RENORMALIZED OSCILLATION THEORY FOR DIRAC OPERATORS

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ABSTRACT. Oscillation theory for one-dimensional Dirac operators with separated boundary conditions is investigated. Our main theorem reads: If $\lambda_{0,1} \in \mathbb{R}$ and if u, v solve the Dirac equation $Hu = \lambda_0 u$, $Hv = \lambda_1 v$ (in the weak sense) and respectively satisfy the boundary condition on the left/right, then the dimension of the spectral projection $P_{(\lambda_0,\lambda_1)}(H)$ equals the number of zeros of the Wronskian of u and v. As an application we establish finiteness of the number of eigenvalues in essential spectral gaps of perturbed periodic Dirac operators.

1. INTRODUCTION

In [16] Sturm originated oscillation theory for second-order differential equations one hundred and fifty years ago. Since then numerous extensions have been made (see, e.g., [2],[11],[14],[17], and the references therein). In [24] Weidmann extended results for Sturm-Liouville operators from Hartman [5], [6], Hartman and Putnam [7], and himself [23] to the case of Dirac operators. In particular, he proves Sturm-type comparison theorems and applies the results to investigate the essential spectrum of Dirac operators. With the present paper we want to complement [24] in the sense that we will use oscillation theory to investigate the discrete spectrum.

Using standard oscillation theory would mean to count zeros of components of solutions of the Dirac equation. Unfortunately this approach soon leads into severe troubles:

(i). Components of solutions might vanish identically on some intervals.

(ii). Zeros of components of solutions are not monotone with respect to the spectral parameter. Hence solutions can pick up or loose zeros as the spectral parameter increases which, in general, destroys the connection between zeros and number of eigenvalues (cf. Remark 3.3).

The natural remedy is to look at zeros of the Wronskian instead, that is, use a renormalized version of oscillation theory developed in [4] for the case of Sturm-Liouville operators (see [18] in the case of Jacobi operators). In addition, this approach avoids technical difficulties arising from the fact that Dirac operators, in contradistinction to Sturm-Liouville operators, are not bounded from below.

To set the stage, let $I = (a, b) \subseteq \mathbb{R}$ (with $-\infty \le a < b \le \infty$) be an arbitrary interval and consider the Dirac differential expression

$$\tau = \frac{1}{\mathrm{i}} \sigma_2 \frac{d}{dx} + \phi(x). \tag{1.1}$$

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Here

$$\phi(x) = \phi_{\rm el}(x)\mathbb{1} + \phi_{\rm am}(x)\sigma_1 + (m + \phi_{\rm sc}(x))\sigma_3, \tag{1.2}$$

σ_1, σ_2, σ_3 denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.3}$$

and m, $\phi_{\rm sc}$, $\phi_{\rm el}$, and $\phi_{\rm am}$ are interpreted as mass, scalar potential, electrostatic potential, and anomalous magnetic moment, respectively (see [19], Chapter 4). As usual we require $m \in [0, \infty)$ and $\phi_{\rm sc}, \phi_{\rm el}, \phi_{\rm am} \in L^1_{loc}(I, \mathbb{R})$ real-valued. We don't include a magnetic moment

$$\hat{\tau} = \tau + \sigma_2 \phi_{\rm mg}(x) \tag{1.4}$$

since it can be easily eliminated by a simple gauge transformation

$$\tau = \exp\Big(\mathrm{i}\int^x \phi_{\rm mg}(t)dt\Big)\hat{\tau}\exp\Big(-\mathrm{i}\int^x \phi_{\rm mg}(t)dt\Big) \tag{1.5}$$

(there is also a gauge transformation which gets rid of ϕ_{am} or $\phi_{\rm el}$ (see [12], Section 7.1.1)). Explicitly we have

$$\begin{aligned} \tau f &= \begin{pmatrix} \phi_{11} & -\frac{d}{dx} + \phi_{12} \\ \frac{d}{dx} + \phi_{12} & \phi_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ &= \begin{pmatrix} f_2' + \phi_{12} f_2 + \phi_{11} f_1 \\ -f_1' + \phi_{12} f_1 + \phi_{22} f_2 \end{pmatrix}, \quad f \in AC_{loc}(I, \mathbb{C}^2), \end{aligned} \tag{1.6}$$

where primes denote derivatives with respect to x and $\phi_{11} = \phi_{\rm el} + m + \phi_{\rm sc}$, $\phi_{12} = \phi_{21} = \phi_{\rm am}$, $\phi_{22} = \phi_{\rm el} - m - \phi_{\rm sc}$.

If τ is limit point at both $\pm\infty$, then τ gives rise to a unique self-adjoint operator H when defined maximally (cf., e.g., [12], [22], [24]). Otherwise, we fix a boundary condition at each endpoint where τ is limit circle.

