Intermittency and Regularized Fredholm Determinants

Hans Henrik Rugh
(University of Warwick, Coventry CV4 7AL, UK)

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Abstract

We consider real-analytic maps of the interval $I = [0,1]$ which are expanding everywhere except for a neutral fixed point at 0. We show that on a certain function space the spectrum of the associated Perron-Frobenius operator $\mathcal{M}$ has a decomposition $\mathrm{sp}(\mathcal{M}) = \sigma_c \cup \sigma_p$ where $\sigma_c = [0,1]$ is the continuous spectrum of $\mathcal{M}$ and σ_p is the pure point spectrum with no points of accumulation outside 0 and 1. We construct a regularized Fredholm determinant $d(\lambda)$ which has a holomorphic extension to $\lambda \in -\sigma_c$ and can be analytically continued from each side of σ_c to an open neighborhood of $\sigma_c - \{0,1\}$ (on different Riemann sheets). In $-\sigma_c$ the zero-set of $d(\lambda)$ is in one-to-one correspondence with the point spectrum of $\mathcal{M}$. Through the conformal transformation $\lambda(z) = \frac{1}{4z}(1+z)^2$ the function $d \circ \lambda(z)$ extends to a holomorphic function in a domain which contains the unit disc.

Shorttitle : Intermittency and Regularized Fredholm Determinants.

1 Assumptions and statement of results.

Finding analytic continuations of functions, holomorphic in some a priori given domains, is a challenging and rewarding mathematical task in itself. In some cases it also provides elegant and non-trivial solutions to other problems in mathematics or physics. Thus, in dynamical systems theory the spectral properties of transfer operators and the zero sets of analytically extended holomorphic functions are related through Ruelle's generalized Fredholm determinants and dynamical zeta functions. In establishing such a relationship uniform hyperbolicity traditionally plays a crucial role and have led to interesting results e.g. in the cases of purely expanding maps or hyperbolic flows, both on compact manifolds and with various degrees of smoothness imposed [8, 9]. In these cases, an associated Perron-Frobenius type positive bounded transfer operator $\mathcal{M}$ has a spectral radius $r(\mathcal{M})$, and an essential spectral radius[1] $r_{\mathrm{ess}}(\mathcal{M})$ strictly smaller than $r(\mathcal{M})$. The part of the spectrum which intersects the annulus $(r_{\mathrm{ess}}, r(\mathcal{M})]$ is then non-empty and isomorphic to the zero-set of a Fredholm-determinant. This in turn yields informations about the decay of correlations and in some cases geometrical properties of the underlying manifold [5]. However, in the presence of neutral fixed points, e.g. [6] of the Pomeau-Manneville intermittency type 1, the two spectral radii coincide and the annulus, whence the result about the spectrum, becomes void. Recently, using the technique of induced mappings, Prellberg [7] and also Isola [3] have overcome part of this problem and obtained some interesting spectral properties. We refer also to Mayer [4] for a closely related analysis of the Gauss map.

Below we shall develop a theory which combines results from complex dynamics, for which we rely on the exposition of Shishikura [10], with the nuclear theory of Grothendieck [1, 2] and the dynamical zeta-function analysis first introduced by Ruelle [8]. We shall consider the intermittent case for certain real-analytic mappings of an interval and associate to such a map a Perron-Frobenius type transfer operator $\mathcal{M}$. We introduce a regularized Fredholm determinant, $d(\lambda)$, and consider the relationship between its analytic properties and the spectrum of $\mathcal{M}$. By first using spectral properties of $\mathcal{M}$ we deduce that $d(\lambda)$ is analytic in the complement of the line segment $\sigma_c = [0, 1]$ (where eigenvalues of $\mathcal{M}$ can be identified with zeroes of $d(\lambda)$) and then by analytically continuing d across the open segment $(0, 1)$ from both sides (onto different Riemann sheets) we deduce that the pure point spectrum of $\mathcal{M}$ can not have points of accumulation apart from the end points 0 and 1. Moreover, we find through a conformal transformation an expression for the Fredholm determinant, accessible to numerical analyses. These results are special cases (section 1.1) of a slightly more general set-up which we shall now describe.

We consider an analytic map $f : \Delta \to \Delta$ of an open, simply connected domain $\Delta \subset {}^* = - \{0\}$. Also, let g be an analytic function on Δ. We shall assume that both f and g have continuous extensions to the boundary of Δ. For $r_0 > 0$, $\pi > \theta_0 > 0$, the set

[1] The spectral 'rug' under which you sweep the part of the spectrum you don't understand.

$\mathcal{S}[r_0, \theta_0] = \{re^{i\theta} \in \ : r \in (0, r_0), \theta \in (-\theta_0, \theta_0)\}$ will denote an open angular sector.

We say that f is an intermittent analytic contraction at 0 of order $1 + p > 1$ and that g is an associated weight provided :

(i) $f : \mathrm{Cl}\ \Delta \to \Delta \cup \{0\}$ and f is univalent in Δ.

(ii) There is $r_0 > 0$, $\theta_0 > \frac{\pi}{2p}$ such that Δ contains the open angular sector $\mathcal{S}[r_0, \theta_0]$.

(iii) There are constants $\epsilon > 0, a > 0$ and $b \in$ such that for $z \in \Delta$
$f(z) = z - az^{1+p} + \mathcal{O}(|z|^{1+p+\epsilon})$,
$g(z) = 1 - bz^p + \mathcal{O}(|z|^{p+\epsilon})$.

In the following let $\mu(d\omega)$ be a finite, positive measure on a measure space Ξ. With respect to this measure let $\{f_\omega : \Delta \to \Delta\}_{\omega \in \Xi}$ be a measurable family of analytic maps with continuous extension to the boundary. We shall assume that the family is uniformly contracting in Δ by which we mean that

(iv) $\mathrm{Cl}\ \bigcup_\omega f_\omega(\Delta) \subset \Delta$.

Also, let $\{g_\omega : \Delta \to \ \}_{\omega \in \Xi}$ be a measurable family of analytic (weight-) functions with continuous extension to the boundary and such that

(v) $\int_\Xi \mu(d\omega) \sup_{z \in \Delta} |g_\omega(z)| < \infty$.

For notational convenience we introduce $\Xi^* = \Xi \cup \{0\}$ and extend μ to Ξ^* by setting $\mu^*(\{0\}) = 1$. We set $f_0 = f$ and $g_0 = g$ and define $\Sigma^* \subset \Delta \cup \{0\}$ to be the compact invariant set under the extended family $\{f_\omega\}_{\omega \in \Xi^*}$. We define for each $\omega \in \Xi^*$ an operator $\mathcal{M}_\omega : \mathcal{H}(\Delta) \to \mathcal{H}(\Delta)$ acting on functions analytic in Δ (cf. section 2 for the definition of $\mathcal{H}(\cdot)$),

$$\mathcal{M}_\omega h(z) = g_\omega(z) h \circ f_\omega(z) \tag{1.1}$$

and we set $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$ where

$$\mathcal{M}_1 = \int_\Xi \mu(d\omega) \mathcal{M}_\omega, \tag{1.2}$$

which is well-defined by properties (iv)-(v). For $\bar{\omega} = (\omega_1, ..., \omega_m) \in (\Xi^*)^m$, $m > 0$ we introduce the abbreviations :

$$f_{\bar{\omega}}(z) \quad = \quad f_{\omega_1} \circ \cdots \circ f_{\omega_m}(z) \tag{1.3}$$

$$g_{\bar{\omega}}(z) = g_{\omega_m}(z) g_{\omega_{m-1}}(f_{\omega_m} z) \cdots g_{\omega_1}(f_{\omega_2,\ldots,\omega_m} z) \quad (1.4)$$

and we let $x(\bar{\omega})$ denote the (necessarily unique) fixed point of $f_{\bar{\omega}}$ in Cl Δ. The regularized m'th direct product of the family Ξ^* consists of $\Xi^*_m = (\Xi^*)^m - \{0 \times \cdots \times 0\}$, i.e. the direct product minus the element consisting of m repeats of the intermittent fixed point. We write

$$\zeta_m = \int_{(\Xi^*)^m} \mu^*(d\omega_1) \cdots \mu^*(d\omega_m) g_{\bar{\omega}}(x(\bar{\omega})), \quad (1.5)$$

$$d_m = \int_{\Xi^*_m} \mu^*(d\omega_1) \cdots \mu^*(d\omega_m) \frac{g_{\bar{\omega}}(x(\bar{\omega}))}{1 - f'_{\bar{\omega}}(x(\bar{\omega}))}. \quad (1.6)$$

