Preservation of the absolutely continuous spectrum of Schrödinger equation under perturbations by slowly decreasing potentials and a.e. convergence of integral operators.

Alexander Kiselev

Mathematical Sciences Research Institute

Berkeley CA 94720

e-mail: kiselev@msri.org

Abstract

We prove a new criteria of stability of the absolutely continuous spectrum of one-dimensional Schrödinger operators under slowly decaying perturbations. As applications, we show that the absolutely continuous spectrum of the free and periodic Schrödinger operators is preserved under perturbations by all potentials $V(x)$ satisfying $|V(x)| \leq C(1+x)^{-\frac{2}{3}-\epsilon}$. The main new technique includes an a.e. convergence theorem for a class of integral operators.

1. Introduction

In this paper we study the stability of the absolutely continuous spectrum of one-dimensional Schrödinger operators under perturbations by slowly decaying potentials. The most general formulation of the problem we study is as follows. Let H_U be a Schrödinger operator acting on $L^2(0,\infty)$ and given by the differential expression

$$-\frac{d^2}{dx^2} + U(x). \tag{1}$$

Suppose for simplicity that the potential $U(x)$ is bounded (more general assumptions on $U(x)$ will be specified below) and fix some self-adjoint boundary condition at zero. It is a well-known fact that under the above conditions the differential expression (1) defines a unique self-adjoint operator. Let us assume that the absolutely continuous spectrum of the operator H_U is not empty. We perturb the operator H_U by a decaying potential $V(x)$:

$$H_{U+V} = H_U + V(x).$$

We assume that $V(x)$ is locally integrable and $V(x) \stackrel{x\to\infty}{\longrightarrow} 0$. Since the potential decaying at infinity constitutes a relatively compact perturbation of the unperturbed part, the essential spectrum of the operator H_{U+V} coincides with that of H_U. We would like to study which conditions on the rate of decay of V are sufficient to ensure that the absolutely continuous spectrum of H_U is preserved.

The main results of our study is the derivation of the new criteria on the stability of the absolutely continuous spectrum of Schrödinger operators under perturbations by the general classes of slowly decaying potentials. We apply these criteria, in particular, to find new classes of slowly decaying potentials preserving the absolutely continuous spectrum of the free and periodic Schrödinger operators.

Over the years, there has been much attention to the subject and we briefly recall main results. Suppose that the potential V is of short range, by which we mean that $|V(x)| \leq C(1+x)^{-1-\epsilon}$, or, more generally, suppose that $V(x)$ is absolutely integrable. Then for a wide class of background potentials $U(x)$ a number of techniques may be used to show that the absolutely continuous spectrum of H_U is preserved. In many situations the trace class theorems of scattering theory may be used to show that, moreover, the wave operators exist and are complete in this case. This short-range result is basically all what is known on the preservation of the absolutely continuous spectrum of Schrödinger operators under decaying perturbations in general situation. Essentially more information is available in the case when $U(x) = 0$. There has been much work on proving the absolute continuity of the spectrum for the Schrödinger operators with potentials of slower decay satisfying some additional special structural assumptions. It is a classic result, going back to Weidmann [30] that if a potential V may be represented as a sum of a function of bounded variation and an absolutely integrable function, then the spectrum of the operator H_V on $R^+ = (0, \infty)$ is purely absolutely contimuous. Many authors developed a scattering theory for potentials satisfying certain conditions on its derivatives [2], [1], [12]. These results hold in any dimension and the proofs use the method of approximating the scattering trajectories by the solutions of the classical Hamilton-Jacobi equation. This method is applicable only when there are certian conditions on the decay of the derivatives of the potential. The weakest conditions on the long-range part of the potentials, under which the wave operators exist, are given in [12]. For potentials satisfying $|V(x)| \leq C(1+|x|)^{-\frac{1}{2}-\epsilon}$, for instance, one can infer the existence of the wave operators if also $|D^\alpha V(x)| \leq C_1(1+|x|)^{-\frac{3}{2}-\epsilon}$ for every multiindex α with $|\alpha| = 1$.

Another class of results describe spectral behavior of the spherically symmetric (i.e. essentially one-dimensional) specific oscillating potentials, the typical example being potentials of type $V(x) = \frac{\sin x^\alpha}{x^\beta}$ with α, β positive. Clearly, potentials of this type may not satisfy in general conditions on the decay of the derivatives which are needed for the method of the above works to be applicable. We mention the papers of [3], [4], [11], and [31] in which further references may be found. The spectrum of the operator H_V for such potentials turns out to be absolutely continuous with perhaps some isolated embedded eigenvalues when $\alpha = 1$. Such situations generalize the celebrated Wigner-von Neumann example [20]. Wigner and von Neumann were first to construct an example which shows that already for potentials deacying as a Coulomb potential at infinity, i.e. $V(x) = O(\frac{1}{1+|x|})$, the spectrum need not to be purely absolutely continuous and positive embedded eigenvalues may occur. Naboko [19] and Simon [24] found different constructions which show that for potentials decaying arbitrarily slower than at a Coulomb rate, one can already have a rather striking spectral phenomena. The essential result is that for every function $C(x)$ going monotonically to infinity as x goes to infinity at an arbitrarily slow rate, and for every sequence of positive energies $\{\lambda_i\}_{i=1}^{\infty}$ one can find a potential $V(x)$ satisfying

$$|V(x)| \leq \frac{C(x)}{1+|x|},$$

such that the corresponding Schrödinger operator H_V has the sequence $\{\lambda_i\}_{i=1}^{\infty}$ among its eigenvalues. The earlier work of Naboko had also an additional assumption of rational independence of the sequence $\{\sqrt{\lambda_i}\}_{i=1}^{\infty}$, but Simon employs a different method which does away with this condition. In particular, these examples show that, in general, already for a potential decaying only arbitrary slower than a Coulomb potential, the corresponding Schrödinger operator can have a dense set of eigenvalues on the positive semi-axis. In such situation the Weyl criteria of the preservation of essential spectrum does not tell us anything about whether there is any other kind of the spectrum on R^+. ¿From the constructions of Naboko and Simon, it was also not clear whether the eigenvalues they construct are really embedded or, in fact, they are the only spectrum the operator H_V has on R^+, and hence one can have dense pure point spectrum on R^+ for the potentials which decay so fast.

Until recently, even for the basic case $U = 0$ the absolutely integrable class of potentials remained the only class defined purely in terms of the rate of decay, which was known to preserve the absolutely continuous spectrum of the free Hamiltonian. A new general class of potentials preserving the absolutely continuous spectrum of the free Schrödinger operator was found in [15]. Namely, if the potential V satisfies $|V(x)| \leq C(1+|x|)^{-\frac{3}{4}-\epsilon}$ with some $\epsilon > 0$, with no additional assumptions, then the absolutely continuous spectrum fills the whole R^+. Of course, as the examples of Naboko and Simon show, rich embedded singular spectrum may occur; however it is indeed embedded in the sense that there is an underlying absolutely continuous spectrum. One can decribe the set where the singular part of the spectral measure might be supported in R^+ rather explicitly in terms of the properties of the Fourier transform of V [15].

We should mention that there exists a work on random potentials by Kotani and Ushiroya [17] which provides in a sense a bound for the best possible result that one can hope to prove in the deterministic case. They consider the random potentials of type

$$V(x) = a(x)F(Y_x(\omega)),$$

where $Y_x(\omega)$ is a Brownian motion on a compact Riemannian manifold M, F is a C^∞ function which maps M to the real axis, satisfying certain additional assupmtions (see [17]), and $a(x)$ is a deterministic factor which is in particular taken to be power-decaying: $a(x) = (1+x)^{-\alpha}$. The main result of [17] shows a sharp transition in the spectral properties as α passes $\frac{1}{2}$: for $\alpha < \frac{1}{2}$, the spectrum on the positive half-axis is dense pure point with probability one; for $\alpha > \frac{1}{2}$ it is almost surely purely absolutely continuous. For $\alpha = \frac{1}{2}$ one may have a mixture of pure point and singular continuous spectrum for different regions of energies in R^+ with probability one. Hence, [17] implies that there exist potentials $V(x)$ satisfying $V(x) \leq C(1+x)^{-\frac{1}{2}}$, which lead to the purely singular spectrum on R^+. Therefore, the largest general class of power rate decaying potentials for which one can hope to prove the preservation of the absolutely continuous spectrum of the free Hamiltonian is the class of potentials satisfying $|V(x)| \leq C(1+x)^{-\frac{1}{2}-\epsilon}$ with some $\epsilon > 0$.

In this paper we study the problem of the preservation of the absolutely continuous spectrum of the Schrödinger operators in more general setting. We formulate two general criteria, which may be applied to any operator H_U with non-empty absolutely continuous spectrum, provided it satisfies a mild additional assumption. These criteria give conditions which are sufficient for the absolutely continuous spectrum to be stable under perturbations by all potentials satisfying $|V(x)| \leq C(1+x)^{-\frac{3}{4}-\epsilon}$ and $|V(x)| \leq C(1+x)^{-\frac{2}{3}-\epsilon}$ respectively.

Shortly, the first criterion works as follows. Suppose that a certain set S belongs to the essential support of the a.c. part of the spectral measure of the operator H_U. Suppose in addition that all solutions of the equation

$$-\frac{d^2y}{dx^2} + U(x)y = \lambda y. \tag{2}$$

for the energies from the set S are bounded. This is a rather natural assumption. In most situations of interest for almost every energy from the essential support of the absolutely continuous spectrum all solutions of (2) are bounded. For every energy $\lambda \in S$, choose two linearly independent uniformly bounded solutions $\theta(x,\lambda)$ and $\overline{\theta}(x,\lambda)$ of the equation (2). The key object for our study turns out to be the operator T, defined on the bounded functions of compact support in the following way:

$$(Tf)(\lambda) = \frac{\chi(S)}{\Im(\theta\overline{\theta}')} \int_0^\infty \theta^2(x,\lambda) f(x)\, dx. \tag{3}$$

Here $\chi(S)$ denotes the characteristic function of the set S. Note that the factor in front of the integral does not depend on x since the denominator is the Wronskian of the two linearly independent solutions of (2), θ and $\overline{\theta}$. Suppose that we can pick such $\theta(x,\lambda)$ so that the operator T satisfies an $L^2 - L^2$ bound, i.e. for all measurable bounded functions f of compact support we have

$$\|Tf\|_2 \leq C\|f\|_2,$$

where $\|f\|_2$ denotes the L^2 norm of the function f. Then the absolutely continuous spectrum of H_U, supported on the set S, is stable under all perturbations by potentials $V(x)$ satisfying $V(x) \leq C(1+x)^{-\frac{3}{4}-\epsilon}$. In other words, the set S belongs to the essential support of the absolutely continuous part of the spectral measure of the Schrödinger operator H_{U+V}. We remark that the verification of the $L_2 - L_2$ bound on the operator T is, in general, not trivial. Intutively, $\theta(x,\lambda)$ constitute a generalized continuous orhtogonal system, and hence the $L_2 - L_2$ bound would be natural if we had $\theta(x,\lambda)$, not $\theta(x,\lambda)^2$ in the definition of the operator T. In this respect, the problem of the perturbation of the free Schrödinger operator has "special status": indeed, we can take in this case $\theta(x,\lambda) = \exp(i\sqrt{\lambda}x)$. The free exponent $\exp(i\sqrt{\lambda}x)$ has a unique property that its square is also a free exponent and hence the $L_2 - L_2$ bound for the operator T comes for free in this case. But in general, the square in (3) makes the problem more delicate.

