THE CAUCHY PROBLEM FOR ABSTRACT EVOLUTION EQUATIONS WITH GHOST AND FERMION DEGREES OF FREEDOM

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ABSTRACT. We consider a class of abstract nonlinear evolution equations in supermanifolds (smf's) modelled over $\mathbb{Z}_2$-graded locally convex spaces. We show uniqueness, local existence, smoothness, and an abstract version of causal propagation of the solutions. If an a-priori estimate prevents the solutions from blowing-up then an infinite-dimensional smf of "all" solutions can be constructed.

We apply our results to a class of systems of nonlinear field equations with anticommuting fields which arise in classical field models used for realistic quantum field theoretic models. In particular, we show that under suitable conditions, the smf of smooth Cauchy data with compact support is isomorphic with an smf of corresponding classical solutions of the model.

CONTENTS

1. Introduction and preliminaries

1.1. **Introduction.** The investigation of the field equations belonging to a quantum-field theoretical model as classical nonlinear wave equations has a long history, dating back to Segal [17], [16]; cf. also [2], [5], [6], [18]. Usually, Dirac fields have been considered in the obvious way as sections of a spinor bundle, as e. g. in [2].

On the other hand, the rise of supersymmetry made the question of an adequate treatment of the fermion fields urgent — supersymmetry and supergravity do not work with commuting fermion fields. The same applies to ghost fields: BRST symmetry, which now arouses a considerable interest among mathematicians (cf. e. g. [9]), simply does not exist with commuting ghost fields.

The anticommutivity required from fermion and ghost fields is often implemented by letting these fields have their values in the odd part of an auxiliary Grassmann algebra, as e. g. in [3]. However, in [13], we have raised our objections against the use of such an algebra, at least as a fundamental tool.

As we have argued in [13], a satisfactory description of fermion and ghost fields is possible in the framework of infinite-dimensional supergeometry: the totality of configurations on space-time should not be considered as a set but as an infinite-dimensional supermanifold (smf), and the totality of classical solutions should be a sub-supermanifold. While in [12], [13], we have developed the necessary supergeometric machinery, this paper will combine it with old and new techniques in non-linear wave equations in order to implement this point of view.

Our motivating example is the standard Lagrangian of quantum chromodynamics, which is a SU(3) Yang-Mills theory coupled with spinorial fields in the fundamental representation (in this paper, we will not really study any example; a systematic application of our results to a large class of classical field theories will be given in the successor paper):

$$\mathcal{L}[A, \Psi] = -\frac{1}{4}F^{ab}F_{ab} + \frac{\mathrm{i}}{2}\left(\overline{\Psi}\gamma^a D_a^A\Psi - \overline{D_a^A\Psi}\gamma^a\Psi\right) - m\overline{\Psi}\Psi \tag{1.1.1}$$

with $F_{ab}[A] := \partial_b A_a - \partial_a A_b + [A_a, A_b]$, and $D_a^A := \partial_a + \mathrm{i}/2A_a^i\lambda_i$, $\lambda_1, \ldots, \lambda_8$ are the Gell-Mann matrices which realize the fundamental representation of su(3), and we are using Einstein's summation convention, with suppressing spinor indices as well as the coupling constant.

It is well-known that in order to get a well-posed Cauchy problem, we have to break the gauge symmetry. Although this is rather unphysical, we pass here to the temporal gauge $A_0 = 0$. (Unfortunately, this breaks Poincaré invariance; for the treatment with gauge-breaking term and ghosts preferred in physicist's textbooks, we do not yet have the necessary a priori estimates to show completeness; cf. Thm. 2.3.1).

It is reasonable to conjecture that the arising equations of motion

$$\partial_a F_i^{ab} - \left[A_a, F^{ab}\right]_i = \frac{1}{2}\overline{\Psi}\gamma^b\lambda_i\Psi, \quad \mathrm{i}\,\gamma^a D_a^A\Psi = m\overline{\Psi} \tag{1.1.2}$$

are all-time solvable. For the pure Yang-Mills case $\Psi = 0$, this is is already a highly non-trivial result proven in [5]; cf. also [18].

Thus, for any $k > 3/2$, let

$$B'_k := H_k(\mathbb{R}^3) \otimes \mathbb{R}^{24} \quad \oplus \quad H_{k-1}(\mathbb{R}^3) \otimes \mathbb{R}^{24} \quad \oplus \quad H_{k-1}(\mathbb{R}^3) \otimes \mathbb{C}^{12}$$

be the Banach space of Cauchy data $(A^{\mathrm{Cau}}, \dot{A}^{\mathrm{Cau}}, \Psi^{\mathrm{Cau}})$; here H_k is the usual Sobolev space W_k^2. Also, let $C(\mathbb{R}, B'_k)$ denote the space of continuous functions $\mathbb{R} \to B'_k$. Then a precise formulation of the conjecture above states that there should exist a map

$$B'_k \to C(\mathbb{R}, B'_k), \qquad (A^{\mathrm{Cau}}, \dot{A}^{\mathrm{Cau}}, \Psi^{\mathrm{Cau}}) \mapsto (A^{\mathrm{sol}}, \partial_t A^{\mathrm{sol}}, \Psi^{\mathrm{sol}}) \tag{1.1.3}$$

such that $(A^{\mathrm{sol}}, \partial_t A^{\mathrm{sol}}, \Psi^{\mathrm{sol}})|_{t=0} = (A^{\mathrm{Cau}}, \dot{A}^{\mathrm{Cau}}, \Psi^{\mathrm{Cau}})$, and the equations (1.1.2) are satisfied. (It follows from the results presented below that this map, once its existence can be proven, will be uniquely determined, and in fact real-analytic, and thus the Cauchy problem will have the best stability property one can want.)

However, it is well-known (at least in the physical literature) that the classical field Ψ should be treated as *anticommuting*, i. e.

$$[\Psi_i(x), \Psi_j(y)]_+ = 0 \tag{1.1.4}$$

for all $x, y \in \mathbb{R}^4$ and indices i, j. This requirement is not satisfied by modelling Ψ as a function on space-time; in fact, it drastically changes the meaning of (1.1.2). It is even problematic what a configuration should be.

As the author argued in [13], the conceptually best answer to the problem of satisfying (1.1.4) is the following: the totality of configurations of the classical fields should not be modelled as a set (in our example the set $C(\mathbb{R}, B_k'))$ but as an *infinite-dimensional supermanifold.* Roughly speaking, the coordinates of this supermanifold are the degrees of freedom of the model: the bosonic field strengthes $A_a^n(x)$ for all $x \in \mathbb{R}^4$ are the even coordinates, the fermionic field strengthes $\Psi_\alpha^n(x)$ are the odd ones.

This implies that the meanwhile well-established framework of finitedimensional supergeometry (cf. [10], [8], [11]) has to be extended to the infinite-dimensional case. (Cf. also [13] for a discussion why we prefer the Berezin-Leites-Kostant approach to supermanifolds to the deWitt-Rogers one.)

A calculus of real-analytic supermanifolds (smf's) modelled over locally convex spaces, suitable for our purposes, has been constructed by the present author in [15], [12]; cf. the remarks in the next section. Thus, we replace B_k' by the $\mathbb{Z}_2$-graded Banach space

$$B_k := H_k(\mathbb{R}^3) \otimes \mathbb{R}^{24} \quad \oplus \quad H_{k-1}(\mathbb{R}^3) \otimes \mathbb{R}^{24} \quad \oplus \quad \Pi H_{k-1}(\mathbb{R}^3) \otimes \mathbb{C}^{12},$$

where, as usual in supergeometry, Π is a formal odd symbol, and we assign to it the corresponding *supermanifold of Cauchy data* $\mathrm{L}(B_k)$, which is the linear (or "affine") supermanifold with model space B_k. Also, the Frèchet space $C(\mathbb{R}, B_k)$ inherits a $\mathbb{Z}_2$-grading, and the associated linear smf $\mathrm{L}(C(\mathbb{R}, B_k))$ is in our approach the *supermanifold of configurations.*

Instead of the map (1.1.3), the results of this paper combined with that of [5] yield a morphism

$$(A^{\mathrm{sol}}, \partial_t A^{\mathrm{sol}}, \Psi^{\mathrm{sol}}) : \mathrm{L}(B_k) \to \mathrm{L}(C(\mathbb{R}, B_k)) \tag{1.1.5}$$

such that (1.1.5) solves (1.1.2), and its time zero Cauchy datum, $(A^{\mathrm{sol}}, \partial_t A^{\mathrm{sol}}, \Psi^{\mathrm{sol}})(0) \in \mathcal{O}^{B_k}(\mathrm{L}(B_k))$, is just the standard coordinate superfunction $(A^{\mathrm{Cau}}, \dot{A}^{\mathrm{Cau}}, \Psi^{\mathrm{Cau}})$. Moreover, it turns out that the image of (1.1.5) exists as a sub-smf $\mathrm{L}(C(\mathbb{R}, B_k))^{\mathrm{sol}} \subseteq \mathrm{L}(C(\mathbb{R}, B_k))$; we call $\mathrm{L}(C(\mathbb{R}, B_k))^{\mathrm{sol}}$ the *supermanifold of classical solutions of* (1.1.2) *within* $\mathrm{L}(C(\mathbb{R}, B_k))$.

However, viewing $\mathrm{L}(C(\mathbb{R}, B_k))^{\mathrm{sol}}$ as "the" manifold of classical solutions has the severe defect that we do not know whether it is Lorentz invariant in a reasonable sense; probably, it is not. At any rate, there is no reasonable action of the Lorentz group on $\mathrm{L}(C(\mathbb{R}, B_k))$. (Of course, in this particular example, Lorentz invariance is spoiled anyway by the temporal gauge condition. But the objection stands for many other models.)

An obvious proposal for improvement is to use smooth Cauchy data and configurations. Thm. 3.3.3 below yields the following variant of (1.1.5):

$$(A^{\mathrm{sol}}, \partial_t A^{\mathrm{sol}}, \Psi^{\mathrm{sol}}) : \mathrm{L}\big(C^\infty(\mathbb{R}^3) \otimes \mathbb{R}^{48} \;\oplus\; \Pi C^\infty(\mathbb{R}^3) \otimes \mathbb{C}^{12}\big) \to$$
$$\to \mathrm{L}\big(C^\infty(\mathbb{R}^4) \otimes \mathbb{R}^{24} \;\oplus\; \Pi C^\infty(\mathbb{R}^4) \otimes \mathbb{C}^{12}\big).$$

(Actually, in order to derive this, one has to use the formulation of the Yang-Mills equation given originally by Segal, since that used by [5], although better reflecting the degrees of smoothness, obscures the causal properties. A systematic discussion will be given in a successor paper.)

Again, this possesses an image sub-smf, the *smf of smooth solutions of* (1.1.2).

However, while the absence of any growth condition in spatial direction does not cause trouble in the construction, due to finite propagation speed, it causes difficulties in the subsequent investigation of differential-geometric structures on the image $M^{\rm sol}_{C^\infty}$: Roughly spoken, any superfunction $K[\Phi|\Psi]$ on the Cauchy smf is influenced only by the "values" of the fields on the finite region Ω. In particular, the energy at a given time instant is not a well-defined superfunction; only the energy in a finite space-time region is so. What is still worse, the symplectic structure on the solution smf which one expects (cf. [13, 1.12.4]), and which we will study in subsequent papers, simply does not make sense; only the corresponding Poisson structure does.

Thus, it seems reasonable to use only smooth Cauchy data with compact support, i. e. of test function quality. However, we have to be careful in the choice of the model space for the target smf: simply taking all smooth functions on $\mathbb{R}^4$ which are spatially compactly supported would violate Lorentz invariance. However, if we additionally suppose that the spatial support grows only with light speed then everything is OK: Let $C^\infty_c(\mathbb{R}^4)$ denote the space of all $f \in C^\infty(\mathbb{R}^4)$ such that there exists $R > 0$ with $f(t,x) = 0$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^3$ with $|x| \geq |t| + R$. (Note that this is only a strict inductive limes of Frèchet spaces.) Thm. 3.3.2 now yields that (1.1.5) restricts to a morphism

$$(A^{\rm sol}, \partial_t A^{\rm sol}, \Psi^{\rm sol}) : \mathrm{L}\big(C^\infty_0(\mathbb{R}^3)\otimes\mathbb{R}^{48} \;\oplus\; \Pi C^\infty_0(\mathbb{R}^3)\otimes\mathbb{R}^{12}\big) \to$$
$$\to \mathrm{L}\big(C^\infty_c(\mathbb{R}^4)\otimes\mathbb{R}^{24} \;\oplus\; \Pi C^\infty_c(\mathbb{R}^4)\otimes\mathbb{R}^{12}\big)$$

Again, this possesses an image sub-smf, the *smf of smooth solutions of* (1.1.2) *with causally growing spatially compact support*. In a subsequent paper, we will show that for suitable models, this smf is acted upon by the Poincaré group and carries an invariant symplectic structure.

1.2. **Infinite-dimensional supergeometry.** Let us shortly recall some notions and conventions from [12], [13]. We follow the usual conventions of $\mathbb{Z}_2$-graded algebra: All vector spaces will be $\mathbb{Z}_2$-graded, $E = E_{\mathbf{0}} \oplus E_{\mathbf{1}}$ (decomposition into *even* and *odd* part); for the *parity* of an element, we will write $|e| = \mathbf{i}$ for $e \in E_{\mathbf{i}}$. In multilinear expressions, parities add up; this fixes parities for tensor product and linear maps. (Note that space-time, being not treated as vector space, remains ungraded. On the other hand, "classical" function spaces, like Sobolev spaces, are treated as purely even.)

First Sign Rule: Whenever in a complex multilinear expression two adjacent terms a, b are interchanged the sign $(-1)^{|a||b|}$ has to be introduced.

In order to get on the classical level a correct model of operator conjugation in the quantized theory we also have to use the additional rules of the hermitian calculus developed in [11]. That is, the role of real supercommutative algebras is taken over by *hermitian supercommutative algebras*, i. e. complex supercommutative algebras R together with an involutive antilinear map $\bar{\cdot} : R \to R$ (*hermitian conjugation*) such that $\overline{rs} = \overline{s}\cdot\overline{r}$ for $r, s \in R$ holds. Note that this rule does not contradict the first sign rule since $\overline{rs}$ is not complex multilinear in r, s. Also, *the real elements of a hermitian algebra do in general not form a subalgebra*, i. e. R is not just the complexification of a real algebra. More general, all real vector spaces have to be complexified before its elements may enter multilinear expressions. The essential ingredient of the hermitian framework is the

Second Sign Rule: If conjugation is applied to a bilinear expression in the terms a, b (i. e. if conjugation is resolved into termwise conjugation), either a, b have to be rearranged backwards, or the expression acquires the sign factor $(-1)^{|a||b|}$. Multilinear terms have to be treated iteratively.

A calculus of real-analytic infinite-dimensional supermanifolds (smf's) has been constructed by the present author in [15], [12]. Here we note that it assigns to every real $\mathbb{Z}_2$-graded locally convex space (henceforth abbreviated $\mathbb{Z}_2$-lcs) $E = E_{\mathbf{0}} \oplus E_{\mathbf{1}}$ a *linear supermanifold* $\mathrm{L}(E)$ which is essentially a ringed space $\mathrm{L}(E) = (E_{\mathbf{0}}, \mathcal{O})$ with underlying topological space $E_{\mathbf{0}}$ while the structure sheaf $\mathcal{O}$ might be thought very roughly of as a kind of completion of $\mathcal{A}(\cdot)\otimes\Lambda E^*_{\mathbf{1},\mathbb{C}}$; here $\mathcal{A}(\cdot)$ is the sheaf of real-analytic functions on the even part $E_{\mathbf{0}}$ while $\Lambda E^*_{\mathbf{1},\mathbb{C}}$ is the exterior algebra over the complexified dual of the odd part of E.

The actual definition of the structure sheaf treats even and odd sector much more on equal footing than the tensor product ansatz above: Given a second real $\mathbb{Z}_2$-lcs F, one defines the $\mathbb{Z}_2$-graded complex vector space $\mathcal{P}_{\mathrm{f}}(E;\ F)$ of *F-valued formal power series on E* as the set of all formal sums $u = \sum_{k,l\geq 0} u_{(k|l)}$ where $u_{(k|l)} : \prod^k E_{\mathbf{0}} \times \prod^l E_{\mathbf{1}} \to F \otimes_{\mathbb{R}} \mathbb{C}$ is a jointly continuous, multilinear map which is symmetric on $E_{\mathbf{0}}$ and alternating on $E_{\mathbf{1}}$. This space has a natural hermitian conjugation, and, by usual multilinear techniques, one constructs an associative bilinear pairing $\mathcal{P}_{\mathrm{f}}(E;\ F) \times \mathcal{P}_{\mathrm{f}}(E;\ F') \to \mathcal{P}_{\mathrm{f}}(E;\ F \otimes F')$; in particular, $\mathcal{P}_{\mathrm{f}}(E;\ \mathbb{R})$ becomes a $\mathbb{Z}_2$-commutative hermitian algebra.

Recall that, assigning to a seminorm p its unit ball $\{e \in E :\ p(e) \leq 1\}$, we get a bijection between the set $\mathrm{CS}(E) \ni p$ of continuous seminorms on E, and the set $\mathrm{CB}(E)$ of convex balanced closed neighbourhoods of the origin.