By $u_\pm(z, x)$ we will denote (non identically vanishing) solutions of the differential equation $\tau u = zu$, $z \in \mathbb{C}$, which satisfy the following requirements (whenever such solutions exist).

(i). $u_\pm(z, .) \in AC_{loc}(I, \mathbb{C}^2)$ and $\tau u_\pm(z) = z u_\pm(z)$.
(ii). $u_+(z, .)$ (resp. $u_-(z, .)$) is square integrable near b (resp. a) and fulfills the boundary condition of H at b (resp. a) if any (i.e., if τ is limit circle at b (resp. a)).

Explicitly, H is given by

$$\begin{array}{rccl} H: & \mathfrak{D}(H) & \to & L^2(I, \mathbb{C}^2) \\ & f & \mapsto & \tau f \end{array}, \tag{1.7}$$

where

$$\mathfrak{D}(H) = \{f \in L^2(I, \mathbb{C}^2) |\ f \in AC_{loc}(I, \mathbb{C}^2),\ \tau f \in L^2(I, \mathbb{C}^2), \\ W_a(u_-(\lambda_0), f) = W_b(u_+(\lambda_0), f) = 0\} \tag{1.8}$$

with

$$W_x(f, g) = f_1(x) g_2(x) - f_2(x) g_1(x) \tag{1.9}$$

the usual Wronskian (we remark that the the limit $W_{a,b}(.,..) = \lim_{x\to a,b} W_x(.,..)$ exists for functions as in (1.8)). The resolvent of H can be expressed in terms of $u_\pm(z)$ as follows

$$(H-z)^{-1} f(x) = \int_a^b G(z,x,y) f(y) dy, \tag{1.10}$$

where

$$G(z,x,y) = \frac{u_\pm(z,x) \otimes u_\mp(z,y)}{W(u_+(z), u_-(z))}, \quad \pm(x-y) > 0. \tag{1.11}$$

Recall that $W_x(u_+(z), u_-(z))$ is independent of x (cf. (2.1)). In addition, we set $G(z,x,x) = \lim_{\varepsilon\to 0}(G(z,x+\varepsilon,x) + G(z,x-\varepsilon,x))/2$.

Denote by $H_{x,-}$ (resp. $H_{x,+}$), $x \in I$ self-adjoint operators associated with τ on $L^2((a,x),\mathbb{C}^2)$ (resp. $L^2((x,b),\mathbb{C}^2)$) obtained from H by imposing the additional boundary condition $f_1(x) = 0$. Then $H_{x,-} \oplus H_{x,+}$ is a rank one resolvent perturbation of H and hence $\sigma_{ess}(H) = \sigma_{ess}(H_{x,-}) \cup \sigma_{ess}(H_{x,+})$ (cf. [25], Korollar 6.2). Here $\sigma_{ess}(.)$ denotes the essential spectrum. If $G_{x,\pm}(z,.,..)$ denotes the resolvent kernel of $H_{x,\pm}$ we define the Weyl m-functions $m_{x,\pm}(z)$ (w.r.t. the base point x) by

$$G_{x,\pm}(z,x,x) = \begin{pmatrix} 0 & \pm\frac{1}{2} \\ \pm\frac{1}{2} & m_{x,\pm}(z) \end{pmatrix}. \tag{1.12}$$

The first resolvent identity shows that $m_{x,\pm}(z)$ are Herglotz functions (cf., e.g., [20]).

Lemma 1.1. *Let $\lambda \in \mathbb{R}\backslash\sigma_{ess}(H_{x_0,\pm})$. Then the solutions $u_\pm(\lambda,x)$ exist and are real analytic with respect to λ.*

Proof. If $U(z,x,x_0)$, $z \in \mathbb{C}$ is a fundamental matrix solution for $\tau u = zu$ (i.e., $U(z,x_0,x_0) = \mathbb{1}$, $x_0 \in I$) and $m_{x_0,\pm}(z)$ are the Weyl m-functions with respect to the base point x_0. Then we can choose

$$u_\pm(\lambda,x) = \frac{\mp m_{x_0,\pm}(\lambda)}{1 + m_{x_0,\pm}(\lambda)^2} U(\lambda,x,x_0) \begin{pmatrix} 1 \\ \pm m_{x_0,\pm}(\lambda) \end{pmatrix}. \tag{1.13}$$

□

A finite end point is called regular if $\phi_{11}, \phi_{12}, \phi_{22}$ are integrable near this end point. In this case boundary values for all functions exist at this end point. In particular, τ is called regular if both end points a, b are regular, that is, $a, b \in \mathbb{R}$ and $\phi_{11}, \phi_{12}, \phi_{22} \in L^1(I,\mathbb{R})$. In the regular case the resolvent of H is Hilbert-Schmidt and hence the spectrum is purely discrete (i.e., $\sigma_{ess}(H) = \emptyset$).