The main new technique we develop to prove the criteria inculdes a condition ensuring the a.e. convergence of the linear integral operators of rather general form. The result we prove seems quite natural. It generalizes some well-known results on the a.e. convergence of Fourier integrals.

Applying the new criteria, we show that every potential $V(x)$, verifying $|V(x)| \leq C(1+x)^{-\frac{2}{3}-\epsilon}$, preserves the absolutely continuous spectrum of the free Hamiltonian, relaxing the condition given in [15]. The new technique naturally has a wider range of applications than perturbations of the free Schrödinger operator. We also consider the perturbations of the periodic Schrödinger operators and find that the new criteria may be applied in this case.

We also show that all results we prove have natural analogs for the whole-axis problem.

The paper is organized as follows: in the first section we formulate the main results and recall the general framework of the approach: the consequences of the Gilbert-Pearson subordinacy theory and the Harris-Lutz mehtod. In the second section we develop the basic new technique: a theorem on the a.e. convergence of integral operators. In the third section, we complete the proofs of the abstract criteria for the preservation of the absolutely continuous spectrum and discuss applications.

1. Main results and basic technique.

Let us denote by $d\rho^{H_V}$ the spectral measure associated with the Schrödinger operator H_V in a usual way (see, e.g., [6]). We denote the components in a standart decomposition of this measure into absolutely continuous, singular continuous and pure point parts by $d\rho_{\mathrm{ac}}^{H_V}$, $d\rho_{\mathrm{sc}}^{H_V}$ and $d\rho_{\mathrm{pp}}^{H_V}$ respectively. The singular part of the spectral measure, which is a sum of two latter components, we denote by $d\rho_{\mathrm{s}}^{H_V}$. We use notation $m(E)$ and $\chi(E)$ for the Lebesgue measure and the characteristic function of the measurable set E. One of our goals in this paper is to prove the following theorem.

Theorem 1.1. *Suppose that the potential $V(x)$ satisfies $|V(x)| \leq C(1+x)^{-\frac{2}{3}-\epsilon}$, where ϵ is an arbitrary small positive number. Then the absolutely continuous part, $\rho_{\mathrm{ac}}^{H_V}$, of the spectral* *measure of the operator H_V, fills the whole positive semi-axis (i.e. we have $\rho_{\mathrm{ac}}^{H_V}(T) > 0$ for every measurable $T \subset R^+$ with $m(T) > 0$).*

The second theorem we prove here has to do with the slowly decreasing perturbations of the Schrödinger operators with periodic potentials. Suppose that $U(x)$ is a piecewise continuous periodic function on the real axis and H_U is defined, as before, on the half-axis. It is a well-known fact that the spectrum of H_U consists of the bands $\{[a_n, b_n]\}_{n=0}^{\infty}$, where the spectrum is simple and purely absolutely continuous and perhaps single eigenvalues in the gaps of the spectrum. This follows easily, for example, from the considerations in [23]. We have

Theorem 1.2. *Let $U(x)$ be piecewise continuous and periodic and let the set $S = \bigcup_{n=0}^{\infty}[a_n, b_n]$ be the absolutely continuous part of the spectrum of the operator H_U. Then the absolutely continuous spectrum o f H_U is stable under all perturbations $V(x)$ satisfying $|V(x)| \leq C(1+x)^{-\frac{2}{3}-\epsilon}$ (i.e. $\rho_{\mathrm{ac}}^{H_U+V}(T) > 0$ for every $T \subset S$ with $m(T) > 0$).*

Of course, it may be more natural to consider an analog of Theorem 1.2 for the whole line problem. This poses no difficulty, as well as an extension of Theorem 1.1 to this case. Let us denote the Schrödinger operator given by the differential expression $-\frac{d^2}{dx^2} + V(x)$ on the whole axis by $\tilde{H}_V$. Then the following statements hold true:

Theorem 1.3. *Suppose that the potential $V(x)$ satisfies $|V(x)| \leq C(1+x)^{-\frac{2}{3}-\epsilon}$. Then the absolutely continuous spectrum of the operator $\tilde{H}_V$ fills the whole positive semi-axis with multiplicity two.*

Theorem 1.4. *Let $U(x)$ be piecewise continuous and periodic and let $S = \bigcup_{n=0}^{\infty}[a_n, b_n]$ be the spectrum of the operator $\tilde{H}_U$. Then for every perturbation $V(x)$, satisfying $|V(x)| \leq C(1+x)^{-\frac{2}{3}-\epsilon}$, the absolutely continuous spectrum of the operator $\tilde{H}_{U+V}$ fills S with multi-*

plicity two.

These theorems will follow from the general criteria for the preservation of the absolutely continuous spectrum under slowly decreasing perturbations. We will give a precise formulation of these criteria in the end of this section when all relevant no tation will be introduced in the cousre of work and some of the technique will be developed which makes the results more transparent.

The first important ingredient of the approach to the general slowly decaying perturbations of the Schrödinger operators which we develop here is the relation between the behavoir of the generalized eigenfunctions and the spectral properties of the Schrödinger operators. Namely, we use the fact that for a large class of potentials to show that a certain set S belongs to the essential support of the absolutely continuous part of the spectrum of the operator H_W it suffices to show that for every energy from the set S, all solutions of the equation

$$\left(-\frac{d^2}{dx^2} + W(x)\right) u = \lambda u \tag{4}$$

are bounded. This result was first proven by Stolz [28] for the potentials $W(x)$ satisfying

$$W \in L^{1,\mathrm{loc}} \text{ and } \sup_{\mathrm{x}} |\int_{|\mathrm{x}-\mathrm{y}|\leq 1} \mathrm{W}_-(\mathrm{y})\,\mathrm{dy}| < \infty, \tag{5}$$

where W_- is the negative part of the potential W. The proof of Stolz relies on the Gilbert-Pearson subordinacy theory [9], a rather recent development in the spectral theory of one-dimensional Schrödinger operators, which is remarkable in a way that it provides a direct and efficient relation between the properties of the solutions of (4) and the spectrum. The results of [9] were elaborated and proofs simplified by Jitomirskaya and Last in [13]. We will need the following

Lemma 1.5. *Suppose that the potential W satisfies* (5) *and for every energy λ from a certain set S all solutions of the equation* (4) *are bounded as $x \to \infty$. Then the absolutely* *continuous spectrum of the operator H_W fills the whole set S, so that we have $\rho_{\mathrm{ac}}^{H_W}(T) > 0$ for every measurable $T \subset S$ with $m(T) > 0$. Moreover, no part of the singular measure is supported on S : $\rho_{\mathrm{s}}^{H_V}(S) = 0$.*

We remark that a simplest proof of this statement (with slightly more restrictive conditions on the potential) may be found in [25]. From the considerations in [25] also follows the analogous result for the whole-line problem:

Lemma 1.6. *Let the potential W satisfy* (5). *Suppose that for every energy from the sets $S_\pm$ all solutions of the generalized eigenfunction equation* (4) *are bounded as $x \to \pm\infty$ respectively. Then the absolutely continuous spectrum of the operator $\tilde{H}_W$ fills the set $S_+ \cup S_-$ with the multiplicity at least one and singular component of the spectral measure gives zero weight to the set $S_+ \cup S_-$: $\rho_s(S_+ \cup S_-) = 0$. Moreover, the absolutely continuous spectrum of the operator $\tilde{H}_W$ fills the set $S_+ \cap S_-$ with multiplicity two.*

Now let us now formulate an additional assumption on the absolutely continuous spectrum of the background operators H_U and $\tilde{H}_U$.

Assumption. *For the half-axis problem, we suppose that there exists a measurable set S of positive Lebesgue measure such that for every energy $\lambda \in S$ all solutions of the equation* (4) *are bounded.*

For the whole axis problem, we assume that there exist two sets $S_\pm$ such that all solutions of the equation (4) *are bounded as $x \to \pm\infty$ for $\lambda \in S_\pm$ respectively, and $m(S_+ \cup S_-) > 0$.*

Notice that Lemmas 1.5 and 1.6 imply that the sets S (respectively $S_\pm$) belong to the support of the absolutely continuous part of the spectral measure of the operator H_U (respectively

$\tilde{H}_U$). Moreover, the spectrum on these sets is purely absolutely continuous, which means that the singular parts of the corresponding spectral measures give no weight to these sets. Here we will study the stability under slowly decaying perturbations of the absolutely continuous spectrum of the operator H_U (or $\tilde{H}_U$) supported on S ($S_+ \cup S_-$) respectively. In other words, the methods we develop here are applicable only to the type of the absolutely continuous spectrum which corresponds to the situation when all solutions for the corresponding energies are bounded (at least on one of the half-axes in the whole-line problem case). This condition is not very restrictive. In most situations when one-dimensional Schrödinger operators are known to have absolutely continuous spectrum, it is exactly of the type described by the assumption. The question whether there exist at all Schrödinger operators with the absolutely continuous spectrum of the different type, i.e. such that for a set of the energies of positive measure from the esential support of $\rho_{\mathrm{ac}}^{H_U}$ there exist unbounded solutions of (4) was open for a long time. Although it was settled in a positive way in [18], the corresponding examples are rather special (for instance, the potential is unbounded both from above and from below).