Now let F be a $\mathbb{Z}_2$-graded Banach space, and $U \in \mathrm{CB}(E)$. For $u \in \mathcal{P}(E;\ F)$, let $\|u_{(k|l)}\|$ be the supremum of $\|u_{(k|l)}(\cdot)\|$ on $\prod^k (U \cap E_{\mathbf{0}}) \times \prod^l (U \cap E_{\mathbf{1}})$. Let $\mathcal{P}(E, U;\ F)$ be the Banach space of all those $u \in \mathcal{P}_{\mathrm{f}}(E;\ F)$ for which $\|u\| := \sum_{k,l\geq 0} \|u_{(k|l)}\|$ is finite. Conforming with [12], we will denote this space also by $\mathcal{P}(E, p;\ F)$ where p is the seminorm with unit ball U.

Conceptually, $\mathcal{P}(E, U;\ F)$ is the *space of power series converging on U*. Indeed, every element $K \in \mathcal{P}(E, U;\ F)$ is "a function element on $U \cap E_{\mathbf{0}}$", i. e. it will be the Taylor expansion at zero of a uniquely determined superfunction $K \in \mathcal{O}^F(U \cap E_{\mathbf{0}})$ within the superdomain $\mathrm{L}(E)$ (cf. [12, Prop. 3.5.2]).

Define the *space $\mathcal{P}(E;\ F)$ of analytic power series from E to F* as the set of all $u \in \mathcal{P}_{\mathrm{f}}(E;\ F)$ such that for all $p \in \mathrm{CS}(F)$ there exists $U \in \mathrm{CB}(E)$ such that $i_p \circ u \in \mathcal{P}(E, U;\ \hat{F}_p)$ where $i_p : F \to \hat{F}_p$ is the canonical map into the completion of F w. r. to p (with the zero space of p factored out).

Given power series $u \in \mathcal{P}(E, U;\ F)$ where F is Banach and $v \in \mathcal{P}(E';\ E)_{\mathbf{0},\mathbb{R}}$ with $v_{(0|0)} \in U$, one defines with some multilinear voodoo the *composition* $u[v] \in \mathcal{P}(E';\ F)$; cf. [13, 2.3] for details.

Now, for any $\mathbb{Z}_2$-lcs F, one defines the sheaf $\mathcal{O}^F(\cdot)$ of *F-valued superfunctions* on $E_{\mathbf{0}}$: an element of $\mathcal{O}^F(U)$ where $U \subseteq E_{\mathbf{0}}$ is open is a map $f : U \to \mathcal{P}(E;\ F)$, $e \mapsto f_e$, which satisfies a certain "coherence" condition which makes it sensible to interpret f_e as the Taylor expansion of f at e: One requires that for all $p \in \mathrm{CS}(F)$ there exists $U \in \mathrm{CB}(E)$ such that $i_p \circ f_{e+e'}[x] = i_p \circ f_e[x + e]$ for $e \in U \cap E_{\mathbf{0}}$. Here $x \in \mathcal{P}(E;\ E)$ is the identity $E \to E$ viewed as power series; it acts as identity under composition.

Now the structure sheaf of our ringed space $\mathrm{L}(E)$ is simply $\mathcal{O}(\cdot) := \mathcal{O}^{\mathbb{R}}(\cdot)$; it is a sheaf of hermitian supercommutative algebras, and each $\mathcal{O}^F(\cdot)$ is a module sheaf over $\mathcal{O}(\cdot)$.

Actually, in considering more general smf's than superdomains, one has to enhance the structure of a ringed space slightly, in order to avoid "fake morphisms". What matters here is that the enhancement is done in such a way that the following holds (cf. [13, Thm. 2.8.1]):

Lemma 1.2.1. *Given an $\mathbb{Z}_2$-lcs F and an arbitrary smf Z, the set of morphisms $Z \to \mathrm{L}(F)$ is in natural 1-1-correspondence with the set*

$$\mathcal{M}^F(Z) := \mathcal{O}^F(Z)_{\mathbf{0},\mathbb{R}}.$$

(Here the subscript stands for the real, even part.) The correspondence works as follows: There exists a distinguished element $x \in \mathcal{M}^F(\mathrm{L}(F))$ called the standard coordinate, *and one assigns to $\mu : Z \to \mathrm{L}(F)$ the pullback $\mu^*(x)$.* □

(In previous papers, we had denoted this pullback by $\hat{\mu}$; in this one, we will abuse notation and drop the hat, thus identifying a superfunction $\mu \in \mathcal{M}^F(Z)$ with its corresponding morphism $\mu : Z \to \mathrm{L}(F)$.)

This is the infinite-dimensional version of the fact that if $F = \mathbb{R}^{m|n} = \mathbb{R}^m \oplus \Pi\mathbb{R}^n$ is the standard $m|n$-dimensional super vector space then a morphism $Z \to \mathrm{L}(\mathbb{R}^{m|n})$ is known by knowing the pullbacks of the coordinate superfunctions, and these can be prescribed arbitrarily as long as parity and reality are OK (cf. e. g. [10, Thm. 2.1.7]).

The most straightforward way to do the enhancement mentioned is a chart approach; since the supermanifolds we are going to use are actually all superdomains, and only the morphisms between them are non-trivial, we need not care here for details.

If E, F are spaces of generalized functions on $\mathbb{R}^d$ which contain the test functions as dense subspace then the Schwartz kernel theorem tells us that the multilinear forms $u_{(k|l)}$ are given by their integral kernels, which are generalized functions. Thus one can apply rather suggestive integral writings (cf. [12]) like e. g. (1.1.1): The general form of a power series becomes

$$\begin{aligned}(1.2.1)\quad K[\Phi|\Psi] = \sum_{k,l\geq 0} \frac{1}{k!l!} \sum_{I,J} \int_{\mathbb{R}^{d(k+l)}} dx_1 \cdots dx_k dy_1 \cdots dy_l \cdot \\ \cdot K^{i_1,\ldots,i_k|j_1,\ldots,j_l}(x_1,\ldots,x_k|y_1,\ldots,y_l)\Phi_{i_1}(x_1)\cdots\Phi_{i_k}(x_k)\Psi_{j_1}(y_1)\cdots\Psi_{j_l}(y_l)\end{aligned}$$

where we have used collective indices $i = 1,\ldots,N_{\mathbf{0}}$ and $j = 1,\ldots,N_{\mathbf{1}}$ for the real components of bosonic and fermionic fields, respectively. The *coefficient functions* $K^{i_1,\ldots,i_k|j_1,\ldots,j_l}(x_1,\ldots,x_k|y_1,\ldots,y_l)$ are distributions which can be supposed to be symmetric in the pairs $(x_1,i_1),\ldots,(x_k,i_k)$ and anti-symmetric in $(y_1,j_1),\ldots,(y_l,j_l)$. Of course, they have to satisfy also certain growth and smoothness conditions. However, what matters here is that the Φ's and Ψ's can be formally treated as commuting and anticommuting fields, respectively; in fact, after establishing the proper calculational framework, the writing (1.2.1) is sufficiently correct. Also, it is possible to substitute power series into each other under suitable conditions. Cf. [12] for a detailed exposition.

We conclude with some additional preliminaries. It will be convenient to work not with the bidegrees $(k|l)$ of forms but with *total degrees*: For any formal power series $K \in \mathcal{P}_f(E;\ F)$ set for $m \geq 0$

$$K_{(m)} := \sum_{k=0}^{m} K_{(k|m-k)}, \qquad K_{(\leq m)} := \sum_{n=0}^{m} K_{(n)}.$$

Thus $K = \sum_{m\geq 0} K_{(m)}$.

Let B be a $\mathbb{Z}_2$-graded Banach space and E any $\mathbb{Z}_2$-lcs. We call a superfunction $f \in \mathcal{O}^E(\mathrm{L}(B))$ *entire* if for every $q \in \mathrm{CS}(E)$ and every $n > 0$ we have $f_0 \in \mathcal{P}(B, nU;\ \hat{E}_q)$ where f_0 is the Taylor expansion at zero, and U is the unit ball. For instance, every $k|l$-form $u_{(k|l)} \in \mathcal{P}_{\mathrm{f}}(B;\ E)$ is the Taylor expansion at zero of a unique entire superfunction.

2. Results in the abstract setting

2.1. **Configuration families.** Through the whole section 2, we fix a real $\mathbb{Z}_2$-graded Banach space B and a strongly continuous group $(\mathrm{A}_t)_{t\in\mathbb{R}}$ of parity preserving bounded linear operators; let $K : \operatorname{dom} K \to B$ denote the generator of this group. Also, let be given an entire even, real superfunction $\Delta \in \mathcal{M}^B(\mathrm{L}(B))$ the Taylor expansion of which in zero has lower degree ≥ 2. Formally, the equation of interest is

$$\frac{d}{dt}\Xi' = K\Xi' + \Delta[\Xi'];\tag{2.1.1}$$

however, this makes sense only if Ξ' takes values in $\operatorname{dom} K$. Therefore we look for the integrated version

$$\Xi'(t) = \mathrm{A}_t\,\Xi'(0) + \int_0^t ds\,\mathrm{A}_{t-s}\,\Delta[\Xi'(s)].\tag{2.1.2}$$

Before embarking in 2.2 into the explanation of the precise meaning of this equation, we first have to clarify the meaning of Ξ'.

For a connected subset $I \subseteq \mathbb{R}$, $I \ni 0$, with non-empty open kernel, let $B(I) := C(I, B)$ equipped with the topology induced by the seminorms $\|\xi\|_{B([a,b])} := \max_{t\in[a,b]} \|\xi(t)\|$ where $a, b \in I$, $a < b$.

Let Z be an arbitrary smf. A *configuration family parametrized by Z with time definition domain I* (or *Z-family*, for short) is an even, real superfunction Ξ' on Z with values in the locally convex space $B(I)$:

$$\Xi' \in \mathcal{M}^{B(I)}(Z)$$

(we recall that $\mathcal{M}$ denotes the real, even part of the sheaf $\mathcal{O}$). Now, given an smf morphism $\pi : Z' \to Z$ we can assign to every Z-family Ξ' its *pullback* $\Xi" := \pi^*(\Xi')$ which is a Z'-family. In fact, the process of passing from Ξ' to $\Xi"$ means in family language nothing but a *change of parametrization* (cf. [13, 1.11]).

Remark. If Z has odd dimension zero, i. e. is effectively an ordinary manifold, then Ξ' is the same as a real-analytic map $\Xi' : Z \to B_{\mathbf{0}}$. In particular, this applies if $Z = P$ is a point; then a Z-family of solutions is just an element $\phi \in B(I)_{\mathbf{0}}$. If also $B = B_{\mathbf{0}}$ is purely even, so that the problem (2.1.2) is a classical one, then this is a configuration in the usual sense.

However, if $B_{\mathbf{1}} \neq 0$, it follows that configuration families with non-trivial odd sector are necessarily parametrized by supermanifolds with non-vanishing odd dimension; in particular, there are no "individual" configurations besides purely even ones.

By Lemma 1.2.1, the smf $\mathrm{L}(B(I))$ is the moduli space for configuration families with time definition domain I, with the standard coordinate

$$\Xi \in \mathcal{M}^{B(I)}(\mathrm{L}(B(I)))$$

being the *universal configuration family with time definition domain I*. Indeed, given an arbitrary Z-family $\Xi' \in \mathcal{M}^{B(I)}(Z)$, it defines by Lemma 1.2.1 a *classifying morphism* $\Xi' : Z \to \mathrm{L}(B(I))$, and Ξ' arises from Ξ just by pullback: $\Xi' = \Xi'^*(\Xi)$.

The *family of Cauchy data* of a family $\Xi' \in \mathcal{M}^{B(I)}(Z)$ with $I \ni 0$ is the element $\Xi'(0) \in \mathcal{M}^{B}(Z)$.

Remark. In the language of category theory, this means that the cofunctor

$$\{\text{supermanifolds}\} \to \{\text{sets}\}, \qquad Z \mapsto \mathcal{M}^{B(I)}(Z),$$

is represented by the object $\mathrm{L}(B(I))$ with the universal element Ξ.

2.2. **Solution families.** We turn to the r. h. s. of (2.1.2): Let $\Xi' \in \mathcal{M}^{B(I)}(Z)$ be a Z-family. Although $\Delta[\Xi'(s)] \in \mathcal{M}^{B}(Z)$ is well-defined for each $s \in I$, we still have to show well-definedness of the integral. We note that, as a consequence of Lemma 2.7.1 below, there exists a unique superfunction $\Delta \in \mathcal{M}^{B(I)}(\mathrm{L}(B(I)))$ (by abuse of notation) such that $\Delta[\Xi'](t) = \Delta[\Xi'(t)]$ for all $t \in I$. Therefore $\Delta[\Xi'] \in \mathcal{M}^{B(I)}(Z)$ is well-defined, and the integral is now simply applied in the target.

It follows that the superfunction symbolically denoted by

$$\mathrm{A}_{\bullet}\, \Xi'(0) + \int_0^{\bullet} ds\, \mathrm{A}_{\bullet - s}\, \Delta[\Xi'(s)] \in \mathcal{M}^{B(I)}(Z)$$

is well-defined (here and in the following, the small bullet stands for the mute time argument).

We call Ξ' a *Z-family of solutions*, or *solution family* for short, if (2.1.2) is satisfied. Trivially, every pullback of a solution family is a solution family.

Thus, to solve the long-time Cauchy problem for the equation (2.1.2) with a given family of Cauchy data $\Xi'^{\mathrm{Cau}} \in \mathcal{M}^{B}(Z)$ means to find a solution family $\Xi' \in \mathcal{M}^{B(\mathbb{R})}(Z)$ with $\Xi'(0) = \Xi'^{\mathrm{Cau}}$.

It turns out that it is sufficient to solve this problem for just one universal family of Cauchy data, in oder to deduce solutions for all other families of Cauchy data:

We call $\mathrm{L}(B)$ the *smf of Cauchy data*, and we denote its standard coordinate by $\Xi^{\mathrm{Cau}} \in \mathcal{M}^{B}(\mathrm{L}(B))$ and call it the *universal family of Cauchy data*. Now suppose we have constructed a solution family $\Xi^{\mathrm{sol}} \in \mathcal{M}^{B(\mathbb{R})}(\mathrm{L}(B))$ with $\Xi^{\mathrm{sol}}(0) = \Xi^{\mathrm{Cau}}$. Then, given an other family of Cauchy data $\Xi'^{\mathrm{Cau}} \in$

$\mathcal{M}^B(Z)$, it determines by Lemma 1.2.1 a morphism $Z \xrightarrow{\Xi'^{\mathrm{Cau}}} \mathrm{L}(B)$, i. e. it is the pullback of the universal family of Cauchy data along this morphism. The solution family Ξ' sought for is then nothing but the pullback of Ξ^{sol} along this same morphism: $\Xi' = (\Xi'^{\mathrm{Cau}})^*(\Xi^{\mathrm{sol}})$.

This justifies it to call Ξ^{sol} the *universal solution family*, and its construction is the main concern of this paper.

In this way, we arrive at the picture sketched in the Introduction.

If $Z = P$ is a point then a Z-family of solutions is just an element $\phi \in B(I)_{\mathbf{0}}$ which solves the *underlying even problem*

$$\phi(t) = \mathrm{A}_t\,|_{B_{\mathbf{0}}}\phi(0) + \int_0^t ds\,\mathrm{A}_{t-s}\,|_{B_{\mathbf{0}}}\tilde{\Delta}[\phi(s)] \tag{2.2.1}$$

in the usual sense. ($\tilde{\Delta}$ is the underlying function of the superfunction Δ).

Remark. If Ξ' takes values in $D := \operatorname{dom} K$ (i. e. $\Xi' \in \mathcal{M}^{C(I,\operatorname{dom} K)}(Z)$ where $\operatorname{dom} K$ is equipped with the graph norm) then (2.1.2) is equivalent to the differentiated form (2.1.1)).

Our first non-trivial result is Cauchy uniqueness (in fact, this will be a special case of the more general result Thm. 2.6.1):

Theorem 2.2.1. *Let be given solution families $\Xi', \Xi" \in \mathcal{M}^{B(I)}(Z)$ such that for some $t_0 \in I$ we have $\Xi'(t_0) = \Xi"(t_0)$. Then $\Xi' = \Xi"$.*

All proofs, as far as not omitted, will be given in section 4.

2.3. **Completeness and solvability.** Loosely said, we call the problem (2.1.2) complete iff the underlying even problem is all-time solvable; it is a standard observation in nonlinear evolution equations that this is equivalent with the existence of a-priori estimates. It turns out that this condition also completely controls the solvability of the all-time Cauchy problem for solution families:

Theorem 2.3.1. *Fix the problem* (2.1.2) *and a subset $A \subseteq B_{\mathbf{0}}$. The following conditions are equivalent:*

(i) For every solution $\phi \in B((a,b))_{\mathbf{0}}$ of the underlying even problem (2.2.1) *on a bounded open time interval $(a,b) \ni 0$ such that $\phi(0) \in A$, we have*

$$\sup_{t\in(a,b)} \|\phi(t)\|_B < \infty.$$

(ii) The underlying even problem is all-time solvable for Cauchy data in A:

> *Given Cauchy data $\phi^{\mathrm{Cau}} \in A$ there exists an element $\phi \in B(\mathbb{R})$ with these Cauchy data which solves the problem* (2.2.1).