The first integral thus extends over all possible periodic orbit of length m of the extended family whereas in the latter the intermittent fixed point itself (and only that) has been excluded. A zeta function and a regularized Fredholm determinant is then defined through the following formal power series in z and λ^{-1} respectively :

$$\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \zeta_m. \quad (1.7)$$

$$d(\lambda) = \exp - \sum_{m=1}^{\infty} \frac{\lambda^{-m}}{m} d_m. \quad (1.8)$$

Theorem 1.1 *Assume that our collection of maps and weights satisfy properties (i) through (v). If also $g \neq 0$ on Σ^* and $Re(b) > pa$ (the constants as in property (iii)) then there is a domain $U \subset \Delta$ which contains the invariant set $\Sigma^* - \{0\}$ and a Banach space $X(U) \subset \mathcal{H}(U)$ of analytic functions in U such that (1.1-1.2) defines a bounded linear operator $\mathcal{M} : X(U) \to X(U)$. Furthermore,*

(a) The spectrum of $\mathcal{M}$ has a decomposition $sp(\mathcal{M}) = \sigma_c \cup \sigma_p$ where $\sigma_c = [0,1]$ (the line-segment) is the continuous spectrum of $\mathcal{M}$ and σ_p is the pure point spectrum of $\mathcal{M}$.

(b) σ_p consists of eigenvalues of finite multiplicity and has no points of accumulations in $- (\{0\} \cup \{1\})$.

(c) $d(\lambda)$ extends to a holomorphic function in$^- - \sigma_c$ where its zeroes counted with order are the same as the eigenvalues of $\mathcal{M}$ counted with multiplicity.

(d) $d(\lambda)$ *has an analytic continuation from each side of* σ_c *to an open neighborhood of* $\sigma_c - \{0,1\}$.

In the following Corollaries 1.1 through 1.3 we shall assume that the conditions of the Theorem are satisfied.

Corollary 1.1 *Through the conformal transformation* $\lambda(z) = \frac{1}{4z}(1+z)^2$ *the function*

$$d(\lambda(z)) = \exp - \sum_{m>0} \frac{z^m}{m} \left(\frac{2}{1+z} \right)^{2m} d_m \tag{1.9}$$

extends to a holomorphic function in a domain which contains the open unit disc. Here its zero-set is in one-to-one correspondence with the point spectrum of $\mathcal{M}$ *in* $- \sigma_c$.

Standard results from nuclear theory yields that the Fredholm determinant d is holomorphic in minus the unit disc (cf. [7]) and hence, that $d \circ \lambda$ is holomorphic in the disc of radius $3 - \sqrt{8} = 0.17...$ The non-trivial content of the Theorem and its Corollary is that the analyticity extends to $- \sigma_c$, respectively to the open unit-disc. We remark, that the function $d \circ \lambda$ is accessible for numerical analysis through formal power series.

Corollary 1.2 *(to the proof of Theorem 1.1) Let* $\nu = p\theta_0 - \frac{\pi}{2}$ $(\in (0, \frac{\pi}{2}))$. *The Fredholm determinant* $d(\lambda)$ *has an analytic continuation to a Riemann surface which is constructed in the following way. Take the slit Riemann sphere,* $\bar{} - \sigma_c$, *and cut it open along the open line segment* $(0,1)$. *To each side of this segment glue a half-infinite logarithmic spiral, turning clockwise, respectively anti-clockwise around the origin and with the radius, r, decreasing as a function of the angle,* ϕ, *as* $r(\phi) = \exp(-\cot(\nu)|\phi|)$. *When* $\nu = \pi/2$ *(and thus* $r(\phi) \equiv 1$*), this surface is the same as the one obtained by taking the universal cover of the open unit disc punctured at the origin* $D - \{0\}$, *pick one of the copies (cut from 0 to 1) and gluing to its circumference (the unit circle minus the point* $\{1\}$*) the rest of the Riemann sphere (minus* $\{1\}$*).*

For the dynamical zeta-function associated with our intermittent system we have :

Corollary 1.3 *Assume that the derivatives* $\{f'_w\}_{w \in \Xi}$ *are uniformly bounded in* Δ. *Then the function* $\zeta(z)$ *extends to a meromorphic function in* $- [1, \infty)$ *and has a meromorphic extension from both sides to an open neighborhood of the segment* $(1, \infty)$.

1.1 *Dynamical systems.*

Let $F: I \to I$ be a piece-wise expanding and real-analytic map of the interval $I = [0,1]$ and such that each of its branches is surjective on I. Let $\Delta \supset I$ be a complex domain to be specified in the following. For simplicity we shall assume that there is only a finite number $(N+1)$ of branches and that each inverse branch, f_α, $\alpha \in \Xi^* = \{0\} \cup \Xi = \{0, 1, ..., N\}$ extends to an analytic function with no critical points in the domain Δ. Let $\beta \in$ be a complex parameter and let $s_\alpha \in \{1, -1\}$ be the sign of f'_α restricted to the real interval I. We choose the particular (well-defined) family of weights $g_\alpha = (s_\alpha f'_\alpha)^\beta$, $\alpha \in \Xi^*$. The domain Δ should be chosen so that $f = f_0$ and the weight $g = g_0$ has the properties (i)-(iii) above for some values of the constants p, r_0, θ_0, a, b and ϵ and also that the other branches and weights $\{f_\alpha, g_\alpha\}_{\alpha \neq 0}$ satisfies (iv) and (v) (μ is then just the counting measure).

One verifies that the hypotheses of the Theorem are satisfied provided Re $\beta > \frac{p}{1+p}$.
Remark : the above can easily be extended to a numerable family of inverse branches, possibly by sharpening the condition on β.

The associated Perron-Frobenius operator

$$\mathcal{P}_\beta h(z) = \sum_\alpha (s_\alpha f'_\alpha(z))^\beta h \circ f_\alpha(z) \tag{1.10}$$

defines a continuous operator on $\mathcal{H}(\Delta)$. Setting

$$\zeta_m = \sum_{x \in \mathrm{Fix} F^m} |DF^m(x)|^{-\beta}, \tag{1.11}$$

$$d_m = \sum_{x \in \mathrm{Fix} F^m - \{0\}} \frac{|DF^m(x)|^{1-\beta}}{|DF^m(x) - 1|}, \tag{1.12}$$

the Theorem with its Corollaries (including 1.2) apply to the operator $\mathcal{P}_\beta$, the associated regularized Fredholm determinant (1.8) and the zeta-function (1.7).
Example : The Farey map is defined by :

$$F(x) = \begin{cases} x/(1-x) & \text{for } x \in (0, \frac{1}{2}) \\ (1-x)/x & \text{for } x \in (\frac{1}{2}, 1) \end{cases} . \tag{1.13}$$

For the inverse map f of the left-most branch we have $1 + p = 2$ (a parabolic fixed point at 0), $a = 1$ and $\epsilon = 1$. The basin of attraction contains an open angular sector for any

$\theta_0 < \pi$ and we may find a domain Δ satisfying the properties (i)-(iii) and such that the inverse of the right-most branch is a strict contraction of Δ (yielding property (iv)). A calculation shows that also property (v) is verified. The Farey map has the Gauss map as induced map. We refer to Prellberg [7] for related results (on induced maps) and to Mayer [4] for a beautiful analysis of the Gauss map.