2. Wronskians

In this section we want to investigate the Wronskian of two solutions u, v. A straightforward calculation gives

$$W_x'(u,v) = (\lambda_1 - \lambda_0) u(x) v(x) \tag{2.1}$$

if $\tau u = \lambda_0 u$ and $\tau v = \lambda_1 v$. Note that (in contradistinction to the Sturm-Liouville case) the Wronskian of two solutions can only have simple zeros (unless $\lambda_0 = \lambda_1, u = v$ or $u \equiv 0$ (resp. $v \equiv 0$) of course). Moreover, $W_x(u,v) = 0$ if $u(x), v(x)$ are parallel and $W_x'(u,v) = 0$ if $u(x), v(x)$ are orthogonal.

Clearly this implies

Lemma 2.1. *Let $\tau u = \lambda_0 u$ and $\tau v = \lambda_1 v$ for some $\lambda_1 \neq \lambda_0$. If $u, v \in L^2((c,d), \mathbb{C}^2)$ and $W_c(u,v) = W_d(u,v)$ for some $a \le c < d \le b$ then u, v are orthogonal on (c,d), that is,*

$$\int_c^d u(t)v(t)dt = 0. \tag{2.2}$$

Proof. Integrating (2.1) we obtain for $c, d \in I$

$$W_d(u,v) - W_c(u,v) = (\lambda_1 - \lambda_0) \int_c^d u(t)v(t)\, dt, \tag{2.3}$$

and hence the result is immediate (take limits if $c = a$ or $d = b$). □

Lemma 2.2. *Let $\lambda \in \mathbb{R} \backslash \sigma_{ess}(H)$. Then*

$$W_x(u_\pm(\lambda), \dot{u}_\pm(\lambda)) = \left\{ \begin{array}{l} -\int_x^b u_+(\lambda,t)^2 dt \\ \int_a^x u_-(\lambda,t)^2 dt \end{array} \right. , \tag{2.4}$$

where the dot denotes a derivative with respect to λ.

Proof. ¿From Lemma 2.1 we know

$$W_x(u_\pm(\lambda), u_\pm(\tilde{\lambda})) = (\tilde{\lambda} - \lambda) \left\{ \begin{array}{l} -\int_x^b u_+(\lambda,t) u_+(\tilde{\lambda},t) dt \\ \int_a^x u_-(\lambda,t) u_-(\tilde{\lambda},t) dt \end{array} \right. . \tag{2.5}$$

Now use this to evaluate the limit

$$\lim_{\tilde{\lambda} \to \lambda} W_x\Big(u_\pm(\lambda), \frac{u_\pm(\lambda) - u_\pm(\tilde{\lambda})}{\lambda - \tilde{\lambda}}\Big). \tag{2.6}$$

□

3. Oscillation theory

We first introduce Prüfer variables for $u \in C(I, \mathbb{R})$ defined by

$$u_1(x) = \rho_u(x) \sin(\theta_u(x)) \qquad u_2(x) = \rho_u(x) \cos(\theta_u(x)). \tag{3.1}$$

If u is never $(0,0)$ and u is continuous, then ρ_u is positive and θ_u is uniquely determined once a value of $\theta_u(x_0)$, $x_0 \in I$ is chosen by the requirement $\theta_u \in C(I, \mathbb{R})$.

Clearly

$$W_x(u,v) = \rho_u(x)\rho_v(x) \sin(\theta_u(x) - \theta_v(x)). \tag{3.2}$$

An important role is played by the following observation.

Lemma 3.1. *Let $\lambda_0 < \lambda_1$, let u, v solve $\tau u = \lambda_0 u$, $\tau v = \lambda_1 v$, and introduce*

$$\Delta_{u,v}(x) = \theta_u(x) - \theta_v(x). \tag{3.3}$$

Then, if $\Delta_{u,v}(x) \equiv 0 \mod \pi$,

$$\lim_{x \to x_0} \frac{\Delta_{u,v}(x) - \Delta_{u,v}(x_0)}{x - x_0} = (\lambda_1 - \lambda_0) > 0. \tag{3.4}$$

Proof. If $\Delta_{u,v}(x_0) \equiv 0 \mod \pi$, then (from (3.2))

$$\lim_{x \to x_0} \frac{\rho_u(x)\rho_v(x) \sin(\Delta_{u,v}(x))}{x - x_0} = W'_{x_0}(u,v) > 0 \tag{3.5}$$

implies the assertion using (2.1). □

Or, put differently, the last proposition implies that the integer part of $\Delta_{u,v}(x)/\pi$ is increasing.

Lemma 3.2. *Let $\lambda_0 < \lambda_1$ and let u, v solve $\tau u = \lambda_0 u$, $\tau v = \lambda_1 v$. Denote by $\#(u,v)$ the number of zeros of $W(u,v)$ inside the interval (a,b). Then*

$$\#(u,v) = \lim_{x\uparrow b} [\![\Delta_{u,v}(x)/\pi]\!] - \lim_{x\downarrow a} [\![\Delta_{u,v}(x)/\pi]\!], \tag{3.6}$$

where $[\![x]\!]$ denotes the integer part of a real number x, that is, $[\![x]\!] = \sup\{n \in \mathbb{Z} | n \leq x\}$.