(iii) The problem (2.1.2) *is all-time solvable for families of Cauchy data "taking values in A":*

> *Whenever we are given an smf Z and an smf morphism $\Xi'^{\mathrm{Cau}} : Z \to \mathrm{L}(B)$ such that the image of the underlying map $\widetilde{\Xi'^{\mathrm{Cau}}} : \mathrm{space}(Z) \to B_{\mathbf{0}}$ is contained in A, there exists a (necessarily uniquely determined) solution family $\Xi' \in \mathcal{M}^{B(\mathbb{R})}(Z)$ with $\Xi'(0) = \Xi'^{\mathrm{Cau}}$.*

If these conditions are satisfied we call the the problem (2.1.2) *(B, A)*-complete. *If the problem* (2.1.2) *is $(B, B_{\mathbf{0}})$-complete we call it simply B*-complete.

Remarks. (1) The notion "completeness" has been chosen by analogy with the usual completeness of flows (i. e. local one-parameter groups of automorphisms) on manifolds. Indeed, the problem (2.1.2) determines a time evolution flow on the smf $\mathrm{L}(B)$, and it is B-complete iff this flow is complete.

However, the problem in making that rigorous is that our smf calculus is real-analytic while this flow in time direction is not.

(2) Let us comment on the fact that completeness depends only on the underlying even problem (2.2.1): Mathematically, this is an analogon of several theorems in supergeometry that differential-geometric tasks, like trivializing a fibre bundle, or presenting a closed form as differential, are solvable iff the underlying smooth tasks are solvable.

Physically, our interpretation is somewhat speculative: In the bosonic sector, the classical field theory approximates the behaviour of coherent states, and completeness excludes that "too many" particles may eventually assemble at a space-time point, making the state non-normable. On the fermionic side, apart from the non-existence of genuine coherent states, it is the Pauli principle which automatically prevents such an assembly.

2.4. **Solvability in function spaces.** We will be interested in subspaces of $B(\mathbb{R})$ which are determined by additional functional-analytic quality conditions. Thus, suppose that we are given a continuous, even inclusion $E \subseteq B(\mathbb{R})$ where E is another $\mathbb{Z}_2$-lcs.

We call any element $\Xi' \in \mathcal{M}^E(Z)$ a *Z-family of configurations of quality E*. By the inclusion $\mathcal{M}^E(Z) \subseteq \mathcal{M}^{B(\mathbb{R})}(Z)$, such an element can be viewed as a Z-family in the previous sense, and so it makes sense to speak of *solution families of quality E*.

Denote by $E^{\mathrm{Cau}} \subseteq B$ the image of E under the projection onto Cauchy data, and equip it with the quotient topology. We call E^{Cau} the *space of Cauchy data belonging to E*.

We call the problem (2.1.2) *solvable in* $\mathrm{L}(E)$ if: there exists a (necessarily uniquely determined) solution family of quality E henceforth denoted by $\Xi^{\mathrm{sol}} \in \mathcal{M}^E(\mathrm{L}(E^{\mathrm{Cau}}))$ the Cauchy data of which is the standard coordinate $\Xi^{\mathrm{Cau}} \in \mathcal{M}^{E^{\mathrm{Cau}}}(\mathrm{L}(E^{\mathrm{Cau}}))$:

$$\Xi^{\mathrm{sol}}(0) = \Xi^{\mathrm{Cau}}.$$

Theorem 2.4.1. *If the problem* (2.1.2) *is solvable in* $\mathrm{L}(E)$ *then the image of the morphism* $\Xi^{\mathrm{sol}} : \mathrm{L}(E^{\mathrm{Cau}}) \to \mathrm{L}(E)$ *is a split sub-smf which we call the* smf of classical solutions within $\mathrm{L}(E)$*, and denote by* $\mathrm{L}(E)^{\mathrm{sol}} \subseteq \mathrm{L}(E)$.

$\mathrm{L}(E)^{\mathrm{sol}}$ has the following universal property: Fixing an smf Z we have a bijection between Z-families Ξ' of configurations of quality E with time definition interval $\mathbb{R}$, and morphisms $\Xi' : Z \to \mathrm{L}(E)$. Now Ξ' is a solution family iff the morphism Ξ' factors to $\Xi' : Z \to \mathrm{L}(E)^{\mathrm{sol}} \subseteq \mathrm{L}(E)$.

In this way, we get a bijection between Z-families Ξ' of solutions of quality E with time definition interval $\mathbb{R}$, and morphisms $\Xi' : Z \to \mathrm{L}(E)^{\mathrm{sol}}$.

Remarks. (1) Suppose that the problem (2.1.2) is both solvable in $\mathrm{L}(E)$ and in $\mathrm{L}(B(\mathbb{R}))$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{L}(E^{\mathrm{Cau}}) & \xrightarrow{\Xi^{\mathrm{sol}}} & \mathrm{L}(E) \\ \downarrow & & \downarrow \\ \mathrm{L}(B) & \xrightarrow{\Xi^{\mathrm{sol}}} & \mathrm{L}(B(\mathbb{R})), \end{array}$$

which justifies it to use the same notation Ξ^{sol} in all cases. Moreover, $\mathrm{L}(E)^{\mathrm{sol}}$ is just the intersection $\mathrm{L}(E) \cap \mathrm{L}(B(\mathbb{R}))^{\mathrm{sol}}$ in the categorial sense, i. e. the pullback of the diagram $\mathrm{L}(E) \xrightarrow{\subseteq} \mathrm{L}(B(\mathbb{R})) \xleftarrow{\supseteq} \mathrm{L}(B(\mathbb{R}))^{\mathrm{sol}}$.

(2) It follows that the underlying manifold $\widetilde{\mathrm{L}(E)^{\mathrm{sol}}}$ identifies with the set of all $\phi \in E_{\mathbf{0}}$ which satisfy (2.2.1).

(3) Note that $\mathrm{L}(E)^{\mathrm{sol}}$ is still a linear smf which is, however, in a non-linear way embedded into $\mathrm{L}(E)$.

An obvious necessary condition for solvability in $\mathrm{L}(E)$ is $(B,(E^{\mathrm{Cau}})_{\mathbf{0}})$-completeness. For the maximal choice $E = B(\mathbb{R})$, it follows from Thm. 2.3.1 that this condition is also sufficient:

Corollary 2.4.2. *The problem* (2.1.2) *is B-complete iff it is solvable in* $\mathrm{L}(B(\mathbb{R}))$. □

Cor. 4.2.5 below gives a general method for showing solvability.

A simple but useful observation is:

Corollary 2.4.3. *Let be given a family $(E_\kappa)_{\kappa\in K}$ of $\mathbb{Z}_2$-lcs and continuous, even inclusions $E_\kappa \subseteq B(\mathbb{R})$ such that the problem* (2.1.2) *is solvable in each $\mathrm{L}(E_\kappa)$. Let $E := \bigcap_{\kappa\in K} E_\kappa$, equipped with the projective limes topology. Then the problem* (2.1.2) *is solvable in* $\mathrm{L}(E)$. □

In the following, we will consider some special cases.

2.5. **Smoothness scales.** Up to now, the solutions the existence of which is asserted in Thm. 2.3.1 and Cor. 2.4.2 are in time direction only continuous. Using smoothness scales we get temporal differentiability properties.

Suppose we are given a sequence of real $\mathbb{Z}_2$-graded Banach spaces and continuous even inclusions

$$B = B_0 \supseteq B_1 \supseteq \cdots \supseteq B_l. \tag{2.5.1}$$

For $j = 0,\dots,l$, set

$$B^j(I) := \{f \in C(I,B_j) : \ f \in C^{j-i}(I,B_i) \ \text{ for } i = 0,\dots,j\} \tag{2.5.2}$$

and equip this space with the corresponding locally convex topology defined by the seminorms $\|f\|_{B^j([a,b])} := \sum_{i=0}^{j} \max_{t\in[a,b]} \left\| \frac{d^{j-i}}{dt^{j-i}} f(t) \right\|_{B_i}$ where $a,b \in I$, $a<b$. Of course, if I is compact then $B^j(I)$ is a Banach space.

We call the sequence (2.5.1) a *smoothness scale* iff

(i) (A_t) descends to a strongly continuous group (A_t) on each B_i;
(ii) B_{i+1} lies in the domain of the generator of $(\mathrm{A}_t\,|_{B_i})$;
(iii) Δ restricts to a (necessarily unique) entire superfunction $\Delta \in \mathcal{M}^{B_i}(\mathrm{L}(B_i))$ for all i.

We now get a temporal smoothness assertion:

Proposition 2.5.1. *Given a smoothness scale* (2.5.1)*, every solution family $\Xi' \in \mathcal{M}^{B_l(I)}(Z)$ satisfies $\Xi' \in \mathcal{M}^{B^l(I)}(Z)$.*

In case of B_l-completeness, it follows that the universal solution family for the Banach space B_l, $\Xi^{\mathrm{sol}} \in \mathcal{M}^{B_l(\mathbb{R})}(\mathrm{L}(B_l)))$, satisfies $\Xi^{\mathrm{sol}} \in \mathcal{M}^{B^l(\mathbb{R})}(\mathrm{L}(B_l)))$, and hence:

Corollary 2.5.2. *If the problem* (2.1.2) *is B_l-complete then it is solvable in* $\mathrm{L}(B^l(\mathbb{R}))$. □

We call an infinite sequence $B = B_0 \supseteq B_1 \supseteq \dots$ an *smoothness scale of infinite length* if for each l, the sequence $B = B_0 \supseteq \cdots \supseteq B_l$ is a smoothness scale in the sense of above. Given a smoothness scale of infinite length, set $B^\infty(\mathbb{R}) := \bigcap_{i>0} B^i(\mathbb{R})$, equipped with the projective limes topology. The space of Cauchy data belonging to this is $B_\infty := \bigcap_{i>0} B_i$, again with the projective limes topology. Now Cor. 2.5.2 and Cor. 2.4.3 together yield:

Corollary 2.5.3. *Let be given a smoothness scale of infinite length. If the problem* (2.1.2) *is $(B_i,(B_\infty)_{\mathbf{0}})$-complete for $i \gg 0$ then it is solvable in* $\mathrm{L}(B^\infty(\mathbb{R}))$. □

2.6. **Support scales.** Here we give an abstract version of causal propagation of perturbations.

A family $(S_t)_{t\in I}$ of closed $\mathbb{Z}_2$-graded subspaces of B where $I \ni 0$ is an interval is called a *support scale* if

(ii) we have $S_t \subseteq S_{t'}$ for $0 \leq t \leq t'$ or $t' \leq t \leq 0$;
(ii) the free evolution "stays within the scale": If $\xi \in S_0$ then $\mathrm{A}_t\,\xi \in S_t$ for all $t \in I$;
(iii) the interaction "is local": For all $t \in I$, $\Delta \in \mathcal{M}^B(\mathrm{L}(B))$ restricts to a (necessarily unique) superfunction $\Delta \in \mathcal{M}^{B/S_t}(\mathrm{L}(B/S_t))$, i. e. we have a commutative diagram

$$\begin{array}{ccc} \mathrm{L}(B) & \xrightarrow{\Delta} & \mathrm{L}(B) \\ \big\downarrow & & \big\downarrow \\ \mathrm{L}(B/S_t) & \xrightarrow{\Delta} & \mathrm{L}(B/S_t); \end{array}$$

(iv) "Splitting property": There exists a strongly continuous family $(\mathrm{E}_t)_{t\in I}$ of operators $\mathrm{E}_t : B \to B$ such that for all $\xi \in B$ and $t \in I$ we have $\mathrm{E}_t\,\xi - \xi \in S_t$ and $\|\mathrm{E}_t\,\xi\|_B \leq C(t)\,\|\xi\|_{B/S_t}$ with some constant $C(t) > 0$ which is bounded on bounded intervals. (Thus, E_t factors to a bounded operator $B/S_t \to B$ which is a right inverse to the projection $B \to B/S_t$.)

For two superfunctions $K, K' \in \mathcal{O}^B(Z)$, we will write for shortness $K \equiv_t K'$ iff $K - K' \in \mathcal{O}^{S_t}(Z)$.

We call a Z-family $\Xi' \in \mathcal{M}^{B(I)}(Z)$ of configurations a *relative solution family* (with respect to the support scale $(S_t)_{t\in I}$) iff

$$\Xi'(t) \equiv_t \mathrm{A}_t\,\Xi'(0) + \int_0^t ds\,\mathrm{A}_{t-s}\,\Delta[\Xi'(s)]$$

for all $t \in I$. With this notion, we get a refined Uniqueness Theorem:

Theorem 2.6.1. *Fix the problem* (2.1.2) *and a support scale* $(S_t)_{t\in I}$. *Let be given two Z-families* $\Xi', \Xi'' \in \mathcal{M}^{B(I)}(Z)$ *which are both relative solution families, and suppose that* $\Xi'(0) \equiv_0 \Xi''(0)$. *Then* $\Xi'(t) \equiv_t \Xi''(t)$ *for all* $t \in I$.

Now Thm. 2.2.1 follows by taking here the trivial support scale $S_t := 0$ for all t.

2.7. **Variants and generalizations.**

2.7.1. *Time-dependent interaction.* Consider the problem

$$\Xi'(t) = \mathrm{A}_t\,\Xi'(0) + \int_0^t ds\,\mathrm{A}_{t-s}\,\Delta_s[\Xi'(s)] \tag{2.7.1}$$

where $\Delta_s \in \mathcal{M}^B(\mathrm{L}(B))$ for each $s \in \mathbb{R}$.

An obvious idea is the reduction onto the time-independent form (2.1.2) by passing to the enlarged Banach space $B^{\mathrm{ext}} := B \oplus \mathbb{R} \oplus \mathbb{R}$ and forming a new one parameter group $(\mathrm{A}_t^{\mathrm{ext}})$ in B^{ext} which acts as A_t in B and as $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ on $\mathbb{R} \oplus \mathbb{R}$. Setting also

$$\Delta^{\mathrm{ext}}[\Xi^{\mathrm{Cau}}, \theta, \nu] := \big(\Delta_\theta[\Xi^{\mathrm{Cau}}], 0, 0\big) \in \mathcal{M}^{B^{\mathrm{ext}}}(\mathrm{L}(B^{\mathrm{ext}})_{\Xi^{\mathrm{Cau}},\theta,\nu}),$$

the original problem becomes equivalent with the problem

$$\Xi'^{\mathrm{ext}}(t) = \mathrm{A}_t^{\mathrm{ext}}\,\Xi'^{\mathrm{ext}}(0) + \int_0^t ds\,\mathrm{A}_{t-s}^{\mathrm{ext}}\,\Delta^{\mathrm{ext}}[\Xi'^{\mathrm{ext}}(s)],$$

together with the initial conditions $\Xi'^{\mathrm{ext}}(0) = (\Xi'(0), 0, 1)$. Indeed, these enforce every solution family to have the form $\Xi'^{\mathrm{ext}} = (\Xi', t, 1)$.

However, this reduction works only if Δ_s depends real-analytically on s, which makes it unapplicable in the classical field models of quantum field theory for constraining the interaction onto a finite space-time domain with the aid of a buffer function g ("adiabatically switching the interaction").

It is a better idea to generalize the theory by considering an interaction term to be given as an entire superfunction $\Delta \in \mathcal{M}^{B(I)}(\mathrm{L}(B))$ the Taylor expansion of which at the origin has lower degree ≥ 2. Let δ_s denote evaluation at $s \in I$, and $\Delta_s := \delta_s \Delta \in \mathcal{M}^{B}(\mathrm{L}(B))$. Then the equation (2.7.1) makes sense for each s. Moreover, it is not hard to show that if leaving s unfixed, the r. h. s. defines an element of $\mathcal{M}^{B(I)}(Z)$. This follows from the following general fact:

Lemma 2.7.1. *Let M be a finite-dimensional smooth (non-super) manifold, and B a $\mathbb{Z}_2$-graded Banach space. For each finite $l \geq 0$, equip $C^l(M,B)$ with the topology of convergence of derivatives up to l-th order on compacta. Also, equip $C^\infty(M,B) = \bigcap_l C^l(M,B)$ with the projective limes topology.*

Let be given a superfunction $\Delta \in \mathcal{M}^{C^l(M,B)}(\mathrm{L}(B))$ where $0 \leq j \leq \infty$. Then there exists a unique superfunction $\Delta \in \mathcal{M}^{C^l(M,B)}(\mathrm{L}(C^l(M,B)))$ (by abuse of notation) which makes the diagram

$$\begin{array}{ccc}
\mathrm{L}(C^l(M,B)) & \xrightarrow{\Delta} & \mathrm{L}(C^l(M,B)) \\
\delta_t \Big\downarrow & & \Big\downarrow \delta_t \\
\mathrm{L}(B) & \xrightarrow{\Delta_t} & \mathrm{L}(B)
\end{array}$$

commutative where δ_t denotes evaluation at $t \in M$, and $\Delta_t := \delta_t \Delta \in \mathcal{M}^{B}(\mathrm{L}(B))$.

