On the Right Hamiltonian for Singular Perturbations: General Theory

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Abstract

Let a pair of self-adjoint operators $\{A \geq 0, W \leq 0\}$ be such that

(a) there is a dense domain $\mathcal{D} \subseteq \mathrm{dom}(A) \cap \mathrm{dom}(W)$ that $\dot{H} = (A + W)|\mathcal{D}$ is semibounded from below (stability domain),

(b) the symmetric operator $\dot{H}$ is not essentially self-adjoint (singularity of the perturbation),

(c) the Friedrichs extension $\hat{A}$ of $\dot{A} = A|\mathcal{D}$ is maximal with respect to W, i.e., $\mathrm{dom}(\sqrt{-W}) \cap \ker(\dot{A}^* - \eta I) = \{0\}$, $\eta < 0$.

Let $\{W_n\}_{n=1}^{\infty}$ be a regularizing sequence of bounded operators which tends in the strong resolvent sense to W. The abstract problem of the right Hamiltonian is:

(i) to give conditions such that the limit H of self-adjoint regularized Hamiltonians $\tilde{H}_n = \tilde{A} + W_n$ exists and is *unique* for any self-adjoint extension $\tilde{A}$ of $\dot{A}$,

(ii) to describe the limit H.

We show that under the conditions (a) - (c) there is a regularizing sequence $\{W_n\}_{n=1}^{\infty}$ such that $\tilde{H}_n = \tilde{A} + W_n$ tends in the strong resolvent sense to unique (right Hamiltonian) $\hat{H} = \hat{A} \dot{+} W$, otherwise the limit is not unique.

1 Introduction

History of the definition of the "right Hamiltonian" is rather long (see e.g. Simon [12]). But we would like to mention that the "physical approach" to definition of the Hamiltonian in the case of singular interactions is always based on a paradigm of removing cut-offs or *regularization*. The papers [6], [7], [11], [12], [13], [15] scrutinize this approach in the framework of the abstract scheme of functional analysis.

¿From this point of view all singular perturbations could be divided into two complementary classes: positive and non-positive.

The "right Hamiltonian" $\hat{H}$ proposed in [11] and [15] for positve perturbations is a straightforward application of the property of semiboundedness of the perturbation W and the monotonicity of the regularizing sequence of cut-off potentials. In this case one gets automatically the form sum $\hat{H} = \hat{A} \dot{+} W$ as the right Hamiltonian which is expected.

The situation drastically changes when we come to non-positive perturbations which are singular enough that the operator $A+W$ on $\mathrm{dom}(A)$ is not semibounded from below. The first abstract result in this direction is due to [7]. The important (for the mathematical and physical comprehension of the situation) notion of the *stability domain* $\mathcal{D} \subseteq \mathrm{dom}(A) \cap \mathrm{dom}(W)$ with the property (1.1) (see below) was introduced there. Then, in the paper [8] the notion of an *admissible* regularizing sequence was added. These two notions are crucial for understanding how the regularizing sequence chooses itself the "right" (i.e. unique) regularized Schrödinger operator. In [8] the phenomenon was clarified on an abstract operator-theoretical level. Moreover, the conditions formulated in [7] were essentially relaxed which improves the main result of [7] for finite deficiency indices case and gives a natural sufficient condition for the case of infinite deficiency indices.

More precisely the situation can be described as follows. Let $A \geq 0$ be a self-adjoint operator on some separable Hilbert space $\mathcal{H}$ which can be regarded as the Hamiltonian of some quantum system. Let this operator be perturbed by self-adjoint operator W. With respect to quantum mechanics the operator A can be thought as the Laplacian $-\Delta$ on $L^2(\mathbf{R}^n)$ and W as a multiplication operator which arises from some potential.

We assume that the perturbation W is form bounded with respect to A on some dense subset $\mathcal{D}$ (called *stability domain*) of $\mathrm{dom}(A) \cap \mathrm{dom}(W)$, i.e.

$$\begin{aligned} |(Wf,f)| &\leq a(Af,f) + b(f,f), \qquad 0 < a < 1, \quad 0 < b, \\ f &\in \mathcal{D} \subseteq \mathrm{dom}(A) \cap \mathrm{dom}(W). \end{aligned} \tag{1.1}$$

Then let us introduce the symmetric operator $\dot{H}$,

$$\dot{H} f = Af + Wf, \qquad f \in \mathrm{dom}(\dot{H}) = \mathcal{D}, \tag{1.2}$$

which is semibounded from below by (1.1). If $\dot{H}$ is *not essentially self-adjoint*, then W is called *singular perturbation* of A.