Now suppose that H_U is a Schrödinger operator with part of the absolutely continuous spectrum supported on the set S as in assumption. Lemmas 1.5 and 1.6 reduce the problem of spectral analysis of the absolutely continuous spectrum of H_{U+V} to studing the asymptotics of the solutions of the equation

$$-\phi'' + (U+V)\phi = \lambda\phi. \tag{6}$$

Namely, if we could show that all solutions of this equation are still bounded for $\lambda \in S$, we could apply again Lemma 1.5 to infer that the absolutely continuous spectrum on S is preserved. It is easy to show that for the short range potentials this idea may be realized. However, as we mentioned in the introduction, already for potentials decaying at a Coulomb rate imbedded singular spectrum may appear, and it may be dense already for potentials decaying arbitrarily slower than Coulomb. Hence the boundedness of all solutions may not hold already in this case for rather rich set. The main idea is now to show that for *almost every* energy λ from the set S, all solutions of the equation (6) are bounded.

The second essential component of the approach is a certain asymptotic integration method. We use it in a form proposed originally by Harris and Lutz [10] for studing the asymptotics of the solutions of Schrödinger equation with some particular oscillating potentials. We rewrite the equation (6) as a system

$$y' = \begin{pmatrix} 0 & 1 \\ U+V-\lambda & 0 \end{pmatrix} y.$$

We apply a variation of the parameters transformation

$$y = \begin{pmatrix} \theta(x,\lambda) & \overline{\theta}(x,\lambda) \\ \theta'(x,\lambda) & \overline{\theta}'(x,\lambda) \end{pmatrix} z,$$

to bring the system to the more symmetric form

$$z' = \frac{1}{\Im(\theta\overline{\theta}')} \begin{pmatrix} V(x)|\theta(x,\lambda)|^2 & V(x)\overline{\theta}(x,\lambda)^2 \\ -V(x)\theta(x,\lambda)^2 & -V(x)|\theta(x,\lambda)|^2 \end{pmatrix} z.$$

Let us introduce a short-hand notation for the functions appearing in a latter system, namely, let us write the system as

$$z' = \begin{pmatrix} D & L \\ \overline{L} & \overline{D} \end{pmatrix} z, \tag{8}$$

where $D = \frac{1}{\Im(\theta\overline{\theta}')}V(x)|\theta(x,\lambda)|^2$ and $L(x) = \frac{1}{\Im(\theta\overline{\theta})}V(x)\overline{\theta}(x,\lambda)^2$. The main approach to the study of the asymptotics of solutions for systems similar to (7) is to attempt to find some transformation which will reduce the off-diagonal terms so that they will become absolutely integrable and then try to apply Levinson's theorem [6] on the L^1-perturbations of the systems of linear differential equations. Indeed, if we could assume that the off-diagonal terms of the system (7) are absolutely integrable, then in many cases the main term of the asymptotics of the solutions of the system (7) would, by Levinson's theorem (see, e.g.

[6]), coincide with the solution of the system (7) with only diagonal part present. These solutions are all bounded and going back to the original Schrödinger equation we also find the asymptotics of its solutions and see that they are also bounded.

It was discovered by Harris and Lutz [10] that when $W(x)$ is a conditionally integrable function, the following simple transformation of the system (8) works in some cases. We let

$$z(x) = (1 - |q|^2)^{-\frac{1}{2}}(I + \mathcal{Q})\omega(x) \tag{9}$$

where I is an identity matrix, while $\mathcal{Q}$ is given by

$$\mathcal{Q}(x) = \begin{pmatrix} 0 & q(x) \\ \overline{q}(x) & 0 \end{pmatrix}$$

where $q(x) = -\int_x^\infty L(y)\,dy$. In this case $q(x) \overset{x\to\infty}{\longrightarrow} 0$, so that for large enough x the transformation (9) is non-singular and preserves the asymptotics of the solutions. For the new variable $\omega(x)$ we obtain a system

$$\omega' = \left(\begin{pmatrix} D & 0 \\ 0 & \overline{D} \end{pmatrix} + (1 - |q|^2)^{-1}) \begin{pmatrix} i\Im(q\overline{L}) + 2|q|^2 D & 2\overline{qD} - \overline{q}^2 L \\ 2qD - q^2\overline{L} & -i\Im(q\overline{L}) + 2|q|^2\overline{D} \end{pmatrix} \right)\omega. \tag{10}$$

Because of the way the function q was defined, we see that the terms in the second summand, which we would like to treat as a perturbation, decay faster than the elements of the original system. In particular, for the system (10) that we consider, for the energies for which we manage to define the function $q(x, \lambda)$ as above, every entry of the second matrix is a product of the function

$$V(x)\int_x^\infty V(t)\theta(x,\lambda)^2\,dt = V(x)q(x,\lambda)$$

and some bounded function of x.

It was shown in [15] that in the case of the background potential U equal to zero if we take $\theta(x, \lambda) = \exp(i\sqrt{\lambda})x$ and the potential $V(x)$ satisfies $|V(x)| \le C(1 + x)^{-\frac{3}{4}-\epsilon}$, then for a.e. $\lambda \in (0, \infty)$ we have $q(x, \lambda)V(x) \in L^1$. Hence, for these λ we can apply the Levinson's theorem (or, in our particular case, more straightforward integral equation technique as in [22]), to find the asymptotics of the solutions and see that all solutions are bounded for a.e. positive λ. The problem of proving that $V(x)q(x,\lambda) \in L^1$ reduces to studing the function $q(x,\lambda)$, in particular the a.e. λ existence and rate of the convergence of the integral defining $q(x,\lambda)$. We remark that when $U=0$, the function $q(x,\lambda)$ is just the "tail" of the Fourier transform of the potential $V(x)$. Parseval formula may be used in this case not only to show that $V(x)q(x,\lambda) \in L^1$ for a.e. λ, but also to describe rather explicitely in terms of the Fourier transform of $x^{\frac{1}{4}}V(x)$ the exceptional divergence set where the singular part of the spectral measure might be supported.

If we study the structure of the perturbation in the system (10), we see that the slowest decaying off-diagonal terms are of type $2qD$ or $2\overline{q}\overline{D}$. If we had $D=0$, the slowest decaying terms would be $q^2\overline{L}$ and $\overline{q}^2 L$, which contain the decaying function q in the power two. Now let us perform one more transformation with the system (8):

$$z = \begin{pmatrix} \exp\left(-\frac{1}{\Im(\theta\overline{\theta}')}\int_0^x V(t)|\theta(t,\lambda)|^2\,dt\right) & 0 \\ 0 & \exp\left(\frac{1}{\Im(\theta\overline{\theta}')}\int_0^x V(t)|\theta(t,\lambda)|^2\,dt\right) \end{pmatrix} z_1.$$

Let us denote by $L_1(x,\lambda)$ the function

$$\frac{1}{\Im(\theta\overline{\theta}')}V(x)\overline{\theta}(x,\lambda)^2 \exp\left(-\frac{2}{\Im(\theta\overline{\theta}')}\int_0^x V(t)|\theta(t,\lambda)|^2\,dt\right). \tag{11}$$

For the variable z_1 we have

$$z_1' = \begin{pmatrix} 0 & L_1(x,\lambda) \\ \overline{L}_1(x,\lambda) & 0 \end{pmatrix} z_1.$$

Define

$$q_1(x,\lambda) = -\int\limits_x^\infty L_1(s,\lambda)\,ds. \tag{12}$$

(for those λ for which such integral will exist) and perform an $I+Q_1$ transformation

$$z_1 = (1-|q_1|^2)^{-\frac{1}{2}}\begin{pmatrix} 1 & q_1 \\ \overline{q}_1 & 1 \end{pmatrix}\omega_1.$$

We get for the new variable ω_1

$$\omega_1' = (1-|q_1|^2)^{-1}\left(\begin{pmatrix} \frac{1}{2}(q_1\overline{L}_1 - \overline{q}_1 L_1) & 0 \\ 0 & -\frac{1}{2}(q_1\overline{L}_1 - \overline{q}_1 L_1) \end{pmatrix} + \begin{pmatrix} 0 & \overline{q}_1^2 L_1 \\ q_1^2\overline{L}_1 & 0 \end{pmatrix}\right)\omega_1. \quad (13)$$

Here the first diagonal term has purely imaginary entries and hence leads to bounded solutions, while the second term decays as $|q_1(x,\lambda)|^2 V(x)$. This computation suggests that we should study the question of existence and convergence of the integrals like

$$q(x,\lambda) = \int_x^\infty V(t)\overline{\theta}(t,\lambda)^2\, dt$$

and

$$q_1(x,\lambda) = \int_x^\infty V(t)\overline{\theta}(t,\lambda)^2 \exp\left(-\frac{2}{\Im(\theta\overline{\theta}')}\int_0^x V(t)|\theta(t,\lambda)|^2\, dt\right).$$

In particular if we could get the a.e. convergence estimates for latter integrals similar to the case of Fourier integral, then we could study the asymptotics of solutions of Schrödinger operators with potentials decaying slower than at $x^{-\frac{3}{4}-\epsilon}$ rate.

The next section is devoted to the handling of the a.e. convergence questions and reduction to a simpler issue of norm estimates.

We conclude this section with the formulation of the two general criteria for the preservation of the absolutely continuous spectrum, both for half-line and full line problems.

Let H_U, $\theta(x,\lambda)$ and S be as above. It will be convenient for us to choose a certain basis of solutions for every $\lambda \in S$. Namely, for any $\lambda \in S$ we choose two linearly independent, measurable in (x,λ) solutions $\theta(x,\lambda)$ and its complex conjugate $\overline{\theta}(x,\lambda)$ such that:

$$|\theta(x,\lambda)| \le C \text{ for every } \lambda \text{ uniformly in } \lambda \in \mathrm{S}. \quad (14)$$

Of course we can always choose such basis $\theta(x,\lambda)$, $\overline{\theta}(x,\lambda)$ if all solutions of (4) are bounded for every $\lambda \in S$.

For the whole-axis problem, for every $\lambda \in S_\pm$ we choose a basis consisting of solutions $\theta_\pm(x,\lambda)$, $\overline{\theta_\pm}(x,\lambda)$ satisfying (14) when $x \in (0,\pm\infty)$ respectively.

For example, treating the free case we will have $S = (0, \infty)$ and $\theta(x, \lambda) = \exp(i\sqrt{\lambda}x)$ and considering the periodic case we will have $S = \cup_{n=1}^{\infty}(a_n, b_n)$ and $\theta(x, \lambda)$ will be chosen to be Bloch functions.

