sect. 9.4–9.6, [22], Ch. 1, [25], Ch. IV, [33], Ch. 2, and [35]. ¿From a historical perspective, Sturm's original memoir [32] of 1836 and Bôcher's monograph [2] of 1917 played a premier role in the development of the field. References intimately related to the topics we discuss in this paper are Kneser's original work [21] in 1893, its generalization due to Weber [34] from 1912 and the rediscovery of the latter by Hartman [12] and Hille [14] in 1948.

In connection with the special case $p_0 = \psi_0 = 1$, $q_0 = \lambda_0 = 0$ and Kneser's (non)oscillation criterion

$$\lim_{x\uparrow b} \genfrac{}{}{0pt}{}{\inf}{\sup}\{x^2 q(x)\} \gtrless -1/4 \text{ implies } \genfrac{}{}{0pt}{}{\text{nonoscillation}}{\text{oscillation}} \text{ of } \tau \text{ near } b \tag{1.7}$$

one should mention the possibility of factoring

$$\tau_\mu^{(0)} = -\frac{d^2}{dx^2} + \frac{\mu}{(x-a)^2}, \quad x > R > a,\ \mu \in \mathbb{R} \tag{1.8}$$

for $a < R$ large enough into a product of first–order differential expressions $A_\alpha^{(0)}$ and $A_\alpha^{(0)+}$. Consider

$$A_\alpha^{(0)} = \frac{d}{dx} + \frac{\alpha}{x-a}, \quad A_\alpha^{(0)+} = -\frac{d}{dx} + \frac{\alpha}{x-a}, \quad x > R,\ \alpha \in \mathbb{C}. \tag{1.9}$$

Then one verifies

$$\tau_\mu^{(0)} = A_\alpha^{(0)+} A_\alpha^{(0)} \text{ if and only if } \mu = \alpha(\alpha - 1). \tag{1.10}$$

In particular, since for $\alpha \in \mathbb{R}$, $\alpha(\alpha-1) \geq -1/4$, $\tau_\mu^{(0)}$ in (1.8) admits the factorization (1.10) with $\alpha \in \mathbb{R}$ if and only if $\mu \geq -1/4$. Intimately connected with the existence of the factorization (1.10) for $\alpha \in \mathbb{R}$ and the associated borderline $\mu_0 = -1/4$ is (a special case of) Hardy's inequality (see, e.g., [9], [10], [11], p. 240)

$$\int_a^\infty |\phi'(x)|^2\, dx > \frac{1}{4} \int_a^\infty \frac{|\phi(x)|^2}{(x-a)^2}\, dx, \quad 0 \neq \phi \in C_0^\infty\big((a,\infty)\big). \tag{1.11}$$

In connection with the general Sturm–Liouville differential expressions (1.2) and (1.3) we shall show in Section 3 that

$$\tau_\mu = -\frac{d}{dx} p_0(x) \frac{d}{dx} + q_0(x) + \mu p_0(x)^{-1} \psi_0(\lambda_0, x)^{-4} \Big(\int^x p_0(t)^{-1} \psi_0(\lambda_0, t)^{-2}\, dt \Big)^{-2}, \\ x > R > a,\ \mu \in \mathbb{R}, \tag{1.12}$$

for $a < R$ sufficiently close to b, admits the factorization

$$\tau_\mu = A_\alpha^+ A_\alpha + \lambda_0 \text{ if and only if } \mu = \alpha(\alpha - 1), \tag{1.13}$$

where

$$
\begin{aligned}
A_\alpha &= p_0(x)^{1/2}\frac{d}{dx} - p_0(x)^{1/2}\psi_0(\lambda_0,x)^{-1}\psi_0'(\lambda_0,x) - \alpha p_0(x)^{-1/2}\psi_0(\lambda_0,x)^{-2} \\
&\qquad \times \Big(\int^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-1}, \\
A_\alpha^+ &= -\frac{d}{dx}p_0(x)^{1/2} - p_0(x)^{1/2}\psi_0(\lambda_0,x)^{-1}\psi_0'(\lambda_0,x) - \alpha p_0(x)^{-1/2}\psi_0(\lambda_0,x)^{-2} \\
&\qquad \times \Big(\int^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-1}, \quad x \in (R,b).
\end{aligned} \tag{1.14}
$$

Moreover, as our second major result we shall prove in Theorem 3.1 the following natural generalization (from the point of view of quadratic form perturbations) of Hardy's inequality (1.11):

Assume $\psi_0(\lambda_0,x) > 0$ *for* $x \in (a,b)$, $|\int_a p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt| < \infty$, *and* $|\int^b p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt| = \infty$. *Then*

$$
\begin{aligned}
\int_a^b p_0(x)|\phi'(x)|^2\,dx &> \int_a^b \Big[\lambda_0 - q_0(x) + 4^{-1}p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4} \\
&\quad \times \Big(\int_a^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-2}\Big]|\phi(x)|^2\,dx, \quad 0 \neq \phi \in \mathcal{D}_0((a,b)),
\end{aligned} \tag{1.15}
$$

where

$$\mathcal{D}_0((a,b)) = \{f \in L^2((a,b)) | f \in AC_{loc}((a,b));\ supp(f) \subset (a,b)\ compact;\ p_0^{1/2}f' \in L^2((a,b))\}. \tag{1.16}$$

(Here $AC_{loc}(\Omega)$ *denotes the set of locally absolutely continuous functions on* $\Omega \subset \mathbb{R}$.*) The constant* $1/4$ *in* (1.15) *(and* (1.11)*) is optimal.*

A look at (1.8) and (1.12) reveals that the role of the basic comparison potential

$$q_\mu^{(0)}(x) = \mu(x-a)^{-2}, \quad x > a, \tag{1.17}$$

in the special case where $p_0 = \psi_0 = 1$, $q_0 = \lambda_0 = 0$, $\tau_0^{(0)} = -\frac{d^2}{dx^2}$, is played by

$$q_\mu^0(x) = \mu p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4}\Big(\int_a^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-2}, \quad x \in (a,b) \tag{1.18}$$

in the general Sturm–Liouville case $\tau = \tau_0 + q(x)$, where $\tau_0 = -\frac{d}{dx}p_0(x)\frac{d}{dx} + q_0(x)$.

We close this introduction with a heuristic comment on why the comparison potential (1.18) is a natural generalization of (1.17). Taking $a = 0$, $b = \infty$, $\lambda_0 = 0$ for simplicity, one considers

$$\tau_0^{(0)} = -\frac{d^2}{dx^2} \quad \text{and} \quad \mu x^{-2}, \quad x > 0 \tag{1.19}$$

and hence gets a fundamental system of solutions

$$\psi_0(x) = 1, \quad \widehat{\psi_0}(x) = x \tag{1.20}$$

of $\tau_0\psi = 0$, where $\psi_0(x)$ is principal near ∞ and nonprincipal near 0 whereas $\widehat{\psi_0}(x)$ is principal near 0 but nonprincipal near ∞. In particular, one observes that

$$\mu x^{-2} = \mu(\psi_0(x)\widehat{\psi_0}(x))^{-2}, \quad x > 0. \tag{1.21}$$

This trivial fact is no accident. Indeed, consider

$$\tau_0 = -\frac{d}{dx}p_0(x)\frac{d}{dx} + q_0(x), \quad x > 0 \tag{1.22}$$

and assume $\psi_0(x) > 0$ is a principal solution of $\tau_0\psi = 0$ near ∞, but nonprincipal near 0. Then it is a well–known fact (see, e.g., [13], Sect. XI.6) that

$$\widehat{\psi_0}(x) = \psi_0(x) \int_0^x p_0(t)^{-1}\psi_0(t)^{-2}\, dt \tag{1.23}$$

is another linearly independent solution of $\tau_0\psi = 0$ which is principal near 0 and nonprincipal near ∞. Moreover,

$$\mu p_0(x)^{-1}(\psi_0(x)\widehat{\psi_0}(x))^{-2} = \mu p_0(x)^{-1}\psi_0(x)^{-4}\Big(\int_0^x p_0(t)^{-1}\psi_0(t)^{-2}\, dt\Big)^{-2} \tag{1.24}$$

is precisely of the form (1.18).