Since for singular perturbations the operator $\dot{H}$ is not essentially self-adjoint, it is uncomprehensible which self-adjoint operator is a *right Hamiltonian* for the perturbed quantum system. The only requirement is that the right Hamiltonian has to be a semibounded self-adjoint extension of $\dot{H}$. Since it is *a priori* unclear which of them is the right we have to apply additional arguments to find out this extension, see [8].

A standard procedure goes as follows [6], [7], [8], [9], [11], [15]. We choose a sequence of bounded self-adjoint operators $\{W_n\}_{n=1}^{\infty}$ which tends in the strong resolvent sense to W as $n \longrightarrow \infty$. Since the sequence $\{W_n\}_{n=1}^{\infty}$, which is called a *regularizing sequence* for the singular perturbation W, consists of bounded operators we have no problem to associate with it a sequence of self-adjoint operators $\{H_n\}_{n=1}^{\infty}$:

$$H_n f = Af + W_n f, \qquad f \in \mathrm{dom}(H_n) = \mathrm{dom}(A). \tag{1.3}$$

The sequence $\{H_n\}_{n=1}^{\infty}$ is called the *approximating sequence.* If the approximating sequence tends in the strong resolvent sense to some semibounded self-adjoint extension H of $\dot{H}$, then one believes that this H is the *right* Hamiltonian for the *singular perturbed system.*

However, if $\dot{A} = A|\mathcal{D}$ is not essentially self-adjoint, then there are a lot of self-adjoint extensions $\tilde{A}$ of $\dot{A} = A|\mathcal{D}$ which represent different *unperturbed systems.* Moreover, replacing A by $\tilde{A}$ one again satisfies condition (1.1). Since

$$\dot{H} f = Af + Wf = \tilde{A} f + Wf, \qquad f \in \mathcal{D}, \tag{1.4}$$

the operator W is singular not only for A but for *all* extensions $\tilde{A}$. Hence, the problem of finding out the right Hamiltonian arises for *all* extensions $\tilde{A}$. Therefore, using the same method as above one has to investigate the convergence of the approximating sequences $\{\tilde{H}_n\}_{n=1}^{\infty}$,

$$\tilde{H}_n f = \tilde{A} f + W_n f, \qquad f \in \mathrm{dom}(\tilde{H}_n) = \mathrm{dom}(\tilde{A}) \tag{1.5}$$

and to identify their limits $\tilde{H}$ for different $\tilde{A}$.

It is clear that the perturbation W is singular as well for the Friedrichs extension $\hat{A}$ of $\dot{A} = A|\mathcal{D}$. Using (1.1) one can prove [8] that the approximating sequence

$$\hat{H}_n f = \hat{A} f + W_n f, \qquad f \in \mathrm{dom}(\hat{H}_n) = \mathrm{dom}(\hat{A}), \tag{1.6}$$

converges in the strong resolvent sense to the so-called *form extension*

$$\hat{H} = \hat{A} \dot{+} W, \tag{1.7}$$

which means the extension given by the KLMN-Theorem [10].

Then the question which arises is to describe *all* extensions of $\dot{H}$ which can be obtained by the above procedure. Surprisingly it turns out that under some natural conditions the form sum $\hat{H}$ is the only *one* which is available in this way.

In the present paper this result is proven for non-positive singular perturbations, i.e., for

$$W \leq 0, \tag{1.8}$$

under the assumption that the Friedrichs extension $\hat{A}$ is *maximal* with respect to W. This means that if for some semibounded self-adjoint extension $\tilde{A}$ we have

$$\mathrm{dom}(\tilde{\nu}) \subseteq \mathrm{dom}(\sqrt{-W}), \tag{1.9}$$

where $\tilde{\nu}$ is the quadratic form associated with $\tilde{A}$, then $\tilde{A}$ coincides with the Friedrichs extension $\hat{A}$ and, obviously, $\tilde{\nu} = \hat{\nu}$. Note that one always has $\mathrm{dom}(\hat{\nu}) \subseteq \mathrm{dom}(\sqrt{-W})$ by (1.1).