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2 Proof of Theorem 1.1

For an open connected domain $\Omega \subset$ let $\mathcal{H}(\Omega)$ be the Fréchet space of analytic functions in Ω with the topology generated by the family of sup-norms on compact subsets of Ω. For $K \subset \Omega$ closed and $f \in \mathcal{H}(\Omega)$ we set $p_K(f) = \sup_{z \in K} |f(z)|$. By $A(\Omega) = \mathcal{H}(\Omega) \cap C^0(\bar{\Omega})$ we denote the subset of analytic functions in Ω having a continuous extension to the boundary. $A(\Omega)$ is a Banach space with the norm given by the supremum norm on Ω.

For D a subset of Ω we let r_D (omitting the explicit reference to Ω) denote the restriction map. The continuity properties with respect to the underlying function spaces will be clear from the context.

For any real number a we let $H_a = \{w \in \ : \text{Re } w > a\}$ denote the open half plane of complex numbers with real part greater than a.

We shall first consider the parabolic case where $p + 1 = 2$ and with the special choice of weight $g = (f')^\beta$ with $\beta \in$ (a restriction on β is specified below). A reduction to this case is given in section 2.6.

2.1 *Complex dynamics and Fatou coordinates. The parabolic case.*

The conditions on f implies that its dynamics may be described through Fatou coordinates. Let $s(w) = w+1$, $w \in$ be the translation map. Using properties (i)-(iii) of the intermittent contraction we have the following

Lemma 2.1 *There is an open simply connected domain $\Omega \subset$ and an analytic bijection $\phi : \Omega \to \Delta$ such that ϕ conjugates $f : \Delta \to \Delta$ with the (well-defined) translation $s : \Omega \to \Omega$:*

$$f \circ \phi(w) = \phi(w + 1), \ w \in \Omega. \tag{2.14}$$

For any $\alpha < \theta_0$, with $\theta_0 > \frac{\pi}{2}$ as in property (ii), there is $\rho_\alpha \in$ such that Ω contains a translated angular sector S_α of the form

$$\{\rho_\alpha + re^{i\theta} : r > 0, |\theta| < \alpha\}. \tag{2.15}$$

In any such sector, the functions $\phi'(w)$ and $w^2\phi'(w)$ are uniformly bounded.

proof :

The map f is a self-map of Δ and it fixes the boundary point $0 \in \partial\Delta$. It also attracts at least one interior point (just choose a point on the positive real axis close to 0). As a consequence every point $z \in \Delta$ converges to 0 under iteration by f.

[To see this, let z^* be an accumulation point of the sequence $f^k(z)$, i.e. there is a subsequence $\psi_n = f^{k_n}$ such that $\psi_n(z)$ converges to $z^* \in \mathrm{Cl}(\Delta)$. As ψ_n is a normal family it has itself a convergent subsequence. The limit function $\tilde{\psi}$ is analytic in Δ, and thus either open or constant. Since $\tilde{\psi}(c) = 0$ for an interior point c, $\tilde{\psi}$ is not open and is therefore the constant map. In particular, $z^* = \tilde{\psi}(z) = \tilde{\psi}(c) = 0$. Thus the sequence $f^k(z)$ has 0 as its only point of accumulation, hence it converges to 0.]

From the asymptotic form of f one sees that when $f^k z$ converges to 0 the argument of $f^k z$ tends to 0 as well, i.e. the convergence happens tangentially to the positive real axis.

Let $\sigma : {}^- \to {}^-$ be the involutory conformal map $\sigma(u) = 1/au$ and define for $u \in \sigma(\Delta)$ the 'almost translation', $\hat{f}(u) = \sigma \circ f \circ \sigma(u)$. One verifies that the discrepancy, $\vartheta(u) = \hat{f}(u) - u - 1$, satisfies a bound,

$$|\vartheta(u)| < C|u|^{-\epsilon}, \ u \in \sigma(\Delta) \tag{2.16}$$

with $C < \infty$.

For $r > 0$ small enough the disc $B = B(r, r)$, centered at $z = r$ and of radius r, is contained in Δ (since $\theta_0 > \frac{\pi}{2}$) and its closure is mapped strictly into its interior (union $\{0\}$). For any $z \in \Delta$ there is a unique $n \in Z$ such that $f^n(z) \in \bar{B} - f(\bar{B})$. Choosing r small enough we may ensure that the discrepancy satisfies the bounds

$$|\vartheta(u)| < 1/4 \ , \quad |\vartheta'(u)| < 1/4 \ , \quad u \in \sigma(B). \tag{2.17}$$

Using these bounds the 'almost' translation map $\hat{f}$ can be straightened out K-quasi-conformally with $K < 2$ on the cylinder $\Phi = \sigma(\bar{B} - f(B))$ ([10], proof of Proposition A.2.1) and hence by uniformization also conformally, i.e. $\psi(\hat{f}(u)) = \psi(u) + 1$, where ψ is univalent and holomorphic in IntΦ. It extends homeomorphically, whence also holomorphically, to the whole of $\sigma(\Delta)$. Let $\Omega = \psi \circ \sigma(\Delta)$ and let $\phi : \Omega \to \Delta$ be the inverse of $\psi \circ \sigma$. One has $\phi(w+1) = \sigma(\hat{f}\sigma\phi^{-1}(w)) = f(w)$ for $w \in \Omega$. From [10] (Lemma A.2.4, case (i)) and our previous estimates it follows that for $R_1 > 0$ big enough there are constants $C_1, C_2 > 0$ such that if the disc $B(u_0, R)$, centered at u_0 and of radius $R > R_1$, is contained in $\sigma(\Delta)$ then $|\psi'(u_0) - 1| \leq C_1/R + C_2|u_0|^{-\epsilon}$. As $\sigma(\Delta)$ contains the complex numbers of the form $re^{i\theta}$ with $r > 1/ar_0$ and $|\theta| < \theta_0$ it is verified that Ω contains angular (of any angle α strictly smaller than θ_0) sectors S_α as described in the Lemma. Finally, from the estimate on ψ' we obtain that $\lim_{|u|\to\infty} \psi'(u) = 1$, whence $\lim_{|w|\to\infty} aw^2\phi'(w) = -1$ where the convergence is uniform in any of the translated sectors. □

2.2 *The continuous spectrum.*

Let $\kappa(t) = \exp(-t)$. A continuous linear operator $M_\kappa : L^1(_+) \to L^1(_+)$ is defined by :

$$M_\kappa \psi(t) \equiv \kappa(t)\psi(t) \tag{2.18}$$

Denote by sp(M_κ) the spectrum of M_κ and for $\lambda \notin$ sp(M_κ), let $R(\lambda, M_\kappa) = (\lambda - M_\kappa)^{-1}$ be the associated resolvent operator. Define also $\sigma_c = [0, 1]$ (the line segment in the complex plane). One has

Lemma 2.2 *The spectrum of M_κ is continuous and equals $\sigma_c = [0, 1]$.*

proof : As $\kappa(t)$ is continuous and bounded by 1, the operator M_κ is a bounded linear operator. Whenever λ is not in the above mentioned line segment σ_c the function $(\lambda - e^{-t})^{-1}$ is continuous and bounded by dist$(\lambda, \sigma_c)^{-1}$ thus providing an upper bound for the norm of the resolvent operator. By approximation one shows that it gives a lower bound as well. As it diverges for λ approaching σ_c we have sp$(M_\kappa) = \sigma_c$. For $\lambda \in$, $(\lambda - M_\kappa)u = 0$ forces $u = 0$ a.e. and hence, the point spectrum of M_κ is empty. □

Let m be a real number to be fixed below. The (shifted) Laplace transform of a function ψ in $L^1(_+)$ is then given by :

$$\mathcal{L}_m\psi(w) = \int_0^\infty dt\ \psi(t)e^{-(w-m)t} \tag{2.19}$$

For $w \in H_m$ the integral converges absolutely and uniformly and is bounded by the L^1 norm of ψ. By standard arguments $\mathcal{L}_m\psi$ is analytic for $w \in H_m$ and has a continuous extension to the boundary, its absolute value being bounded by the L^1 norm of ψ. It is also clear that the map $\mathcal{L}_m : L^1(_+) \to \mathcal{H}(H_m)$ is injective. Hence we have shown the following :

Lemma 2.3 *The map $\mathcal{L}_m : L^1(_+) \to A(H_m)$ is bounded and injective.* $\Box$

Denote by $X(H_m)$ the isometric (with induced norm) image of $L^1(_+)$ in $\mathcal{H}(H_m)$ under the Laplace transform.