2. A Generalization of Kneser's Theorem

In order to set the stage we assume

$$0 < p_0^{-1} \in L^1_{loc}((a,b)),\ q_0, q \in L^1_{loc}((a,b)) \text{ real–valued},\ -\infty \leq a < b \leq \infty \tag{2.1}$$

and introduce the unperturbed (quasi) differential expression

$$\tau_0 = -\frac{d}{dx}p_0(x)\frac{d}{dx} + q_0(x), \quad x \in (a,b) \tag{2.2}$$

and its perturbation τ,

$$\tau = \tau_0 + q(x) = -\frac{d}{dx}p_0(x)\frac{d}{dx} + q_0(x) + q(x), \quad x \in (a,b). \tag{2.3}$$

We recall that $(\tau - \lambda)$, $\lambda \in \mathbb{R}$ is called nonoscillatory near b if and only if $\tau\psi = \lambda\psi$ has a real–valued solution $\psi(\lambda, x)$ which has finitely many zeros near b (in this case all real–valued solutions of $\tau\psi = \mu\psi$ with $\mu \leq \lambda$ share this property). Otherwise, that is, if one (and hence all) real–valued solution of $\tau\psi = \lambda\psi$ has b as an accumulation point of zeros, $(\tau - \psi)$ is called oscillatory. (In this case all real–valued solutions ψ of $\tau\psi = \mu\psi$ with $\mu \geq \lambda$ share this property.) Similarly, τ is called to be in the limit point ($l.p.$) case at b if and only if there exists a $z \in \mathbb{C}$ such that not all solutions ψ of $\tau\psi = z\psi$ are L^2 near b. Otherwise, τ is called to be in the limit circle ($l.c.$) case at b. Finally, a solution ψ_0 of $\tau\psi = \lambda_0\psi$ with $\psi_0 > 0$ near b is called principal near b if and only if $|\int^b p_0(x)^{-1}\psi_0(x)^{-2}\, dx| = \infty$. Every linearly independent solution $\widehat{\psi_0}$ of $\tau\psi = \lambda_0\psi$ then satisfies $|\int^b p_0(x)^{-1}\widehat{\psi_0}(x)^{-2}\, dx| < \infty$ and is called nonprincipal (see, e.g., [13], Sect. XI.6). Analogous definitions apply at the point a.

In addition to (2.1), we shall need a further spectral hypothesis in connection with τ_0 and hence introduce our basic hypothesis

(H.2.1). *Suppose p_0, q_0, and q satisfy* (2.1). *Moreover, assume that $(\tau_0 - \lambda_0)$ is nonoscillatory near b (resp. a) for some $\lambda_0 \in \mathbb{R}$, and let $\psi_0(\lambda_0, x)$ denote a solution of $\tau_0\psi = \lambda_0\psi$ which is positive in a neighborhood of b (resp. a).*

Occasionally we shall consider maximal and minimal operators H_0, H and $\check{H}_0$, $\check{H}$ in $L^2((a,b))$ associated with τ_0 and τ defined as follows,

$$H_{(0)}f = \tau_{(0)}f, \quad f \in \mathcal{D}(H_{(0)}) = \{g \in L^2((a,b)) |\ g,\ p_0 g' \in AC_{loc}((a,b)); \tau_{(0)}g \in L^2((a,b))\}, \tag{2.4}$$

$$\check{H}_{(0)}f = \tau_{(0)}f,\ f \in \mathcal{D}(\check{H}_{(0)}) = \{g \in \mathcal{D}(H_{(0)}) |\ \mathrm{supp}(g) \subset (a,b) \text{ compact}\}, \tag{2.5}$$

where $H_{(0)}$, $\check{H}_{(0)}$ abbreviates H_0 or H, $\check{H}_0$ or $\check{H}$. If and only if $\tau_{(0)}$ is in the $l.p.$ case at a and b one obtains

$$\overline{\check{H}_{(0)}} = H_{(0)}, \tag{2.6}$$

where $-$ denotes the operator closure in $L^2((a,b))$, otherwise one has

$$\overline{\check{H}_{(0)}} \underset{\neq}{\subset} H_{(0)}. \tag{2.7}$$

Given these preliminaries, we can now state our generalization of Kneser's [21] criterion.

Theorem 2.2. *Assume Hypothesis (H.2.1) near b. If* $|\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx| = \infty$ *(i.e., if* $\psi_0(\lambda_0,x)$ *is principal near b) choose* $R\in(a,b)$ *in a sufficiently small neighborhood of b such that* $\psi_0(\lambda_0,x)>0$ *on* (R,b)*. Then,*
(i) If

$$\liminf_{x\uparrow b}\,\{p_0(x)\psi_0(\lambda_0,x)^4\Big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^2 q(x)\} > -1/4, \tag{2.8}$$

$(\tau-\lambda_0)$ *is nonoscillatory near b.*
(ii) If

$$\limsup_{x\uparrow b}\,\{p_0(x)\psi_0(\lambda_0,x)^4\Big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^2 q(x)\} < -1/4, \tag{2.9}$$

$(\tau-\lambda_0)$ *is oscillatory near b.*
If $|\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx|<\infty$ *(i.e., if* $\psi_0(\lambda_0,x)$ *is nonprincipal near b) replace* $\int_R^x$ *by* $\int_x^b$ *in* (2.8) *and* (2.9) *to reach the same conclusions.*

Instead of proving Theorem 2.2 at this point we shall actually consider a generalization, implying a scale of new oscillation criteria. We start with a bit of notation. Define

$$\ln_0(x)=|x|,\ \ln_n(x)=\ln(\ln_{n-1}(x)),\quad n\in\mathbb{N},\ x\in\mathbb{R}\backslash\{0\}, \tag{2.10}$$

$$\begin{aligned} q_{1,\mu}(x) &= \mu x^{-2},\\ q_{n,\mu}(x) &= x^{-2}\Big[-4^{-1}\sum_{k=0}^{n-2}\prod_{j=1}^{k}\big(\ln_j(x)\big)^{-2}+\mu\prod_{j=1}^{n-1}\big(\ln_j(x)\big)^{-2}\Big],\\ &= x^{-2}[-4^{-1}+q_{n-1,\mu}(\ln|x|)],\quad \mu\in\mathbb{R},\ n\geq 2, \end{aligned} \tag{2.11}$$

$$\begin{aligned} Q_{n,\mu}(x) &= p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4}\\ &\times\begin{cases} q_{n,\mu}(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt) & \text{if } |\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx|=\infty\\ q_{n,\mu}(\int_x^b p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt) & \text{if } |\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx|<\infty \end{cases},\\ &\qquad n\in\mathbb{N}. \end{aligned} \tag{2.12}$$