Under these assumptions we shall show that if the approximating sequence $\{\tilde{H}_n\}_{n=1}^{\infty}$ is such that

$$\sup_n \|\sqrt{|W_n|}(\tilde{H}_n - z)^{-1}\sqrt{|W_n|}\| < +\infty, \qquad \Im\mathrm{m}(z) \neq 0, \tag{1.10}$$

and if it tends in the strong resolvent sense to some semi-bounded self-adjoint extension $\tilde{H}$ of $\dot{H}$, then $\tilde{H}$ coincides with the form sum $\hat{H}$ (1.7). Conversely, for *any* semibounded self-adjoint extension $\tilde{A}$ of $\dot{A}$ there is a regularizing sequence $\{W_n\}_{n=1}^{\infty}$ such that the corresponding approximating sequence $\{\tilde{H}_n\}_{n=1}^{\infty}$ tends in the strong resolvent sense to the form sum (1.7) and the condition (1.10) is satisfied. In other words, if one *assumes* that for a regularizing sequence and for a semibounded self-adjoint extension $\tilde{A}$ of $\dot{A}$ the corresponding approximating sequence $\{\tilde{H}_n\}_{n=1}^{\infty}$ satisfies the condition (1.10), then either this sequence converges in the strong resolvent sense to the form sum (1.7) (if it happens the regularizing sequence is called *well-chosen*) or there is no convergence at all. It turns out, that for each semibounded self-adjoint extension of $\dot{A}$ a *well-chosen* regularizing sequence *always exists*. Therefore, for any self-adjoint extension $\tilde{A}$ the well-chosen regularization "picks out" the right Hamiltonian which is in this case (1.7).

It remains open the question whether there always exists a *non-increasing* (see section 4) regularizing sequence such that $\{\tilde{H}_n\}_{n=1}^{\infty}$ tends in the strong resolvent sense to $\hat{H}$ and the condition (1.10) is satisfied.

We would like to note that if the *maximality* condition is *omitted* then there are semibounded self-adjoint extensions of $\dot{A}$ such that for a given regularizing sequence the corresponding approximating sequences satisfy the condition (1.10) but converge to *different* self-adjoint extensions of $\dot{H}$. Therefore, we lose the uniqueness of the limit with the pleasant property of the regularization to *choose* the right Hamiltonian.

The paper is organized as follows. In the Section 2 we recall some necessary facts from the Krein's extension theory. The aim of Section 3 is to find an explicite formula for the resolvent of the perturbed operator which links the Friedrichs extension $\hat{A}$, the extension parameter B which coresponds to $\tilde{A}$ and a *bounded* perturbation W. The problem of convergence of approximating sequences is solved in the whole generality in Section 4. In Section 5 we discuss the consequences of Section 4, in particular, we prove the above mentioned results.

2 Extensions

Let $A_0 \geq 0$ be some non-negative closed symmetric operator which can be regarded as the closure of $\dot{A} = A|\mathcal{D}$, where A is a self-adjoint operator and $\mathcal{D} \subseteq \mathrm{dom}(A)$. To describe all semibounded self-adjoint extensions $\tilde{A}$ of A_0 obeying $\tilde{A} \geq \eta I$, $\eta < 0$, we introduce the deficiency subspace $\mathcal{N}_\eta = \ker(A_0^* - \eta I)$. By Theorem 2.9 of [2] there is a one-to-one correspondence between *semibounded* self-adjoint extensions $\tilde{A}$ obeying $\tilde{A} \geq \eta I$ and *non-negative* closed quadratic forms $\tilde{q}$ on $\mathcal{N}_\eta$. This correspondence is given in terms of quadratic forms. Denoting by $\tilde{\nu}$ the quadratic form which is associated with $\tilde{A}$, i.e.

$$\tilde{\nu}(f,f) = \left(\sqrt{\tilde{A} - \eta} f, \sqrt{\tilde{A} - \eta} f\right) + \eta(f,f), \quad f \in \mathrm{dom}\left(\sqrt{\tilde{A} - \eta}\right), \tag{2.1}$$

and by $\hat{\nu}$ the form which corresponds to Friedrichs extension $\hat{A}$ of A_0 we have

$$\mathrm{dom}(\tilde{\nu}) = \mathrm{dom}(\hat{\nu}) \dotplus \mathrm{dom}(\tilde{q}) \tag{2.2}$$