For functions $h \in \mathcal{H}(H_m)$ we define the translation operator : $Sh(w) = h(w+1)$. Its restriction to $X(H_m)$ is conjugated to the multiplication operator M_κ (hence they have identical spectral properties) under the isometry given by the Laplace transform :

$$\begin{array}{ccc} L^1(_+) & \xrightarrow{M_\kappa} & L^1(_+) \\ \mathcal{L}_m \ \downarrow & & \downarrow \ \mathcal{L}_m \\ X(H_m) & \xrightarrow{S} & X(H_m) \end{array} \tag{2.20}$$

This follows since for $h = \mathcal{L}_m\psi$, $\psi \in L^1(_+)$ and $w \in \bar{H}_m$ we have :

$$S\mathcal{L}_m\psi(w+1) = \int_0^\infty dt\ e^{-(w-m)t-t}\psi(t) = \mathcal{L}_m M_\kappa\psi(w).$$

Let $K = \bigcup_{\omega\in\Xi} \mathrm{Cl}\ f_\omega(\Delta) \subset \Delta$ be the closed union of the images of Δ under the uniformly contracting family. We have

Lemma 2.4 *There is an open connected domain $N \subset \Omega$ and real numbers $m_1 > m_2$ such that $K \subset \phi(N)$, $s(N) \subset N$ and $\bar{H}_{m_1} \subset N \subset H_{m_2}$.*

proof : By Lemma 2.1 there is a sector $S_{\alpha=0} \supset \bar{H}_{m_1}$ (for some $m_1 \in R$) contained in Ω. Let γ be a finite path connecting $\phi^{-1}K$ and H_{m_1} in the open set Ω. By compactness there is an n such that $s^n(\gamma \cup \phi^{-1}K) \in H_{m_1}$. Let $\epsilon > 0$ be such that the $\epsilon/(n+1-k)$ neighborhood of $s^k(\gamma \cup \phi^{-1}K)$, $k \le n$ all are in Ω. The union of these and H_{m_1} will satisfy our requirements with $m_2 = m_1 - n$. □

With the notation of the above Lemma choose $m > m_1$ and let it be fixed in the following. Let $\delta = m - m_1 > 0$ and let n be an integer such that $m_2 + n > m$. The set H_m shifted n times to the left covers the domain N. Thus, we have

$$s^{-n} H_m \supset N \supset H_{m-\delta} \supset H_m. \tag{2.21}$$

Let $X(N)$ be the subset of functions in $X(H_m)$ which have an analytic (necessarily unique) continuation to N with a continuous extension to the boundary. It is a Banach space under the following norm :

$$\|h\|_{X(N)} \equiv \|r_{H_m} h\|_{X(H_m)} + \|h\|_{A(N)}. \tag{2.22}$$

As $s(N) \subset N$ we may define the translation operator $T : X(N) \to X(N)$ as $Th(w) = h(w+1)$. The inclusions (2.21) give rise (by analytic continuation and Lemma 2.3) to bounded linear operators :

$$X(H_m) \xrightarrow{r_N S^n} X(N) \xrightarrow{r_{H_m}} X(H_m). \tag{2.23}$$

Denote $r = r_{H_m}$ and $q = r_N S^n$. One verifies that $q \circ S = T \circ q$, $r \circ T = S \circ r$, $r \circ q = S^n$, $q \circ r = T^n$ and therefore for $\lambda \neq 0$

$$R(\lambda, S) = \frac{1}{\lambda} + \frac{S}{\lambda^2} + \cdots \frac{S^{n-1}}{\lambda^n} + \lambda^{-n} r \circ R(\lambda, T) \circ q \tag{2.24}$$

(and similarly with S and T interchanged). We conclude that $\mathrm{sp}(T) = \mathrm{sp}(S) = \sigma_c$.

Remark : Life would be somewhat simpler if K was contained in a half plane H_{m_2} which in turn was contained in Ω. However, this might not be the case (the set $s^{-n} H_m$ in (2.21) might not be included in Ω) in which case the above distinction between the two translation operators S and T proves necessary.

The conjugating map ϕ (Lemma 2.1) is univalent and Ω is simply connected. The image of Ω under ϕ' is thus contractible in $^* = - \{0\}$. Hence $\log(-\phi'(w))$ is uniquely determined when specifying its asymptotic behavior in H_m, $\lim_{w \to \infty} \log(-a w^2 \phi'(w)) = 0$. For $\beta \in$ a complex parameter we set $(-\phi')^\beta = \exp(\beta \log(-\phi'))$. The same arguments apply to

$f : \Delta \to \Delta$ and $(f')^\beta$ is uniquely determined when specifying $\log(f'(0)) = 0$. Given any open set $\Upsilon \subset \Omega$ we may define the following isomorphism $J : \mathcal{H}(\phi\Upsilon) \to \mathcal{H}(\Upsilon)$:

$$Jv(w) \equiv v \circ \phi(w) \; (-\phi'(w))^\beta. \tag{2.25}$$

Let $U = \phi N \subset \Delta$ and denote by $X(U) \subset \mathcal{H}(U)$ the isometric (with induced norm) image of $X(N)$ under J^{-1}. The translation operator $T : X(N) \to X(N)$ is then conjugated to the Perron-Frobenius type operator, $\mathcal{M}_0 : X(U) \to X(U)$:

$$\mathcal{M}_0 v(z) = v \circ f(z) \; (f'(z))^\beta. \tag{2.26}$$

This is seen from $\phi' \circ s = f' \circ \phi \; \phi'$ and hence

$$J\mathcal{M}_0 v = (v \circ f \cdot (f')^\beta) \circ \phi \cdot (-\phi')^\beta = (v \circ \phi \cdot (-\phi')^\beta) \circ s = TJv. \tag{2.27}$$

$$\begin{array}{ccc} X(N) & \xrightarrow{T} & X(N) \\ J^{-1} \downarrow & & \downarrow J^{-1} \\ X(U) & \xrightarrow{\mathcal{M}_0} & X(U) \end{array} \tag{2.28}$$

In particular, it follows that the spectrum of $\mathcal{M}_0 : X(U) \to X(U)$ is the line-segment σ_c. One verifies that the point spectrum of $\mathcal{M}_0$ is void.
Remark : The function spaces $X(\cdot)$ are in some sense quite abstract as they are given through isometric injections into Fréchet spaces. In contrast, the spaces $A(\cdot)$ are very easy to handle. The following two Lemmas provide the crucial relationship between these function spaces.