(As usual we denote $\prod_{j=1}^0(\cdot)=1$ in (2.11).) Explicitly, one obtains

$$\begin{aligned} Q_{1,\mu}(x) &= \mu p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4}\\ &\times\begin{cases} \big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)^{-2} & \text{if } |\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx|=\infty\\ \big(\int_x^b p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)^{-2} & \text{if } |\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx|<\infty \end{cases}, \end{aligned}$$

$$Q_{2,\mu}(x) = p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4}\begin{Bmatrix} \big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)^{-2} \\ \big(\int_x^b p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)^{-2} \end{Bmatrix}$$
$$\times\left\{-4^{-1}+\mu\left(\ln\begin{Bmatrix} \big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big) \\ \big(\int_x^b p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big) \end{Bmatrix}\right)^{-2}\right\} \begin{matrix} \text{if } |\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx| = \infty \\ \text{if } |\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx| < \infty \end{matrix}, \tag{2.13}$$

etc.

Our first major result then reads as follows.

Theorem 2.3. *Assume Hypothesis (H.2.1) near b. If $|\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx| = \infty$ choose $R \in (a,b)$ sufficiently close to b such that $\psi_0(\lambda_0,x) > 0$ on (R,b). Suppose*

$$\begin{aligned} &\lim_{x\uparrow b}\Big\{p_0(x)\psi_0(\lambda_0,x)^4\Big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}dt\Big)^2 \\ &\times\prod_{j=1}^{n-2}\Big[\ln_j\big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)\Big]^2\Big[q(x)+4^{-1}p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4} \\ &\times\Big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-2}\sum_{k=0}^{n-3}\prod_{j=1}^{k}\Big(\ln_j\big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)\Big)^2\Big]\Big\} = -1/4, \quad n\geq 3. \end{aligned}\tag{2.14}$$

Then,
(i) If

$$\begin{aligned} &\liminf_{x\uparrow b}\Big\{p_0(x)\psi_0(\lambda_0,x)^4\Big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}dt\Big)^2\prod_{j=1}^{n-1}\Big[\ln_j\big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}dt\big)\Big]^2\Big[q(x) \\ &+4^{-1}p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4}\Big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-2}\sum_{k=0}^{n-2}\prod_{j=1}^{k}\Big(\ln_j\big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)\Big)^2\Big]\Big\} \\ &> -1/4, \end{aligned}\tag{2.15}$$

$(\tau-\lambda_0)$ is nonoscillatory near b.
(ii) If

$$\begin{aligned} &\limsup_{x\uparrow b}\Big\{p_0(x)\psi_0(\lambda_0,x)^4\Big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}dt\Big)^2\prod_{j=1}^{n-1}\Big[\ln_j\big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)\Big]^2\Big[q(x) \\ &+4^{-1}p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4}\Big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-2}\sum_{k=0}^{n-2}\prod_{j=1}^{k}\Big(\ln_j\big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)\Big)^2\Big]\Big\} \\ &< -1/4, \end{aligned}\tag{2.16}$$

$(\tau-\lambda_0)$ is oscillatory near b.

If $|\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx| < \infty$, replace $\int_R^x$ by $\int_x^b$ in (2.14)–(2.16) *in order to reach the same conclusions.*

Proof. By Sturm's comparison theorem (see, e.g., [4], Lemma XIII.7.35) it suffices to describe solutions of

$$\tau_{n,\mu}\psi = \lambda_0\psi, \quad \tau_{n,\mu} = \tau_0 + Q_{n,\mu}(x), \quad n\geq 2. \tag{2.17}$$

Assuming $|\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx| = \infty$, one verifies inductively that fundamental systems of solutions of (2.17) are given by (*a*) $\mu = -1/4$:

$$\psi_1(\lambda_0,x) = \psi_0(\lambda_0,x)\Big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{1/2}\prod_{j=1}^{n-1}\Big(\ln_j\big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)\Big)^{1/2},$$
$$\psi_2(\lambda_0,x) = \psi_1(\lambda_0,x)\ln_n\Big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big). \tag{2.18}$$

(*b*) $\mu > -1/4$:

$$\psi_p(\lambda_0,x) = \psi_0(\lambda_0,x)\Big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{1/2}\prod_{j=1}^{n-1}\Big(\ln_j\big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)\Big)^{1/2}$$
$$\times\Big(\ln_{n-1}\big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)\Big)^{(-1)^p(\mu+4^{-1})^{1/2}}, \quad p=1,2. \tag{2.19}$$

(*c*) $\mu < -1/4$:

$$\psi_p(\lambda_0,x) = \psi_0(\lambda_0,x)\Big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{1/2}\prod_{j=1}^{n-1}\Big(\ln_j\big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)\Big)^{1/2}$$
$$\times\exp\Big[(-1)^p i(-\mu-4^{-1})^{1/2}\ln_n\big(\int_R^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)\Big], \quad p=1,2. \tag{2.20}$$

This proves (2.15) and (2.16). If $|\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx| < \infty$, consistently replacing $\int_R^x$ by $\int_x^b$ in (2.14)–(2.16) and (2.18)–(2.20) completes the proof. $\square$

The special case $p_0 = \psi_0 = 1$, $q_0 = \lambda_0 = 0$ in Theorem 2.2 is Kneser's [21] original criterion, while that special case in Theorem 2.3 represents a generalization of Kneser's result originally due to Weber [34], p.53–62 and later rediscovered by Hartman [12] and Hille [14].

Theorem 2.2 (and 2.3) can also be proved by reducing it to Kneser's criterion (1.7) using an appropriate change of variables (Kummer–type transformations). Moreover, a variety of extensions of Kneser's original criterion (1.7) corresponding to $p_0 = 1$, $q_0 = \lambda_0 = 0$ can be generalized to our perturbative setting which relates p_0, q_0 and $p_0, q_0 + q$. Since a comprehensive treatment is beyond the scope of the present paper we omit further details.