Proof. We start with an interval $[x_0, x_1]$ containing no zeros of $W(u,v)$. Hence $[\![\Delta_{u,v}(x)/\pi]\!] = [\![\Delta_{u,v}(x)/\pi]\!]$. Now let $x_0 \downarrow a$, $x_1 \uparrow b$ and use Lemma 3.1. □

If $\lambda \in \mathbb{R}\backslash\sigma_{ess}(H)$ holds, then equation (2.4) clearly implies

$$\dot\theta_+(\lambda,x) = \frac{\int_x^b u_+(\lambda,t)^2 dt}{\rho_+(\lambda,x)^2} > 0, \qquad \dot\theta_-(\lambda,x) = -\frac{\int_a^x u_-(\lambda,t)^2 dt}{\rho_-(\lambda,x)^2} < 0, \tag{3.7}$$

where we have abbreviated $\rho_\pm(\lambda,x) = \rho_{u_\pm(\lambda)}(x)$ and $\theta_\pm(\lambda,x) = \theta_{u_\pm(\lambda)}(x)$.

Remark 3.3. *We remark that linking zeros of u_j to the rotation number θ_u is not possible since (unlike in the Sturm-Liouville case) the integer part of θ_u does not count zeros of u_j. Indeed, (assuming ϕ continuous for a moment) shows*

$$u_1(x) = 0 \quad \Rightarrow \quad \lim_{x\to x_0} \frac{\theta_u(x) - \theta_u(x_0)}{x - x_0} = \phi_{22}(x_0) - \lambda_0 \tag{3.8}$$

which is not necessarily positive. Hence the integer part of θ_u/π can increase or decrease (or stay the same) at zeros of u_1 (cf. the discussion at the end of Section 2 in [24]). In addition, this implies that zeros of $u_{\pm,1}(\lambda,.)$, are not monotone with respect to λ. Hence solutions can pick up or loose zeros as λ increases. This, in general, destroys the connection between zeros and number of eigenvalues. Moreover, if $\phi_{22}(x) - \lambda_0$, vanishes on a subinterval of I, then $u_1(x)$ can vanish on the same interval (without u being identically zero).

However, if ϕ_{22}, is bounded from above (resp. below), we can apply standard oscillation theory for values of λ with $\phi_{22}(x) - \lambda < 0$ (resp. $\phi_{22}(x) - \lambda > 0$) for all $x \in I$ (cf. Remark 4.10 (ii)). Similar for u_2.

To further illustrate these problems we consider the following example with

$$\phi = \begin{pmatrix} \theta' & 0 \\ 0 & \theta' \end{pmatrix}. \tag{3.9}$$

We will normalize $\theta(x_0) = 0$ for some $x_0 \in I$. The solution u of $\tau u = \lambda_0 u$ satisfying the initial condition $u(x_0) = \rho_0(\sin\theta_0, \cos\theta_0)$ is given by

$$u(x) = \rho_0 \begin{pmatrix} \sin(\theta_0 - \lambda_0 x + \theta(x)) \\ \cos(\theta_0 - \lambda_0 x + \theta(x)) \end{pmatrix}. \tag{3.10}$$

Clearly, if $\theta'(x) = \lambda_0$ for $x \in (x_0, x_0+\varepsilon)$ and $\theta_0 = 0$, then $u(x) = (0, \rho_0)$ for $x \in (x_0, x_0+\varepsilon)$.

To get more specific, let $I = (0,1)$, $\theta(x) = 4x(x-1)$, $x_0 = 0$, and impose the boundary conditions $f_1(0) = f_1(1) = 0$. We easily obtain $\sigma(H) = \pi\mathbb{Z}$ and

$$\theta_-(\lambda,x) = \theta(x) - \lambda x, \quad \theta_+(\lambda,x) = \theta(x) - \lambda(x-1). \tag{3.11}$$

This implies the following for the zeros of $u_{-,1}(\lambda,.)$ as λ increases. At $\lambda = 0 \in \sigma(H)$ there are no zeros. At $\lambda = 4(\sqrt{\pi} - 1) \notin \sigma(H)$ we pick up two zeros one of which *gets lost again at* $\lambda = \pi \in \sigma(H)$*. As soon as* $\lambda > 4$ *we have* $\theta'(x) - \lambda > 0$ *for all* $x \in I$ *and from now on* $u_{-,1}(\lambda, .)$ *picks up precisely one zero whenever* λ *hits an eigenvalue (and no zeros get lost).*

To end this remark we compute

$$\Delta_{u_-(\lambda_0),u_+(\lambda_1)}(x) = \lambda_1(x-1) - \lambda_0 x, \tag{3.12}$$

where all unpleasant factors cancel.