We now perturb the operator H_U by a decaying potential $V(x)$. Define the linear operators T_1 and T_2 acting on the bounded functions of compact support by

$$(T_1 f)(\lambda) = \frac{\chi(S)}{\Im(\theta\overline{\theta}')} \int_0^\infty \overline{\theta}(x, \lambda)^2 f(x)\, dx \tag{15}$$

and

$$(T_2 f)(\lambda) = \frac{\chi(S)}{\Im(\theta\overline{\theta}')} \int_0^\infty \overline{\theta}(x, \lambda)^2 \exp\left(-\frac{2}{\Im(\theta\overline{\theta}')} \int_0^x V(t)|\theta(t, \lambda)|^2\, dt\right) f(x)\, dx. \tag{16}$$

We have

Theorem 1.7. *Suppose that there exists the partition of the set S, $S = \cup_{i=1}^{\infty} S_i$, such that for every i we have the bounds*

$$\|\chi(S_i) T_1 f\|_2 \le C_{1i} \|f\|_2.$$

Then the absolutely continuous spectrum of the operator H_U, supported on S, is stable under perturbations by all potentials $V(x)$ satisfying $|V(x)| \le C(1+x)^{-\frac{3}{4}-\epsilon}$ (i.e. $\rho_{\text{ac}}^{H_{U+V}}(S_1) > 0$ for every $S_1 \subset S$ with $m(S_1) > 0$). Moreover, for a.e. $\lambda \in S$ we have two linearly independent solutions $\phi(x, \lambda)$, $\overline{\phi}(x, \lambda)$ of the equation $(H_{U+V} - \lambda)\phi = 0$ with the asymptotics

$$\phi(x, \lambda) = \theta(x, \lambda) \exp\left(-\frac{2}{\Im(\theta\overline{\theta}')} \int_0^x V(t)|\theta(t, \lambda)|^2\, dt\right) \left(1 + O(x^{-\epsilon})\right). \tag{17}$$

Theorem 1.8. *Assume that a potential $V(x)$, verifies $|V(x)| \le C(1+x)^{-\frac{2}{3}-\epsilon}$. Suppose that there exists a partition of the set S into the sets $\{S_i\}_{i=1}^{\infty}$ such that for every i we have the bounds*

$$\|\chi(S_i) T_2 f\|_2 \le C_{2i} \|f\|_2.$$

Then the set S remains in the support of the absolutely continuous spectrum of the operator H_{U+V} (i.e. $\rho_{\text{ac}}^{H_{U+V}}(S_1) > 0$ for every measurable $S_1 \subset S$ with $m(S_1) > 0$). Moreover, *for a.e.* $\lambda \in S$ *we have two linearly independent solutions* $\phi(x,\lambda)$, $\overline{\phi}(x,\lambda)$ *of the equation* $(H_{U+V} - \lambda)\phi = 0$ *with the asymptotics*

$$\begin{aligned}\phi(x,\lambda) &= \theta(x,\lambda)\exp\left(-\frac{2}{\Im(\theta\overline{\theta}')}\int_0^x V(t)|\theta(t,\lambda)|^2\,dt + i\int_0^x \Im(q_1(t,\lambda)\overline{L}_1(t,\lambda))\,dt\right)\times \\ &\quad\times\left(1+O(x^{-\epsilon})\right),\end{aligned} \tag{18}$$

where L_1 *and* q_1 *are as in* (11) *and* (12).

The whole-axis analogs of Theorems 1.7 and 1.8 are formulated as follows. Let, as before, $\tilde{H}_U$ be a Schrödinger operator with potential U defined on the whole axis. Let S_-, S_+, $\theta_-(x,\lambda)$, and $\theta_+(x,\lambda)$ be as above. Let the operators $T_1^{\pm}$, $T_2^{\pm}$ be defined on bounded functions of compact support by

$$(T_1^- f)(\lambda) = \frac{\chi(S_-)}{W[\theta_-,\overline{\theta}_-]}\int_{-\infty}^0 \overline{\theta}_-(x,\lambda)^2 f(x)\,dx$$

and

$$(T_2^- f)(\lambda) = \frac{\chi(S_-)}{\Im(\theta_-\overline{\theta}_-')}\int_{-\infty}^0 \overline{\theta}_-(x,\lambda)^2\exp\left(-\frac{2}{\Im(\theta_-\overline{\theta}_-')}\int_0^x V(t)|\theta_-(t,\lambda)|^2\,dt\right) f(x)\,dx.$$

In the definitions of $T_{1,2}^+$ we just replace all signs "$-$" by the "$+$" signs in the right-hand side.

Then we have the following criteria:

Theorem 1.9. *Suppose that there exist partitions of the sets* $S_\pm$ *into countable unioins of sets* $\{S_{\pm i}\}_{i=1}^\infty$ *respectively such that the following bounds hold for every* i : $\|\chi(S_{+i})T_1^+ f\|_2 \le C_{+i}\|f\|_2$ *and* $\|\chi(S_{-i})T_1^- f\| \le C_{-i}\|f\|_2$ *for every* i. *Then the absolutely continuous spectrum of the operator* H_U, *supported on* S_- *and* S_+, *is stable under perturbations by all potentials* $V(x)$ *verifiyng* $|V(x)| \le C(1+|x|)^{-\frac{3}{4}-\epsilon}$. *Namely, for every such* V *the operator* $\tilde{H}_{U+V}$ *has absolutely continuous spectrum of multiplicity at least one on* $S_- \cup S_+$ *and of multiplicity two* *on $S_+ \cap S_-$.*

Theorem 1.10. *Let potential $V(x)$ satisfy $|V(x)| \le C(1+|x|)^{-\frac{2}{3}-\epsilon}$. Suppose that there exist the partitions of the sets $S_\pm$ into the countable unions of the sets $\{S_{\pm i}\}_{i=1}^\infty$ respectively so that the following bounds hold for every i : $\|\chi(S_{+i})T_2^{+}f\|_2 \le C_{+i}\|f\|_2$ and $\|\chi(S_{-i})T_2^{-i}f\| \le C_{-i}\|f\|_2$ for every i. Then the absolutely continuous spectrum of the operator $\tilde{H}_U$, supported on S_- and S_+, is preserved under pertubation by the potential $V(x)$. Namely, the operator $\tilde{H}_{U+V}$ has absolutely continuous spectrum of multiplicity at least one on $S_+ \cup S_-$ and of multiplicity two on $S_+ \cap S_-$.*

All these criteria show that instead of studying the rate of a.e. convergence of certian integral operators all we have to do is to check a certain $L_2 - L_2$ estimate, which is in many situations much simpler. The passage from a.e. convergence questions to norm estimates is the third ingredient of our approach. This is a crucial element and the main new idea in the context of the spectral study of Schrödinger operators. We treat this subject in the next section.

2. A.e. convergence for integral operators.

Let the operator T be defined on the measurable bounded functions f of compact support by

$$(Tf)(k) = \int\limits_0^\infty A(k,x)f(x)\,dx, \tag{19}$$

where $A(k,x)$ is a measurable and bounded function on R^2. Let us denote by A the upper bound on the kernel $A(k,x)$. Denote by $Mf(k)$ the corresponding maximal function

$$Mf(k) = \sup_N \left| \int\limits_0^N A(k,x)f(x)\,dx \right|. \tag{20}$$

We are interested in studying the a.e. convergence questions for the integral operators of type (19), i.e. in finding more or less simple conditions implying that the integral (19) converges for a.e. k if, say, f belongs to some L^p space. The natural class of conditions to look at are norm estimates on the operator T. When we deal with Fourier integrals, in many instances a Parseval equality with explicit kernel helps us to get some information about a.e. convergence (see, e.g., [29]). Of course, there is rarely such thing as the Parseval equality for the integrals of type (19). However, a weaker tool - norm estimates - turns out to be sufficient for our needs. We use a traditional method of maximal function estimates to infer the a.e. convergence. The principal result of this section is the derivation of the estimates on the maximal function (20) given certian norm estimates on the operator itself. This reduces the proof of the a.e. convergence of the operator T given by (19) for the functions from the certain class to establishing the appropriate norm estimates for T.

To formulate the main result of this section in the natural form, it is useful to introduce the scale of Lorentz spaces L_{pq}. We remind here the basic definitions and properties of these spaces. For more details and proofs we refer to [26]. The function f belongs to L_{pq} iff

$$\|f\|^*_{pq} = \left(\frac{q}{p} \int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty,$$

where $f^*(t)$ is a non-decreasing rearrangement of the function f, defined by

$$f^*(t) = \inf\{s : m\{x|\,|f(x)| > s\} \leq t\}.$$

The expression $\|f\|^*_{pq}$ does not generally define a norm since it does not generally satisfy the triangle inequality. However, if $1 \leq p \leq \infty$ and $1 < q \leq \infty$, there exists a norm $\|\ \|_{pq}$ on L_{pq} which is equivalent to $\|\ \|^*_{pq}$. In particular, $\|f\|^*_{pq} \leq \|f\|_{pq} \leq \frac{p}{p-1}\|f\|^*_{pq}$, see [26]. The L_{pq} spaces are of "L_p- type", i.e. for every characteristic function of the measurable set E, $\chi(E)$, we have $\|\chi(E)\|_{pq} = (m(E))^{\frac{1}{p}}$. Also $\|f\|^*_{pq_1} \leq \|f\|^*_{pq_2}$ if $q_1 > q_2$, so that for fixed p, the Lorentz space extends as q grows. Finally, we note that the L_{pp}-space coincides with the usual L_p since $\|f\|_{pp} = \|f\|_p$.

We will also need the following well-known interpolation theorem, so-called generalized Marcinkiewicz theorem. We refer to [26] or [5] for a proof. The operator B is called subadditive if it satisfies $|B(f_1+f_2)(k)| \leq |Bf_1(k)|+|Bf_2(k)|$. We say that B is of restricted weak type (r,p) if its domain $D(B)$ contains all finite linear combinations of characteristic functions of sets of finite measure and all truncations of its members and satisfies $\|Tf\|_{p\infty} \leq C\|f\|_{r1}$ for all $f \in D \cap L_{r1}$.

Theorem. *Suppose that T is a subadditive operator of restricted weak types (r_j, p_j), $j = 1, 2$ with $r_0 < r_1$ and $p_0 \neq p_1$, then there exists constant B_θ such that*

$$\|Tf\|_{pq} \leq B_\theta \|f\|_{rq}$$

for all f from the domain of T and L_{rq}, where $1 \leq q \leq \infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, $0 < \theta < 1$.

Now we are in a position to formulate the main result that we need. The theorem below generalizes the classical theorem of Zygmund [32] for the case of Fourier transform.

Theorem 2.1. *Suppose that an operator T, defined by (19) with bounded kernel $A(k,x)$, satisfies the norm estimate $\|Tf\|_2 \leq C_q \|f\|_2$ for all bounded functions of compact support. Then for every $q > 2$ and p such that $q^{-1} + p^{-1} = 1$, we have the following estimate for the maximal function:*

$$\|Mf\|_q \leq C_q \|f\|_{pq} \;\; for \;\; every \;\; f \in L_{pq} \tag{21}$$

(hence, in particular, $\|Mf\|_q \leq C_q \|f\|_p$). As a consequence, the integral

$$\int_0^N A(k,x) f(x)\, dx$$

converges as $N \to \infty$ for almost every value of k if $f \in L_{pq}$ (and in particular if $f \in L_p$, $1 \leq p < 2$).