Next we describe a simple application of Theorem 2.2 to the case where τ_0 is periodic, $(a,b) = \mathbb{R}$, $p_0 = 1$, and $q(x) \underset{x\to\infty}{=} O(x^{-2})$. In addition to assuming that q_0 satisfies Hypothesis (H.2.1) near $\pm\infty$ we suppose that q_0 is periodic of period one, that is,

$$p_0(x) = 1, \quad q_0(x+1) = q_0(x), \quad x \in \mathbb{R}, \tag{2.21}$$

and that λ_0 is the lowest periodic eigenvalue associated with $\tau_0|_{[0,1]}$ and $\psi_0 > 0$ the corresponding unique eigenfunction extended to all of $\mathbb{R}$ by periodicity, that is,

$$\psi_0(x+1) = \psi_0(x), \quad x \in \mathbb{R}. \tag{2.22}$$

Introducing the operator H_0 in $L^2(\mathbb{R})$ associated with τ_0 as in (2.4) one infers that H_0 is self–adjoint and bounded from below. In fact,

$$\sigma(H_0) = \sigma_{\rm ess}(H_0) \subseteq [\lambda_0,\infty), \tag{2.23}$$

where $\sigma(\cdot)$ and $\sigma_{\mathrm{ess}}(\cdot)$ abbreviate the spectrum and essential spectrum, respectively. Concerning the perturbation q we assume

$$q \in L^\infty(\mathbb{R}) \text{ real–valued}, \quad q(x) \underset{|x|\to\infty}{=} c|x|^{-2}[1+o(1)], \quad c>0, \tag{2.24}$$

introduce $\tau = \tau_0 + q(x)$, and define the self–adjoint operator H associated with τ as in (2.4). (One could easily admit local singularities of q as long as q stays locally integrable but this is not the point of our investigation below.) Since $q(x) \underset{|x|\to\infty}{\to} 0$ one obtains

$$\sigma_{\mathrm{ess}}(H) = \sigma(H_0). \tag{2.25}$$

Finally we abbreviate

$$\min_{x\in[0,1]} \psi_0(\lambda_0, x) = \psi_{0,\min}, \quad \max_{x\in[0,1]} \psi_0(\lambda_0,x) = \psi_{0,\max} \tag{2.26}$$

and

$$A_0 = \int_0^1 \psi_0(\lambda_0,t)^{-2}\, dt. \tag{2.27}$$

Given these preliminaries one easily obtains

Lemma 2.4. *Introduce the notation established in* (2.21)–(2.27)*. Then,*
(i) If

$$c > -4^{-1}\psi_{0,\max}^{-4} A_0^{-2} \ (\geq -4^{-1}), \tag{2.28}$$

$(\tau - \lambda_0)$ *is nonoscillatory near* $\pm\infty$.
(ii) If

$$c < -4^{-1}\psi_{0,\min}^{-4} A_0^{-2} \ (\leq -4^{-1}), \tag{2.29}$$

$(\tau - \lambda_0)$ *is oscillatory near* $\pm\infty$.
Equality with $-1/4$ *on the right–hand–sides of* (2.28) *and* (2.29) *holds if and only if* $q_0(x) = \textit{constant}$.

Proof. Consider $x > 0$. (2.28) and (2.29) immediately follow from (2.8) and (2.9) upon decomposing $x = [x] + r(x)$, $0 \leq r(x) < 1$, $[x] = \inf_{n\in\mathbb{N}}\{n \leq x\}$, the integer part of $x > 0$, and noting that

$$\int_0^x \psi_0(\lambda_0,t)^{-2}\, dt = A_0[x] + s(x), \quad 0 \leq s(x) < A_0,\ x > 0. \tag{2.30}$$

Thus

$$\lim_{x\to+\infty} \frac{\inf}{\sup}\{\psi_0(\lambda_0,x)^4 \Big(\int_0^x \psi_0(\lambda_0,t)^{-2}\, dt\Big)^2 c\, x^{-2}\} = c\,\psi^4_{0,{\max \atop \min}} A_0^2. \tag{2.31}$$

The proof for $x < 0$ is similar. Equality with $-1/4$ on the right–hand–sides of (2.28) and (2.29) holds if and only if $\psi_0(\lambda_0, x)$ is constant, that is, if and only if $q(x)$ is constant. $\square$

In order to fill the gap left between (2.28) and (2.29), that is, to determine the (non)oscillation status of $\tau = \tau_0 + c(1+|x|^2)^{-1}$ for

$$c \in [-4^{-1}\psi_{0,\min}^{-4} A_0^{-2}, -4^{-1}\psi_{0,\max}^{-4} A_0^{-2}], \tag{2.32}$$

one needs to invoke more detailed information on the periodic background operator H_0. The result stated in [20] (without proof) requires the additional knowledge of the effective mass m_0 associated with $\lambda_0 = \inf \sigma(H_0)$ (and similar for higher spectral gaps of H_0).

Interest in the periodic operator H_0 and its perturbation H stems from possible applications in solid state physics. In fact, q_0 models one–dimensional periodic structures such as crystals and q describes additional impurities or defects (responsible, e.g., for the color of crystals). Moreover, as a further illustration of (2.28) and (2.29) it should be mentioned that Rofe–Beketov [27] proved that H has finitely many eigenvalues in each spectral gap of H_0 whenever $q \in L^1(\mathbb{R}; (1+|x|)dx)$.

3. Hardy-Type Inequalities and Birman–Schwinger Bounds

In our final section we discuss factorizations of the type (1.13), (1.14) and use them to prove a natural generalization of Hardy's inequality (1.11) from the point of view of quadratic form perturbations. At the end we illustrate some explicit bounds on the number of bound states of H below a fixed real number $\lambda_0 \leq \inf \sigma(H_0)$.

We start with a factorization of our comparison differential expression on (a,b),

$$\tau_\mu = -\frac{d}{dx} p_0(x) \frac{d}{dx} + q_0(x) + \mu p_0(x)^{-1} \psi_0(\lambda_0,x)^{-4} \Big(\int_{x_0}^x p_0(t)^{-1} \psi_0(\lambda_0,x)^{-2}\, dt \Big)^{-2} \quad \text{for some } a \leq x_0 \leq b,\ \mu \in \mathbb{R}. \tag{3.1}$$

Introducing

$$A_\alpha = p_0(x)^{1/2} \frac{d}{dx} - p_0(x)^{1/2} \psi_0(\lambda_0,x)^{-1} \psi_0'(\lambda_0,x) - \alpha p_0(x)^{-1/2} \psi_0(\lambda_0,x)^{-2} \times \Big(\int_{x_0}^x p_0(t)^{-1} \psi_0(\lambda_0,x)^{-2}\, dt \Big)^{-1}, \quad \alpha \in \mathbb{C}, \tag{3.2}$$

$$A_\alpha^+ = -\frac{d}{dx} p_0(x)^{1/2} - p_0(x)^{1/2} \psi_0(\lambda_0,x)^{-1} \psi_0'(\lambda_0,x) - \alpha p_0(x)^{-1/2} \psi_0(\lambda_0,x)^{-2} \times \Big(\int_{x_0}^x p_0(t)^{-1} \psi_0(\lambda_0,x)^{-2}\, dt \Big)^{-1}, \tag{3.3}$$

one verifies that

$$\tau_\mu = A_\alpha^+ A_\alpha + \lambda_0 \quad \text{if and only if} \quad \mu = \alpha(\alpha - 1). \tag{3.4}$$

Moreover, solutions of $\tau_\mu \psi = \lambda_0 \psi$ are given by

$$\psi_\mu(\lambda_0,x) = \psi_0(\lambda_0,x) \Big(\int_{x_0}^x p_0(t)^{-1} \psi_0(\lambda_0,x)^{-2}\, dt \Big)^{\frac{1}{2} \pm (\mu + \frac{1}{4})^{\frac{1}{2}}}. \tag{3.5}$$

The factorization (3.4) of τ is possible for $\alpha \in \mathbb{R}$ if and only if $\mu \geq -1/4$. In this case A_α^+ is the formal adjoint of A_α, $\alpha \in \mathbb{R}$.