4. Number of eigenvalues and zeros of Wronskians

The objective of this section is to establish the connection between zeros of the Wronskian and spectra of Dirac operators. As a warm up we considers the regular case.

Theorem 4.1. *Suppose* τ *is regular. Denote by* $P_\Omega(H)$ *the family of spectral projections for* H*. Then we have for* $\lambda_0 < \lambda_1$

$$\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H) = \#(u_-(\lambda_0), u_+(\lambda_1)) = \#(u_+(\lambda_0), u_-(\lambda_1)), \tag{4.1}$$

where $\#(u, v)$ *is the number of zeros of* $W(u, v)$ *inside* (a, b)*.*

Proof. We only carry out the proof for the $\#(u_-(\lambda_0), u_+(\lambda_1))$ case. Defining $\#(u_-(\lambda_0), u_+(\lambda_1))$ as in (3.6) shows that our claim is true for λ_1 close to λ_0. Abbreviate $\Delta(\lambda, x) = \Delta_{u_-(\lambda_0),u_+(\lambda)}(x)$. Since $\Delta(\lambda, b)$ is independent of λ it suffices to look at $\Delta(\lambda, a)$. As λ increases from λ_0 to λ_1, $-\Delta(\lambda, a)$ increases by (3.7) and is 0 mod π if and only if λ is an eigenvalue of H (Lemma 3.2, equation (1.8)) completing the proof. □

Next, we want to prove Theorem 4.1 in the general case. This will be done in two parts.

Theorem 4.2. *Let* $\lambda_0 < \lambda_1$ *and* $\sigma_0, \sigma_1 \in \{\pm\}$*. Suppose* $u_{\sigma_j}(\lambda_j, .)$*,* $j = 0, 1$ *exist. Then*

$$\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H) \geq \#(u_{\sigma_0}(\lambda_0), u_{\sigma_1}(\lambda_1)). \tag{4.2}$$

Proof. Again the proof is only done for $\sigma_0 = -$. Abbreviate $u = u_-(\lambda_0)$ and $v = u_+(\lambda_1)$ and $n = \#(u, v)$. Suppose n finite, otherwise the following argument works for arbitrary large n. Let $x_1, \dots, x_n$ be the zeros of $W_x(u, v)$. Since $W_{x_j}(u, v) = 0$ there exists constants γ_j such that

$$\eta_j(x) = \left\{ \begin{array}{ll} u(x), & x \leq x_j \\ \gamma_j v(x) & x > x_j \end{array} \right. ,1 \leq j \leq n \tag{4.3}$$

is in the domain of H (i.e., $u(x_j) = \gamma_j v(x_j)$). Furthermore, set

$$\tilde{\eta}_j(x) = \left\{ \begin{array}{ll} -u(x), & x \leq x_j \\ \gamma_j v(x) & x > x_j \end{array} \right. ,1 \leq j \leq n. \tag{4.4}$$

If λ_1 is an eigenvalue of H we define in addition $\eta_0 = v = \tilde{\eta}_0$, $x_0 = a$ and if λ_0 is an eigenvalue of H, $\eta_{n+1} = u = -\tilde{\eta}_{n+1}$, $x_{n+1} = b$. Lemma 2.1 implies $\int_{x_j}^{x_k} uv\, dx = 0$ and hence $\int_b^a \eta_j \eta_k\, dx = \int_b^a \tilde{\eta}_j \tilde{\eta}_k\, dx$ for all j, k. Using

$$(H - \frac{\lambda_1 + \lambda_0}{2})\eta_j = \frac{\lambda_1 - \lambda_0}{2}\tilde{\eta}_j \tag{4.5}$$

we obtain

$$\|(H - \frac{\lambda_1 + \lambda_0}{2})\eta\| = \frac{\lambda_1 - \lambda_0}{2}\|\eta\| \tag{4.6}$$

for any η in the span of the η_j's. Thus, $\dim \mathrm{Ran}\ P_{[\lambda_0,\lambda_1]}(H) \geq \dim(\mathrm{span}\{\eta_j\})$. But u and v are independent on each interval (since their Wronskian is non-constant) and so the η_j are linearly independent. This proves the theorem in the $u = u_-(\lambda_0)$, $v = u_+(\lambda_1)$ case.