Remark. The result we prove here is suited for the applications we make in this paper. In fact, similar results hold in greater generality for more general integral operators with natural defintions of the maximal function. The proof is more involved and we plan to devote a separate publication [16] to this problem.

Proof. First we remark that from the estimate on the maximal function the a.e. convergence for operator T in classes L_{pq}, $p < 2$, follows in a standart way (see, e.g., [8]). We sketch here this simple argument for the sake of completness. Indeed, suppose that there exists a function $f \in L_{pq}$ with p and q as in the theorem and a set S of positive measure where the integral defining $Tf(k)$ diverges. Then we can find $\epsilon > 0$ and a set $S_\epsilon \subset S$, such that $m(S_\epsilon) > 0$, and for every $k \in S_\epsilon$ and every positive number N_1 there exists a larger number N_2 such that $|\int_{N_1}^{N_2} A(k,x)f(x)\,dx| > \epsilon$. Hence, for every N_1, we have $\|Mf\chi(N_1,\infty)\|_q \geq \epsilon m(S_\epsilon)^{\frac{1}{q}}$. On the other hand, clearly $\|f\chi(N_1,\infty)\|_{pq} \stackrel{N_1\to\infty}{\longrightarrow} 0$. This contradicts the bound (21).

We now come to a proof of the first assertion. The first step is the decompostion of the support of the function f into dyadic pieces and estimates on the certain auxilliary maximal functions. A similar idea was used already by Paley [21] in hi s proof of a.e. convergence for the series of orhtogonal functions. Let f be a measurable bounded function of compact support and choose n so that $2^{n-1} \leq m(\mathrm{supp(f)}) \leq 2^{\mathrm{n}}$. Let the measurable set E be the support of the function f : $E = \{x| \ |f(x)| > 0\}$. For every integer $m < n$, we consider a partition of the set E into the following sets $E_{m,l}$:

$$E_{m,l} = (a_{m,l}, a_{m,l+1}) \cap E,$$

where $a_{m,l}$ is defined by a condition

$$a_{m,l} = \inf\{a|m((0,a)\cap E)\} = 2^m l.$$

The number of the sets $E_{m,l}$ is between 2^{n-m} and 2^{n-m-1}. For notational convenience, we will assume that this number is always 2^{n-m} and will define the missing $E_{m,l}$ as empty sets.

Let us define functions $M_{m,l}f$ and $M_m f$ by

$$M_m f(k) = \sup_l |M_{m,l} f(k)|$$

and

$$M_{m,l} f(k) = \left| \int\limits_{E_{m,l}} A(k,x) f(x)\, dx \right|.$$

Considering a dyadic development of every real number N, it is easy to see that

$$Mf(k) \leq \sum_{m=-\infty}^{n} M_m f(k). \tag{22}$$

Indeed, suppose that for a given value of k, the supremum in (20) is reached when the upper limit is N (clearly if f has compact support, the supremum is reached for some value of N). Define the real number s by $s = m(E \cap (0,N))$ and consider the dyadic development $s = \sum_{m=-\infty}^{n} s_m 2^m$, where s_m is equal to 0 or 1 for every m. Then by construction, we can find disjoint sets $E_{m,l}$, at most one for each value of m, so that $m((E \cap (0,N))/(\cup_m E_{m,l})) = 0$. In fact, for each m the set $E_{m,l}$ belongs to the union iff $s_m = 1$. The corresponding value of l then may be found by the formula $l = \sum_{j=m+1}^{n} s_j 2^{j-m}$.

Obviously, we also have

$$M_m^q f(k) \leq \sum_{l=1}^{2^{n-m}} M_{m,l}^q f(k), \tag{23}$$

for every $q > 0$. Fix now any $q > 2$ and let q' satisfy $q > q' > 2$. Under the conditions of the theorem, we have that $\|Tf\|_\infty \leq C_1 \|f\|_1$ for all $f \in L_1$ and $\|Tf\|_2 \leq C_2 \|f\|$ is satisfied for all measurable bounded functions of compact support. By interpolation, we have

$$\|Tf\|_{q'} \leq C_{q'} \|f\|_{p'q'}, \;\; (p')^{-1} + (q')^{-1} = 1.$$

Noting that

$$\|f\|_{p'q'}^* = \left(\frac{q'}{p'} \int\limits_0^\infty |f^*(t)|^{q'} t^{\frac{q'}{p'}-1}\, dt \right)^{\frac{1}{q'}},$$

and using the equivalence of $\|\cdot\|^*$ and $\|\cdot\|$, we see that in particular

$$\|M_{m,l} f(k)\|_{q'}^{q'} = \|T(f\chi(E_{m,l}))(k)\|_{q'}^{q'} \leq C_{q'}^{q'} \left(\frac{q'}{p'} \int\limits_0^\infty |(f\chi(E_{m,l}))^*|^{q'} t^{\frac{q'}{p'}-1}\, dt \right) \leq$$

$$\leq C_{q'}^{q'} 2^{m(\frac{q'}{p'}-1)} \frac{q'}{p'} \|f\chi(E_{m,l})(t)\|_{q'}^{q'}.$$

Summing over l and using (23), we obtain

$$\|M_m f\|_{q'}^{q'} \leq C_{q'}^{q'} \frac{q'}{p'} 2^{m(\frac{q'}{p'}-1)} \|f\|_{q'}^{q'}.$$

By (22), we have that

$$\|Mf\|_{q'} \leq C_{q'} \left(\frac{q'}{p'}\right)^{\frac{1}{q'}} \|f\|_{q'} \sum_{m=-\infty}^{n} 2^{m(\frac{1}{p'}-\frac{1}{q'})} = B_{q'} 2^{n(\frac{1}{p'}-\frac{1}{q'})} \|f\|_{q'}. \tag{24}$$

Now we note that in a particular case when f is a characteristic function of a set, $f = \chi(E)$, (24) means

$$\|M\chi(E)\|_{q'} \leq B_{q'} 2^{n(\frac{1}{p'}-\frac{1}{q'})} 2^{\frac{n}{q'}} = B_{q'} 2^{\frac{n}{p'}} \leq 2^{\frac{1}{p'}} B_{q'} \|\chi(E)\|_{p'1}. \tag{25}$$

It is easy to check that the operator Mf, defined originally on the measurable bounded functions of compact support, is a sublinear operator. It is well-known fact that from the inequality (25) for sublinear operator it follows that $\|Mf\|_{q'} \leq C\|\chi(E)\|_{p',1}^{*}$ holds for all finite combinations of simple functions (see [26]) and hence by simple limiting argument for all measurable bounded functions of compact support. Interpolating with an obvious relation $\|Mf\|_{\infty} \leq A\|f\|_1$, we obtain that $\|Mf\|_{\tilde{q}} \leq C_{\tilde{q}}\|f\|_{\tilde{p}\tilde{q}}$ for every $\tilde{p}$, $\tilde{q}$ such that $q' > \tilde{q} > 2$ and $\tilde{p}^{-1} + \tilde{q}^{-1} = 1$ and for every function f bounded and of compact support. In particular, this relation holds for the value of q we fixed in the beginning of the proof (and hence for every $q > 2$). It is straightforward to see that this inequality is then extended to all functions $f \in L_{pq}$. $\square$

3. Proofs of criteria and applications.

Now we prove all theorems formulated in the first section. First we give proofs of the general criteria.

Proof of Theorem 1.7. By assumption, the operator T_{1i} defined by

$$(T_{1i}f)(\lambda) = \frac{\chi(S_i)}{\Im(\theta\overline{\theta}')} \int_0^\infty \overline{\theta}(x,\lambda)^2 f(x)\, dx,$$

satisfies an $L_2 - L_2$ bound on the bounded functions of compact support. We may also assume that on every set S_i we have the Wronskian $W[\theta, \overline{\theta}]$ bounded away from zero by some constant c_i, or else subdivide the S_i so that it holds true. Since the functions $\theta(x,\lambda)$ are uniforml y bounded when $\lambda \in S$, we also have an obvious $L_1 - L_\infty$ bound for T_{1i}. Hence, by Theorem 2.1, the integral

$$\int_0^x \overline{\theta}(y,\lambda)^2 f(y)\, dy \tag{26}$$

converges as $x \to \infty$ for a.e. $\lambda \in S_i$ for every $f \in L_p$, $1 \le p < 2$. By the assumption on the decay of $V(x)$, the function $x^{\frac{1}{4}}V(x)$ belongs to $L_{2(1-\epsilon)}$. Therefore, we find that

$$q(x,\lambda) = \int_x^\infty \overline{\theta}(y,\lambda)^2 V(y)\, dy = \int_x^\infty \overline{\theta}(y,\lambda)^2 (V(y)y^{\frac{1}{4}})y^{-\frac{1}{4}}\, dy =$$

$$= x^{-\frac{1}{4}} \int_0^x \overline{\theta}(y,\lambda)^2 (y^{\frac{1}{4}}V(y))\, dy + \frac{1}{4}\int_x^\infty y^{-\frac{5}{4}} \int_0^y \overline{\theta}(t,\lambda)^2 (t^{\frac{1}{4}}V(t))\, dt.$$

We conclude that for all $\lambda \in S_i$ such that the integral (26) converges, and hence for a.e. $\lambda \in S_i$, the function q satisfies $q(x,\lambda) = O(x^{-\frac{1}{4}})$ as $x \to \infty$. Since this holds for any i, we have that this estimate is also true for a.e. $\lambda \in S$. This implies that $q(x,\lambda)V(x) \in L_1$ and allows for a.e. $\lambda \in S$ to find the asymptotics of solutions of the perturbed Schrödinger equation. Transforming back via (7) and (9) we find that for a.e. $\lambda \in S$, there exist two solutions $\phi(x,\lambda)$, $\overline{\phi}(x,\lambda)$ of the generalized eigenfunction equation (4) with the following asymptotics:

$$\phi(x,\lambda) = \theta(x,\lambda) \exp\left(-\frac{2}{\Im(\theta\overline{\theta}')} \int_0^x V(t)|\theta(t,\lambda)|^2\, dt \right) \left(1 + O(x^{-\epsilon})\right),$$

$$\phi'(x,\lambda) = \theta'(x,\lambda) \exp\left(-\frac{2}{\Im(\theta\overline{\theta}')} \int_0^x V(t)|\theta(t,\lambda)|^2\, dt \right) \left(1 + O(x^{-\epsilon})\right).$$