Our second main result now reads as follows.

Theorem 3.1. *Assume p_0, q_0, and $\psi_0(\lambda_0)$ satisfy (H.2.1) near a and b and $\psi_0(\lambda_0,x) > 0$ for all $x \in (a,b)$. Moreover, suppose $|\int_a p_0(x)^{-1} \psi_0(\lambda_0,x)^{-2}\, dx| < \infty$ and $|\int^b p_0(x)^{-1} \psi_0(\lambda_0,x)^{-2}\, dx| = \infty$ (i.e., $\psi_0(\lambda_0,x)$ is nonprincipal near a and principal near b). Then for all $0 \neq \phi \in \mathcal{D}_0((a,b))$,*

$$\int_a^b p_0(x) |\phi'(x)|^2\, dx > \int_a^b \big[\lambda_0 - q_0(x) + 4^{-1} p_0(x)^{-1} \psi_0(\lambda_0,x)^{-4} \big(\int_a^x p_0(t)^{-1} \psi_0(\lambda_0,t)\, dt \big)^{-2} \big] |\phi(x)|^2\, dx, \tag{3.6}$$

where

$$\mathcal{D}_0((a,b)) = \{ f \in L^2((a,b)) |\, f \in AC_{loc}((a,b));\ \mathrm{supp}(f) \subset (a,b) \text{ compact},\ p_0^{1/2} f' \in L^2((a,b)) \,\}. \tag{3.7}$$

The constant $1/4$ in (3.6) is optimal.

Proof. Choose $c,\ d \in (a,b)$, $c<d$ and $f \in AC_{loc}((a,b))$. Then

$$\int_c^d \psi_0(\lambda_0,x)^2\big(\int_a^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}dt\big)p_0(x)\Big|\Big[f(x)\psi_0(\lambda_0,x)^{-1}\Big(\int_a^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-1/2}\Big]'\Big|^2\,dx$$
$$= \int_c^d \Big\{p_0(x)|f'(x)|^2 + 4^{-1}p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4}\big(\int_a^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\big)|f(x)|^2$$
$$+ \psi_0(\lambda_0,x)^{-2}p_0(x)\psi_0'(\lambda_0,x)^2|f(x)|^2 + \psi_0(\lambda_0,x)^{-3}\psi_0'(\lambda_0,x)\Big(\int_a^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-1}|f(x)|^2$$
$$- \Big[\overline{f(x)}f'(x) + f(x)\overline{f'(x)}\Big]\Big[\psi_0(\lambda_0,x)^{-1}p_0(x)\psi_0'(\lambda_0,x)$$
$$+ 2^{-1}\psi_0(\lambda_0,x)^{-2}\Big(\int_a^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-1}\Big]\Big\}\,dx$$
$$= \int_c^d \Big\{p_0(x)|f'(x)|^2 + [q_0(x)-\lambda_0]|f(x)|^2$$
$$- 4^{-1}p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4}\Big(\int_a^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-2}|f(x)|^2\Big\}\,dx$$
$$- |f(x)|^2\Big[p_0(x)\psi_0(\lambda_0,x)^{-1}\psi_0'(\lambda_0,x) + 2^{-1}\psi_0(\lambda_0,x)^{-2}\Big(\int_a^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-1}\Big]\Big|_{x=c}^d \geq 0 \tag{3.8}$$

integrating by parts once. Choosing $f \in \mathcal{D}_0((a,b))$ and letting $c\downarrow a$, $d\uparrow b$ then proves

$$\int_a^b p_0(x)|f'(x)|^2\,dx \geq \int_a^b \Big\{\big[\lambda_0 - q_0(x)\big]|f(x)|^2 + 4^{-1}p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4}$$
$$\times\Big(\int_a^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-2}|f(x)|^2\Big\}\,dx. \tag{3.9}$$

Since equality in (3.8) (and hence (3.9)) can only hold for

$$f(x)=0 \quad \text{and} \quad f(x) = \psi_0(\lambda_0,x)\Big(\int_a^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{1/2}, \tag{3.10}$$

(3.6) results. The constant $1/4$ in (3.6) is optimal since (3.5) becomes oscillating for $\mu < -1/4$. $\square$

Note that $\psi_0(\lambda_0,x_0) > 0$ on (a,b) in Theorem 3.1 implies that

$$\overline{\check{H}_0} \geq \lambda_0 \tag{3.11}$$

(cf. (2.6)).

Alternatively, one can prove Theorem 3.1 using

$$\int_c^d |(A_{1/2}f)(x)|^2\,dx = (3.8), \quad x_0 = a \tag{3.12}$$

(cf. (3.1)–(3.5)).

In order to avoid the compact support assumption on ϕ in Theorem 3.1, one needs to guarantee the vanishing of the boundary terms in (3.8) as $c\downarrow a$ and $d\uparrow b$. The following observations illustrate such a possibility.

We need to introduce a bit of notation. Define in $L^2((a,b))$,

$$\check{H}_0^0 f = \tau_0^0 f, \quad \tau_0^0 = -\frac{d}{dx}p_0(x)\frac{d}{dx}, \quad x\in(a,b), \tag{3.13}$$

$$f \in \mathcal{D}(\check{H}_0^0) = \{g \in L^2((a,b))\,|\, g,\ p_0 g' \in AC_{loc}((a,b))\,;\mathrm{supp}(g)\ \subset\ (a,b)\text{ compact}\,;\tau_0^0 g \in L^2((a,b))\},$$

$$(A_0^0 f)(x) = p_0(x)^{1/2} f'(x),\ f \in \mathcal{D}(A_0^0) = \{g \in L^2((a,b)) \,|\, g \in AC_{loc}((a,b));\ p_0^{1/2} g' \in L^2((a,b))\}. \tag{3.14}$$

Then one infers

$$\mathcal{D}(\check{H}_0^0) \subset \mathcal{D}(A_0^0) \tag{3.15}$$

since

$$p_0^{1/2} f' = p_0^{-1/2}(p_0 f') \in L^2((a,b)), \quad f \in \mathcal{D}(\check{H}_0^0) \tag{3.16}$$

using $p_0^{1/2} \in L^2_{loc}((a,b))$ and $p_0^{1/2} f' \in L^p((a,b)) \cap AC_{loc}((a,b))$, $1 \leq p \leq \infty$ since $\operatorname{supp}(f) \subset (a,b)$ is compact. We also introduce

$$(\check{A}_0^0 f)(x) = p_0(x)^{1/2} f'(x), \quad f \in \mathcal{D}(\check{A}_0^0) = \mathcal{D}(\check{H}_0^0). \tag{3.17}$$

Next we define the form $\check{Q}_0^0$ in $L^2((a,b))$,

$$\check{Q}_0^0(f,g) = (f, \check{H}_0^0 g), \quad \mathcal{D}(\check{Q}_0^0) = \mathcal{D}(\check{H}_0^0). \tag{3.18}$$