The case $u = u_-(\lambda_0)$, $v = u_-(\lambda_1)$ is similar. We define

$$\eta_j(x) = \left\{ \begin{array}{cc} u(x) + \gamma_j v(x), & x \leq x_j \\ 0 & x > x_j \end{array} \right., 1 \leq j \leq n \tag{4.7}$$

(with $\eta_j \in \mathfrak{D}(H)$) and

$$\tilde{\eta}_j(x) = \left\{ \begin{array}{cc} -u(x) + \gamma_j v(x), & x \leq x_j \\ 0 & x > x_j \end{array} \right., 1 \leq j \leq n. \tag{4.8}$$

If λ_1 is an eigenvalue of H we define in addition $\eta_0 = v = \tilde{\eta}_0$, $x_0 = b$ and if λ_0 is an eigenvalue of H, $\eta_{n+1} = u = -\tilde{\eta}_{n+1}$, $x_{n+1} = b$. Again, η_j's are linearly independent by considering their supports. And since $\int_a^{x_j} uv\, dx = 0$, $1 \leq j \leq n$ we can proceed as before. □

Fix functions u, v. Pick $a_m \downarrow a$, $b_m \uparrow b$ and set $I_m = (a_m, b_m)$. Define $\tilde{H}_m$: $\mathfrak{D}(\tilde{H}_m) \to L^2(I_m, \mathbb{C}^2)$, $f \mapsto \tau f$ with

$$\begin{aligned} \mathfrak{D}(\tilde{H}_m) = \{f \in L^2(I_m, \mathbb{C}^2) |\ & f \in AC(I_m, \mathbb{C}^2),\ \tau f \in L^2(I_m, \mathbb{C}^2), \\ & W_{a_m}(u, f) = W_{b_m}(v, f) = 0\}. \end{aligned} \tag{4.9}$$

Consider $H_m = \alpha\mathbb{1} \oplus \tilde{H}_m \oplus \alpha\mathbb{1}$ on $L^2(I, \mathbb{C}^2) = L^2((a, a_m), \mathbb{C}^2) \oplus L^2(I_m, \mathbb{C}^2) \oplus L^2((b_m, b), \mathbb{C}^2)$, where α is a fixed real constant. Then we have the following standard result ([22], Chapter 16, [24], Section 1, and [4], Section 5).

Lemma 4.3. *Suppose that either H is limit point at a or that $u = u_-(\lambda_0)$ for some λ_0 and similarly, that either H is limit point at b or $v = u_+(\lambda_1)$ for some λ_1. Then H_m converges to H in strong resolvent sense as $m \to \infty$ and hence*

$$\dim \mathrm{Ran}\ P_{(\lambda_0,\lambda_1)}(H) \leq \liminf\ \dim \mathrm{Ran}\ P_{(\lambda_0,\lambda_1)}(H_m). \tag{4.10}$$

Now we are ready to prove

Theorem 4.4. *If $u = u_\mp(\lambda_0)$ and $v = u_\pm(\lambda_1)$, then*

$$\dim \mathrm{Ran}\ P_{(\lambda_0,\lambda_1)}(H) \leq \#(u, v). \tag{4.11}$$

If H is limit point at b (resp. a) we can replace $u_-(\lambda_j)$ (resp. $u_+(\lambda_j)$) by an arbitrary solution of $\tau u = \lambda_j u$.

Proof. We can assume $\#(u, v) < \infty$ (otherwise there is nothing to prove). Pick $a_m \downarrow a, b_m \uparrow b$. Let H_m be given as in Lemma 4.3 with $\alpha \notin [\lambda_0, \lambda_1]$. If m is so large, that all zeros of $W(u, v)$ are in (a_m, b_m), Theorem 4.1 implies $\#(u, v) = \dim \mathrm{Ran}\ P_{(\lambda_0,\lambda_1)}(\tilde{H}_m) = \dim \mathrm{Ran}\ P_{(\lambda_0,\lambda_1)}(H_m)$ since $\alpha \notin [\lambda_0, \lambda_1]$. Thus by Lemma 4.3, (4.11) holds as was to be proven. □

Combining the last two theorems we get:

Theorem 4.5. *Let $\lambda_0 < \lambda_1$, then*

$$\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H) = \#(u_-(\lambda_0), u_+(\lambda_1)) = \#(u_+(\lambda_0), u_-(\lambda_1)), \tag{4.12}$$

where $\#(u,v)$ denotes the number of zeros of $W(u,v)$ inside (a,b). The result still holds for $u = u_-(\lambda_0)$, $v = u_-(\lambda_1)$ (resp. $u = u_+(\lambda_0)$, $v = u_+(\lambda_1)$) if H is limit point at b (resp. a).