Lemma 2.5 *For Re $\beta > \frac{1}{2}$ the injection map $A(U) \overset{j}{\hookrightarrow} X(U)$ is continuous.*

proof : Let τ = Re β. For $v \in A(U)$ both $v \circ \phi$ and ϕ' belong to $A(N)$, whence $\|Jv\|_{A(N)} \leq \|v\|_{A(U)} \|\phi'\|^\tau_{A(N)}$. Since also $w^2\phi' \in A(N)$ and $H_{m-\delta} \subset N$ we may retrieve (an L^1 representative of) the inverse Laplace transform of Jv. Using a change of coordinates $u \to u - \delta$ and analyticity we obtain :

$$\psi(t) = \int_{\partial H_{m-\delta}} \frac{du}{2\pi i} e^{t(u-m)} Jv(u) = \int_{\partial H_m} \frac{du}{2\pi i} e^{-t\delta + t(u-m)} Jv(u-\delta) \tag{2.29}$$

and since $(1 + |u|^2)|\phi'(u)| \leq \|\phi'\|_{A(N)} + \|w^2\phi'\|_{A(N)} \equiv c < \infty$, for $u \in N$, we have that

$$|\psi(t)| \leq e^{-t\delta} \|v\|_{A(U)} \int_{-\infty}^{\infty} \frac{dx}{2\pi} \frac{c^\tau}{(1+x^2)^\tau} = C(\tau) e^{-t\delta} \|v\|_{A(U)} \tag{2.30}$$

where $C(\tau) < \infty$ since $\tau > \frac{1}{2}$. Integration yields $\|\psi\|_{L^1(+)} \leq C(\tau)\frac{1}{\delta}\|v\|_{A(U)}$. As N is connected, Jv is determined uniquely through its inverse Laplace transform ψ_t and we obtain :

$$\|Jv\|_{X(N)} = \|Jv\|_{A(N)} + \|r_{H_m} Jv\|_{X(H_m)} \leq \|v\|_{A(U)}(\|\phi'\|^{\tau}_{A(N)} + \frac{C(\tau)}{\delta}). \tag{2.31}$$

Composing with the isometry $J^{-1} : X(N) \to X(U)$ yields the Lemma. $\square$

For the other direction we have the following

Lemma 2.6 *For any compact set $K \subset U$ the restriction map $r_K : X(U) \to A(K)$ is continuous.*

proof : Set $K = \phi Q$. For $v \in X(U)$, $h = Jv \in X(N)$ we have $p_Q(h) \leq p_N(h) \leq \|h\|_{X(N)} = \|v\|_{X(U)}$. Hence

$$\|r_K v\|_{A(K)} = p_K(v) \leq p_K(J^{-1}h) \leq p_Q(h) p_Q((-\phi')^{-\beta}) \leq \|v\|_{X(U)} C(\beta) \tag{2.32}$$

where $C(\beta) < \infty$ since Q is compact. $\square$

2.3 *Nuclear theory and regularized Fredholm determinants.*

Lemma 2.7 *For Re $\beta > \frac{1}{2}$ and $\lambda \in -\sigma_c$ the operator $\mathcal{M}_1 R(\lambda, \mathcal{M}_0) : X(U) \to X(U)$ is nuclear of order zero.*

proof : As above let K be the closure of union of the images of Δ under the uniformly contracting family and choose K' compact such that $K \subset \text{Int } K' \subset K' \subset U$. The injection $A(K') \hookrightarrow \mathcal{H}(\text{Int}K')$ is a continuous map and the further restriction $r_K : \mathcal{H}(\text{Int}K') \to A(K)$ is a bounded linear map from the nuclear space $\mathcal{H}(\text{Int}K')$ to the Banach space $A(K)$. It follows that $r_K : A(K') \to A(K)$ is nuclear of order zero [1, 8].

From the integrability condition on the weights the linear operator (from equation 1.2), $\tilde{\mathcal{M}}_1 : A(K) \to A(U)$ is bounded and hence by Lemma 2.5 and 2.6 so is $\mathcal{M}_1 = j\tilde{\mathcal{M}}_1 r_K : X(U) \to X(U)$. For $\lambda \notin \sigma_c$ the resolvent operator $R(\lambda, \mathcal{M}_0) : X(U) \to X(U)$ is bounded and so is $r_{K'} R(\lambda, \mathcal{M}_0) : X(U) \to A(K')$ by Lemma 2.6. Composing with the further (nuclear) restriction $r_K : A(K') \to A(K)$, the bounded operator $\tilde{\mathcal{M}}_1 : A(K) \to A(U)$ and finally the continuous injection (Lemma 2.5), $j : A(U) \hookrightarrow X(U)$ we conclude that

$$\mathcal{M}_1 R(\lambda, \mathcal{M}_0) = j \tilde{\mathcal{M}}_1 r_K r_{K'} R(\lambda, \mathcal{M}_0) : X(U) \to X(U) \tag{2.33}$$

is nuclear of order zero. □

For $\lambda_0 \in -\sigma_c$, Re $\beta_0 > \frac{1}{2}$ the bounds in lemma 2.5, 2.6 and 2.7 can be made uniform in a small complex neighborhood of λ_0 and β_0 (more precisely, one should consider the operator $\tilde{\mathcal{M}}_1 r_K R(\lambda, \mathcal{M}_0) j : A(U) \to A(U)$ for which the function space is kept fixed). It follows that the operator $\mathcal{M}_1 R(\lambda, \mathcal{M}_0) : X(U) \to X(U)$ is a holomorphic family of nuclear operators in $\lambda \in -\sigma_c$, Re $\beta > \frac{1}{2}$ and by [1, 2] it has a Fredholm determinant

$$\hat{d}(z, \lambda, \beta) = \det(1 - z\mathcal{M}_1 R(\lambda, \mathcal{M}_0)) \tag{2.34}$$

which is entire in z and holomorphic in $\lambda \in -\sigma_c$, Re $\beta > \frac{1}{2}$. It corresponds to the Fredholm determinant associated with the so-called induced family of contractions. Setting $z = 1$ we obtain the regularized Fredholm determinant of $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$:

$$d_\beta(\lambda) = \det(1 - \mathcal{M}_1 R(\lambda, \mathcal{M}_0)) \tag{2.35}$$

In order to relate its zeroes to the eigenvalues of the operator $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$ we need the following :

Lemma 2.8 *Let $A : X \to X$ be a bounded linear operator on a Banach space X and let $N : X \to X$ be nuclear of order zero. Assume that $E = -sp(A)$ is connected. Then the part of the spectrum of $A + N$ which intersects E consists of isolated eigenvalues of finite multiplicity only which can not accumulate in E. The Fredholm determinant*

$$d(u) = \det(1 - N(u - A)^{-1})$$

is analytic in $u \in E$. Furthermore in this domain, the zero-set of $d(u)$ counted with order is the same as the eigenvalues of $A + N$ counted with multiplicity.

proof : For finite matrices the result is trivial. The problem arises since the determinant of $(1 - u^{-1}A)$ is not defined. Note first, that for $\lambda \in E$, the resolvent $R(\lambda) = (\lambda - A)^{-1}$ is a (bounded) operator-valued analytic function of λ. The expression

$$(\lambda - A - N)^{-1} = R(\lambda)(1 - NR(\lambda))^{-1}$$

is valid if 1 is not in the spectrum of $NR(\lambda)$. The operator $NR(\lambda)$ is a nuclear operator valued (being a nuclear operator times a bounded operator) analytic function of $\lambda \in E$. Its spectrum is discrete and for $\lambda \to \infty$ all eigenvalues go to zero. In particular as E is connected the value 1 can be an eigenvalue of $NE(\lambda)$ only for a discrete set of λ values in E.