One verifies,

$$\begin{aligned}\check{Q}_0^0(f,g) &= -\lim_{\substack{c\downarrow a\\ d\uparrow b}} \int_c^d \overline{f(x)}(p_0(x) g'(x))'\, dx \ = -\lim_{\substack{c\downarrow a\\ d\uparrow b}} \overline{f(x)} p_0(x) g'(x)|_{x=c}^d \\ &\quad + \lim_{\substack{c\downarrow a\\ d\uparrow b}} \int_c^d p_0(x)\overline{f'(x)} g'(x)\, dx \ = (\check{A}_0^0 f, \check{A}_0^0 g), \quad f,\, g \in \mathcal{D}(\check{H}_0^0).\end{aligned} \tag{3.19}$$

Since $\check{H}_0^0 \geq 0$, the form $\check{Q}_0^0$ is closable ([18], p. 318) and we denote its closure by Q_0^0, that is, $Q_0^0 = \overline{\check{Q}_0^0}$. On the other hand, a form Q_S in some Hilbert space $\mathcal{H}$ of the type

$$Q_S(f,g) = (Sf, Sg), \quad \mathcal{D}(Q_S) = \mathcal{D}(S) \tag{3.20}$$

is closable (closed) if and only if S is closable (closed). In particular, if Q_S is closable, its closure is given by

$$\overline{Q_S}(f,g) = (\overline{S}f, \overline{S}g), \quad \mathcal{D}(\overline{Q_S}) = \mathcal{D}(\overline{S}), \tag{3.21}$$

with $\overline{S}$ the operator closure of S in $\mathcal{H}$ (see [18], p. 311, 317). Applied to (3.19) this yields that $\check{A}_0^0$ is closable and that the Friedrichs extension H_0^0 of $\check{H}_0^0$, that is, the unique self–adjoint nonnegative operator associated with $Q_0^0 = \overline{\check{Q}_0^0}$,

$$Q_0^0(f,g) = (\overline{\check{A}_0^0} f, \overline{\check{A}_0^0} g), \quad f,g \in \mathcal{D}(\overline{\check{A}_0^0}) \tag{3.22}$$

($\overline{\check{A}_0^0}$ the operator closure of $\check{A}_0^0$), is given by

$$H_0^0 = \left(\overline{\check{A}_0^0}\right)^* \overline{\check{A}_0^0} \tag{3.23}$$

(cf. [8], Theorem 2.7, [18], p. 322–326, [29]). We note that

$$\mathcal{D}(\overline{\check{A}_0^0}) \subseteq \{g \in L^2((a,b)) \,|\, g \in AC_{loc}((a,b));\ p_0^{1/2} g' \in L^2((a,b))\} = \mathcal{D}(A_0^0) \tag{3.24}$$

since A_0^0 is a closed operator in $L^2((a,b))$, that is,

$$\overline{A_0^0} = A_0^0. \tag{3.25}$$

(3.25) directly proves that $\check{A}_0^0$ is closable. Even though (3.25) might seem obvious, we prefer to provide a straightforward proof. Pick $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(A_0^0)$ and assume

$$f_n \underset{n\to\infty}{\xrightarrow{s}} f \in L^2((a,b)) \quad \text{and} \quad p_0^{1/2} f_n' \underset{n\to\infty}{\xrightarrow{s}} g \in L^2((a,b)). \tag{3.26}$$

In order to prove (3.25) one needs to show that

$$f \in \mathcal{D}(A_0^0) \quad \text{and} \quad g = p_0^{1/2} f'. \tag{3.27}$$

Since $f_n \in AC_{loc}((a,b))$, we may write

$$f_n(x) = \int_c^x f_n'(t)\,dt + f_n(c), \quad c,\, x \in (a,b). \tag{3.28}$$

Next, since $g \in L^2((a,b))$ and $p_0^{-1/2} \in L^2_{loc}((a,b))$ by (2.1), one infers $p_0^{-1/2} g \in L^1_{loc}((a,b))$ and hence $\int_c^x p_0(t)^{-1/2} g(t)\,dt, \quad c, x \in (a,b)$ is well–defined. One obtains by Cauchy's inequality,

$$\begin{aligned} \Big|\int_c^x \big[f_n'(t) - p_0(t)^{-1/2} g(t)\big]\,dt\Big| &\leq \left(\int_c^x p_0(t)^{-1}\,dt\right)^{1/2} \left(\int_c^x |p_0(t)^{1/2} f_n'(t) - g(t)|^2\,dt\right)^{1/2} \\ &\leq \left(\int_c^x p_0(t)^{-1}\,dt\right)^{1/2} \| p_0^{1/2} f_n' - g \|_2 \underset{n\to\infty}{\longrightarrow} 0, \quad x \in (a,b). \end{aligned} \tag{3.29}$$

Next, using (3.28), one estimates

$$|f_n(c) - f_m(c)|^2 \leq 2|f_n(x) - f_m(x)|^2 + 2\left(\int_c^x |f_n'(t) - f_m'(t)|\,dt\right)^2, \tag{3.30}$$

and choosing $\epsilon > 0$ such that $(c-\epsilon, c+\epsilon) \subset (a,b)$, integrating (3.30) from $c-\epsilon$ to $c+\epsilon$ yields

$$\begin{aligned} &2\epsilon |f_n(c) - f_m(c)|^2 \\ &\leq 2 \| f_n - f_m \|_2^2 + 2 \int_{c-\epsilon}^{c+\epsilon} \left(\int_c^x p_0(t)^{-1}\,dt\right) \left(\int_c^x |p_0(s)^{\frac12} f_n'(s) - p_0(s)^{\frac12} f_m'(s)|^2\,ds\right) dx \\ &\leq 2 \| f_n - f_m \|_2^2 + 2 \| p_0^{1/2} f_n' - p_0^{1/2} f_m' \|_2^2 \int_{c-\epsilon}^{c+\epsilon} \left(\int_c^x p_0(t)^{-1}\,dt\right) dx \underset{n,m\to\infty}{\longrightarrow} 0. \end{aligned} \tag{3.31}$$

Thus $\{f_n(c)\}_{n\in\mathbb{N}}$ is a Cauchy sequence and

$$f_n(c) \underset{n\to\infty}{\longrightarrow} F \in \mathbb{C} \tag{3.32}$$

implying

$$f_n(x) \underset{n\to\infty}{\longrightarrow} \int_c^x p_0(t)^{-1/2} g(t)\,dt + F \quad \text{pointwise} \tag{3.33}$$

for all $x \in (a,b)$. Since $f_n \overset{s}{\underset{n\to\infty}{\longrightarrow}} f$ in $L^2((a,b))$ guarantees the existence of a subsequence $\{f_{n_k}(x)\}_{k\in\mathbb{N}} \subset \{f_n(x)\}_{n\in\mathbb{N}}$ which converges *a.e.* to $f(x)$ on (a,b), that is,

$$f_{n_k}(x) \underset{k\to\infty}{\longrightarrow} f(x) \quad \text{for a.e. } x \in (a,b), \tag{3.34}$$

uniqueness of the limit yields

$$f(x) = \int_c^x p_0(t)^{-1/2} g(t)\,dt + F \quad \text{for a.e. } x \in (a,b). \tag{3.35}$$

In particular, this proves

$$f \in AC_{loc}((a,b)),\ f' = p_0^{-1/2} g \quad \text{a.e. on } (a,b), \text{ and } p_0^{1/2} f' \in L^2((a,b)) \tag{3.36}$$

and hence (3.27).