With obvious modifications, our notions and results now carry over to problems of the form (2.7.1). In particular, in the definition of a smoothness scale, condition (iii) has to be replaced by a condition on temporal smoothness of Δ:

(iii)' $\Delta \in \mathcal{M}^{B(I)}(\mathrm{L}(B))$ restricts to an entire superfunction $\Delta \in \mathcal{M}^{B^i(I)}(\mathrm{L}(B_i))$ for all i.

In adapting the proof of Prop. 2.5.1, one uses Lemma 2.7.1 with $l > 0$.

2.7.2. *Source terms.* Another generalization arises by allowing source terms in (2.1.1):

$$\frac{d}{dt}\Xi' = K\Xi' + \Delta[\Xi'] + J',$$

or, in integral form,

$$\Xi'(t) = \mathrm{A}_t\, \Xi'(0) + \int_0^t ds\, \mathrm{A}_{t-s}(\Delta[\Xi'(s)] + J'(s)). \tag{2.7.2}$$

We suppose the source term J' to be given as superfunction on a parameter smf S (this allows sources also for the anticommuting degrees of freedom).

Thus, a senseful Cauchy problem for (2.7.2) is to look for $\Xi' \in \mathcal{M}^{B(I)}(Z \times S)$ with given Cauchy data $\Xi'(0) \in \mathcal{M}^{B}(Z \times S)$ and given source $J' \in \mathcal{M}^{B(I)}(S)$ which satisfies (2.7.2) within $\mathcal{M}^{B}(Z \times S)$ for $t \in I$.

For technical simplification, we may assume the source to take values in the Banach space $B_b(\mathbb{R})$ of bounded continuous functions $\mathbb{R} \to B$ equipped with the sup norm.

Now there is a universal formulation for this problem which includes all possible Cauchy data and all possible sources: given (A_t) and Δ, we have to find a superfunction $\Xi^{\mathrm{sol}} \in \mathcal{M}^{B(I)}(\mathrm{L}(B) \times \mathrm{L}(B_b(\mathbb{R})))$ such that

$$\Xi^{\mathrm{sol}}(t) = \mathrm{A}_t\, \Xi^{\mathrm{Cau}}(0) + \int_0^t ds\, \mathrm{A}_{t-s}(\Delta[\Xi^{\mathrm{sol}}(s)] + J(s)) \tag{2.7.3}$$

where Ξ^{Cau}, J are the standard coordinates on the factors.

This problem is easily reduced to our standard form (2.1.2): we form a new one parameter group $(\mathrm{A}_t^{\mathrm{ext}})$ in the enlarged Banach space $B^{\mathrm{ext}} := B \oplus B_b(\mathbb{R})$,

$$\mathrm{A}_t^{\mathrm{ext}}(\xi^{\mathrm{Cau}}, j) := (\mathrm{A}_t\, \xi^{\mathrm{Cau}} + \int_0^t ds\, \mathrm{A}_{t-s}\, j(s),\ j(t+\cdot)).$$

Setting also $\Delta^{\mathrm{ext}}[\Xi^{\mathrm{Cau}}, J] := \big(\Delta[\Xi^{\mathrm{Cau}}], 0\big) \in \mathcal{M}^{B^{\mathrm{ext}}}(\mathrm{L}(B^{\mathrm{ext}})_{\Xi^{\mathrm{Cau}},J})$, the problem (2.7.3) becomes equivalent with the problem

$$\Xi'^{\mathrm{ext}}(t) = \mathrm{A}_t^{\mathrm{ext}}(\Xi^{\mathrm{Cau}}, J) + \int_0^t ds\, \mathrm{A}_{t-s}^{\mathrm{ext}}\, \Delta^{\mathrm{ext}}[\Xi'^{\mathrm{ext}}(s)],$$

which has our standard form (2.1.2).

With a similar trick, one can also treat non-dynamical fields.

2.7.3. *Semigroups.* An obvious way to generalize (2.1.2) is to replace the strongly continuous group $(\mathrm{A}_t)_{t\in\mathbb{R}}$ on B by a strongly continuous semigroup $(\mathrm{A}_t)_{t\geq 0}$. In that case, only configuration families $\Xi' \in \mathcal{M}^{B(I)}(Z)$ with $I \subseteq \mathbb{R}_+ := \{t \geq 0\}$ are to be taken into account.

All our results generalize mutatis mutandis onto this case; if the problem is complete we get a universal solution supermanifold $\mathrm{L}(B(\mathbb{R}_+))^{\mathrm{sol}} \subseteq \mathrm{L}(B(\mathbb{R}_+))$. (Note, however, that anticommuting degrees of freedom occur mainly in classical field models of quantum field theory, where the time evolution is always time-reversible.)

2.7.4. *Non-entire interaction: Cauchy uniqueness.* The reader will note that in the original problem (2.1.2), the entireness hypothesis on Δ will be not needed for showing Cauchy uniqueness; it will be used only for the construction of the short-time solution.

In order to formulate Cauchy uniqueness in its most general form, we go a step further and consider a generalization of the problem (2.1.2) by supposing only $\Delta = \Delta[\Xi] \in \mathcal{M}^B(U)$ where $U \subseteq \mathrm{L}(B)$ is an open subset of $B_{\mathbf{0}}$ which contains 0, considered as sub-superdomain. However, we keep the requirement that the Taylor expansion of Δ in zero has lower degree ≥ 2.

The notion of a configuration family has to be modified: we require additionally that for the underlying function $\widetilde{\Xi'} : \mathrm{space}(Z) \to B(I)_{\mathbf{0}}$ of $\Xi' \in \mathcal{M}^{B(I)}(Z)$, we have $\widetilde{\Xi'}(t) \in \mathrm{space}(U)$ for all $t \in I$.

For any compact interval $I \ni 0$, set $U(I) := C(I, U)$; this is open in $B(I)_{\mathbf{0}}$, and hence is the underlying space of an open sub-superdomain in $\mathrm{L}(B(I))$ which we abusively denote by $\mathrm{L}(U(I))$. Now a configuration family is the same as a morphism $\Xi' : Z \to \mathrm{L}(U(I))$.

For such a configuration family, the r. h. s. of (2.1.2) is now well-defined. Of course, we call Ξ' again a *solution family* iff (2.1.2) holds.

The Cauchy uniqueness still generalizes to this situation. The proof of Thm. 2.2.1 actually yields:

Corollary 2.7.2. *Fix the problem* (2.1.2) *where B and $(\mathrm{A}_t)_{t\geq 0}$ are as in 2.1, and $\Delta = \Delta[\Xi] \in \mathcal{M}^B(U)$, $U \subseteq \mathrm{L}(B)$ open. Suppose that $0 \in U$, and that the Taylor expansion Δ_0 has lower degree ≥ 2. Given solution families $\Xi', \Xi" : Z \to \mathrm{L}(U(I))$ where $I \ni 0$ is connected such that for some $t_0 \in I$ we have $\Xi'(t_0) = \Xi"(t_0)$, we have $\Xi' = \Xi"$.* □

2.7.5. *Non-entire interaction: Short-time existence.* For Δ defined only on some open U, looking for all-time existence is not very senseful. However, the approach to short-time existence given in Prop. 4.1.1 below generalizes: the assertion (i) on the existence of a formal solution remains unchanged (it only uses the formal power series Δ_0), while for analyticity we have to make a certain trade-off in the domain of definition (which is clearly necessary since the free evolution has to stay at least for a short time in the domain of definition of the interaction):

Corollary 2.7.3. *Let $U' \in \mathrm{CB}(B)$ such that $\Delta_0 \in \mathcal{P}(B, U';\ B)$. For each $c < 1/\limsup_{t\to 0} \|\mathrm{A}_t\|$ there exists θ such that $\Xi^{\mathrm{sol}} \in \mathcal{P}(B, cU';\ B([-\theta, \theta]))$.* □

2.7.6. *Grassmann-valued solutions.* The most naive notion of a configuration in a classical field model with anticommuting fields arises by replacing the domain $\mathbb{R}$ for the real field components by a finite-dimensional Grassmann algebra $\Lambda_n = \mathbb{C}[\zeta_1, \ldots, \zeta_n]$ (we recall that, in accordance with our hermitian framework, only complex Grassmann algebras should be used). Thus, a *Λ_n-valued configuration* is an element $\xi \in (\Lambda_n \otimes B(I))_{\mathbf{0},\mathbb{R}}$.

Now denote by Z_n the unique connected $0|n$-dimensional smf, which is just a point together with the Grassmann algebra $\mathcal{O}(Z_n) = \Lambda_n$. Because of $\Lambda_n \otimes B(I) = \mathcal{O}^{B(I)}(Z_n)$, such an element ξ is the same as a Z_n-family. Also, ξ is a solution family in our sense iff the equation (2.1.2) is satisfied within $\Lambda_n \otimes B$. We now get an overview over all Λ_n-valued solutions:

Corollary 2.7.4. *Suppose that the problem* (2.1.2) *is complete, and let be given Λ_n-valued Cauchy data $\xi^{\mathrm{Cau}} \in (\Lambda_n \otimes B)_{\mathbf{0},\mathbb{R}}$. Then there exists a unique solution ξ with these Cauchy data. It is given by*

$$\xi = \Xi^{\mathrm{sol}}[\xi^{\mathrm{Cau}}] = \Xi^{\mathrm{sol}}_{b(\phi^{\mathrm{Cau}})}[s(\xi^{\mathrm{Cau}})]$$

where $b(\cdot) : \Lambda_n \to \mathbb{C}$ denotes the body map, and $s(\cdot) = 1 - b(\cdot)$ the soul map. □

(For a discussion in the context of evolution PDEs as well as of solutions in the infinite-dimensional Grassmann algebra Λ_∞ of supernumbers introduced by deWitt [4], cf. [14].)

3. Application to systems of evolution equations

3.1. **The setting.** Here we fix a class of systems of classical nonlinear wave equations in Minkowski space $\mathbb{R}^{d+1}$ which is wide enough to describe the field equations of many usual models, like e. g. Φ^4, quantum electrodynamics, Yang-Mills theory with usual gauge-breaking term, Faddeev-Popov ghosts, and possibly minimally coupled fermionic matter. The novelty in our equations is the appearance of anticommuting fields; in describing the system, they simply appear as anticommuting variables generating a differential power series algebra. However, it is no longer obvious what a solution of our system should be. In fact, as argued in [13], there are no longer "individual" solutions (besides purely bosonic ones, with all fermionic components put to zero); but it is sensible to look for *families* of solutions parametrized by supermanifolds. In particular, solutions with values in Grassmann algebras can be reinterpreted as such families (cf. 2.7.6 and [14]).

We will consider the system of partial differential equations in $\mathbb{R}^{d+1}$

$$L_i[\Xi'] \equiv \partial_t \Xi'_i - \sum_{j=1}^{N} K_{ij}(\partial_x)\Xi'_j - \Delta_i[\Xi'] = 0 \quad (i = 1, \ldots, N = N_{\mathbf{0}} + N_{\mathbf{1}}). \tag{3.1.1}$$

Here $\Xi' = (\Xi'_1, \ldots, \Xi'_N) = (\Phi'_1, \ldots, \Phi'_{N_{\mathbf{0}}} | \Psi'_1, \ldots, \Psi'_{N_{\mathbf{1}}})$ is a tuple of $N_{\mathbf{0}}$ commuting, ordinary, "bosonic" fields as well as of $N_{\mathbf{1}}$ anticommuting, "fermionic" fields. The *kinetic operator* $K_{ij}(\partial_x)$ is a real differential operator with constant coefficients and containing only spatial derivatives. We demand that parities are preserved, i. e. $K_{ij}(\partial_x) = 0$ if $|\Xi_i| \neq |\Xi_j|$; additional requirements will be specified below.

The *interaction terms* $\Delta_i[\underline{\Xi}]$ are real, entire differential power series (in the finite-dimensional sense) of lower degree ≥ 2, i. e.

$$\Delta_i[\underline{\Xi}] = \Delta_i[\underline{\Phi}|\underline{\Psi}] \in \mathbb{C}[[(\partial^\nu \underline{\Xi}_i)_{i=1,\ldots,N,\ \nu \in \mathbb{Z}^d_+,\ |\nu| \leq n}]]$$

for some $n \geq 0$, where, as usual, $\partial^\nu := \partial_1^{\nu_1} \cdots \partial_d^{\nu_d}$. As in [12], the underlined letters $\underline{\Phi}, \underline{\Psi}$ denote the even and odd indeterminates of an algebra of differential polynomials or differential power series, while the non-underlined letters Ξ, Φ, Ψ denote superfunctions or their Taylor expansions. (As usual, a power series in a finite number of even and odd variables, $P[y|\eta] = \sum P_{\mu\nu} y^\mu \eta^\nu \in \mathbb{C}[[y_1, \ldots, y_m | \eta_1, \ldots, \eta_n]]$ is entire iff for all $R > 0$ there exists $C > 0$ such that $|P_{\mu\nu}| \leq CR^{-|\mu|}$ for all μ, ν.) Of course, we also require that Δ_i is even for $i = 1, \ldots, N_{\mathbf{0}}$ and odd for $i = N_{\mathbf{0}} + 1, \ldots, N_{\mathbf{0}} + N_{\mathbf{1}}$.

We require that there exist integers $\tau_1, \ldots, \tau_N$, called *smoothness offsets*, with the following properties:

I. There exist $t_0 > 0$, $C > 0$ such that the matrix-valued function

$$\hat{A} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^{N\times N}, \quad \hat{A}(t,p) := (2\pi)^{-d/2} \exp(K(\mathrm{i}\,p)t) 1_{N\times N} \tag{3.1.2}$$

satisfies the estimate

$$\left\| \hat{A}_{ij}(t,p) \right\| \leq C(1+|p|)^{\tau_j - \tau_i}, \tag{3.1.3}$$

for $p \in \mathbb{R}^d$, $t \in [-t_0, t_0]$ with suitable $t_0 > 0$, $C > 0$.

II. For all $i, k = 1, \ldots, N$, $\nu \in {\mathbb{Z}_+}^n$, we have

$$\frac{\partial}{\partial(\partial^\nu \Xi_k)} \Delta_i[\Xi] \neq 0 \qquad \Longrightarrow \qquad \max(0,\ \tau_i) \leq \tau_k - |\nu| \,. \tag{3.1.4}$$

Remarks. (1) The function $\hat{A}$ satisfies the spatially Fourier-transformed and complexified free field equations, $\frac{d}{dt}\hat{A}(t,p) - K(\mathrm{i}\,p)\hat{A} = 0$, $\hat{A}(0,p) = (2\pi)^{-d/2} 1_{N\times N}$. (Our convention for Fourier transforms is $\hat{f}(p) = \mathcal{F}_{x\to p} f(p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} dx\, \mathrm{e}^{-\mathrm{i}\,px}\, f(x)$ for $f \in \mathcal{S}(\mathbb{R}^d)$.)

(2) Obviously, the estimate (3.1.3) implies hyperbolicity of the kinetic operators, i. e. for all $p \in \mathbb{R}^d$, the matrix $K(\mathrm{i}\,p)$ has only imaginary eigenvalues.

(3) Usually, the smoothness offsets save that smoothness information which would be otherwise lost in reducing a temporally higher-order system to a temporally first-order one.

(4) The smoothness condition (3.1.4) is rather constraining; it excludes e. g. the Korteweg-de Vries equation as well as the nonlinear Schrödinger equations. Fortunately, it is satisfied for apparently all wave equations occurring in quantum-field theoretical models. (Of course, the smoothness offsets have to be chosen suitably: usually, one for second-order fields, and zero for their derivatives as well as for first-order fields.)

(5) In [14], we had constrained the smoothness offsets to be nonnegative.