Clearly the solutions ϕ, $\overline{\phi}$ are linearly independent, since the Wronskian $W[\phi, \overline{\phi}] = W[\theta, \overline{\theta}] \neq 0$. This concludes the proof, given Lemma 1.5. □

Proof of Theorem 1.8. Similarly to the previous proof, we infer that under the assumption of the theorem, for every $f \in L_p$, $1 \leq p < 2$, the integral

$$\int_0^x \overline{\theta}(y,\lambda)^2 \exp\left(-\frac{1}{\Im(\theta\overline{\theta}')}\int_0^y V(t)|\theta(t,\lambda)|^2\,dt\right) f(y)\,dy$$

converges for a.e. $\lambda \in S$. As before, integrating by parts, we find that if $|V(x)| \leq C(1+x)^{-\frac{2}{3}-\epsilon}$, then the function $q_1(x,\lambda)$ given by (12) satisfies

$$q_1(x,\lambda) = O(x^{-\frac{1}{6}})$$

for a.e. $\lambda \in S$. Therefore, for a.e. $\lambda \in S$ we also have $|q_1(x,\lambda)^2 V(x)| \leq C(1+x)^{-1-\epsilon}$. This allows us to find the asymptotics of the solutions of the system (13) and then of the original Schrödinger equation. As in the previous proof, the Wronskian argument shows linear independence of the solution with asymptotics (18) and its complex conjugate. □

The proofs of the whole line analogs of the criteria, Theorems 1.9 and 1.10, follow in the same way given Lemma 1.7.

Now we come to a final goal of this paper - concrete applications to the preservation of the absolutely continuous spectrum of the Schrödinger operators. We first discuss the free case: $U(x) = 0$. We remark that the criterion given by Theorem 1.7 applies trivially since the operator T_1 in question is just a rescaled Fourier transform and hence satisfies the $L_2 - L_2$ estimate. This gives the stability of the absolutely continuous spectrum of the free Schrödinger operators under perturbations by all potentials V satisfying $V(x) \leq C(1+x)^{-\frac{3}{4}-\epsilon}$. This has been proven in [15] using a more direct method rather than Theorem 1.7, which is possible becuase the integral operator is just a Fourier transform in this case.

To prove Theorem 1.1, we would like to apply the criterion of Theorem 1.8. This leads to the consideration of the operator T_2 given by

$$(T_2 f)(\lambda) = \frac{i\chi(a,b)(\lambda)}{2\sqrt{\lambda}} \int\limits_0^\infty \exp\left(-2i\sqrt{\lambda}x + \frac{i}{\sqrt{\lambda}} \int\limits_0^x V(t)\,dt\right) f(x)\,dx, \tag{27}$$

where $(a,b) \subset (0,\infty)$ and $a > 0$, $b < \infty$. We seek to show that the operator T_2 satisfies the $L_2 - L_2$ bound for every choice of a, b (although the value of the constant in the estimate may of course depend on this choice). Theorem 1.1 will then follow immediately from Theorem 1.8 given that (a,b) is an arbitrary proper subinterval of $(0,\infty)$.

For the proof of the $L_2 - L_2$ estimate it suffices to assume that the potential V satisfies $|V(x)| \leq C(1+x)^{-\frac{1}{2}-\epsilon}$, with some $\epsilon > 0$. Let $2\sqrt{\lambda} = k$. It is clear that it is sufficient to show the $L_2 - L_2$ for the operator T_2' given by

$$(T_2' f)(k) = \chi(a,b) \int\limits_0^\infty \exp\left(-ikx + \frac{2i}{k} \int\limits_0^x V(t)\,dt\right) f(x)\,dx, \tag{28}$$

The operator T_2' looks like a pseudodifferential operator (restricted to the interval (a,b)) with a symbol

$$a(k,x) = \exp\left(\frac{i}{k} \int\limits_0^x V(t)\,dt\right).$$

We remind that a symbol $a(k,x)$ belongs to an exotic class $S_{\rho,\sigma}$ if $a(k,x)$ is an infinitely differentiable function satisfying

$$|\partial_k^n \partial_x^m a(k,x)| \leq C_{mn}(1+|x|)^{\sigma n - \rho m} \tag{29}$$

for every m, n. For the symbol classes $S_{\rho,\sigma}$, $1 > \rho \geq \sigma \geq 0$, the $L_2 - L_2$ estimate is well-known (see, e.g., [27]). However, for our purpose, although we may without loss of generality assume that $V \in C^\infty$ (absorbing all lack of smoothness into short range correction which is easy to treat), there is no hope in general that an estimate like (29) holds for all integer m, n. Already taking the second derivative in x, we should derivate V, while under our assumptions we have absolutely no control over its derivative. However, Coifman and Meyer [7] have studied the question what is the minimal number of derivative estimates in (29) one has to ask for in order to have an $L_2 - L_2$ bound. In particular, from their results it follows (Theorem 7 on page 30) that it suffices to check (29) for m, $n = 0, 1$ for some $1 > \rho \geq \sigma \geq 0$ in order to ensure an $L_2 - L_2$ bound on T. It is straightforward to check that for our symbol we have these estimates for every V satisfying $|V(x)| \leq C(1+x)^{-\frac{1}{2}-\epsilon}$ (and hence in particular for every V satisfying $|V(x)| \leq C(1+x)^{-\frac{2}{3}-\epsilon}$) :

$$|\partial_x a(k,x)| \leq C(1+x)^{-\frac{1}{2}-\epsilon};$$

$$|\partial_k a(k,x)| \leq \frac{2C}{(1-2\epsilon)a}(1+x)^{\frac{1}{2}-\epsilon};$$

$$|\partial_x \partial_k a(k,x)| \leq C_{11}(1+x)^{-2\epsilon}.$$

In particular, $a(k,x) \in$ "$S_{\frac{1}{2},\frac{1}{2}}$" (and even "$S_{\frac{1}{2}+\epsilon,\frac{1}{2}-\epsilon}$") where quotations mean the reduced number of conditions on the derivatives, i.e. m and n are not greater than 1 in (29). Hence we have a theorem:

Theorem 3.1. *The operator T_2, given by* (27), *satisfies the $L_2 - L_2$ bound $\|T_2 f\|_2 \leq C_2 \|f\|_2$ if $V(x)$ verifies $|V(x)| \leq C(1+x)^{-\frac{1}{2}-\epsilon}$. Also, the bound $\|T_2 f\|_\infty \leq C_1 \|f\|_1$ holds trivially.*

This theorem together with criterion given by Theorem 1.8 implies Theorem 1.1.

We also sketch an alternative proof of Theorem 3.1 which uses only an $L_2 - L_2$ bound for the usual "exotic" symbol class $S_{\frac{1}{2},\frac{1}{2}}$ with (29) true for any number of derivatives. For this we need the following lemma:

Lemma 3.2. *Let $V(x)$ satisfy $|V(x)| \leq C(1+x)^{-\frac{1}{2}-\epsilon}$; then we can represent a function $V(x)$ as a sum $V(x) = V_1(x) + V_2(x)$, where $V_1(x)$ satisfies*

$$|V_1^{(m)}(x)| \leq C_m (1+x)^{-\frac{1}{2}(m+1)-\epsilon}$$

for every integer $m \geq 0$, and $V_2(x)$ is conditionally integrable: $\int_0^x V(t)\,dt$ converges as

$x \to \infty$.

Proof. Define an increasing sequence $\{a_n\}_{n=1}^{\infty}$ by the conditions $a_0 = 1$, $a_n - a_{n-1} = a_{n-1}^{\frac{1}{2}}$. Let ξ be a C^∞ function such that ξ vanishes on the interval $(-\delta, \delta)$, δ small positive number, and $\xi = 1$ outside $(-2\delta, 2\delta)$. Let us define $V_1(x)$ by

$$V_1(x) = \sum_{n=1}^{\infty} C_n \chi(a_n, a_{n+1}) \xi \left(\frac{x - a_n}{a_n^{\frac{1}{2}}} \right) \xi \left(\frac{x - a_{n+1}}{a_{n+1}^{\frac{1}{2}}} \right).$$

We choose each C_n by a condition that $\int_{a_n}^{a_{n+1}} (V - V_1)(t)\, dt = 0$ for every n. It is easy to check that $V_1 \in C^\infty$ and for $x \in (a_n, a_{n+1})$ we have

$$|V_1^{(m)}(x)| \le \sup_{x \in (a_n, a_{n+1})} |V(x)| C_\xi a_n^{-\frac{1}{2}m},$$

where C_ξ depends only on L_∞ norms of the derivatives of ξ up to the m-th order. It is easy to see that $\frac{a_{n+1}}{a_n} \overset{n \to \infty}{\longrightarrow} 0$ and hence we obtain $|V_1^{(m)}(x)| \le C_m (1 + x)^{-\frac{1}{2}(m+1) - \epsilon}$. On the other hand, $\int_0^{a_n} V_2(t)\, dt = 0$ for every n and therefore it is easy to see that V_2 is conditionally integrable and in fact $|\int_x^\infty V_2(t)\, dt| \le C x^{-\epsilon}$. □

Now let us write the symbol $a(k, x)$ as follows:

$$a(k, x) = i\chi(a, b)(k) \exp \left(\frac{i}{2k} \int_0^x V_1(t)\, dt \right) \exp \left(\frac{i}{k} \int_0^x V_2(t)\, dt \right).$$

The first two factors constitute a symbol from the $S_{\frac{1}{2}, \frac{1}{2}}$ class by inspection. Denote this symbol by $a_1(k, x)$. For every bounded function f of compact support a pseudodifferential operator T_1, associated with the symbol $a_1(k, x)$, satisfies $\|T_1 f\|_2 \le C_1 \|f\|_2$. Since V_2 is a conditionally integrable function, there exists a constant C_2 such that $|\int_0^x V_2(t)\, dt| \le C_2$ for every x. Write the action of the operator T as

$$Tf(k) = T_1 \left(\sum_{j=1}^{\infty} \frac{1}{j!} \left(\frac{i}{k} \int_0^x V_2(t)\, dt \right)^j f(x) \right) =$$

$$= \sum_{j=1}^{\infty} \frac{1}{j!} \left(\frac{i}{k} \right)^j T_1 \left(\left(\int_0^x V(t)\, dt \right)^j f(x) \right),$$

where the change of the orders of the action of T_1 and summation is justified by the absolute convergence of the series. Hence, for every bounded function of compact support, we have

$$\|Tf\|_2 \leq C_1 \sum_{j=1}^{\infty} \frac{1}{j! a^j} C_2^j \|f\|_2 \leq C_1 \exp(\frac{C_2}{a}) \|f\|_2. \quad \square$$

We now consider slowly decaying perturbations of Schrödinger operators with periodic potentials. Let $U(x)$ be a periodic, piecewise continuous function of period T. It is a well-known fact (see, e.g., [23]) that the spectrum of the operator

$$\tilde{H}_U = -\frac{d^2}{dx^2} + U(x)$$

acting on $L^2(-\infty, \infty)$ is purely absolutely continuous of multiplicity two and consisits of bands $[a_n, b_n]$, n=1,..., where $a_n < b_n \leq a_{n+1}$ for every n. First we will consider slowly decaying perturbations for the case of an operator H_U, $U(x+T) = U(x)$ for every $x > 0$, defined on the semi-axis with some boundary condition at zero. It is easy to see that the absolutely continuous spectrum of this operator is of multiplicity one and coincides as a set with the absolutely continuous spectrum of the corresponding whole-axis operator. This follows, for example, from the existence of the Bloch solutions, which we will discuss shortly.