From the above it follows [1, 2] that the Fredholm determinant $d(u)$ is analytic in $u \in E$. A root of d, $d(\lambda) = 0$, corresponds to an eigenvalue 1 of $NR(\lambda)$. Let $\lambda \in E$ be a (necessarily isolated) zero of d of order p and let $\mathcal{C}$ be the boundary of a small closed disc E which intersects the spectrum of $A + N$ only in λ. Through analytic continuation one obtains from $d(u) = \exp(\mathrm{tr}(\log(1 - NR(u))))$ the expression

$$\frac{d}{du} \log d(u) = \mathrm{tr}[(1 - NR(u))^{-1} NR(u)^2].$$

Integrating along the contour $\mathcal{C}$ we get

$$\begin{aligned} p &= \oint_{\mathcal{C}} \frac{du}{2\pi i} \frac{d}{du} \log d(u) \\ &= \mathrm{tr} \oint_{\mathcal{C}} \frac{du}{2\pi i} [(1 - NR(u))^{-1} NR(u)^2] \qquad (2.36) \\ &= \mathrm{tr} \oint_{\mathcal{C}} \frac{du}{2\pi i} [(1 - NR(u))^{-1} R(u)] \qquad (2.37) \\ &= \mathrm{tr} \oint_{\mathcal{C}} \frac{du}{2\pi i} [(u - A - N)^{-1}] \qquad (2.38) \\ &= \mathrm{tr} P_\lambda \end{aligned}$$

where P_λ is the projection onto the generalized eigenspace of $A + N$ corresponding to the eigenvector λ. The validity of the above calculations is justified by the nuclearity of N (eq. 2.36), the identity

$$(1 - NR(u))^{-1} NR(u)^2 = -R(u) + (1 - NR(u))^{-1} R(u)$$

and the analyticity of $R(u)$ (thus yielding a vanishing contribution) inside the contour (eq. 2.37) and the finite dimensionality of the resulting projection operator (eq. 2.38). $\square$

For $|\lambda| > \|\mathcal{M}_0\| + \|\mathcal{M}_1\|$ we have the uniformly convergent series :

$$\log(1 - \mathcal{M}_1 (\lambda - \mathcal{M}_0)^{-1}) = \mathcal{M}_1 \sum_{m>0} \lambda^{-m} A_m \qquad (2.39)$$

where A_m is a polynomial in $\mathcal{M}_0$ and $\mathcal{M}_1$ for which $\|A_m\| < (\|\mathcal{M}_0\| + \|\mathcal{M}_1\|)^{m-1}$. We note that the right hand side is nuclear of order zero and by the formal calculation $(1 - \mathcal{M}_1(\lambda - \mathcal{M}_0)^{-1}) = (1 - \lambda^{-1}(\mathcal{M}_0 + \mathcal{M}_1))(1 - \lambda^{-1}\mathcal{M}_0)^{-1}$ we derive the formula :

$$\mathrm{tr}\ \log(1 - \mathcal{M}_1(\lambda - \mathcal{M}_0)^{-1}) = \sum_{m>0} \frac{\lambda^{-m}}{m} \mathrm{tr}\ ((\mathcal{M}_0 + \mathcal{M}_1)^m - \mathcal{M}_0{}^m). \tag{2.40}$$

Defining $d_m = \mathrm{tr}\ ((\mathcal{M}_0 + \mathcal{M}_1)^m - \mathcal{M}_0{}^m)$ we conclude that the Fredholm determinant :

$$d(\lambda) = \det(1 - \mathcal{M}_1(\lambda - \mathcal{M}_0)^{-1}) = \exp - \sum_{m>0} \frac{\lambda^{-m}}{m} d_m \tag{2.41}$$

is holomorphic for $|\lambda| > \|\mathcal{M}_0\| + \|\mathcal{M}_1\|$. By Lemma 2.7 it has a unique analytic extension to $-\sigma_c$. Inserting (1.1-1.2) and using the uniform contraction and the uniform convergence of the integrals (hence uniform nuclearity) we get for the trace :

$$d_m = \mathrm{tr} \int_{\Xi^*_m} \mu^*(\bar{\omega})\ \mathcal{M}^m_{\bar{\omega}} = \int_{\Xi^*_m} \mu^*(\bar{\omega})\ \mathrm{tr}\ \mathcal{M}^m_{\bar{\omega}}, \tag{2.42}$$

where the multiple-integral extends over all $\bar{\omega} = (\omega_1, ..., \omega_m) \in \Xi^*_m = \Xi^m - \{(0, ..., 0)\}$ and $\mathcal{M}^m_{\bar{\omega}} = \mathcal{M}_{\omega_m} \cdots \mathcal{M}_{\omega_1} : X(U) \to X(U)$ is given by $\mathcal{M}^m_{\bar{\omega}} v(z) = g_{\bar{\omega}}(z)\ v(f_{\bar{\omega}}(z))$. As $f_{\bar{\omega}} : \mathrm{Cl}\ U \to \mathrm{Int}\ U$ standard arguments show that this map has a unique fixed point $x(\bar{\omega}) \in U$ where its multiplier is strictly smaller than one in modulus. Furthermore, $\mathcal{M}^m_{\bar{\omega}}$ is a nuclear operator on $X(U)$ and its trace does not change if we replace $X(U)$ by $A(D)$ where the domain D is such that for some $k > 0$, $\mathrm{Cl}\ f^k_{\bar{\omega}} U \subset \mathrm{Cl}\ f_{\bar{\omega}} D \subset D \subset U$. Choosing for D a small disc centered at $x(\bar{\omega})$ we obtain by standard Fredholm theory and residue calculus,

$$\mathrm{tr}\mathcal{M}^m_{\bar{\omega}} = \oint_{\partial D} \frac{dz}{2\pi i} \frac{g_{\bar{\omega}}}{z - f_{\bar{\omega}}(z)} = \frac{g_{\bar{\omega}}(x(\bar{\omega}))}{1 - f'_{\bar{\omega}}(x(\bar{\omega}))}, \tag{2.43}$$

from which the formula (1.6) for the traces follows.

2.4 *Crossing the continuous spectrum.*

With θ_0 as in property (ii) and for $\alpha \in (\frac{\pi}{2}, \theta_0)$ let N_α be the union of N and the sector S_α (equation 2.15, Lemma 2.1). A Banach space $X(N_\alpha)$ and a norm $\|\cdot\|_{X(N_\alpha)}$ is defined as in equation (2.22). Let $U_\alpha = \phi N_\alpha$ and denote again by $X(U_\alpha)$ the isometric image of $X(N_\alpha)$ under J^{-1}. As ϕ' and $w^2\phi'$ are uniformly bounded in N_α, a repetition of the proof of Lemma 2.5 shows that

Lemma 2.9 *For Re $\beta > \frac{1}{2}$ the injection map $A(U_\alpha) \stackrel{j_\alpha}{\hookrightarrow} X(U_\alpha)$ is continuous.* $\Box$

For $\nu \in (\frac{\pi}{2} - \alpha, \alpha - \frac{\pi}{2})$ set $\kappa = \tan \nu$ and define Y^ν as the set of complex numbers γ where either $\mathrm{Re}\, \gamma < 0$ and $\mathrm{Im}\, \gamma \in [0, 2\pi]$ or $\mathrm{Re}\, \gamma \geq 0$ and $\mathrm{Im}\gamma - \kappa \mathrm{Re}\gamma \in (0, 2\pi)$. Under the exponential map $\lambda = \exp(-\gamma)$, $2\pi i$ separated points are identified and the strip Y^0 maps to the sliced complex plane $-\, \sigma_c$ and $Y^{\nu \neq 0}$ maps to $-\, \{$a logarithmic spiral$\}$.

We let Y_α denote the union of all Y^ν, $|\nu| < \alpha - \frac{\pi}{2}$ (which then maps to a multi-sheeted Riemann surface under the exponential map). Finally, define the domain $T_\alpha = \{re^{i\theta} : r > 0, \theta \in (\frac{\pi}{2} - \alpha, \alpha - \frac{\pi}{2})\}$.

Lemma 2.10 *Let $K' \subset U$ be a compact set. The operator $Q(\gamma) : A(U_\alpha) \to A(K')$ given by :*

$$Q(\gamma) = r_{K'} \circ R(e^{-\gamma}, \mathcal{M}_0) \circ r_U \circ j_\alpha. \tag{2.44}$$

has a unique holomorphic extension to the domain Y_α.