, (where $\mathcal{M} \dot{+} \mathcal{N}$ means that $\mathcal{M} \cap \mathcal{N} = \{0\}$) and

$$\tilde{\nu}(g+h, g+h) = \hat{\nu}(g,g) + 2\eta \Re\mathrm{e}(g,h) + \eta(h,h) + \tilde{q}(h,h) \tag{2.3}$$

for $g \in \mathrm{dom}(\hat{\nu})$ and $h \in \mathrm{dom}(\tilde{q})$. Notice that it is not necessary that the domain of the quadratic form $\tilde{q}$ should be dense in $\mathcal{N}_\eta$ or should be a closed subspace of $\mathcal{N}_\eta$. The Friedrichs extension $\hat{A}$ of A_0 corresponds to the quadratic form $\tilde{q} = \hat{q} = 0$ with $\mathrm{dom}(\hat{q}) = \{0\}$ ($\hat{q} = +\infty$ in the terminology of [2]). The smallest self-adjoint extension $\check{A}$ of A_0 obeying $\check{A} \geq \eta I$ is called the Krein extension with respect to η. It is given by the trivial quadratic form $\tilde{q} = \check{q} = 0$ with $\mathrm{dom}(\check{q}) = \mathcal{N}_\eta$. All other extensions obeying $\tilde{A} \geq \eta I$ are between these two, i.e. $\check{A} \leq \tilde{A} \leq \hat{A}$.

Since $\tilde{q}$ is a *non-negative* closed quadratic form on the subspace $\tilde{\mathcal{Q}}_\eta = \overline{\mathrm{dom}(\tilde{q})} \subseteq \mathcal{N}_\eta$, there exist a *non-negative* self-adjoint operator $B \geq 0$ on $\tilde{\mathcal{Q}}_\eta$ such that the representation

$$\tilde{q}(h,h) = (\sqrt{B}h, \sqrt{B}h), \qquad h \in \mathrm{dom}(\tilde{q}) = \mathrm{dom}(\sqrt{B}), \tag{2.4}$$

holds. As far as relation (2.3) is equivalent to

$$\left\| \sqrt{\tilde{A} - \eta}\,(g+h) \right\|^2 = \left\| \sqrt{\hat{A} - \eta} g \right\|^2 + \left\| \sqrt{B} h \right\|^2, \tag{2.5}$$

we see that $\ker(\tilde{A} - \eta) = \{0\}$ is equivalent to $\ker(B) = \{0\}$. Assuming $\ker(B) = \{0\}$ which implies the existence of B^{-1} we find the representation

$$(\tilde{A} - \eta)^{-1} f = (\hat{A} - \eta)^{-1} f + B^{-1} P f, \qquad f \in \mathcal{H}, \quad Pf \in \mathrm{ran}(B), \tag{2.6}$$

where P denotes the orthogonal projection from $\mathcal{H}$ onto $\tilde{\mathcal{Q}}_\eta$.

It turns out, however, that the relation (2.6) defines a self-adjoint extension $\tilde{A}$ of A_0 even in the case when one assumes *only* the invertibility of B. In this case the formula (2.6) defines a self-adjoint extension of A_0 which is not necessarily *semibounded*. In particular, if $B \leq 0$ and invertible we get an extension for which the interval $(\eta, 0)$ is a spectral gap, i.e. $(\eta, 0) \subseteq \rho(\tilde{A})$ where $\rho(\cdot)$ denotes the resolvent set.