To prove the stabilty of the absolutely continuous spectrum of periodic Schrödinger operators under a new class of slowly decaying perturbations, we would like to apply Theorem 1.8. For this we need to establish an $L_2 - L_2$ bound for an appropriate operator T_2. Rather detailed knowledge of the properties of the solutions $\theta(x, \lambda)$, which we choose to be the Bloch functions, is important to achieve this goal.

Let us recall the basic facts about the spectrum and the eigenfunctions of the one-dimensional Schrödinger operators with periodic potentials; for the missing proofs we refer to [23].

For every band $[a_n, b_n]$ there exists a real analytic function $\gamma(\lambda)$, which is called quasi-momentum, such that $\gamma(\lambda)$ changes monotonically on (a_n, b_n) from 0 to π if n is odd , and from π to 0 if n is even. The derivative $\gamma'(\lambda)$ might only vanish at the points a_n or b_n and in this case respectively $b_{n-1} = a_n$ or $b_n = a_{n+1}$, i.e. there is no gap between the bands. For every energy $\lambda \in (a_n, b_n)$ there exists a solution $\theta(x, \lambda)$, which is called a Bloch function, such that

$$\theta(x + T, \lambda) = \exp(i\gamma(\lambda))\theta(x, \lambda)$$

and

$$\theta'(x + T, \lambda) = \exp(i\gamma(\lambda))\theta'(x, \lambda).$$

The following lemma holds:

Lemma 3.3. *For every $\lambda \in (a_n, b_n)$, the solutions $\theta(x, \lambda)$ and $\overline{\theta}(x, \lambda)$ are linearly independent.*

Proof. Indeed, suppose that $\overline{\theta}(x, \lambda) = c\theta(x, \lambda)$; then we must have

$$\overline{\theta}(x + T, \lambda) = \exp(-i\gamma(\lambda))\overline{\theta}(x, \lambda)$$

and

$$\overline{\theta}(x + T, \lambda) = c\theta(x + T, \lambda) = c\exp(i\gamma(\lambda))\theta(x, \lambda).$$

Together this implies $\sin\gamma(\lambda) = 0$, which is not possible when $\lambda \in (a_n, b_n)$. $\square$

Hence, the Wronskian $W[\theta, \overline{\theta}] = \Im(\theta\overline{\theta}') \neq 0$ when $\lambda \in (a_n, b_n)$. Next, we remind that the function $\theta(x, \lambda)$, normalized by a condition $\|\theta(x, \lambda)\|_{L_2(0,T)} = 1$, is real analytic in λ as a function in $L_2(0, T)$ when λ belongs to (a_n, b_n). Moreover, we have

Lemma 3.4. *The solution $\theta(x, \lambda)$ is analytic in λ when $\lambda \in (a_n, b_n)$ for every fixed $x \in [0, T]$. Moreover, the functions $\theta(x, \lambda)$, $\partial_x\theta(x, \lambda)$, $\partial_\lambda\theta(x, \lambda)$ and $\partial^2_{x\lambda}\theta(x, \lambda)$ are continuous functions in every rectangle $[a'_n, b'_n] \times [0, T]$, where $a_n < a'_n < b'_n < b_n$.*

Proof. Let us consider two solutions of (4), $y_1(x, \lambda)$ and $y_2(x, \lambda)$, satisfying $y_1(x, \lambda) = 0$, $y'_1(x, \lambda) = 1$ and $y_2(x, \lambda) = 1$, $y'_2(x, \lambda) = 0$. Functions y_1, y_2 satisfy the properties claimed for θ in the lemma by standart calculations using integral equations. Let us represent the function $\theta(x,\lambda)$ as a linear combination of these functions:

$$\theta(x,\lambda) = c_1(\lambda)y_1(x,\lambda) + c_2(\lambda)y_2(x,\lambda).$$

Consider now a vector

$$v(\lambda) = y_2(x,\lambda) - \frac{\langle y_1(x,\lambda), y_2(x,\lambda)\rangle_{L_2(0,T)}}{\|y_1(x,\lambda)\|_{L_2(0,T)}} y_1(x,\lambda)$$

for $\lambda \in [a_n, b_n]$. This is an analyitc in $L_2(0,T)$ vector with norm bounded away from zero on $[a_n, b_n]$ (we remind that solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ satisfy different boundary conditions and their derivatives are in x are bounded by some constant in $[a_n, b_n] \times [0,T]$). We have

$$\langle \theta(x,\lambda), v(\lambda)\rangle_{L_2(0,T)} = c_2(\lambda)\left(\|y_2\|_{L_2(0,T)} - \frac{|\langle y_1, y_2\rangle_{L_2(0,T)}|^2}{\|y_1\|_{L_2(0,T)}}\right).$$

Hence

$$c_2(\lambda) = \frac{\langle \theta(x,\lambda), v(\lambda)\rangle_{L_2(0,T)}}{\|v(\lambda)\|^2_{L_2(0,T)}}.$$

The denominator of the last expression is bounded away from zero and all functions on the right hand side are real analytic when $\lambda \in (a_n, b_n)$, hence, $c_2(\lambda)$ is analytic in this interval. Similarly, we show the analyticity of $c_1(\lambda)$. The statement of the lemma now follows from the properties of $c_1(\lambda)$, $c_2(\lambda)$, $y_1(x,\lambda)$ and $y_2(x,\lambda)$. $\Box$

Proof of Theorem 1.2. Let us consider some band $[a_n, b_n]$. Pick an arbitrary interval $(a'_n, b'_n) \subset [a_n, b_n]$, such that $a_n < a'_n$, $b'_n < b_n$. This interval will serve us as a set S_i from Theorem 1.8. To prove the Theorem 1.2 it suffices to show that an operator T_2 defined by

$$(T_2 f)(\lambda) = \frac{\chi(a'_n, b'_n)}{2\Im(\theta\overline{\theta}')} \int_0^\infty \overline{\theta}^2(x,\lambda) \exp\left(-\frac{2}{\Im(\theta\overline{\theta}')}\int_0^x V(t)|\theta(t,\lambda)|^2\,dt\right) f(x)\,dx$$

satisfies the bound $\|T_2 f\|_{L_2(a'_n, b'_n)} \le C\|f\|_2$ for every bounded function f of compact support.

First we will show that the $L_2 - L_2$ bound holds for an operator T_1 defined by (13):

$$(T_1 f)(\lambda) = \frac{\chi(S)}{\Im(\theta\overline{\theta}')} \int_0^\infty \overline{\theta}(x,\lambda)^2 f(x)\,dx.$$

This will only prove that the absolutely continuous spectrum of Schrödinger operators with periodic potentials is stable under perturbations $V(x)$ satisfying $|V(x)| \le C(1+x)^{-\frac{3}{4}-\epsilon}$. However, later we will see that it is easy to adapt the proof to obtain the $L_2 - L_2$ bound for the operator T_2.

¿From the discussion of the properties of Bloch functions it follows that the Wronskian $W[\theta, \overline{\theta}] = \Im(\theta \overline{\theta}')$ is bounded away from zero on $[a_n', b_n']$. Indeed, the Wronskian is continuous (and, in fact, real analytic) inside each band and vanishes only at a_n or b_n by Lemma 3.3. Let us denote

$$\omega_n = \inf_{\lambda \in (a_n', b_n')} \Im(\theta \overline{\theta}').$$

Also, the module of the derivative of the quasimomentum, $|\gamma'(\lambda)|$, is bounded away from zero on (a_n', b_n'). Let

$$\eta_n = \inf_{\lambda \in (a_n', b_n')} |\theta'(\lambda)|.$$

Next, we note that the function

$$\sigma(x, \lambda) = \left(\exp(-i\gamma(\lambda)\frac{x}{T})\theta(x, \lambda) \right)^2$$

is a periodic function with period T. Let us consider the Fourier series for $\sigma(x, \lambda)$:

$$\sigma(x, \lambda) = \sum_j \exp\left(2\pi i j \frac{x}{T}\right) \hat{\sigma}_j(\lambda).$$

By $\hat{f}_j$ or $\hat{f}(k)$ we denote the Fourier transform of the function f in the discrete and continuous case respectively. ¿From the properties of $\theta(x, \lambda)$, it follows that $\partial_x \sigma(x, \lambda)$ is a continuous function on $[a_n', b_n'] \times [0, T]$; let us denote

$$\sigma_n = \sup_{[a_n', b_n'] \times [0, T]} |\partial_x \sigma(x, \lambda)|.$$

Now note that

$$\|T_1 f\|_{L_2(a_n', b_n')} \le \frac{1}{2\omega_n} \|\chi(a_n', b_n') \int_0^\infty \exp(2i\gamma(\lambda)\frac{x}{T})\sigma(x, \lambda) f(x)\, dx\|_{L_2(a_n', b_n')} =$$

$$= \frac{1}{2\omega_n} \| \int_0^\infty \exp(2i\gamma(\lambda)\frac{x}{T}) \sum_j \left(\exp(2\pi i j \frac{x}{T} \hat{\sigma}_j(\lambda) \right) f(x)\, dx \|_{L_2(a'_n, b'_n)}.$$

Since $|\partial\sigma(x,\lambda)| \le \sigma_n$ for all $x \in [0,T]$ uniformly in $\lambda \in (a'_n, b'_n)$, it is a standart fact that the Fourier series (in x) for $\sigma(x,\lambda)$ converges absolutely. In fact, even for Lipshitz-continuous with power $\alpha > \frac{1}{2}$ function $f(x)$ one has $\sum_n |\hat{f}(n)| \le C\|f\|_{\Lambda_\alpha}$, see, for example, [14].