3.2. **Basic results.** We use the standard Sobolev spaces: For real $k > d/2$, let $H_k(\mathbb{R}^d)$ be the space of all $f \in L_2(\mathbb{R}^d)$ for which $(1+|p|)^k \hat{f}(p)$ is square-integrable. Our basic Banach space of Cauchy data is

$$\mathcal{H}_k^{\mathrm{V}} := \bigoplus_{i=1}^{N_0} H_{k+\tau_i}(\mathbb{R}^d) \quad \oplus \quad \bigoplus_{i=1}^{N_1} \Pi H_{k+\tau_i}(\mathbb{R}^d).$$

Because of (3.1.3) we can take the inverse spatial Fourier transform $A(t,x)$ of the function $\hat{A}(t,p)$ defined in (3.1.2), and it follows that $K(\partial_x)$ is the generator of the continuous one-parameter group (A_t) in $\mathcal{H}_k^{\mathrm{V}}$ given by

$$\mathrm{A}_t\, \xi(x) := \int_{\mathbb{R}^d} dy A(t, x-y)\xi(y), \quad \text{i. e.} \quad \widehat{\mathrm{A}_t\, \xi}(p) := (2\pi)^{d/2} \hat{A}(t,p)\hat{\xi}(p). \tag{3.2.1}$$

In order to assign to the Δ_i an entire superfunction $\Delta[\Xi^{\mathrm{Cau}}]$, we split them by degree: $\Delta_i = \sum_{l\geq 2} \Delta_{i,(l)}$. Thus, $\Delta_{i,(l)}[\Xi]$ is a differential polynomial, and, due to the condition (3.1.4), the substitution $\partial^\nu \Xi_i \mapsto \partial^\nu \Xi_i^{\mathrm{Cau}}$ yields a polynomial superfunction $\Delta_{i,(l)}[\Xi^{\mathrm{Cau}}] \in \mathcal{O}^{H_{k+\tau_i}(\mathbb{R}^d)}(\mathrm{L}(\mathcal{H}_k^{\mathrm{V}}))$. On the other hand, we have a Frèchet topology on the subspace $\mathcal{O}^{H_{k+\tau_i}(\mathbb{R}^d)}(\mathrm{L}(\mathcal{H}_k^{\mathrm{V}}))_{\mathrm{ent}}$ of entire superfunctions (cf. 1.2) by the seminorms

$$f \mapsto \|f_0\|_{\mathcal{P}(\mathcal{H}_k^{\mathrm{V}}, nU;\ H_{k+\tau_i}(\mathbb{R}^d))}$$

where $n = 1, 2, \ldots,$ f_0 is the Taylor expansion at zero, and U is the unit ball. In this topology, the series $\Delta_i[\Xi^{\mathrm{Cau}}] := \sum_{l\geq 2} \Delta_{i,(l)}[\Xi^{\mathrm{Cau}}]$ converges, and hence

$$\Delta[\Xi^{\mathrm{Cau}}] := (\Delta_1[\Xi^{\mathrm{Cau}}], \ldots, \Delta_{N_{\mathbf{0}}}[\Xi^{\mathrm{Cau}}], \Pi\Delta_{N_{\mathbf{0}}+1}[\Xi^{\mathrm{Cau}}], \ldots, \Pi\Delta_{N_{\mathbf{0}}+N_{\mathbf{1}}}[\Xi^{\mathrm{Cau}}]) \in \mathcal{M}^{\mathcal{H}_k^{\mathrm{V}}}(\mathrm{L}(\mathcal{H}_k^{\mathrm{V}}))$$

is a well-defined entire superfunction. Thus, we can rewrite (3.1.1) into integral form:

$$\Xi'(t,x) = \int_{\mathbb{R}^d} dy A(t, x-y)\Xi'(0,y) + \int_0^t ds \int_{\mathbb{R}^d} dy A(t-s, x-y)\Delta[\Xi'(s,\cdot)](y). \tag{3.2.2}$$

This has the form of the abstract problem (2.1.2) with $B := \mathcal{H}_k^{\mathrm{V}}$, and A_t being given by (3.2.1). A superfunction $\Xi' \in \mathcal{M}^{\mathcal{H}_k^{\mathrm{V}}(I)}(Z)$ is a solution family if and only if it satisfies (3.1.1) within $\mathcal{M}^{\mathcal{D}'(I\times\mathbb{R}^d)}(Z)$. On the other hand, an element $\phi = (\phi_1, \ldots, \phi_{N_{\mathbf{0}}}) \in \mathcal{H}_k^{\mathrm{V}}(I)_{\mathbf{0}}$ satisfies the underlying system (2.2.1) iff the functions $\phi_i \in C(I,\ H_{k+\tau_i}(\mathbb{R}^d))$ fulfill

$$\partial_t \phi_i - \sum_{j=1}^{N_{\mathbf{0}}} K_{ij}(\partial_x)\phi_j - \Delta_i[\phi|0] = 0 \quad (i = 1, \ldots, N_{\mathbf{0}}). \tag{3.2.3}$$

Note that $\phi \mapsto \Delta_i[\phi|0] = \tilde{\Delta}_i[\phi]$ is the underlying function of the superfunction Δ. Now Thm. 2.3.1 specializes to:

Corollary 3.2.1. *Fix some $k > d/2$. For a subset $A \subseteq (\mathcal{H}_k^{\mathrm{V}})_{\mathbf{0}}$, the following conditions are equivalent:*

(i) For every solution $\phi \in \mathcal{H}_k^{\mathrm{V}}((a,b))_{\mathbf{0}}$ of the underlying system (3.2.3) *on a bounded open time interval $(a,b) \ni 0$ such that $\phi(0) \in A$, we have*

$$\sup_{t\in(a,b)} \|\phi(t)\|_{\mathcal{H}_k^{\mathrm{V}}} < \infty.$$

(ii) The underlying system (3.2.3) *is all-time solvable for Cauchy data in A.*

(iii) Whenever we are given an smf Z and a superfunction ${\Xi'}^{\mathrm{Cau}} \in \mathcal{M}^{\mathcal{H}_k^{\mathrm{V}}}(Z)$ such that the image of the underlying function $\widetilde{{\Xi'}^{\mathrm{Cau}}} : \mathrm{space}(Z) \to (\mathcal{H}_k^{\mathrm{V}})_{\mathbf{0}}$ is contained in A, there exists a (necessarily uniquely determined) solution family $\Xi' \in \mathcal{M}^{\mathcal{H}_k^{\mathrm{V}}(\mathbb{R})}(Z)$ of (3.1.1) *with $\Xi'(0) = {\Xi'}^{\mathrm{Cau}}$.*

If these conditions are satisfied we call the the problem (3.1.1) $(\mathcal{H}_k^{\mathrm{V}}, A)$-complete. *If it is* $(\mathcal{H}_k^{\mathrm{V}}, (\mathcal{H}_k^{\mathrm{V}})_{\mathbf{0}})$-*complete we call it simply* $\mathcal{H}_k^{\mathrm{V}}$-complete. □

It follows from Cor. 2.4.2 that if the problem (3.1.1) is $\mathcal{H}_k^{\mathrm{V}}$-complete it defines an smf of classical solutions $\mathrm{L}(\mathcal{H}_k^{\mathrm{V}}(\mathbb{R}))^{\mathrm{sol}} \subseteq \mathrm{L}(\mathcal{H}_k^{\mathrm{V}}(\mathbb{R}))$.

Proposition 3.2.2. *If the problem* (3.1.1) *is $(\mathcal{H}_k^{\mathrm{V}}, A)$-complete with a subset $A \subseteq (\mathcal{H}_{k+l}^{\mathrm{V}})_{\mathbf{0}}$ where $l > 0$ is integer then it is $(\mathcal{H}_{k+l}^{\mathrm{V}}, A)$-complete.*

Set

$$\mu := \max\left\{1, \max_{i,j=1,\ldots,N} (\tau_i - \tau_j + \operatorname{ord} K_{ij}(\partial_x))\right\}$$

where $\operatorname{ord} K_{ij}(\partial_x)$ is the order of the differential operator ($= -\infty$ if $K_{ij} = 0$). Then, fixing $k > d/2$, the sequence $\mathcal{H}_k^{\mathrm{V}} \supseteq \mathcal{H}_{k+\mu}^{\mathrm{V}} \supseteq \mathcal{H}_{k+2\mu}^{\mathrm{V}} \supseteq \ldots$ forms an infinite smoothness scale, and Prop. 2.5.1 and Cor. 2.5.2 apply.

Set $\mathcal{H}_\infty^{\mathrm{V}} := \bigcap_{k>d/2} \mathcal{H}_k^{\mathrm{V}}$. This is the space of Cauchy data belonging to $C^\infty(\mathbb{R}, \mathcal{H}_\infty^{\mathrm{V}})$, and Cor. 2.5.3 yields:

Corollary 3.2.3. *If the system* (3.1.1) *is $(\mathcal{H}_k^{\mathrm{V}}, (\mathcal{H}_\infty^{\mathrm{V}})_{\mathbf{0}})$-complete for some $k > d/2$ then it is solvable in $\mathrm{L}(C^\infty(\mathbb{R}, \mathcal{H}_\infty^{\mathrm{V}}))$.* □

The space $\mathcal{H}_\infty^{\mathrm{V}}$ lies between the Schwartz space $\mathcal{S}(\mathbb{R}^d) \otimes \mathbb{R}^{N_{\mathbf{0}}|N_{\mathbf{1}}}$ and $C^\infty(\mathbb{R}^d) \otimes \mathbb{R}^{N_{\mathbf{0}}|N_{\mathbf{1}}}$. It would be interesting to know how to descend to the Schwartz space.

3.3. **Causality.** In this section, we study the consequences of finite propagation speed, as it holds in classical field theories used in quantum field theory.

For $(s,x),(t,y)\in\mathbb{R}^{d+1}$ we will write $(s,x)\prec(t,y)$ iff (t,y) lies in the forward light cone of (s,x), i. e. $|t-s|\geq|y-x|$. We call the system (3.1.1) *causal* iff we have (cf. (3.2.1))

$$\operatorname{supp} A\subseteq\{(t,x)\in\mathbb{R}^{d+1}:\quad |x|\leq|t|\}.$$

Given a point $p=(s,x)\in\mathbb{R}^{d+1}$ with $s\neq 0$, write

$$\Omega(p):=\begin{cases}\{(s',x')\in\mathbb{R}^{d+1}:\ (s',x')\prec(s,x),\ 0<s'\} & \text{if } s>0,\\ \{(s',x')\in\mathbb{R}^{d+1}:\ (s,x)\prec(s',x'),\ s'<0\} & \text{if } s<0,\end{cases}$$

$$\mathcal{J}(p):=\{x'\in\mathbb{R}^d:\ |x'-x|<|s|\}.$$

As to be expected, causality implies that perturbations of solution families propagate within the light cone:

Theorem 3.3.1. *Suppose that the system is causal and $k>d/2$.*

(i) Let be given a point $p=(s,x)\in\mathbb{R}^{d+1}$ with $s\neq 0$, and let $I=[0,s]$ if $s>0$ and $I=[s,0]$ if $s<0$, respectively.

Let be given two Z-families $\Xi',\Xi"\in\mathcal{M}^{\mathcal{H}_k^{\mathrm{V}}(I)}(Z)$, and suppose that

$$L_i[\Xi']|_{\Omega(p)}=L_i[\Xi"]|_{\Omega(p)}=0\quad(i=1,\dots,N),$$
$$(\Xi'(0)-\Xi"(0))\,|_{\mathcal{J}(p)}=0.$$

Then $(\Xi'-\Xi")\,|_{\Omega(p)}=0$.

(ii) Suppose that for a solution family $\Xi'\in\mathcal{M}^{\mathcal{H}_k^{\mathrm{V}}(I)}(Z)$ with $I\ni 0$ satisfies $\Xi'(0,x)=0$ for $|x|>r$ with some $r>0$. Then $\Xi'(t,x)=0$ for $|x|>|t|+r$ and $t\in I$.

For $r\geq 0$, let

$$\mathbb{V}_r:=\{(t,x)\in\mathbb{R}^{d+1}:\quad |x|\leq r+|t|\},\tag{3.3.1}$$

and set

$$C_c^\infty(\mathbb{R}^{d+1},\mathbb{R}^{N_0|N_1})=\bigcup_{r>0}\{f\in C^\infty(\mathbb{R}^{d+1},\mathbb{R}^{N_0|N_1}):\ \operatorname{supp} f\subseteq\mathbb{V}_r\}.\tag{3.3.2}$$

Equipping each item of the union with the closed subspace topology and (3.3.2) with the arising inductive limit topology, this is a strict inductive limes of Frèchet spaces, and hence complete. Also, $\mathcal{D}(\mathbb{R}^{d+1})$ is dense in (3.3.2); hence (3.3.2) is admissible in the sense of [12, 3.1]. Moreover, it is important for field-theoretical applications that the Poincaré group acts continuously on (3.3.2). Of course, the space of Cauchy data belonging to (3.3.2) is the testfunction space $\mathcal{D}(\mathbb{R}^d,\mathbb{R}^{N_0|N_1})$.

Our main result for the causal case is:

Theorem 3.3.2. *If the system (3.1.1) is both causal and $(\mathcal{H}_k^{\mathrm{V}},\mathcal{D}(\mathbb{R}^d,\mathbb{R}^{N_0|0}))$-complete for some $k>d/2$, then it is solvable in $\mathrm{L}(C_c^\infty(\mathbb{R}^{d+1},\mathbb{R}^{N_0|N_1}))$.*

We want to show also solvability in smooth functions,

$$\mathcal{C}_\infty^{\mathrm{V}}:=C^\infty(\mathbb{R}^{d+1},\mathbb{R}^{N_0|N_1}).$$

However, this does not quite fit into our general scheme since there is no Banach space B of functions on $\mathbb{R}^d$ such that $\mathcal{C}_\infty^{\mathrm{V}}\subseteq B(\mathbb{R})$ (indeed, there is no continuous norm on $\mathcal{C}_\infty^{\mathrm{V}}$). Therefore we note that if the system (3.1.1) is causal then for $\Xi'\in\mathcal{M}^{\mathcal{C}_\infty^{\mathrm{V}}}(Z)$, both the system (3.1.1) and the integrated version (3.2.2) make sense and are equivalent; if they are satisfied we call Ξ' a *smooth solution family.* Of course, the appropriate space of Cauchy data is $\mathcal{C}_\infty^{\mathrm{Cau,V}}:=C^\infty(\mathbb{R}^d,\mathbb{R}^{N_0|N_1})$.

Theorem 3.3.3. *If the system* (3.1.1) *is both causal and* $(\mathcal{H}_k^{\mathrm{V}}, \mathcal{D}(\mathbb{R}^d, \mathbb{R}^{N_0|0}))$*-complete for some* $k > d/2$*, then it is solvable in* $\mathrm{L}(\mathcal{C}_\infty^{\mathrm{V}})$ *in the following sense:*

There exists a (necessarily uniquely determined) superfunction $\Xi^{\mathrm{sol}} \in \mathcal{M}^{\mathcal{C}_\infty^{\mathrm{V}}}(\mathrm{L}(\mathcal{C}_\infty^{\mathrm{Cau,V}}))$ *such that* Ξ^{sol} *is a smooth solution family, and* $\Xi^{\mathrm{sol}}(0) = \Xi^{\mathrm{Cau}}$ *where* $\Xi^{\mathrm{Cau}} \in \mathcal{M}^{\mathcal{C}_\infty^{\mathrm{Cau,V}}}(\mathrm{L}(\mathcal{C}_\infty^{\mathrm{Cau,V}}))$ *is the standard coordinate. Moreover, the image of morphism* $\Xi^{\mathrm{sol}} : \mathrm{L}(\mathcal{C}_\infty^{\mathrm{Cau,V}}) \to \mathrm{L}(\mathcal{C}_\infty^{\mathrm{V}})$ *is a split sub-smf which we call the* smf of smooth classical solutions, *and denote by* $\mathrm{L}(\mathcal{C}_\infty^{\mathrm{V}})^{\mathrm{sol}}$.

Of course, the consequences of solvability are the same as in 2.4. In particular, the underlying manifold $\widetilde{\mathrm{L}(\mathcal{C}_\infty^{\mathrm{V}})^{\mathrm{sol}}}$ identifies with the set of all $\phi \in C^\infty(\mathbb{R}^{d+1}, \mathbb{R}^{N_0})$ which satisfy (2.2.1). Also, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{L}(\mathcal{D}(\mathbb{R}^d, \mathbb{R}^{N_0|N_1})) & \xrightarrow{\Xi^{\mathrm{sol}}} & \mathrm{L}(C_c^\infty(\mathbb{R}^{d+1}, \mathbb{R}^{N_0|N_1})) \\ \downarrow & & \downarrow \\ \mathrm{L}(\mathcal{C}_\infty^{\mathrm{Cau,V}}) & \xrightarrow{\Xi^{\mathrm{sol}}} & \mathrm{L}(\mathcal{C}_\infty^{\mathrm{V}}). \end{array}$$

For a further variant, which considers spatially compactly carried excitations of solutions, and therefore is interesting in the context of spontaneous symmetry breaking, cf. [14].

4. Proofs

4.1. **Short-time results.** We will need the following standard fact on strongly continuous operator groups: There exists a constant $C_1 > 0$ such that we have for $\theta \in (0, 1]$ and $\xi \in B$

$$\|\mathrm{A}_\bullet\, \xi]\|_{B([-\theta,\theta])} \le C_1\, \|\xi\|_B\,. \tag{4.1.1}$$

It follows that for $\theta \in (0, 1]$, $g \in B(\mathbb{R})$ we have

$$\left\| \int_0^\bullet ds\, \mathrm{A}_{\bullet - s}\, g(s) \right\|_{B([-\theta,\theta])} \le C_1 \theta\, \|g\|_{B([-\theta,\theta])}\,. \tag{4.1.2}$$

In solving the problem (2.1.2), we first construct the Taylor expansion at zero of the superfunctional Ξ^{sol} sought for; we will denote it by Ξ^{sol} again.

Proposition 4.1.1. *(i) There exists a uniquely determined formal power series*

$$\Xi^{\mathrm{sol}} = \Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}] \in \mathcal{P}_{\mathrm{f}}(B;\ B(\mathbb{R})) \tag{4.1.3}$$

which solves (2.1.2) *within* $\mathcal{P}_{\mathrm{f}}(B;\ B)$*. Explicitly, we have*

$$\Xi^{\mathrm{sol}}_{(\le 1)}(t) = \mathrm{A}_t\, \Xi^{\mathrm{Cau}}, \qquad \Xi^{\mathrm{sol}}_{(n+1)}(t) = \int_0^t ds\, \mathrm{A}_{t-s}\, \Delta[\Xi^{\mathrm{sol}}_{(\le n)}]_{(n+1)}(s)$$

for $n \ge 1$*. We call* (4.1.3) *the* formal solution *of the problem* (2.1.2).