[Remark : Our previous analysis shows already that $Q(\gamma)$ is a bounded operator for $\gamma \in Y^0$, i.e. for $\lambda = e^{-\gamma} \notin \sigma_c$. What is new is that the borders $_+$ and $_+ + 2\pi i$ may be crossed (within some neighborhood) without encountering any singularities from the continuous spectrum of $\mathcal{M}_0$.]

proof : We may assume that $K' \in H_m$ or else use the technique leading to equation (2.24) to achieve this situation. Assume first that $\lambda = e^{-\gamma} \notin \sigma_c$. For a function $v \in A(U_\alpha)$ we retrieve the inverse Laplace transform of $h = Jv$ as in Lemma 2.5 by integrating along the boundary of $H_{m-\delta}$. Since ϕ', $w^2\phi'$ and $j_\alpha v$ are uniformly bounded in N_α this integration may be pushed to a contour $\mathcal{C}_\alpha$ which is formed by taking the boundary of the union of $H_{m-\delta}$ and the angular sector S_α of Lemma 2.1 (note that $\mathrm{Re}(w - u) \geq \delta > 0$ for $w \in H_m$ and $u \in \mathcal{C}_\alpha$). Let $k_\gamma(w, t) = e^{-wt}/(e^{-\gamma} - e^{-t})$ be the kernel (cf. the proof of Lemma 2.2) for the resolvent operator $R(e^{-\gamma}, \mathcal{M}_0)$. For $\gamma \in Y^0$ and $w \in H_m$ the action of $Q(\gamma)$ on $v \in A(U_\alpha)$ is given by :

$$[JQ(\gamma)v](w) = \int_0^\infty dt \int_{\mathcal{C}_\alpha} \frac{du}{2\pi i} k_\gamma(w - u, t) Jv(u). \tag{2.45}$$

[Remark : The point in pushing the u-integration to the contour $\mathcal{C}_\alpha$ is that when $w \in H_m$ and $C > 0$ is arbitrary the exponential term in the kernel, $\exp(-(w - u)t)$, remains uniformly bounded for all (now complex) values of $t \in T_\alpha$ with $\mathrm{Im}\ t \in (-C, C)$ (since u asymptotically behaves like $re^{\pm i\alpha}$, $r \to \infty$). This provides us with some flexibility in

deforming the integration path for the t-variable.]
For $\epsilon \in (0, \frac{\pi}{4})$, $W_\gamma^\epsilon = \bigcup_{k \in Z} B(\gamma + 2\pi i k, \epsilon)$ will denote the union of disjoint ϵ-balls centered around the singularities of $t \mapsto k_\gamma(w-u,t)$ (the zeroes of $e^{-\gamma} - e^{-t}$).

Let $\mathcal{P}_\gamma^\epsilon$ be the set of smooth (denoted admissible) paths in $T_\alpha - W_\gamma^\epsilon$ which starts at the origin and tends to $+\infty$ asymptotically parallel to the real axis. For any such path $\Gamma \in \mathcal{P}_\gamma^\epsilon$, the kernel $t \mapsto k_\gamma(w-u,t)$ stays uniformly bounded (avoiding the singularities) for $t \in \Gamma$, $w \in H_m$, $u \in \mathcal{C}_\alpha$ and furthermore, it tends to zero exponentially fast as $\mathrm{Re}\, t \to \infty$. For any $\gamma \in Y_\alpha$ we can find a suitable $\epsilon > 0$ and connect γ by a path to some fixed $\gamma_0 \in Y^0$ and by following this connection in the opposite direction we deform at the same time smoothly the integration path $\Gamma(\gamma_0) = {}_+ \in \mathcal{P}_{\gamma_0}^\epsilon$ to an integration path $\Gamma(\gamma) \in \mathcal{P}_\gamma^\epsilon$ through admissible paths only and in the process obtain an analytic continuation $Q(\gamma)$ of $Q(\gamma_0)$. As Y_α is simply connected this extension is unique.□

Lemma 2.11 *For Re $\beta > \frac{1}{2}$ the regularized Fredholm determinant extends holomorphically through the map $\gamma \mapsto d(\lambda = e^{-\gamma})$ to the domain Y_α described above.*

proof : Noting that $\mathcal{M}_1$ extends to a continuous operator $\mathcal{M}_1 : A(K) \to A(U_\alpha)$ the sequence,

$$A(U_\alpha) \xrightarrow{Q(\gamma)} A(K') \xrightarrow{r_K} A(K) \xrightarrow{\mathcal{M}_1} A(U_\alpha), \tag{2.46}$$

defines a nuclear operator valued holomorphic function in $\gamma \in Y_\alpha$ (cf. the proof of 2.7). One verifies that its Fredholm determinant on the strip $\gamma \in Y^0$ coincides with $d(\lambda = e^{-\gamma}, z, \beta)$ from equation (2.34). □

For $\nu \in (\frac{\pi}{2} - \alpha, \alpha - \frac{\pi}{2})$ set $\eta = e^{i\nu}$ and consider for $\psi \in L^1(_+)$ its rotated and shifted Laplace transform :

$$\mathcal{L}_{m,\eta}\psi(w) = \int_0^\infty dt\ \psi(t) e^{-(w-m)\eta t}. \tag{2.47}$$

It is holomorphic in

$$H_{m,\eta} = \{w : \mathrm{Re}((w-m)\eta) > 0\} \tag{2.48}$$

and has a continuous extension to the boundary. The translation operator S_η on the space $X(H_{m,\eta})$ is now conjugated to a multiplication with $\exp(-\eta t)$ on $L^1(_+)$ and we deduce that the continuous spectrum now becomes a logarithmic spiral :

$$\mathrm{sp}(S_\eta) = \mathrm{Cl}\{e^{-\eta t} : t \in {}_+\}. \tag{2.49}$$

Replacing H_m by $H_{m,\eta}$, our previous analysis carries over and provides us with a rotated domain N_η (Lemma 2.4, possibly with other choices of constants, m_1 and m_2) and a Banach space $X(U_\eta)$ on which $\mathcal{M}_0$ and $\mathcal{M}_1$ act as bounded linear operators such that $\mathrm{sp}(\mathcal{M}_0) = \mathrm{sp}(S_\eta)$ and $\mathcal{M}_1$ is nuclear as before. Then $\mathcal{M}_1 R(\lambda, \mathcal{M}_0)$ has a Fredholm determinant $d_\eta(\lambda)$ which is analytic in $\; - \mathrm{sp}(S_\eta)$ where its zero-set is in one-to-one correspondence with the eigenvalues of $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1 \in L(X(U_\eta))$. As the formula (2.43) for the traces remains the same, the analytic continuation of $\gamma \to d_\eta(e^{-\gamma})$, $\gamma \in Y_\alpha$ is independent of η. We use this to deduce the following

Lemma 2.12 *For $\lambda_0 \in sp(S_\eta) - \{0,1\}$ let q be the dimension of the generalized eigenspace $\bigcup_{k>0} \ker(\lambda_0 - \mathcal{M})^k$ (in $X(U_\eta)$). Then the Fredholm determinant d when analytically continued from either side of $sp(S_\eta) - \{0,1\}$ will have a zero of at least the same order q. In particular, the point spectrum of $\mathcal{M}$ (in $X(U_\eta)$) can not accumulate on points other than $\{0\}$ and $\{1\}$.*

proof : Write $\lambda_0 = e^{-\eta\gamma}$ with $\gamma \in (0, \infty)$ and let $v \in X(U)$ with $v \in \ker(\lambda_0 - \mathcal{M})^k$, for some $k \geq 1$. Put $v_n = (\lambda_0 - \mathcal{M})^n v \in X(U)$ for $n \geq 0$ so that $v_k = 0$ and

$$(\lambda_0 - \mathcal{M}_0) v_n = v_{n+1} + \mathcal{M}_1 v_n. \tag{2.50}$$

Choose $\epsilon > 0$ small enough so that $H_{m-\delta,\eta+\tau} \subset N_\alpha$ for some $\delta > 0$ and all $|\tau| < \epsilon$ and such that the function $\lambda_0 - e^{-\eta t}$ is non zero for $t \in \mathrm{Cl}\, T_\delta$ except at $t = \gamma$. The operator $\mathcal{M}_1$ maps $A(K)$ and therefore also $X(U)$ into $X(U_\alpha)$ and like in Lemma 2.5 and 2.9, $\mathcal{M}_1 v_n$ has an inverse Laplace transform (an $L^1({}_+)$ representative) $\hat{\rho}_n(t)$ which has an analytic continuation to $t \in T_\delta$ and decays exponentially as $|t| \to \infty$. Let $v_n = \mathcal{L}_{m,\eta} \hat{v}_n(t)$. Then