Hence, we can change the order of summation and integration in the previous formula. We have

$$\|T_1 f\|_{L_2(a'_n, b'_n)} \le \frac{1}{2\omega_n} \| \sum_j \hat{\sigma}_j(\lambda) \int_0^\infty \exp\left(2i\frac{x}{T})(\gamma(\lambda) + j\pi)\right) f(x)\, dx \|_{L_2(a'_n, b'_n)} \le$$

$$\frac{1}{2\omega_n} \left(\int_{a'_n}^{b'_n} \left(\sum_j |\hat{f}\left(\frac{2(\gamma(\lambda)+j\pi)}{T}\right) \hat{\sigma}_j(\lambda)| \right)^2 d\lambda \right)^{\frac{1}{2}} \le$$

$$\le \frac{1}{2\omega_n} \left(\int_{a'_n}^{b'n} \left(\sum_j |\hat{\sigma}_j|^2(\lambda) \right) \left(\sum_j \left| \hat{f}\left(\frac{2(\gamma(\lambda)+j\pi)}{T}\right) \right|^2 \right) d\lambda \right)^{\frac{1}{2}} \le$$

$$\le \frac{\|\sigma^2(x,\lambda)\|_{L_2(0,T)} T^{\frac{1}{2}}}{2\omega_n \eta_n^{\frac{1}{2}}} \left(\sum_j \left| \int_{j\pi+\gamma(a'_n)}^{j\pi+\gamma(b'_n)} |\hat{f}(y)|^2\, dy \right| \right)^{\frac{1}{2}}.$$

To obtain the last inequlity we changed the orders of summation and integration and introduced for each j a new variable $y = \frac{2(\gamma(\lambda)+j\pi)}{T}$. We also note that from Lemma 3.4 it follows that $\sup_{\lambda\in[a'_n,b'_n]} \|\sigma^2(x,\lambda)\|_{L_2(0,T)} \le C_n < \infty$. Hence, the last expression we obtained is estimated by

$$\frac{C_n T^{\frac{1}{2}}}{2\omega_n \eta_n^{\frac{1}{2}}} \|\hat{f}\|_2$$

since the function γ maps the interval (a'_n, b'_n) into the interval $(0,\pi)$.

Therefore, we get the desired bound

$$\|T_1 f\|_{L_2(a'_n, b'_n)} \le C\|f\|_2.$$

Now note that we can write the action of T_2 in a way similar to that of T_1 :

$$(T_2 f)(\lambda) = \frac{\chi(a'_n, b'_n)}{W[\theta, \overline{\theta}]} \sum_j \hat{\sigma}_j(\lambda) \int_0^\infty \exp\left(2i\frac{x}{T}(\gamma(\lambda)+j\pi)\right) a(\lambda,x) f(x)\, dx, \tag{30}$$

where

$$a(\lambda, x) = \exp\left(\frac{1}{W[\theta, \overline{\theta}]} \int\limits_0^x |\sigma(t,\lambda)|^2 V(t)\, dt\right).$$

Proceeding with the estimation of the L_2 norm of the right-hand side of (30) exactly as we did it before, we find that to establish the $L_2 - L_2$ bound, it is sufficient to show that it holds for an operator $\tilde{T}$, defined by

$$(\tilde{T}f)(y) = \int\limits_0^\infty \exp(iyx)\tilde{a}(y,x)\, dx,$$

with a "symbol" $\tilde{a}(y,x)$ defined by

$$\tilde{a}(y,x) = \chi(2\gamma(a_n'), 2\gamma(b_n'))\times$$

$$\times \exp\left(\frac{1}{W[\theta(x,\gamma^{-1}(yT/2)), \overline{\theta}(x,\gamma^{-1}(yT/2)]} \int\limits_0^x |\theta(t,\gamma^{-1}(yT/2))|^2 V(t)\, dt\right)$$

if $y \in [0, \frac{2\pi}{T}]$, and periodic in $y : \tilde{a}(y+\frac{2\pi}{T}, x) = \tilde{a}(y,x)$. The operator $\tilde{T}$ replaces the Fourier transform which appeared in the estimate of T_1. We note that the discontinuity in y due to the presence of characteristic functions is artificail. We can always replace the characteristic functions by smooth functions of compact support equal to 1 when $y \in (2\gamma(a_n', \gamma(b_n'))$ and vanishing outside $(0,\pi)$. Form the $L_2 - L_2$ bound for such operator would follow the bound for the original one. Now it is straightforward to check, using Lemma 3.4 and properties of the quasimomentum $\gamma(\lambda)$, that we have

$$|\partial_x^\alpha \partial_y^\beta \tilde{a}(y,x)| \le C_{\alpha\beta}(1+x)^{(-\frac{1}{2}-\epsilon)\alpha + (\frac{1}{2}-\epsilon)\beta}$$

for all α, β taking values in $\{0,1\}$. Hence, the "symbol" $\tilde{a}(\lambda, x)$ belongs to the "$S_{\frac{1}{2},\frac{1}{2}}$" class with the reduced number of conditions on derivatives. By the Coifman-Meyer criterion [7] it follows that the operator $\tilde{T}$ satisfies an $L_2 - L_2$ bound and therefore this bound also holds for T_2. $\square$

To prove the whole-axis analog of Theorem 1.2, Theorem 1.4, we apply the whole axis criterion formulated in Theorem 1.10. The needed $L_2 - L_2$ bounds are obtained similarly to the semi-axis case.

As a final remark we note that the results parallel to those we show here also hold for Jacobi matrices case. The role of key Theorem 2.1 is played by its discrete analog (which in particular follows from considerations in [16]). We plan to further develop this theme in a subsequent publication.

Acknowledgment.

I would like to thank Prof. B. Simon for stimulating discussions and valuable comments. I am very grateful to Prof. S. Semmes for inspiring and informative conversations on harmonic analysis and to Prof. F. Gesztesy for asking me questions which largely motivated this work.

I gratefully acknowledge hospitality of IHES, where part of this work was done. Research at MSRI supported in part by NSF grant DMS 9022140.

References

[1] P.K. Alsholm and T. Kato, *Scattering with long range potentials,* Partial Diff. Eq., Proc. Symp. Pure Math. Vol. **23**, Amer. Math. Soc., Providence, Rhode Island, 1973, 393–399.

[2] V.S. Buslaev and V.B. Matveev, *Wave operators for Schrödinger equation with a slowly decreasing potentials,* Theor. Math. Phys. **2** (1970), 266–274.

[3] H. Behncke, *Absolute continuity of Hamiltonians with von Neumann-Wigner potentials,* Proc. Amer. Math. Soc. **111** (1991), 373–384.

[4] M. Ben Artzi and A. Devinatz, *Spectral and scatering theory for adiabatic oscilator and related potentials,* J. Math. Phys. **20** (1979), 594–607.

[5] J. Bergh and J. Löfström, *Interpolation Spaces: An Introduction,* Springer-Verlag, Berlin Heidelberg 1976.

[6] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations,* McGraw-Hill, New York, 1955.

[7] R.R. Coifman and Y. Meyer, *Au-delà des opérateurs pseudo-diffèrentiels,* Astèrisque **57** (1978).

[8] A. Garsia, *Topics in a.e. Convergence,* Markham Pub. Company, Chicago 1970.

[9] D.J. Gilbert and D.B. Pearson, *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators,* J. Math. Anal. Appl. **128** (1987), 30–56.

[10] W.A. Harris and D.A. Lutz, *Asymptotic integration of adiabatic oscillator,* J. Math. Anal. Appl. **51** (1975), 76–93.

[11] D.B. Hinton and J.K. Shaw, *Absolutely continuous spectra of second-order differential operators with short and long range potentials,* SIAM J. Math. Anal. **17** (1986), 182–196.

[12] L. Hörmander, *The existence of wave operators in scattering theory,* Math. Z. **146** (1976), 69–91.

[13] S. Jitomirskaya and Y. Last, *Dimensional Hausdorff Properties of Singular Continuous Spectra,* Caltech preprint.

[14] Y. Katznelson, *An Introduction to Harmonic Analysis,* Dover Publications Inc., New York 1976.

[15] A. Kiselev, *Absolutely continuous spectrum of one-dimensional Schrödinger operators and Jacobi matrices with slowly decaying potentials,* Commun. Math. Phys. **179** (1996), 377–400.

[16] A. Kiselev, *A note on almost everywhere convergence of integral operators,* in preparation.

[17] S. Kotani and N. Ushiroya, *One-dimensional Schrödinger operators with random decaying potentials,* Commun. Math. Phys. **115** (1988), 247–266.

[18] V.P. Maslov, S.A. Molchanov, and A.Ya. Gordon, *Behavior of generalized eigenfunctions at infinity and the Schrödinger conjecture,* Russian J. Math. Phys. **1** (1993), 71–104.

[19] S.N. Naboko, *Dense point spectra of Schrödinger and Dirac operators,* Theor.-math. **68** (1986), 18–28.

[20] J. von Neumann and E.P. Wigner, *Über merkwürdige diskrete eigenwerte - Z. Phys.* **30***(1929), 465–467.*

[21] R.E.A.C. Paley, Some theorems on orthogonal functions, Studia Math. **3** (1931), 226–245.

[22] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, III. Scattering Theory,* Academic Press, New York, 1975.

[23] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, IV. Analysis of Operators,* Academic Press, New York, 1978.

[24] B. Simon, *Some Schrödinger operators with dense point spectrum,* Proc. Amer. Math. Soc. (to appear).

[25] B. Simon, *Bounded eigenfunctions and absolutely continuous spectra for one-dimensional Schrödinger operators,* Proc. Amer. Math. Soc. (to appear).

[26] E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces,* Princeton Univ. Press, Princeton 1971.

[27] E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscilatory Integrals,* Princeton Univ. Press, Princeton 1993.

[28] G. Stolz, *Bounded solutions and absolute continuity of Sturm-Liouville operators,* J. Math. Anal. Appl. **169** (1992), 210–228.

[29] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals,* Oxford Univ. Press, Oxford 1948.

[30] J. Weidmann, *Spectral Theory of Ordinary Differential OPerators,* Lecture Nores in Mathematics **1258**, Springer-Verlag, Berlin, 1987.

[31] D.A.W. White, *Schrödinger operators with rapidly oscilating central potentials,* Trans. Amer. Math. Soc. **275** (1983), 641–677.

[32] A. Zygmund, *A remark on Fourier transforms,* Proc. Camb. Phil. Soc. **32** (1936), 321–327.