(ii) The formal solution is "short-time analytic": For any $c > 0$ *there exists* $\theta > 0$ *such that* $\Xi^{\mathrm{sol}} \in \mathcal{P}(B, cU;\ B([-\theta, \theta]))$ *where* $U \subseteq B$ *is the unit ball.*

(Of course, in (ii) we have silently applied the restriction map $B(\mathbb{R}) \to B([-\theta, \theta])$ in the target.)

Proof. Ad (i). This follows by splitting (2.1.2) into degrees.

Ad (ii). For $n \ge 0$, we have from (2.1.2)

$$\Xi^{\mathrm{sol}}_{(\le n+1)}(t) = \mathrm{A}_t\, \Xi^{\mathrm{Cau}} + \int_0^t ds\, \mathrm{A}_{t-s}\, \Delta[\Xi^{\mathrm{sol}}_{(\le n)}]_{(\le n+1)}(s). \tag{4.1.4}$$

We will show that for sufficiently small $\theta > 0$ we have for all $n \ge 0$ the estimate

$$\left\| \Xi^{\mathrm{sol}}_{(\le n)} \right\| \le 2 C_1 c \quad \text{within } \mathcal{P}(B, cU;\ B([-\theta, \theta])). \tag{4.1.5}$$

Passing to the limit $n \to \infty$ we get the assertion.

From the hypothesis on entireness and the absence of a constant term in Δ, we have:

Lemma 4.1.2. *Given $C' > 0$, there exists $C" > 0$ with the following property: If E is a $\mathbb{Z}_2$-lcs, $p \in \mathrm{CS}(E)$ and the power series $\Xi' \in \mathcal{P}(E, p;\ B([-\theta, \theta]))$ satisfies $\|\Xi'\| < C'$, then*

$$\|\Delta[\Xi']\| \leq C" \quad \text{within } \mathcal{P}(E, p;\ B([-\theta, \theta])).$$

□

We now prove (4.1.5) by induction on n. The start of induction, $n = 1$, follows from (4.1.1). Now, for $n \geq 1$, we find from (4.1.4) that within $\mathcal{P}(B, cU;\ B([-\theta, \theta]))$

$$\left\|\Xi^{\mathrm{sol}}_{(\leq n+1)}\right\| \leq \left\|\mathrm{A}_{\bullet}\, \Xi^{\mathrm{Cau}}\right\| + \left\|\int_0^{\bullet} ds\, \mathrm{A}_{\bullet - s} \left(\Delta[\Xi^{\mathrm{sol}}_{(\leq n)}]_{(\leq n+1)}\right)\right\|.$$

Using (4.1.1) and (4.1.2), this becomes $\leq C_1 c + C_1 \theta \left\|\Delta[\Xi^{\mathrm{sol}}_{(\leq n)}]_{(\leq n+1)}\right\|$. Because of the hypotheses of induction, the preceding Lemma applies with $C' := 2C_1 c$, yielding

$$\left\|\Xi^{\mathrm{sol}}_{(\leq n+1)}\right\| \leq C_1 c + C_1 \theta C",$$

and the assertion of induction, $\left\|\Xi^{\mathrm{sol}}_{(\leq n+1)}\right\| \leq 2C_1 c$, is satisfied for $\theta \leq c/C"$. The Proposition is proved. □

The condition (2.1.2) makes also sense if Ξ' is only a power series $\Xi' \in \mathcal{P}(F;\ B(I))$ where F is an arbitrary $\mathbb{Z}_2$-lcs; if it is satisfied we call Ξ' a *solution power series.*

Of course, the Taylor expansions $\Xi"_z$ $(z \in Z)$ of any solution family $\Xi" \in \mathcal{M}^{B(I)}(Z)$ are solution power series. Conversely, if I is compact then the target $B(I)$ is a Banach space, too, and hence for any element $\Xi' \in \mathcal{P}(F;\ B(I))$ there exists some $p \in \mathrm{CS}(F)$ such that $\Xi' \in \mathcal{P}(F, p;\ B(I))$; by [12, Prop. 3.5.2], it follows that any solution power series defines a solution family $\Xi' \in \mathcal{M}^{B(I)}(U)$ on the open unit ball U within the superdomain $\mathrm{L}(\hat{F}_p)$. Hence we can switch rather freely between solution families and solution power series.

4.2. **Long-time results.** The existence of long-time solutions for the underlying even problem implies the existence of long-time solution families for the original problem:

Lemma 4.2.1. *Let $\phi \in B_{\mathbf{0}}$ be such that there exists a solution of the underlying even problem $\phi' \in B([0, b))_{\mathbf{0}}$ with $\phi = \phi'(0)$ and*

$$\sup_{t \in [0, b)} \|\phi'(t)\|_B < c' < \infty.$$

Then there exists some $\epsilon > 0$ which depends only on c and a (necessarily uniquely determined) solution power series $\Xi^{\mathrm{sol}}_{\phi} = \Xi^{\mathrm{sol}}_{\phi}[\Xi^{\mathrm{Cau}}] \in \mathcal{P}(B;\ B([0, b+\epsilon)))_{\mathbf{0},\mathbb{R}}$ such that $\Xi^{\mathrm{sol}}_{\phi}(0) = \Xi^{\mathrm{Cau}} + \phi$.

Remark. $\Xi^{\mathrm{sol}}_{\phi}$ will become the Taylor expansion of the superfunction Ξ^{sol} at ϕ, motivating the notation.

Proof. First we note that there exists a solution power series $\Xi' \in \mathcal{P}(B;\ B([0, b')))_{\mathbf{0},\mathbb{R}}$ with some $b' > 0$ such that $\Xi'(0) = \Xi^{\mathrm{Cau}} + \phi$ (indeed, using Prop. 4.1.1.(ii) with $c := \|\phi\| + 1$, the translation (cf. [12, 3.3]) $\Xi' := \mathrm{t}_{\phi}\left(\Xi^{\mathrm{sol}}|_{[0,\theta)}\right)$ of Ξ^{sol} by ϕ has this property with $b' := \theta$). By Cauchy uniqueness (cf. Thm. 2.2.1), such a solution power series exists either for each b', or there is a maximal b' such that such a solution power series exists (roughly spoken, this b' is just the forward lifetime for the Cauchy datum ϕ). If the assertion is wrong then such a maximal b' exists and is $\leq b$. Now, since the absolute term $\Xi'[0] \in B([0, b'))$ is a solution of the underlying even problem, Thm. 2.2.1 implies $\Xi'[0] = \phi'|_{[0,b')}$.

By Prop. 4.1.1.(ii), there exists $\theta > 0$ such that $\Xi^{\mathrm{sol}} \in \mathcal{P}(B, c'U;\ B([-\theta,\theta]))$ where U is the unit ball. Composing Ξ^{sol} with $\Xi'(b'-\theta/2) \in \mathcal{P}(B;\ B)_{\mathbf{0},\mathbb{R}}$ yields a solution power series $\Xi" := \Xi^{\mathrm{sol}}[\Xi'(b'-\theta/2)] \in \mathcal{P}(B;\ B([-\theta,\theta]))$. We perform a time shift: $\Xi"(\bullet - b' + \theta/2) \in \mathcal{P}(B;\ B([b'-3\theta/2, b'+\theta/2]))$. Now $\Xi"(\bullet - b' + \theta/2)$ and Ξ', being solution power series with the same Cauchy data at time $b'-\theta/2$, join together to a solution power series $\Xi^{\mathrm{sol}}_\phi \in \mathcal{P}(B;\ B([0, b'+\theta/2)))$ which extends Ξ', in contradiction to our assumption. $\square$

Proof of Thm. 2.3.1. (iii)⇒(ii) is obvious, and (ii)⇒(i) is clear from Thm. 2.2.1 applied to Z being a point. For (i)⇒(ii), one uses the preceding Lemma. Turning to (ii)⇒(iii), we may assume that $Z \subseteq \mathrm{L}(F)$ is a superdomain. We will show that the assignment

$$\mathrm{space}(Z) \ni z \mapsto \Xi'_z := \Xi^{\mathrm{sol}}_{\lambda(z)} \in \mathcal{P}(F;\ B(\mathbb{R}))$$

where $\lambda := \widetilde{\Xi'^{\mathrm{Cau}}}$, and $\Xi^{\mathrm{sol}}_{\lambda(z)}$ is defined by Lemma 4.2.1, is a superfunction $\Xi' \in \mathcal{M}^{B(\mathbb{R})}(Z)$.

Recalling the definition of the topology of $B(\mathbb{R})$, it is sufficient to show that for $a < 0 < b$, the assignment $z \mapsto \Xi'_z|_{[a,b]}$ is an element of $\mathcal{M}^{B([a,b])}(Z)$. Indeed, fix z, and choose a continuous seminorm p on F with $\Xi'_z|_{[a,b]} \in \mathcal{P}(F, p;\ B([a,b]))$. For $z' \in F_{\mathbf{0}}$ with $p(z') < 1$, $\Xi'_{z+z'}|_{[a,b]}$ is a solution power series with the same Cauchy data as the translation $\mathrm{t}_{z'}\, \Xi'_z|_{[a,b]}$, hence they coincide, proving our assertion. $\square$

We remark that without any completeness hypothesis, one gets from Prop. 4.1.1.(ii) at least an existence result for short-time solution families:

Corollary 4.2.2. *Let be given an smf Z and an smf morphism $\Xi'^{\mathrm{Cau}} : Z \to \mathrm{L}(B)$. Suppose that the image of the underlying map $\widetilde{\Xi'^{\mathrm{Cau}}} : \mathrm{space}(Z) \to B_{\mathbf{0}}$ is bounded. Then there exists an open interval $I \ni 0$ and a solution family $\Xi' \in \mathcal{M}^{B(I)}(Z)$ with $\Xi'(0) = \Xi'^{\mathrm{Cau}}$.* $\square$

The following (more or less standard) method allows to conclude from B-completeness to B'-completeness where B' is a "smaller" Banach space.

Lemma 4.2.3. *(i) Let be given continuous seminorms $p, q \in \mathrm{CS}(B)$ with $p \le q$, and assume that there exists a function $K(\cdot)$ which is bounded on finite intervals such that $q(\mathrm{A}_t(b)) \le K(t) q(b)$ for all $b \in B_{\mathbf{0}}$. Let be given a solution $\phi \in B'(I)_{\mathbf{0}}$ of the underlying problem with $I = (t_1, t_2) \ni 0$, such that both $p(\phi(t))$ and $p(\tilde{\Delta}(\phi(t)))$ are bounded (we recall that $\tilde{\Delta}$ is the underlying function of the superfunction Δ). Suppose that there exists a function $C(\cdot)$ which is bounded on finite intervals such that*

$$q(\tilde{\Delta}(b)) \le C(p(b))(1 + q(b))$$

for $b \in \mathrm{Im}\,\phi$. Then $q(\phi(t))$ will be bounded, too.

(ii) Let be given a continuous, even inclusion $B' \subseteq B$ where B' is another $\mathbb{Z}_2$-graded Banach space. Suppose that (A_t) restricts to a strongly continuous group (A_t) on B', and that $\Delta \in \mathcal{M}^B(\mathrm{L}(B))$ restricts to an entire superfunction $\Delta \in \mathcal{M}^{B'}(\mathrm{L}(B'))$. Suppose also that there exists a function $C(\cdot)$ which is bounded on finite intervals such that

$$\left\| \tilde{\Delta}(b) \right\|_{B'} \le C(\|b\|_B)(1 + \|b\|_{B'}) \tag{4.2.1}$$

holds for all $b \in B'_{\mathbf{0}}$, and that the problem (2.1.2) is B-complete. Then it is B'-complete, too.

Proof. Ad (i). For any $t_0 \in I$ we have

$$\phi(t) = \mathrm{A}_{t-t_0}\, \phi(t_0) + \int_{t_0}^{t} ds\, \mathrm{A}_{t-s}\, \tilde{\Delta}[\phi(s)]. \tag{4.2.2}$$

Choose $K > 0$ such that $q(\mathrm{A}_t(b)) \le Kq(b)$ for all $b \in B_{\mathbf{0}}$, $t \in I$. Setting

$$K' := \sup C(p(\tilde{\Delta}(\phi(t)))), \quad t_0 := \max\{t_2 - 1/(2KK'),\ (t_1 + t_2)/2\},$$

(4.2.2) implies for $t \in [t_0, t_2)$

$$q(\phi(t)) \le Kq(\phi(t_0)) + 1/2 \cdot \big(1 + \max_{s \in [t_0, t]} q(\phi(s))\big)$$

and hence $1/2 \cdot \max_{s \in [t_0,t]} q(\phi(s)) \le Kq(\phi(t_0)) + 1/2$, showing that $\sup_{s \in [t_0, t_2)} q(\phi(s)) < \infty$. The lower interval boundary is done analogously.

Ad (ii). This is an obvious corollary. □

Proof of Thm. 2.4.1. Looking at the linear term of the Taylor expansion of Ξ^{sol} at the origin we get that $\mathrm{A}_\bullet$ maps continuously $E^{\mathrm{Cau}} \to E$. Now it is easy to check that the smf morphism $\alpha : \mathrm{L}(E) \to \mathrm{L}(E)$ given by

$$\alpha[\Xi] := \Xi + \Xi^{\mathrm{sol}}[\Xi(0)] - \mathrm{A}_\bullet\, \Xi(0)$$

makes the diagram

$$\begin{array}{ccc} & \mathrm{L}(E^{\mathrm{Cau}}) & \\ \mathrm{A}_\bullet\, \Xi^{\mathrm{Cau}} \ \swarrow & & \searrow\ \Xi^{\mathrm{sol}} \\ \mathrm{L}(E) & \xrightarrow{\ \ \alpha\ \ } & \mathrm{L}(E) \end{array}$$

commutative. Also, we have a decomposition $E = E^{\mathrm{zero}} \oplus E^{\mathrm{free}}$ with

$$E^{\mathrm{free}} := \{\xi \in E : \quad \xi(t) = \mathrm{A}_t\, \xi(0)\}, \quad E^{\mathrm{zero}} := \{\xi \in E : \quad \xi(0) = 0\},$$

(both terms are equipped with the subspace topology) with the corresponding continuous projections given by $\mathrm{pr}^{\mathrm{free}}(\xi) := \mathrm{A}_\bullet\, \xi(0)$, $\mathrm{pr}^{\mathrm{zero}} := 1 - \mathrm{pr}^{\mathrm{free}}$. Therefore, the assertion follows once we have shown that α is an automorphism.

We get an identification $\mathrm{L}(E) = \mathrm{L}(E^{\mathrm{free}}) \times \mathrm{L}(E^{\mathrm{zero}})$, with the corresponding projection morphisms being $\mathrm{L}(\mathrm{pr}^{\mathrm{free}})$, $\mathrm{L}(\mathrm{pr}^{\mathrm{zero}})$, and α becomes the composite

$$\mathrm{L}(E) = \mathrm{L}(E^{\mathrm{free}}) \times \mathrm{L}(E^{\mathrm{zero}}) \xrightarrow{(\Xi^{\mathrm{sol}} \circ \pi) \times \mathrm{L}(\subseteq)} \mathrm{L}(E) \times \mathrm{L}(E) \xrightarrow{\mathrm{L}(+)} \mathrm{L}(E)$$

where π is the projection onto Cauchy data. As often in supergeometry, it is convenient to look at the point functor picture, i. e. we look how α acts on Z-families of configurations: For any smf Z we get a map

$$\mathrm{Mor}(Z, \mathrm{L}(E)) \to \mathrm{Mor}(Z, \mathrm{L}(E)), \quad \xi \mapsto \alpha \circ \xi, \tag{4.2.3}$$

and our assertion follows once we have shown that this is always an isomorphism. (Indeed, it is sufficient to take $Z := \mathrm{L}(E)$, $\xi := \mathrm{Id}$.)

Now α acts on $\xi \in \mathcal{M}^E(Z) = \mathrm{Mor}(Z, \mathrm{L}(E))$ by

$$\xi = \xi^{\mathrm{free}} + \xi^{\mathrm{zero}} \mapsto \Xi^{\mathrm{sol}}[\xi^{\mathrm{free}}(0)] + \xi^{\mathrm{zero}}.$$

We show injectivity of (4.2.3): If $\alpha \circ \xi = \alpha \circ \xi'$ then, taking Cauchy data at both sides, we get that ξ^{free}, $(\xi')^{\mathrm{free}}$ have the same Cauchy data; hence $\xi^{\mathrm{free}} = (\xi')^{\mathrm{free}}$, and the hypothesis now implies $\xi = \xi'$.

We show surjectivity of (4.2.3): Given $\xi \in \mathcal{M}^E(Z)$, its preimage is given by $\xi^{\mathrm{free}} + \xi^{\mathrm{zero}}$ with

$$\xi^{\mathrm{zero}} := \xi - \Xi^{\mathrm{sol}}[\xi(0)], \qquad \xi^{\mathrm{free}} := \mathrm{A}_\bullet\, \xi(0).$$

The Theorem is proved. □

The following is an abstract version of [14, Thm. 3.4.3]. The proof relies on [13, Prop. 2.4.2].