The relation (2.6) can be used to compute the resolvent $(\tilde{A} - \eta - z)^{-1}$ in terms of the Friedrichs extension $\hat{A}$ and the operator B. To this end we introduce the analytic operator-valued function

$$M(z) = z^2 P(\hat{A} - \eta - z)^{-1} P + zI : \tilde{\mathcal{Q}}_\eta \longrightarrow \tilde{\mathcal{Q}}_\eta, \qquad \Im\mathrm{m}(z) \neq 0. \tag{2.7}$$

which is usually called the *abstract Weyl function* [3],[4]. Notice that the imaginary part of the Weyl function has a fixed sign for z in the upper or lower half plane, i.e.

$$\Im\mathrm{m}(M(z)) = \Im\mathrm{m}(z) P(\hat{A} - \eta)|\hat{A} - \eta - z|^{-2}(\hat{A} - \eta)P. \tag{2.8}$$

Moreover, since $\Im\mathrm{m}(M(z))$ has bounded inverse we immediately find that the operator-valued function $(B - M(z))$ has the same property for each $\Im\mathrm{m}(z) \neq 0$. If B is bounded and has a bounded inverse, from (2.6) we derive the *Krein representation* [1]:

$$\begin{aligned}(\tilde{A} - \eta - z)^{-1} &= (\hat{A} - \eta - z)^{-1} + \\ &(\hat{A} - \eta)(\hat{A} - \eta - z)^{-1} P (B - M(z))^{-1} P (\hat{A} - \eta)(\hat{A} - \eta - z)^{-1}\end{aligned} \tag{2.9}$$

for $\Im m(z) \neq 0$. Notice that the right-hand side of (2.9) makes sense if B is any self-adjoint operator.

A straightforward computation shows that for any self-adjoint operator B the relation (2.9) defines a self-adjoint extension $\tilde{A}$ of A_0. Conversely, if $\tilde{A}$ is any self-adjoint extension of A_0, then there is a unique subspace $\tilde{\mathcal{Q}}_\eta$ of the deficiency subspace $\mathcal{N}_\eta$ and a self-adjoint operator B acting on $\tilde{\mathcal{Q}}_\eta$ such that the Krein representation (2.9) holds for any non-real z. In the following the self-adjoint operator B defined on the subspace $\tilde{\mathcal{Q}}_\eta = \overline{\mathrm{dom}(B)} \subseteq \mathcal{N}_\eta$ is called the *extension parameter* of the extension $\tilde{A}$.

3 Perturbations

Let W be a *bounded* self-adjoint operator on $\mathcal{H}$. We consider the self-adjoint operator $\tilde{H}$,

$$\tilde{H} = \tilde{A} + W, \tag{3.1}$$

where $\tilde{A}$ is a self-adjoint extension of the non-negative closed symmetric operator A_0 with extension parameter B. Our goal is to compute the resolvent $(\tilde{H} - z)^{-1}$, $\Im m(z) \neq 0$, in terms of the Friedrichs extension $\hat{A}$, the extension parameter B and the perturbation W. It turns out that for this purpose the *perturbed abstract Weyl function* $L(z)$, cf. (2.7),

$$L(z) = P(W - z)(\hat{A} + W - \eta - z)^{-1}(W - z)P + zI : \tilde{\mathcal{Q}}_\eta \longrightarrow \tilde{\mathcal{Q}}_\eta, \qquad \Im m(z) \neq 0, \tag{3.2}$$

plays a very important rôle. A straightforward computation shows that

$$\Im m(L(z)) = \Im m(z) P(\hat{A} - \eta)|\hat{H} - \eta - z|^{-2}(\hat{A} - \eta)P. \tag{3.3}$$

Hence the function $L(z)$ is a so-called operator-valued Nevanlinna function, which means that $\Im m(L(z))$ is positive for positive $\Im m(z)$. Since the operator $(\hat{A} - \eta)|\hat{H} - \eta - z|^{-1}$ has a bounded inverse it is easy to see that the operator-valued function $\Gamma(z)$,

$$\Gamma(z) = B + PWP - L(z) : \tilde{\mathcal{Q}}_\eta \longrightarrow \tilde{\mathcal{Q}}_\eta \tag{3.4}$$

has also a bounded inverse for each non-real z.

Now we introduce important for below operator-valued function $\Omega(z)$,

$$\Omega(z) = \hat{A} + W - (W - z)P(B + PWP - z)^{-1}P(W - z) : \mathrm{dom}(\hat{A}) \longrightarrow \mathcal{H}, \tag{3.5}$$

$\Im m(z) \neq 0$, which can be regarded as a non-selfadjoint perturbation of $\hat{H} = \hat{A} + W$. Introducing the operator-valued function $K(z)$,

$$K(z) = (W - z)P(B + PWP - z)^{-1}P(W - z) + zI : \mathcal{N}_\eta \longrightarrow \mathcal{N}_\eta, \tag{3.6}$$