The inclusion (3.24) yields

$$Q_0^0(f,g) = \int_a^b p_0(x) \overline{f'(x)} g'(x)\,dx, \quad f,\, g \in \mathcal{D}(\overline{\breve{A}_0^0}) = \mathcal{D}(Q_0^0). \tag{3.37}$$

Finally we need to introduce the following closed forms in $L^2((a,b))$,

$$Q_{q_0}(f,g)=(|q_0|^{1/2}f,\operatorname{sgn}(q_0)|q_0|^{1/2}g),\quad \mathcal{D}(Q_{q_0})=\mathcal{D}(|q_0|^{1/2}), \tag{3.38}$$

$$Q_{q_{0,\pm}}(f,g)=(|q_{0,\pm}|^{1/2}f,\operatorname{sgn}(q_{0,\pm})|q_{0,\pm}|^{1/2}g),\quad \mathcal{D}(Q_{q_{0,\pm}})=\mathcal{D}(|q_{0,\pm}|^{1/2}), \tag{3.39}$$

where

$$q_{0,\pm}(x)=\big[|q_0(x)|\pm q_0(x)\big]/2, \tag{3.40}$$

and

$$Q_{q^0_\mu}(f,g)=(|q^0_\mu|^{1/2}f,\operatorname{sgn}(q^0_\mu)|q^0_\mu|^{1/2}g),\quad \mathcal{D}(Q_{q^0_\mu})=\mathcal{D}(|q^0_\mu|^{1/2}),\quad \mu\in\mathbb{R}, \tag{3.41}$$

where

$$q^0_\mu(x)=\mu p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4}\left(\int_a^x p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\right)^{-2}. \tag{3.42}$$

Given these preliminaries we finally obtain the following generalization of Theorem 3.1.

Theorem 3.2. *Assume p_0, q_0 and $\psi_0(\lambda_0)$ satisfy (H.2.1) near a and b, $\psi_0(\lambda_0,x)>0$ for all $x\in(a,b)$, and suppose $|\int_a p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx|<\infty$ and $|\int^b p_0(x)^{-1}\psi_0(\lambda_0,x)^{-2}\,dx|=\infty$. In addition, assume that $\mathcal{D}(|q_{0,+}|^{1/2})\supseteq\mathcal{D}(Q^0_0)$. Then for all $\phi\in\mathcal{D}(Q^0_0)=\mathcal{D}(\overline{\check{A}^0_0})$,*

$$\begin{aligned}\int_a^b p_0(x)|\phi'(x)|^2\,dx\geq\int_a^b\Big[\lambda_0-q_0(x)+4^{-1}p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4}\\ \times\Big(\int_a^b p_0(x)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{-2}\Big]|\phi(x)|^2\,dx.\end{aligned} \tag{3.43}$$

Moreover, if $\psi_0(\lambda_0,\cdot)\Big(\int_a^\bullet p_0(t)^{-1}\psi_0(\lambda_0,t)^{-2}\,dt\Big)^{1/2}\notin\mathcal{D}(\overline{\check{A}^0_0})$, one obtains strict inequality in (3.43) for $\phi\neq0$.

Proof. By (3.6) and (3.15) we have

$$\int_a^b p_0(x)|\phi'(x)|^2\,dx\geq\int_a^b\big[\lambda_0-q_{0,+}(x)+q_{0,-}(x)+|q^0_{-1/4}(x)|\big]|\phi(x)|^2\,dx,\quad \phi\in\mathcal{D}(\check{Q}^0_0)=\mathcal{D}(\check{H}^0_0). \tag{3.44}$$

Since $\mathcal{D}(q_{0,+}^{1/2})\supseteq\mathcal{D}(\overline{\check{A}^0_0})$ implies

$$\| q_{0,+}^{1/2}f\|\leq C\,\|\overline{\check{A}^0_0}f\|+D\,\|f\|,\quad f\in\mathcal{D}(\overline{\check{A}^0_0}) \tag{3.45}$$

(cf. [18], p. 191), and $\mathcal{D}(\check{H}^0_0)$ is a form core of Q^0_0, (3.44) extends to all $\phi\in\mathcal{D}(Q^0_0)$. (3.37) then yields (3.43). The last part is clear from the proof of Theorem 3.1. □

Applied to the special case where $q_0(x)=\lambda_0=0$, $p_0(x)=\psi_0(0,x)=1$, (3.43) shows that (1.11) extends to all

$$\phi\in\{g\in L^2((a,\infty))\,|\,g\in AC_{loc}([a,R]),\ R>a;\ \lim_{\epsilon\downarrow0}g(a+\epsilon)=0;\ g'\in L^2((a,\infty))\}, \tag{3.46}$$

which is of course well–known (cf. [9], [10], [11], p. 240). A detailed history of Hardy's inequality can be found in [17]. For generalizations of Hardy's inequality in different directions and a detailed bibliography see, for instance, [5], [6], [17], [23] and the references therein.

We note that a condition of the type (3.45) is necessary to ensure that each individual term of the right–hand–side of (3.43) is finite for $\phi\in\mathcal{D}(\overline{\check{A}^0_0})$. For subtle cancellation effects and/or divergence of energy integrals of the type $\int_a^b p_0(x)|f'(x)|^2\,dx$, $\int_a^b q(x)|f(x)|^2\,dx$ for elements $f\in\mathcal{D}(H_0^0)$, the Friedrichs extension of $\check H_0^0$, see, for instance, [6], [16], [26], and [28].

Our final result below indicates how to bound the number of eigenvalues of the operator H in $L^2((a,b))$ associated with $\tau=\tau_0+q(x)$, $x\in(a,b)$ below the infimum λ_0 of the spectrum of H_0 (associated with τ_0). We shall assume that τ_0 is in the l.p. case at a and b.

We start with a few preparations concerning Green's functions associated with τ_0 on (a,b) and (x_0,b), (a,x_0), $x_0\in(a,b)$, respectively.

First, assume that $(\tau_0-\lambda_0)$ is subcritical on (a,b) (cf. [8]) and denote by $\psi_{0,\pm}(\lambda_0,x)>0$ the corresponding principal solutions of $\tau_0\psi=\lambda_0\psi$ near a and b, that is,

$$\Big|\int_a p_0(x)^{-1}\psi_{0,-}(\lambda_0,x)^{-2}\,dx\Big|=\infty=\Big|\int^b p_0(x)^{-1}\psi_{0,+}(\lambda_0,x)^{-2}\,dx\Big|. \tag{3.47}$$

Since τ_0 is assumed to be in the l.p. case at a and b, denote by $G_0(z,x,x')$ the Green's function of the uniquely associated operator H_0 (cf. (2.4)), that is, the integral kernel of $(H_0-z)^{-1}$. For $z=\lambda_0$ one obtains

$$G_0(\lambda_0,x,x')=\begin{cases}\psi_{0,+}(\lambda_0,x)\psi_{0,+}(\lambda_0,x')\int_a^{\min(x,x')}p_0(y)^{-1}\psi_{0,+}(\lambda_0,y)^{-2}\,dy\\ \psi_{0,-}(\lambda_0,x)\psi_{0,-}(\lambda_0,x')\int_{\max(x,x')}^b p_0(y)^{-1}\psi_{0,-}(\lambda_0,y)^{-2}\,dy\end{cases},\quad x,\,x'\in(a,b). \tag{3.48}$$