Remark 4.6. *The limit point assumption in the case $u = u_\mp(\lambda_0)$, $v = u_\mp(\lambda_1)$ is clearly crucial, since the Wronskian contains no information about the boundary condition at a respectively b in this case.*

Finally we state

Theorem 4.7. *Let $\lambda_0 \neq \lambda_1$. Let $\tau u = \lambda_0 u$, $\tau v = \lambda_1 v$, and $\tau \tilde{v} = \lambda_1 \tilde{v}$ with v independent of $\tilde{v}$. Then the zeros of $W(u,v)$ interlace the zeros of $W(u,\tilde{v})$ (in the sense that there is exactly one zero of one function in between two zeros of the other). In particular, $|\#(u,v) - \#(u,\tilde{v})| \le 1$.*

Proof. The result is immediate from $0 < \Delta_{v,\tilde{v}}(x) < \pi$ (for a suitable normalization of $\Delta_{v,\tilde{v}}(x)$) which follows from constancy of $W(v,\tilde{v})$. □

By applying this theorem twice, we conclude

Theorem 4.8. *Let $\lambda_0 \neq \lambda_1$. Let $u, \tilde{u}$ and $v, \tilde{v}$ be the linearly independent solutions of $\tau u = \lambda_0 u$ and $\tau v = \lambda_1 v$, respectively. Then*

$$|W(u,v) - W(\tilde{u},\tilde{v})| \le 2. \tag{4.13}$$

Moreover, we infer the following useful result.

Corollary 4.9. *Let u, v satisfy $\tau u = \lambda_0 u$, $\tau v = \lambda_1 v$. Then*

$$\#(u,v) < \infty \quad \Leftrightarrow \quad \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H) < \infty. \tag{4.14}$$

Proof. Using the split up $H_{x_0,-} \oplus H_{x_0,+}$ reduces the problems to the case with one regular endpoint. Thus the solutions $u_\pm(\lambda)$ exist at least at one end point. Using first Theorem 4.5 and then Theorem 4.8 finishes the proof. □

Remark 4.10. *(i). We remark that all results obtained thus far also hold for the more general system*

$$\tau = k(x)^{-1}\Big(\frac{1}{\mathrm{i}}\sigma_2(p(x)\frac{d}{dx} + \frac{d}{dx}p(x)) + \phi(x)\Big), \tag{4.15}$$

where $p \in AC_{loc}(I,(0,\infty))$ and k is a symmetric positive definite matrix with coefficients $k_{ij} \in L^1_{loc}(I,\mathbb{R})$. The necessary modifications are straightforward (see also [24]*, Section 5).*
(ii).In the case of supersymmetric Dirac operators (i.e., $\phi_{11} = \phi_{22} = 0$)

$$H = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}, \quad A = \frac{d}{dx} + \phi_{12}(x) \tag{4.16}$$

(note that H and $-H$ are unitarily equivalent) we have

$$H^2 = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \quad H_1 = AA^*,\ H_2 = A^*A. \tag{4.17}$$

Moreover,

$$\tau u = \lambda u \quad \Rightarrow \quad \tau_j u_j = -u_j'' + (\phi_{12}^2 - (-1)^j \phi_{12}')u_j = \lambda u_j, \quad j = 1,2, \tag{4.18}$$

where τ_j is the differential expression corresponding to H_j. This says that all oscillation theoretic results for supersymmetric Dirac operators follow immediately from the corresponding results for (semi-bounded) one-dimensional Schrödinger operators.

5. Applications

In our final section we want to apply our results to investigate the spectra of short-range perturbations of periodic Dirac operators. Our objective is to prove the analog of the Theorem by Rofe-Beketov [15] about the finiteness of the number of eigenvalues in essential spectral gaps of the perturbed Hill operator. The reader might find some results for the special case of perturbed constant operators in [10], [3] and for the general case in [8], [9].

We first recall some basic facts from the theory of periodic Dirac operators (cf., e.g., [1], [21], [22], Chapter 12). Let H_p be a Dirac operator associated with periodic potential ϕ_p, that is, $\phi_p(x+1) = \phi_p(x)$, $x \in I = \mathbb{R}$. The spectrum of H_p is purely absolutely continuous and consists of a countable number of gaps, that is,

$$\sigma(H_p) = \bigcup_{j\in\mathbb{Z}} [E_{2j}, E_{2j+1}] \tag{5.1}$$

with $\cdots E_{2j} < E_{2j+1} \le E_{2g+2} < E_{2g+3} \cdots$. Moreover, Floquet theory implies the existence of solutions $u_{p,\pm}(z,.)$ of $\tau_p u = zu$, $z \in \mathbb{C}$ (τ_p the differential expression corresponding to H_p) satisfying

$$u_{p,\pm}(z,x) = p_\pm(z,x) m(z)^{\pm x}, \quad p_\pm(z,x+1) = p_\pm(z,x), \tag{5.2}$$

where $m(z) \in \mathbb{C}$ is called Floquet multiplier. $m(z)$ satisfies $m(z)^2 = 1$ for $z \in \{E_j\}_{j=0}^{2g+1}$, $|m(z)| = 1$ for $z \in \sigma(H_p)$, and $|m(z)| < 1$ for $z \in \mathbb{C}\backslash\sigma(H_p)$. (This says in particular, that $u_{p,\pm}(z,.)$ are bounded for $z \in \sigma(H_p)$ and linearly independent for $z \in \mathbb{C}\backslash\{E_j\}_{j\in\mathbb{Z}}$.)