$\Im m(z) \neq 0$, one gets that

$$\Omega(z) - (\eta + z)I = \hat{A} + W - K(z) - \eta I. \tag{3.7}$$

However, $K(z)$ is as well the operator-valued Nevanlinna function. This follows from the relation

$$\begin{aligned}&\Im m(K(z)) = \\ &\quad \Im m(z)\{[WP - (B + PWP)]|B + PWP - z|^{-2}[PW - (B + PWP)] + (I - P)\},\end{aligned} \tag{3.8}$$

$\Im\mathrm{m}(z) \neq 0$. Hence $\Omega(z) - z - \eta$ is a dissipative operator for each non-real z.

Let us prove that the operator $\Omega(z) - \eta - z$ has a bounded inverse for each non-real z. By (3.7) and (3.8) we obtain that $\ker(\Omega(z) - \eta - z) = \{0\}$ for each non-real z. Moreover, the relation

$$f = (\Omega(z) - \eta - z)\{(\hat{H} - \eta - z)^{-1} + (\hat{H} - \eta - z)^{-1}(W - z)P(B + PWP - L(z))^{-1}P(W - z)(\hat{H} - \eta - z)^{-1}\}f, \tag{3.9}$$

which holds for each $f \in \mathcal{H}$ and any non-real z, implies that $\mathrm{ran}(\Omega(z) - \eta - z) = \mathcal{H}$. This yields that $(\Omega(z) - \eta - z)$ has a bounded inverse for each non-real z.

The operator-valued functions $\Gamma(z)$ and $\Omega(z)$ are related by

$$(\Omega(z) - \eta - z)^{-1} = (\hat{H} - \eta - z)^{-1} + (\hat{H} - \eta - z)^{-1}(W - z)P\Gamma(z)^{-1}P(W - z)(\hat{H} - \eta - z)^{-1} \tag{3.10}$$

and

$$\Gamma(z)^{-1} = (B + PWP - z)^{-1} + (B + PWP - z)^{-1}P(W - z)(\Omega(z) - \eta - z)^{-1}(W - z)P(B + PWP - z)^{-1}, \tag{3.11}$$

$\Im\mathrm{m}(z) \neq 0$. Both of these formulae we need in the following

Proposition 3.1 *Let $\tilde{A}$ be a self-adjoint extension of A_0 given by the extension parameter B defined on $\tilde{\mathcal{Q}}_\eta = \overline{dom(B)} \subseteq \mathcal{N}_\eta$. Further, let W be a bounded operator on $\mathcal{H}$ and let P be the orthogonal projection from $\mathcal{H}$ onto $\tilde{\mathcal{Q}}_\eta$. Then the resolvent $(\tilde{H} - \eta - z)^{-1}$ admits the representations*

$$(\tilde{H} - \eta - z)^{-1} = (\hat{H} - \eta - z)^{-1} + [I - (\hat{H} - \eta - z)^{-1}(W - z)]P\Gamma(z)^{-1}P[I - (W - z)(\hat{H} - \eta - z)^{-1}], \tag{3.12}$$

and

$$(\tilde{H} - \eta - z)^{-1} = P(B + PWP - z)^{-1}P + [I - P(B + PWP - z)^{-1}P(W - z)](\Omega(z) - \eta - z)^{-1} \times [I - (W - z)P(B + PWP - z)^{-1}P], \tag{3.13}$$

for $\Im m(z) \neq 0$.

Proof: Let $\Im\mathrm{m}(z) \neq 0$. Using (3.1) we get

$$(\tilde{H} - \eta - z)^{-1}f = (I + W(\tilde{A} - \eta - z)^{-1})(\tilde{A} - \eta - z)f, \qquad f \in \mathrm{dom}(\tilde{A}). \tag{3.14}$$

Taking into account (2.9) we find

$$(\tilde{H} - \eta - z)f = \{(\hat{H} - \eta - z)(\hat{A} - \eta - z)^{-1} + W(\hat{A} - \eta)(\hat{A} - \eta - z)^{-1}P(B - M(z))^{-1}P(\hat{A} - \eta)(\hat{A} - \eta - z)^{-1}\}(\tilde{A} - \eta - z)f. \tag{3.15}$$