Next, assume that $(\tau_0-\lambda_0)$ is critical on (a,b). Then $G_0(\lambda_0,x,x')$ does not exist, however, the restrictions $(\tau_0-\lambda_0)|_{\substack{(x_0,b)\\(a,x_0)}}$ are subcritical and the corresponding Green's functions $G^D_{0,\pm}(\lambda_0,x,x')$ associated with the Dirichlet operators $H^D_{0,+}$ (resp. $H^D_{0,-}$) in $L^2\big((x_0,b)\big)$ (resp. $L^2((a,x_0))$),

$$H^D_{0,\pm}f=\tau_0 f, \tag{3.49}$$

$$f\in\mathcal{D}(H^D_{0,\pm}):=\Big\{g\in L^2\big(\substack{(x_0,b)\\(a,x_0)}\big)\,|\,g,p_0g'\in AC\big(\substack{[x_0,R]\\ [R,x_0]}\big),\ R\in\substack{(x_0,b)\\(a,x_0)},\ g(x_{0,\pm})=0,\ \tau_0 g\in L^2\big(\substack{(x_0,b)\\(a,x_0)}\big)\Big\},$$

then read

$$G^D_{0,+}(\lambda_0,x,x')=\psi_{0,+}(\lambda_0,x)\psi_{0,+}(\lambda_0,x')\int_{x_0}^{\min(x,x')}p_0(y)^{-1}\psi_{0,+}(\lambda_0,y)^{-2}\,dy,\quad x,\,x'\in(x_0,b), \tag{3.50}$$

$$G^D_{0,-}(\lambda_0,x,x')=\psi_{0,-}(\lambda_0,x)\psi_{0,-}(\lambda_0,x')\int_{\max(x,x')}^{x_0}p_0(y)^{-1}\psi_{0,-}(\lambda_0,y)^{-2}\,dy,\quad x,\,x'\in(a,x_0). \tag{3.51}$$

Next, consider $q\in L^1_{loc}(\mathbb{R})$ real–valued with

$$\int_a^b\psi_{0,\pm}(\lambda_0,x)^2\Big|\int_{\substack{a\\b}}^x p_0(y)^{-1}\psi_{0,\pm}(\lambda_0,y)^{-2}\,dy\Big|q_-(x)\,dx<\infty \tag{3.52}$$

if $(\tau_0-\lambda_0)$ is subcritical and

$$\int_a^b\psi_{0,\pm}(\lambda_0,x)^2\Big|\int_{x_0}^x p_0(y)^{-1}\psi_{0,\pm}(\lambda_0,y)^{-2}\,dy\Big|q_-(x)\,dx<\infty \tag{3.53}$$

if $(\tau-\lambda_0)$ is critical. Here we abbreviated, as usual,

$$q_\pm(x)=\big[\,|q(x)|\pm q(x)\big]/2. \tag{3.54}$$

Introducing

$$\tau=\tau_0+q(x),\quad x\in(a,b), \tag{3.55}$$

one concludes that τ is in the l.p. case at a and b because of (3.52) and (3.53). This follows from (3.52) in the subcritical case since

$$q_-^{1/2}\big(H_0-\lambda_0\big)^{-1/2} \in \mathcal{B}_2\big(L^2((a,b))\big) \tag{3.56}$$

and hence

$$q_-^{1/2}\big(H_0-\lambda_0\big)^{-1} q_-^{1/2} \in \mathcal{B}_1\big(L^2((a,b))\big) \tag{3.57}$$

as a consequence of the fact that

$$\|q_-^{1/2}\big(H_0-\lambda_0\big)^{-1/2}\|_2^2 = Tr\big[q_-^{1/2}\big(H_0-\lambda_0\big)^{-1} q_-^{1/2}\big] = \int_a^b G_0(\lambda_0,x,x)q_-(x)\,dx < \infty. \tag{3.58}$$

Here $\mathcal{B}_p\big(L^2((a,b)\big)$, $p=1,2$ denote the usual trace class and Hilbert–Schmidt ideals and $\|\cdot\|_2$ abbreviates the Hilbert–Schmidt norm. In the critical case one replaces H_0 by $H_{0,\pm}^D$ and uses (3.53) instead.

Given these preliminaries, a standard application of Birman–Schwinger techniques (see, e.g., [7], [19], [24], Sect. XIII.3, [30], Ch. III, [31]) yields

Theorem 3.3. *Suppose p_0, q_0, q satisfy (H.2.1) near a and b and q satisfies* (3.52) *in the subcritical case and* (3.53) *in the critical case. In addition, assume that $\lambda_0 \le \inf\sigma(H_0)$, and let H be the unique operator associated with τ in $L^2((a,b))$ according to* (2.4)*. If $E_H(\Omega)$, $\Omega\subseteq\mathbb{R}$ denotes the family of spectral projections of H, then*

$$\dim\operatorname{Ran}\big\{E_H\big((-\infty,\lambda_0]\big)\big\} \le \int_a^b \psi_{0,\pm}(\lambda_0,x)^2\Big|\int_{\mp\infty}^x p_0(y)^{-1}\psi_{0,\pm}(\lambda_0,y)^{-2}\,dy\Big| q_-(x)\,dx \tag{3.59}$$

if $(\tau_0-\lambda_0)$ is subcritical on (a,b), and

$$\dim\operatorname{Ran}\big\{E_H\big((-\infty,\lambda_0]\big)\big\} \le 1+\int_a^b \psi_{0,\pm}(\lambda_0,x)^2\Big|\int_{x_0}^x p_0(y)^{-1}\psi_0(\lambda_0,y)^{-2}\,dy\Big| q_-(x)\,dx \tag{3.60}$$

if $(\tau_0-\lambda_0)$ is critical on (a,b).

Proof. If $(\tau_0-\lambda_0)$ is subcritical it suffices to consider the trace of $q_-^{1/2}(H_0-\lambda_0)^{-1}q_-^{1/2}$ (cf. (3.58)). If $(\tau_0-\lambda_0)$ is critical one considers the half–line operators $q_-^{1/2}(H_{0,\pm}^D-\lambda_0)^{-1}q_-^{1/2}$ and then notes that

$$\big(H_0-z\big)^{-1} - \big[\big(H_{0,-}^D-z\big)^{-1}\oplus\big(H_{0,+}^D-z\big)^{-1}\big] \tag{3.61}$$

is a rank–one operator producing the 1 on the right–hand–side of (3.60). The estimates are first obtained for $E_H\big((-\infty,\lambda_0)\big)$ and then extended to $E_H\big((-\infty,\lambda_0]\big)$ by a standard limiting argument (see, e.g., [30], Ch. III). □

We note that our comparison potential (cf. (1.18) or (3.42))

$$q(x) = \mu p_0(x)^{-1}\psi_0(\lambda_0,x)^{-4}\Big(\int_{x_0}^x p_0(y)^{-1}\psi_0(\lambda_0,y)^{-2}\,dy\Big)^{-2}, \quad \mu<0 \tag{3.62}$$

is a border–line case since it yields a logarithmically divergent expression on the right–hand–sides of (3.53), (3.60) (and similarly in the subcritical case of $(\tau_0-\lambda_0)$ on (a,b)).

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