$$\int_0^\infty dt\; e^{-(w-m)\eta t}((\lambda_0 - e^{-\eta t})\hat{v}_n(t) - \hat{\rho}_n(t) - \hat{v}_{n+1}(t)) = 0 \text{ (a.e.)} \tag{2.51}$$

implies that $((\lambda_0 - e^{-\eta t})\hat{v}_n(t) - \hat{\rho}_n(t)) - \hat{v}_{n+1}(t) = 0$ a.e. By induction (starting from v_k) we see that all $\hat{v}_n$ extends analytically to T_δ, possibly with pole singularities at $t = \gamma$. If however, $\hat{\rho}_n(\gamma) + \hat{v}_{n+1}(\gamma)$ did not vanish, then $\hat{v}_n$ would not be in $L^1({}_+)$. Hence, the apparent singularity at $t = \gamma$ is removable and $\hat{v}_n(t)$ has an analytic continuation to T_δ. As $\hat{v}_n(t)$ also decays exponentially for $|t| \to \infty$ it follows that v_n is a generalized eigenvector also for the operator $\mathcal{M}$ acting on $X(U_{\eta+\tau})$, $|\tau| < \epsilon$. By Lemma 2.8 the Fredholm determinant $d_{\eta+\tau}(\lambda)$ has for $0 < |\tau| < \epsilon$ a zero of order at least q at $\lambda = e^{-\eta\gamma}$ and since independent of η the same is true for $d_\eta(\lambda)$ when analytically continued from either side (corresponding to positive and negative values of τ). By analyticity of $d_\eta(\lambda)$ in a neighborhood of $\lambda = \lambda_0$ the claim about accumulation points for the point spectrum of $\mathcal{M}$ now follows. $\square$

2.5 *Proof of the Theorem and Corollaries. The parabolic case.*

The operator $\mathcal{M}_0$ has a continuous spectrum $\sigma_c = [0,1]$. As $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$ is a perturbation with a nuclear (in particular, compact) operator, we obtain part (a) of the Theorem except that we have only shown it for the part of σ_p which is contained in $-[0,1]$. Lemma 2.7 provides the holomorphic continuation of the regularized Fredholm determinant obtained in section 2.3 and Lemma 2.8 identifies its zero set with the eigenvalues of $\mathcal{M}$ thus proving part (c). By Lemma 2.11, $d(\lambda)$ has analytic continuations from both sides to an open neighborhood of $\sigma_c - \{0,1\}$, proving part (d) and by Lemma 2.12 we see that eigenvalues in $-\sigma_c$ of $\mathcal{M}$ can accumulate only on the points 0 and 1, proving part (b) and thus finishing the proof of the Theorem in the parabolic case with weight $g = (f')^\beta$. $\square$

Composing with $z \mapsto \lambda(z) = \frac{1}{4z}(1+z)^2$ we obtain

$$d(\lambda(z)) = \det(1 - \mathcal{M}_1(\frac{1}{4z}(1+z)^2 - \mathcal{M}_0)^{-1}) = \det(1 - 4z\mathcal{M}_1((1+z)^2 - 4z\mathcal{M}_0)^{-1}), \quad (2.52)$$

holomorphic for $\lambda(z) \in -\sigma_c$ and hence for $0 < |z| < 1$. This extends holomorphically to the origin, proving Corollary 1.1.

Lemma 2.11 shows that $Q(\gamma)$ and hence $d(\lambda = e^\gamma)$ has an analytic continuation to $\gamma \in Y_\alpha$ for any $\alpha < \pi$ and hence to $\gamma \in Y_\pi$. Taking the exponential of Y_π yields the Riemann surface described in Corollary 1.2.

Under the conditions of Corollary 1.3 we may replace the weights g_ω by $g_\omega(f'_\omega)^\nu$ for $\mathrm{Re}\,\nu \geq 0$. Making this ν dependence explicit the operator $\mathcal{M}(\nu)$ gives rise to a holomorphic family of regularized Fredholm determinants for $\mathrm{Re}\,\nu > 0$. Since one has

$$\zeta_m(\nu = 0) = (g(0))^m + d_m(\nu = 0) - d_m(\nu = 1) \quad (2.53)$$

where $g(0) = 1$ we get that

$$\zeta(z) = \frac{1}{\log(1-z)} \frac{d(1/z, \nu = 1)}{d(1/z, \nu = 0)} \tag{2.54}$$

from which the Corollary 1.3 follows.

2.6 *Reduction to the parabolic case.*

As Δ is simply connected in $^* = \; - \{0\}$ the conformal transformation $u = u(z) = z^p$ of Δ is well-defined, (for $p > 1$ possibly with a multi-sheeted image, which, however, is unimportant for the arguments). The map f then conjugates to :

$$u \circ f \circ u^{-1} = u - pau^2 + \mathcal{O}(|u|^{2+\epsilon}) \tag{2.55}$$

and the weight becomes :

$$g \circ u^{-1} = 1 - bu + \mathcal{O}(|u|^{1+\epsilon}) \tag{2.56}$$

(and similar transformations for $\{f_\omega, g_\omega\}_{\omega \in \Xi}$).

In the following we shall consider these conjugated maps (denoted by the same symbols).

As $g \neq 0$ on the invariant set Σ^* it does not vanish in an open neighborhood of Σ^* in $\Delta \cup \{0\}$. Possibly by replacing Δ by a smaller set obtained by iterations under the extended family we may assume that $g \neq 0$ on Δ.

We set (cf. section 2.2)

$$(f'(u))^\beta = \exp(\beta \log(f'(u)), \;\; u \in \Delta \tag{2.57}$$

with $\log(f'(0)) = 0$ and for $u \in U$ (defined as in section 2.2) we have

$$(f'(u))^\beta = 1 - 2ap\beta u + \mathcal{O}(|u|^{1+\epsilon}). \tag{2.58}$$

Choosing now $\beta = \frac{b}{2pa}$, we may write

$$g(u) = (f'(u))^\beta \psi(u) \tag{2.59}$$

where $\psi(u) = 1 + \mathcal{O}(|u|^{1+\epsilon}) \in A(U)$ does not vanish. From the asymptotic behavior of f, $kf^k(u)$ is uniformly bounded for $u \in \mathrm{Cl}(U)$ and $k \geq 0$. It follows that the product

$\prod_{k\geq 0}\psi(f^k(u))$ converges uniformly. The limit $\psi^* \in A(U)$ does not vanish and hence we have also $1/\psi^* \in A(U)$.

The calculation

$$\mathcal{M}_0(\psi^* v) = (\psi^* \circ f)\,(v \circ f)\,(f')^\beta\,\psi = \psi^*\,(v \circ f)\,(f')^\beta \tag{2.60}$$

shows that $\mathcal{M}_0$ is conjugated to an operator where the weight has been replaced by $(f')^\beta$ and the norm on the function space is obtained by scaling with ψ^*. If we assume that $\mathrm{Re}(\beta) > \frac{1}{2}$ then since $\psi^*, 1/\psi^* \in A(U)$ the traces of $\mathcal{M}_{\bar{\omega}}^m$, $\bar{\omega} \in \Xi_m^*$ remains the same. Noting that the condition $\mathrm{Re}(\beta) > \frac{1}{2}$ translates into $\mathrm{Re}\, b > pa$ and θ_0 maps into $p\theta_0$ (cf. condition (ii) and Corollary 1.2), the Theorem and its Corollaries follow.

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