Multiplying on the left by the operator $(\hat{A}-\eta)(\hat{H}-\eta-z)^{-1}$ we get

$$(\hat{A}-\eta)(\hat{H}-\eta-z)^{-1}(\tilde{H}-\eta-z)f = \tag{3.16}$$
$$\{I+(\hat{A}-\eta)(\hat{H}-\eta-z)^{-1}W(\hat{A}-\eta)(\hat{A}-\eta-z)^{-1}P(B-M(z))^{-1}P\}\times$$
$$(\hat{A}-\eta)(\hat{A}-\eta-z)^{-1}(\tilde{A}-\eta-z)f.$$

A straightforward computation shows that

$$\Big(I+(\hat{A}-\eta)(\hat{H}-\eta-z)^{-1}W(\hat{A}-\eta)(\hat{A}-\eta-z)^{-1}P(B-M(z))P\Big)^{-1} \tag{3.17}$$
$$= I-(\hat{A}-\eta)(\hat{H}-\eta-z)^{-1}W(\hat{A}-\eta)(\hat{A}-\eta-z)^{-1}P\Gamma(z)^{-1}P.$$

Setting $(\hat{H}-\eta-z)f = g$ one gets

$$(\hat{A}-\eta)(\hat{A}-\eta-z)^{-1}\{g-W(\tilde{H}-\eta-z)^{-1}g\} = \tag{3.18}$$
$$\{I-(\hat{A}-\eta)(\hat{H}-\eta-z)^{-1}W(\hat{A}-\eta)(\hat{A}-\eta-z)^{-1}P\Gamma(z)^{-1}P\}\times$$
$$(\hat{A}-\eta)(\hat{H}-\eta-z)^{-1}g.$$

for each $g \in \mathcal{H}$. Since $(\hat{A}-\eta)(\hat{H}-\eta-z)^{-1} = I-(W-z)(\hat{H}-\eta-z)^{-1}$ one can simplify (3.18) to

$$(\hat{A}-\eta-z)^{-1}\{g-W(\tilde{H}-\eta-z)^{-1}g\} = (\hat{H}-\eta-z)^{-1}g- \tag{3.19}$$
$$(\hat{H}-\eta-z)W(\hat{A}-\eta)(\hat{A}-\eta-z)^{-1}P\Gamma(z)^{-1}P(\hat{A}-\eta)(\hat{H}-\eta-z)^{-1}g.$$

Using the resolvent identity

$$(\hat{A}-\eta-z)^{-1}-(\hat{H}-\eta-z)^{-1} =$$
$$(\hat{H}-\eta-z)^{-1}W(\hat{A}-\eta-z)^{-1} = (\hat{A}-\eta-z)^{-1}W(\hat{H}-\eta-z)^{-1},$$

we find

$$(\hat{A}-\eta-z)^{-1}\{g-W(\tilde{H}-\eta-z)^{-1}g\} = (\hat{H}-\eta-z)^{-1}g- \tag{3.20}$$
$$(\hat{A}-\eta-z)^{-1}W(\hat{H}-\eta-z)^{-1}(\hat{A}-\eta)P\Gamma(z)^{-1}P(\hat{A}-\eta)(\hat{H}-\eta-z)^{-1}g.$$

Multiplying on the left by $(\hat{A}-\eta-z)$ we obtain

$$W(\tilde{H}-\eta-z)^{-1}g = W\{(\hat{H}-\eta-z)^{-1}g+ \tag{3.21}$$
$$(\hat{H}-\eta-z)^{-1}(\hat{A}-\eta)P\Gamma(z)^{-1}P(\hat{A}-\eta)(\hat{H}-\eta-z)^{-1}g\}.$$

If $\ker(W)=\{0\}$, then the relation (3.12) is proven. If $\ker(W)\neq\{0\}$, then we choose $\tau\in\mathbf{R}$ such that $\ker(W-\tau I)=\{0\}$ and we set $W' = W-\tau I$ and $z' = z-\tau$. Using identities

$$\tilde{H}-(\eta+z)I = \tilde{A}+W'-z'I \quad \text{and} \quad \hat{H}-(\eta+z)I = \hat{A}+W'-z'I,$$