Theorem 4.2.4. *Let be given a continuous, even inclusion $E \subseteq B(\mathbb{R})$ where E is another $\mathbb{Z}_2$-lcs such that the set of all linear forms on E which arise by restricting elements of the dual $B(\mathbb{R})^*$ is strictly separating (cf.* [13, 2.4]*).*

Let be given an smf Z and a superfunction $\Xi'^{\mathrm{Cau}} \in \mathcal{M}^{E^{\mathrm{Cau}}}(Z)$ (i. e. a family of Cauchy data). Suppose that

(i) *for each $z \in Z$, there exists a solution $\phi_z(\cdot) \in E_{\mathbf{0}}(\mathbb{R})$ of the underlying even problem* (2.2.1) *with $\phi_z(0) = \Xi'^{\mathrm{Cau}}(z)$;*
(ii) *for each $z \in Z$, the power series $\Xi^{\mathrm{sol}}_{\phi_z(0)} \in \mathcal{P}(B;\ B(\mathbb{R}))$, as defined by Lemma 4.2.1 and (i), restricts to a power series $\Xi^{\mathrm{sol}}_{\phi_z(0)} \in \mathcal{P}(E^{\mathrm{Cau}};\ E)$.*

Then there exists a unique Z-family of solutions $\Xi' \in \mathcal{M}^E(Z)$ which has Ξ'^{Cau} as its Cauchy data, i. e. $\Xi'(0) = \Xi'^{\mathrm{Cau}}$. The Taylor expansion of Ξ' at z is given by

$$\Xi'_z = \Xi^{\mathrm{sol}}_{\phi_z(0)}[\Xi'^{\mathrm{Cau}}_z - \phi_z(0)] \tag{4.2.4}$$

where $\Xi^{\mathrm{sol}}_{\phi_z(0)}$ is given by Thm. 4.2.1. (Note that the insertion is defined since the power series inserted has no absolute term.)

Also, the underlying map of the arising smf morphism $\Xi' : Z \to \mathrm{L}(E)$ is $z \mapsto \phi_z$. □

One gets a general method for showing solvability in function spaces:

Corollary 4.2.5. *Suppose that we are given a continuous, even inclusion $E \subseteq B(\mathbb{R})$ where E is another $\mathbb{Z}_2$-lcs such that:*

(i) *the problem* (2.1.2) *is $(B, E^{\mathrm{Cau}}_{\mathbf{0}})$-complete;*
(ii) *the set of all linear forms on E which arise by restricting elements of the dual $B(\mathbb{R})^*$ is strictly separating (cf.* [13, 2.4]*);*
(iii) *for $\phi \in E^{\mathrm{Cau}}_{\mathbf{0}}$, the power series $\Xi^{\mathrm{sol}}_\phi \in \mathcal{P}(B;\ B(\mathbb{R}))$, as defined by Lemma 4.2.1 and (i), restricts to a power series $\Xi^{\mathrm{sol}}_\phi \in \mathcal{P}(E^{\mathrm{Cau}};\ E)$.*

Then the problem (2.1.2) *is solvable in* $\mathrm{L}(E)$. □

4.3. **Smoothness and support scales: the proofs.**

Proof of Prop. 2.5.1. It is sufficient to show that for $j = 0, \dots, l-1$, $\Xi' \in \mathcal{M}^{C^j(I, B_{l-j})}(Z)$ implies $\Xi' \in \mathcal{M}^{C^{j+1}(I, B_{l-j-1})}(Z)$.

By the Closed Graph Theorem, the generator K is defined as a bounded operator $K : B_{l-j} \to B_{l-j-1}$ for all j. Differentiation of (2.1.2) yields that (2.1.1) holds within $\mathcal{M}^{B_{l-1}(I)}(Z)$.

Clearly, we have $\Delta[\Xi'] \in \mathcal{M}^{C^j(I, B_{l-j})}(Z)$ and $K\Xi' \in \mathcal{M}^{C^j(I, B_{l-j-1})}(Z)$. Hence, the r. h. s. of (2.1.1) lies in $\mathcal{M}^{C^j(I, B_{l-j-1})}(Z)$, from which the assertion follows. □

Proof of Thm. 2.6.1. W. l. o. g., we may assume $I = [0, t_0]$ with some $t_0 > 0$. Also, we may suppose Z to be a superdomain $Z \subseteq \mathrm{L}(F)$. Supposing that our assertion is wrong, we can pick a $z \in Z$ such that the set $\{t \in [0, t_0] :\ (\Xi' - \Xi")_z(s) \equiv_s 0 \text{ for } s \in [0, t]\}$ is smaller than I. This set is easily seen to be closed; let t_2 be its maximum. From the hypotheses we get with $\Theta := \Xi" - \Xi'$ that

$$\Theta_z(t) \equiv_t \int_{t_2}^t ds\, \mathrm{A}_{t-s}(\Delta[\Xi'_z + \Theta_z] - \Delta[\Xi'_z])(s) \tag{4.3.1}$$

for $t \in I$. Using the operator family (E_t) from the definition of support scales, we get an even continuous linear operator $\mathrm{E} : B([t_2, t)) \to B([t_2, t))$, $(\mathrm{E}\,\xi)(t) := \mathrm{E}_t\,\xi(t)$. Now (4.3.1) yields

$$\Theta_z(t) \equiv_t \int_{t_2}^{t} ds\, \mathrm{A}_{t-s}(\Delta[\Xi'_z + \mathrm{E}\,\Theta_z] - \Delta[\Xi'_z])(s). \tag{4.3.2}$$

Choose some $r \in \mathrm{CS}(F)$ such that the relevant Taylor expansions $\Xi'_z, \Xi"_z$ lie in the Banach space $\mathcal{P}(F, r;\ B(I))$. For shortness, we will write $\|\cdot\|_G$ for the norms in $\mathcal{P}(F, r;\ G)$ where G is one of the Banach spaces B, $B([t_2, t])$, etc. Using (4.1.2) we get that with some $C_1 > 0$

$$\left\| \int_{t_2}^{t} ds\, \mathrm{A}_{t-s}(\Delta[\Xi'_z + \mathrm{E}\,\Theta_z] - \Delta[\Xi'_z])(s) \right\|_B \leq C_1\,|t - t_2|\, \|\Delta[\Xi'_z + \mathrm{E}\,\Theta_z] - \Delta[\Xi'_z]\|_{B([t_2,t])}$$

for $t \in [t_2, t_0]$. Because of (4.3.2)s and the estimate required for E_t, this implies with some $C_2 > 0$

$$\|\mathrm{E}_t\,\Theta_z(t)\|_B \leq C_2\,|t - t_2|\, \|\Delta[\Xi'_z + \mathrm{E}\,\Theta_z] - \Delta[\Xi'_z]\|_{B([t_2,t])}\,. \tag{4.3.3}$$

Let $\phi := \Xi'_z(t_2)[0] \in B_{\mathbf{0}}$, and choose some $c > 0$ such that $\Delta_\phi \in \mathcal{P}(B, c\|\cdot\|;\ B)$. (We deliberately do not make use of entireness of Δ, which entails the validity of this for any $c > 0$). Changing the norm in B, we may for notational convenience assume $c = 1$. Now choose $\epsilon > 0$ such that for $t \in I' := [t_2, t_2 + \epsilon]$ we have

$$\|\Xi'_z(t)[0] - \phi\|_B < \frac{1}{4}, \qquad \|\mathrm{E}\,\Theta_z(t)[0]\|_B < \frac{1}{4} \tag{4.3.4}$$

(this is possible since $\Xi'_z(t)[0], \mathrm{E}\,\Theta_z(t)[0]$ depend continuously on t). Now, by dilating r, we may assume

$$\|\Xi'_z - \Xi'_z[0]\|_{B(I')} < \frac{1}{4}, \qquad \|\mathrm{E}\,\Theta_z - \mathrm{E}\,\Theta_z[0]\|_{B(I')} < \frac{1}{4} \tag{4.3.5}$$

(this is possible since both power series do not have an absolute term). Now, letting Ξ and $\delta\Xi$ be independent functional variables, we may expand into bihomogeneous components:

$$\Delta[\Xi + \phi + \delta\Xi] - \Delta[\Xi + \phi] = \sum_{i,j \geq 0} D_{(i,j)}[\Xi, \delta\Xi] \in \mathcal{P}(B \oplus B, \|\cdot\|;\ B),$$

with $D_{(i,0)} = 0$ for all i. For arbitrary $\Xi", \delta\Xi" \in \mathcal{P}(F, r;\ B)_{\mathbf{0}}$ we get the estimate (cf. [12, Proof of Prop. 3.3])

$$\begin{aligned} &\|\Delta[\Xi" + \phi + \delta\Xi"] - \Delta[\Xi" + \phi]\|_B \\ &\quad \leq \sum_{i \geq 0,\ j \geq 1} \left\| D_{(i,j)}[\Xi, \delta\Xi] \right\|_{\mathcal{P}(B \oplus B, \|\cdot\|;\ B)} \|\Xi"\|_B^i\, \|\delta\Xi"\|_B^j \leq \frac{C_3\,\|\delta\Xi"\|_B}{(1 - \|\Xi"\|_B)(1 - \|\delta\Xi"\|_B)} \end{aligned} \tag{4.3.6}$$

with $C_3 := \|\Delta[\Xi + \phi + \delta\Xi] - \Delta[\Xi + \phi]\|_{\mathcal{P}(B \oplus B, \|\cdot\|;\ B)}$. Taking here $\Xi" := \Xi'_z(t) - \phi$, $\delta\Xi" := \mathrm{E}\,\Theta_z(t)$ with $t \in I'$, we get because of (4.3.4), (4.3.5) that $\|\Xi"\|, \|\delta\Xi"\| < \frac{1}{2}$, and (4.3.6) yields

$$\|\Delta[\Xi'_z(t) + \mathrm{E}\,\Theta_z(t)] - \Delta[\Xi'_z(t)]\|_B \leq 4C_3\, \|\mathrm{E}\,\Theta_z(t)\|_B \tag{4.3.7}$$

for $t \in I'$. Putting (4.3.3), (4.3.7) together we get

$$\|\mathrm{E}\,\Theta_z(t)\|_B \leq 4C_2C_3\,|t - t_2|\, \left\|\mathrm{E}\,\Theta_z|_{[t_2,t]}\right\|_{B([t_2,t])}$$

for $t \in I'$. Now, for (say) $0 < |t - t_2| < 1/(8C_2C_3)$, this estimate implies $\|\mathrm{E}\,\Theta_z(t)\|\|_B = 0$, in contradiction to the choice of t_2. □

Proof of Lemma 2.7.1. Uniqueness is easy to see. We construct $\Delta \in \mathcal{M}^{C^l(M,B)}(\mathrm{L}(C^l(M, B)))$ by specifying its Taylor expansions:

$$C^l(M, B_{\mathbf{0}}) \ni \phi \mapsto \Delta_\phi \in \mathcal{P}(C^l(M, B);\ C^l(M, B)), \tag{4.3.8}$$

$$(\Delta_\phi)_{(r|s)}(\phi_1, \ldots, \phi_r | \psi_1, \ldots, \psi_s)(t) := (\Delta_{\phi(t)})_{(r|s)}(\phi_1(t), \ldots, \phi_r(t) | \psi_1(t), \ldots, \psi_s(t))(t) \tag{4.3.9}$$

for $\phi_i \in C^l(M, B_{\mathbf{0}})$, $\psi_i \in C^l(M, B_{\mathbf{1}})$, $t \in M$.

For showing well-definedness, we remark that the topology of $C^l(M,B)$ is defined by seminorms of the form $\|\phi\|_{K,l} := \sum_{|\nu|\le l} \sup_{t\in K} \|\partial^\nu \phi(t)\| /\nu!$ where $K \Subset M$ is contained in a coordinate patch identifying it with the unit ball of $\mathbb{R}^n$ (the modification for $l=\infty$ is obvious).

For $b \in B$ and K as above, choose $U^b \in \mathrm{CB}(B)$ such that $\Delta_b \in \mathcal{P}(B, 4U^b;\ C^l(K,B))$. Now, given K and $\phi \in C^l(M,B)$, compactness allows to find $t_1, \ldots, t_N \in M$ such that $\phi(K) \subseteq \bigcup_{i=1}^N (\phi(t_i) + U^{\phi(t_i)})$; set $U^{K,\phi} := \bigcap_{i=1}^N U^{\phi(t_i)}$.

Now, given $t \in K$, choose i with $\phi(t) - \phi(t_i) \in U^{\phi(t_i)}$. It follows that

$$\Delta_{\phi(t)} = \mathrm{t}_{\phi(t)-\phi(t_i)}\,\Delta_{\phi(t_i)} \in \mathcal{P}(B, 2U^{\phi(t_i)};\ C^l(K,B)) \subseteq \mathcal{P}(B, 2U^{K,\phi};\ C^l(K,B)).$$

We get a map

$$K \ni t \mapsto \Delta_{\phi(t)} \in \mathcal{P}(B, 2U^{K,\phi};\ C^l(K,B)). \tag{4.3.10}$$

Moreover, for each i, the map

$$\phi(t_i) + U^{\phi(t_i)} \to \mathcal{P}(B, 2U^{\phi(t_i)};\ C^l(K,B)), \quad \phi(t_i) + b \mapsto \Delta_{\phi(t_i)+b} = \mathrm{t}_b\,\Delta_{\phi(t_i)}$$

is real-analytic. It follows that the composite map (4.3.10) is C^l.

Hence, given $r, s, \phi_1, \ldots, \phi_r, \psi_1, \ldots, \psi_s$, the r. h. s. of (4.3.9) depends in a C^l way on $t \in K$; since this is true for all K, it follows that Δ_ϕ is well-defined as a formal power series.

Now, for $|\nu| \le l$ and $\phi_i \in C^l(M, U^{K,\phi} \cap B_{\mathbf{0}})$, $\psi_i \in C^l(M, U^{K,\phi} \cap B_{\mathbf{1}})$, $t \in K$,

$$\frac{1}{\nu!}\left\|\partial^\nu(\Delta_{\phi(t)})_{(r|s)}(\phi_1(t), \ldots, \phi_r(t)|\psi_1(t), \ldots, \psi_s(t))(t)\right\| \le 2^{-r-s}\left\|\Delta_{\phi(\cdot)}\right\|_{K,l},$$

where $\left\|\Delta_{\phi(\cdot)}\right\|_{K,l}$ is the C^l norm of the map (4.3.10). Hence

$$\|\Delta_\phi\|_{\mathcal{P}(C^l(M,B), C^l(M,U^{K,\phi});\ C^l(K,B))} \le \sum_{i=0}^{l} (\dim M)^i \cdot \sum_{r,s} 2^{-r-s}\left\|\Delta_{\phi(\cdot)}\right\|_{K,l} < \infty$$

which proves that Δ_ϕ is an analytic power series, i.e. (4.3.8) is well-defined. Now one applies the strictly separating family (cf. [13, 2.4]) of linear functionals $C^l(M,B) \to \mathbb{R}$, $\phi \mapsto \langle b^*, \phi(t)\rangle$ where $b^* \in B^*$ and $t \in M$ to conclude that the map (4.3.8) is a superfunction. □

4.4. **Application: the proofs.**

Proof of Prop. 3.2.2. Of course, we can assume $l = 1$. Fix $a \in \{1, \ldots, d\}$. Using the algebra property of the Sobolev spaces (cf. [1]), there is a constant K_1 such that

$$\left\|\partial_a \tilde{\Delta}_i[\phi]\right\|_{H_{k+\tau_i}(\mathbb{R}^d)} \le K_1 \sum_{j,\nu} \|\partial_a \partial^\nu \phi_j\|_{H_{k+\tau_i}(\mathbb{R}^d)} \cdot \left\|\frac{\partial}{\partial(\partial^\nu \Xi_j)}\tilde{\Delta}_i[\phi]\right\|_{H_{k+\tau_i}(\mathbb{R}^d)}$$

where, because of (3.1.4), the sum runs over those $j = 1, \ldots, N_{\mathbf{0}}$ and $\nu \in {\mathbb{Z}_+}^d$ for which $\tau_i \le \tau_j - |\nu|$. This restriction implies

$$\|\partial_a \partial^\nu \phi_j\|_{H_{k+\tau_i}(\mathbb{R}^d)} \le \|\partial_a \phi_j\|_{H_{k+\tau_j}(\mathbb{R}^d)} \le \|\phi\|_{\mathcal{H}^{\mathrm{V}}_{k+1}}.$$

Setting $B := \mathcal{H}^{\mathrm{V}}_k$, $B' := \mathcal{H}^{\mathrm{V}}_{k+1}$, and

$$C(r) := K_2 \cdot \sup_{\phi \in \mathcal{H}^{\mathrm{V}}_k,\ \|\phi\| \le r} \left\|\tilde{\Delta}[\phi]\right\|_{\mathcal{H}^{\mathrm{V}}_k},$$

with suitable $K_2 > 0$, (4.2.1) is satisfied, and the assertion follows from Lemma 4.2.3.(ii). □

Proof of Thm. 3.3.1. We first note:

Lemma 4.4.1. *Let be given a Z-family $\Xi' \in \mathcal{M}^{\mathcal{H}_k^{\mathrm{V}}(I)}(Z)$ with $k > d/2$, and suppose that*

$$L_i[\Xi']|_{\Omega(p)} = 0 \quad (i = 1, \dots, N)$$

within $\mathcal{M}^{\mathcal{D}'(\Omega(p))\otimes\mathbb{R}^{N_\mathbf{0}|N_\mathbf{1}}}(Z)$. Then Ξ' satisfies the integral equation

$$\Xi'(t, y) = \mathrm{A}_t\, \Xi'(0)(y) + \int_0^t ds\, \mathrm{A}_{t-s}\, \Delta[\Xi'](y)$$

within $\mathcal{O}^{\mathbb{R}^{N_\mathbf{0}|N_\mathbf{1}}}(Z)$ for all $(t, y) \in \Omega(p)$. □

Let $p = (s, x) \in \mathbb{R}^{d+1}$, and assume $s > 0$ ($s < 0$ is done mutatis mutandis): Within $\mathcal{H}_k^{\mathrm{V}}$, we have a support scale $(S_t)_{t\in[0,s]}$,

$$S_t := \left\{\xi \in \mathcal{H}_k^{\mathrm{V}} : \quad \operatorname{supp}\xi \cap \mathcal{J}((x, s-t)) = \emptyset\right\}.$$

The Theorem now follows from Thm. 2.6.1. □

Before proceeding, we do some technical preparations. We will use the notations

$$\mathcal{E}^{\mathrm{V}} := C^\infty(\mathbb{R}^{d+1}, \mathbb{R}^{N_\mathbf{0}|N_\mathbf{1}}), \quad \mathcal{E}^{\mathrm{Cau,V}} := C^\infty(\mathbb{R}^d, \mathbb{R}^{N_\mathbf{0}|N_\mathbf{1}}),$$

$$\mathcal{E}_c^{\mathrm{V}} := C_c^\infty(\mathbb{R}^{d+1}, \mathbb{R}^{N_\mathbf{0}|N_\mathbf{1}}), \quad \mathcal{E}_c^{\mathrm{Cau,V}} := C_c^\infty(\mathbb{R}^d, \mathbb{R}^{N_\mathbf{0}|N_\mathbf{1}}).$$

We need a technical notion: Given a seminorm $p \in \mathrm{CS}(\mathcal{D}(\mathbb{R}^{d+1}))$, we define the *support of* p, denoted by $\operatorname{supp} p$, as the complement of the set of all x which have a neighbourhood $U \ni x$ such that $\operatorname{supp}\varphi \subseteq U$ implies $p(\varphi) = 0$. Obviously, $\operatorname{supp} p$ is closed; using partitions of unity one shows that $\operatorname{supp}\varphi \subseteq \mathbb{R}^{d+1} \setminus \operatorname{supp} p$ implies $p(\varphi) = 0$.

For every $p \in \mathrm{CS}(C^\infty(\mathbb{R}^{d+1}))$, $\operatorname{supp} p$ is compact (where we have silently restricted p to $\mathcal{D}(\mathbb{R}^{d+1})$). On the other hand, given $p \in \mathrm{CS}(\mathcal{E}_c^{\mathrm{V}})$, the set $\operatorname{supp} p \cap \mathbb{V}_r$ (cf. (3.3.1)) is compact for all $r \geq 0$.

Given a bounded open set $\Omega \Subset \mathbb{R}^{d+1}$, we denote by $\mathcal{J}(\Omega) \Subset \mathbb{R}^d$ the *causal influence domain of* Ω on the Cauchy hyperplane, i. e. the set of all $x \in \mathbb{R}^d$ such that $(0, x)$ lies in the twosided light cone of a point in Ω.

For $\Omega \Subset \mathbb{R}^{d+1}$, $l \geq 0$, define the seminorm $q_{l,\Omega} \in \mathrm{CS}(\mathcal{E}^{\mathrm{V}})$ by

$$q_{l,\Omega}(\xi) = \sum_{i=1}^N \sup_{(t,x)\in\Omega} \sum_{\nu\in\mathbb{Z}_+^{d+1},\ |\nu|\leq l} |\partial^\nu \xi_i(t, x)|\,;$$

thus $\operatorname{supp} q = \Omega$. Also, for $J \Subset \mathbb{R}^d$, $k \geq 0$, define the seminorm $p_{k,J} \in \mathrm{CS}(\mathcal{E}^{\mathrm{Cau,V}})$ by

$$p_{k,J}(\xi^{\mathrm{Cau}}) := \sum_{i=1}^N \sup_{x\in J} \sum_{\nu\in\mathbb{Z}_+^d,\ |\nu|\leq k} \left|\partial^\nu \xi_i^{\mathrm{Cau}}(x)\right|\,;$$

thus, $\operatorname{supp} p_{k,J} = J$.

Proof of Thm. 3.3.2. **Lemma 4.4.2.** *Under the hypotheses of Thm. 3.3.2, fix a Cauchy datum $\phi \in (\mathcal{E}_c^{\mathrm{Cau,V}})_\mathbf{0}$.*

(i) There exists a unique element $\phi' \in (\mathcal{E}_c^{\mathrm{V}})_\mathbf{0}$ with $\phi'(0) = \phi$ which solves the underlying system (3.2.3).

(ii) For $\Omega \Subset \mathbb{R}^{d+1}$, $l \geq 0$, let $k > \mu l + d/2 + \max\{\tau_1, \dots, \tau_N\}$. Then, for all $\epsilon > 0$, the power series $\Xi_\phi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}]$ given by Lemma 4.2.1 satisfies a $(q_{l,\Omega}, C_\epsilon p_{k,J_\epsilon})$-estimate (cf. [12, 3.1]*) with some $C_\epsilon > 0$, where $J_\epsilon = U_\epsilon(\mathcal{J}(\Omega))$ is the ϵ-neighbourhood of $\mathcal{J}(\Omega)$.*

(iii) Let $q \in \mathrm{CS}(\mathcal{E}^{\mathrm{V}})$ be arbitrary. Then there exists $k > 0$ such that for all $\epsilon > 0$, $\Xi_\phi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}]$ satisfies the $(q, C_\epsilon p_{k,J_\epsilon})$-estimate with some $C_\epsilon > 0$, where $J_\epsilon = U_\epsilon(\mathcal{J}(\operatorname{supp} q))$.

(iv) $\Xi^{\mathrm{sol}}_\phi[\Xi^{\mathrm{Cau}}]$ *is an analytic power series from* $\mathcal{E}^{\mathrm{Cau,V}}_c$ *to* $\mathcal{E}^{\mathrm{V}}_c$*:*

$$\Xi^{\mathrm{sol}}_\phi[\Xi^{\mathrm{Cau}}] \in \mathcal{P}(\mathcal{E}^{\mathrm{Cau,V}}_c;\ \mathcal{E}^{\mathrm{V}}_c)_{\mathbf{0},\mathbb{R}}.$$

Proof. Ad (i). From the completeness hypothesis and Cor. 3.2.1, we get a solution $\phi' \in (\mathcal{H}^{\mathrm{V}}_k)_{\mathbf{0}}$ of (3.2.3) with $\phi'(0) = \phi$. Now Prop. 3.2.2 and Thm. 3.3.1.(ii) together with the Sobolev Embedding Theorem yield $\phi' \in (\mathcal{E}^{\mathrm{V}}_c)_{\mathbf{0}}$.

Ad (ii). Let $I \subseteq \mathbb{R}$ be the projection of Ω onto the time axis. By the Sobolev Embedding Theorem, there exists a constant C_1 such that

$$q_{l,\Omega}(\varphi) \le C_1 \cdot \|\varphi|_I\|_{\mathcal{H}^{\mathrm{V},l}_k(I)}$$

for $\phi \in \mathcal{H}^{\mathrm{V},l}_k(I)$ where $\mathcal{H}^{\mathrm{V},l}_k(I)$ is $B^l(I)$ with $B := \mathcal{H}^{\mathrm{V}}_k$ (cf. (2.5.2)). Combining this with the Sobolev analyticity of $\Xi^{\mathrm{sol}}_\phi[\Xi^{\mathrm{Cau}}]$ given by Lemma 4.2.1, there exists a constant C_2 such that we have for $r, s \ge 0$, $\varphi^1, \dots, \varphi^r \in (\mathcal{E}^{\mathrm{Cau,V}}_c)_{\mathbf{0}}$, $\psi^1, \dots, \psi^s \in (\mathcal{E}^{\mathrm{Cau,V}}_c)_{\mathbf{1}}$

$$q_{l,\Omega}\Big(\Big\langle \big(\Xi^{\mathrm{sol}}_\phi\big)_{r|s}, \bigotimes_{m=1}^{r} \varphi^m \otimes \bigotimes_{n=1}^{s} \Pi\psi^n \Big\rangle\Big) \le C_2 \cdot \prod_{m=1}^{r} \|\varphi^m\|_{\mathcal{H}^{\mathrm{V}}_k} \cdot \prod_{n=1}^{s} \|\psi^n\|_{\mathcal{H}^{\mathrm{V}}_k}$$

(cf. [12, 3.1] for the notation on the l. h. s.). Now choose some buffer function $h \in \mathcal{D}'(\mathbb{R}^d)$ with $\operatorname{supp} h \subseteq J_\epsilon$ and $h|_{\mathcal{J}(\Omega)} = 1$. By causality (cf. Thm. 2.6.1.(ii)), we have $\Xi^{\mathrm{sol}}_\phi[\Xi^{\mathrm{Cau}}]|_\Omega = \Xi^{\mathrm{sol}}_\phi[h\Xi^{\mathrm{Cau}}]|_\Omega$, and hence

$$\begin{aligned} q_{l,\Omega}\Big(\Big\langle \big(\Xi^{\mathrm{sol}}_\phi\big)_{r|s}, \bigotimes_{m=1}^{r} \varphi^m \otimes \bigotimes_{n=1}^{s} \Pi\psi^n \Big\rangle\Big) &= q_{l,\Omega}\Big(\Big\langle \big(\Xi^{\mathrm{sol}}_\phi\big)_{r|s}, \bigotimes_{m=1}^{r} (h\varphi^m) \otimes \bigotimes_{n=1}^{s} \Pi(h\psi^n) \Big\rangle\Big) \\ &\le C_2 \cdot \prod_{m=1}^{r} \|h\varphi^m\|_{\mathcal{H}^{\mathrm{V}}_k} \cdot \prod_{n=1}^{s} \|h\psi^n\|_{\mathcal{H}^{\mathrm{V}}_k} \end{aligned}$$

But obviously $\|h\cdot\|_{\mathcal{H}^{\mathrm{V}}_k}$ is estimated from above by $C_\epsilon p_{k,J_\epsilon}(\cdot)$ with some $C_\epsilon > 0$, and the assertion follows.

Ad (iii). Since the collection of all $q_{l,\Omega}$ defines the topology of $\mathcal{E}^{\mathrm{V}}$, there exist l, C',and $\Omega' \Subset \mathbb{R}^{d+1}$ such that $q \le C' q_{l,\Omega'}$. However, Ω' may be larger than $\operatorname{supp} q$. Choose a buffer function $g \in \mathcal{D}(\mathbb{R}^{d+1})$, $g \ge 0$, with $g|_{\operatorname{supp} q} = 1$, $\operatorname{supp} g \subseteq J_{\epsilon/2}$. Then

$$q(\cdot) = q(g\cdot) \le C' q_{l,\Omega'}(g\cdot) \le C'_\epsilon q_{l,J_{\epsilon/2}}(\cdot)$$

with some $C'_\epsilon > 0$. The assertion now follows from (ii).

Ad (iv). Let be given a seminorm $q \in \mathrm{CS}(\mathcal{E}^{\mathrm{V}}_c)$. With standard methods one constructs for $i > 0$ buffer functions $f_i \in C^\infty(\mathbb{R}^{d+1})$ with $f_i|_{\mathbb{V}_{i-1}} = 0$, $f_i|_{\mathbb{R}^{d+1}\setminus\mathbb{V}_i} = 1$ (cf. (3.3.1)). Set for convenience $f_0 := 1$. For the seminorms $q_i := q((f_i - f_{i+1})\cdot) \in \mathrm{CS}(\mathcal{E}^{\mathrm{V}}_c)$ we get

$$q(\varphi) \le \sum_{i\ge 0} q_i(\varphi) \tag{4.4.1}$$

for all $\phi \in \mathcal{E}^{\mathrm{V}}_c$, where in fact only finitely many terms on the r. h. s. are non-zero. Now

$$\operatorname{supp} q_i \subseteq \mathbb{V}_{i+1} \cap \operatorname{supp} q$$

which is compact. Also, for $i \ge 1$, we have $(f_i - f_{i+1})|_{\mathbb{V}_{i-1}} = 0$ and hence

$$\operatorname{supp} q_i \cap \mathbb{V}_{i-1} = \emptyset. \tag{4.4.2}$$

Because of (4.4.2), we have $\mathcal{J}(\operatorname{supp} q_i) \subseteq \{x \in \mathbb{R}^d :\ \|x\| \ge i-1\}$ for $i \ge 1$; hence, setting $J_i := \{x \in \mathbb{R}^d :\ \|x\| \ge i-2\}$, Lemma 4.4.2.(ii) yields for each i numbers $C_i > 0$, $k_i \ge 0$ such that $\Xi^{\mathrm{sol}}_\phi[\Xi^{\mathrm{Cau}}]$ satisfies a $(q_i, C_i p_{k_i,J_i})$-estimate.

It follows that for each $\varphi \in \mathcal{E}_c^{\mathrm{Cau,V}}$, the sum

$$p(\varphi) := \sum_i C_i p_{k_i, J_i}(\varphi)$$

has only finitely many nonvanishing terms; using [7, Thm. 15.4.1], we have $p := p(\cdot) \in \mathrm{CS}(\mathcal{E}_c^{\mathrm{Cau,V}})$. It follows directly from the definition of the (q,p)-estimates (cf. [12, 3.1]) and (4.4.1) that the $(q_i, C_i p_{k_i,J_i})$-estimates for $\Xi_\phi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}]$ imply the (q,p)-estimate wanted.

The Lemma is proved. □

Thm. 3.3.2 now follows from Cor. 4.2.5. □

Proof of Thm. 3.3.3. **Lemma 4.4.3.** *Suppose that the problem* (3.1.1) *is causal. Given a bosonic Cauchy datum* $\phi \in (\mathcal{E}^{\mathrm{Cau,V}})_{\mathbf{0}}$*, there exists a solution power series* $\Xi_\phi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}] \in \mathcal{P}(\mathcal{E}^{\mathrm{Cau,V}};\ \mathcal{E}^{\mathrm{V}})_{\mathbf{0},\mathbb{R}}$ *such that*

$$\Xi_\phi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}](0) = \Xi^{\mathrm{Cau}} + \phi. \tag{4.4.3}$$

Proof. Choose a sequence of compactly supported bosonic Cauchy data $\phi_{(n)} \in (\mathcal{E}_c^{\mathrm{Cau,V}})_{\mathbf{0}}$, $n \in \mathbb{Z}_+$, such that $\phi_{(n)}|_{n\mathbb{B}^d} = \phi|_{n\mathbb{B}^d}$ for all i. Composing the power series $\Xi_{\phi_{(n)}}^{\mathrm{sol}} \in \mathcal{P}(\mathcal{E}_c^{\mathrm{Cau,V}};\ \mathcal{E}_c^{\mathrm{V}})$ given by Lemma 4.4.2 with the projection $\mathcal{E}_c^{\mathrm{V}} \to C^\infty(n\mathbb{B}^{d+1}) \otimes \mathbb{R}^{N_{\mathbf{0}}|N_{\mathbf{1}}}$ we get a sequence of power series

$$\Xi_{(n)} := \Xi_{\phi_{(n)}}^{\mathrm{sol}}|_{n\mathbb{B}^{d+1}} \in \mathcal{P}(\mathcal{E}_c^{\mathrm{Cau,V}};\ C^\infty(n\mathbb{B}^{d+1}) \otimes \mathbb{R}^{N_{\mathbf{0}}|N_{\mathbf{1}}})_{\mathbf{0},\mathbb{R}}.$$

Because of Thm. 3.3.1.(i), the restrictions of $\Xi_{(n+1)}$ and $\Xi_{(n)}$ onto $n\mathbb{B}^{d+1}$ coincide. Hence there exists a power series $\Xi_\phi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}] \in \mathcal{P}(\mathcal{E}_c^{\mathrm{Cau,V}};\ \mathcal{E}^{\mathrm{V}})$ whose restriction onto $n\mathbb{B}^{d+1}$ is $\Xi_{(n)}$. It is clear that this is a solution power series which satisfies (4.4.3); the fact that it is actually analytic with respect to the source space $\mathcal{E}^{\mathrm{Cau,V}}$ follows from Lemma 4.4.2.(iii). □

Now one proves quite analogously to the compactly supported case that the power series $\Xi_\phi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}]$ fit together to the superfunction $\Xi^{\mathrm{sol}} \in \mathcal{M}^{\mathcal{C}_\infty^{\mathrm{V}}}(\mathrm{L}(\mathcal{C}_\infty^{\mathrm{Cau,V}}))$ wanted, as well as the remaining assertions. □

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