As anticipated, we will study perturbations H of H_p associated with potential satisfying $\phi(x) \to \phi_p(n)$ as $|x| \to \infty$. Both H and H_p are limit point (cf. [25], Satz 5.1) and hence give rise to a unique self adjoint operator when defined maximally. Using this notation our theorem reads:

Theorem 5.1. *Suppose ϕ_p is a given periodic potential and H_p is the corresponding Dirac operator. Let H be a perturbation of H_p such that*

$$\int_{\mathbb{R}} (1+|x|)|\phi(x) - \phi_p(x)| dx < \infty. \tag{5.3}$$

Then we have $\sigma_{ess}(H) = \sigma(H_p)$, the point spectrum of H is confined to the spectral gaps of H_p, that is, $\sigma_p(H) \subset \mathbb{R}\backslash\sigma(H_p)$ and finite in each gap. Furthermore, the essential spectrum of H_p is purely absolutely continuous.

Proof. Using (1.10) plus $|u_{p,\pm}(z,x)| \le C_\pm |m(z)|^{\pm x}$ shows that H is relatively compact with respect to H_p, implying $\sigma_{ess}(H) = \sigma_{ess}(H_p)$. To prove the remaining claims it suffices to show the existence of solutions $u_\pm(\lambda,.)$ of $\tau u = \lambda u$ for $\lambda \in \sigma(H_p)$ (continuous w.r.t. λ) satisfying

$$\lim_{n\to\pm\infty} |u_\pm(\lambda,x) - u_{p,\pm}(\lambda,x)| = 0. \tag{5.4}$$

In fact, for $\lambda \in \sigma(H_p)$ there exists at least one bounded solution which is not square integrable and hence there are no eigenvalues in the essential spectrum of H (since the Wronskian of a bounded and a square integrable solution must vanish). Invoking Theorem XIII.20 of [13] shows that the essential spectrum of H is purely absolutely continuous. Moreover, since $W_x(u_{p,-}(E_{2j-1}), u_{p,+}(E_{2j}))$ has no zeros, we infer that $W_x(u_-(E_{2j-1}), u_+(E_{2j}))$ has only finitely many zeros. Thus by Corollary 4.9 there are only finitely many eigenvalues in each gap. It remains to show (5.4). Suppose $u_+(\lambda, .)$, $\lambda \in \sigma(H_p)$ satisfies

$$u_\pm(\lambda, x) = u_{p,\pm}(\lambda, x) - \mathrm{i}\sigma_2 \int_{\pm\infty}^x U_p(\lambda, x, y)(\phi(y) - \phi_p(y)) u_\pm(\lambda, y) dy, \tag{5.5}$$

where $U_p(\lambda, ., y)$ is the fundamental matrix solution of of $\tau_p u = \lambda u$ satisfying the initial conditions $U_p(\lambda, y, y) = \mathbb{1}$. Then $u_\pm(\lambda, .)$ fulfills $\tau u = \lambda u$ and (5.4). Existence of a solution of (5.5) follows upon applying a standard iteration argument (compare also [8] and [20] in the special case $\phi_p = 0$) using

$$|U_p(\lambda, x, y)| \le C(1 + |x - y|), \quad \lambda \in \sigma(H_p),\ C > 0. \tag{5.6}$$

□

Clearly, there are several other strategies to prove Theorem 5.1. The proof given here has the advantage of being rather short and transparent. In addition, the idea of proof applies to much general scattering situations (where H_p is not necessarily periodic) as long as sufficient information about the spectrum of H_p and the asymptotic behavior of (weak) solutions of H_p and H is available.

Remark 5.2. *The fact that the essential spectrum of H is purely absolutely continuous has first been proven by* [9] *under the weaker assumption $\int_{\mathbb{R}} |\phi(x) - \phi_p(x)| dx < \infty$. Since (5.3) is only needed to ensure existence of $u_\pm(\lambda, x)$ for λ at the boundary of $\sigma(H_p)$ (for λ in the interior of $\sigma(H_p)$ we have $|U_p(\lambda, x, y)| \le C$) the weaker assumption above suffices) our proof also covers this situation. However, the following example*

$$\phi(x) = \phi_p(x) + \begin{pmatrix} \frac{x^2-1}{(x^2+1)^2} & 0 \\ 0 & 0 \end{pmatrix}, \quad \phi_p(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.7}$$

shows that (5.3) cannot be replace by $\int_{\mathbb{R}} (1 + |x|^\varepsilon) |\phi(x) - \phi_p(x)| dx < \infty$, $\varepsilon < 1$. Indeed, H has an eigenvalue $1 \in \sigma(H_p) = (-\infty, -1] \cup [1, \infty)$ with corresponding eigenfunction

$$u(1, x) = \frac{1}{(x^2+1)^2} \begin{pmatrix} x^2 + 1 \\ -x \end{pmatrix}. \tag{5.8}$$

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