we find

$$W'(\tilde{H}-\eta-z)^{-1}g = W'\{(\hat{H}-\eta-z)^{-1}g+ \tag{3.22}$$
$$(\hat{H}-\eta-z)^{-1}(\hat{A}-\eta)P\Gamma'(z')^{-1}P(\hat{A}-\eta)(\hat{H}-\eta-z)^{-1}g\},$$

with

$$\Gamma'(z') = B + PW'P - L'(z') = \tag{3.23}$$
$$B + PW'P - P(W' - z')(\hat{H} - \eta - z)^{-1}(W' - z')P - z'I = \Gamma(z),$$

which yields

$$W'(\tilde{H} - \eta - z)^{-1} = W'\{(\hat{H} - \eta - z)^{-1}g + \tag{3.24}$$
$$(\hat{H} - \eta - z)^{-1}(\hat{A} - \eta)P\Gamma(z)^{-1}P(\hat{A} - \eta)(\hat{H} - \eta - z)^{-1}g\}.$$

Since $\ker(W') = \{0\}$, we get (3.12).

To prove (3.13) we insert (3.11) into (3.12) and we use the identity

$$(\hat{H} - \eta - z)^{-1} = (\Omega(z) - \eta - z)^{-1} - \tag{3.25}$$
$$(\hat{H} - \eta - z)^{-1}(W - z)P(B + PWP - z)^{-1}P(W - z)(\Omega(z) - \eta - z)^{-1},$$

$\Im\mathrm{m}(z) \neq 0.\bullet$

At the end of this section let us note that formulae (3.12) and (3.13) have a converse. Straightforward computations prove that

$$P\Gamma(z)^{-1}P = -(\hat{A} - \eta)^{-1} - (\hat{A} - \eta)^{-1}(W - z)(\hat{A} - \eta)^{-1} + \tag{3.26}$$
$$[I + (\hat{A} - \eta)^{-1}(W - z)](\tilde{H} - \eta - z)^{-1}[I + (W - z)(\hat{A} - \eta - z)^{-1}]$$

and

$$(\Omega(z) - \eta - z)^{-1} = (\hat{A} - \eta)^{-1} - (\hat{A} - \eta)^{-1}(W - z)(\hat{A} - \eta)^{-1} + \tag{3.27}$$
$$(\hat{A} - \eta)^{-1}(W - z)(\tilde{H} - \eta - z)^{-1}(W - z)(\hat{A} - \eta).$$

In the next sections we use the above representations to resolve the problem of convergence for regularizing sequences of bounded operators $\{W_n\}_{n=1}^{\infty}$.

4 Convergence

Let us start with definition of regularizing sequence.

Definition 4.1 *A sequence $\{W_n\}_{n=1}^{\infty}$ of bounded self-adjoint operators is called regularizing sequence for the self-adjoint operator W if*

$$s - \lim_{n\to\infty}(W_n - z)^{-1} = (W - z)^{-1}, \qquad \Im m(z) \neq 0. \tag{4.1}$$

If $W \leq 0$, the the sequence $\{W_n\}_{n=1}^{\infty}$ is called non-increasing regularizing sequence for W if

$$0 \leq -W_1 \leq -W_2 \leq \ldots \leq -W_n \leq \ldots \leq -W. \tag{4.2}$$

In the following we restrict ourselves to *non-positive perturbations*. Notice that if $\{W_n\}_{n=1}^{\infty}$ is a non-increasing regularizing sequence for the non-positive perturbation W, then $\{-W_n\}_{n=1}^{\infty}$ is a non-decreasing sequence and by Theorem 3.3 of [5] there is always an operator W_∞ such that

$$s-\lim_{n\to\infty}(W_n - z)^{-1} = (W_\infty - z)^{-1}, \qquad \Im m(z) \neq 0. \tag{4.3}$$

Definition 4.1 means that $W = W_\infty$. By [14] we find that

$$\mathrm{dom}(\sqrt{-W}) = \{f \in \mathcal{H} : \sup_n(-W_n f, f) < +\infty\} \tag{4.4}$$

and

$$(\sqrt{-W}f, \sqrt{-W}f) = \lim_{n\to\infty}(-W_n f, f), \qquad f \in \mathrm{dom}(\sqrt{-W}). \tag{4.5}$$

Then by the Theorem 3.13 of [5] one gets

$$\lim_{n\to\infty}\sqrt{-W_n + I}f = \sqrt{-W + I}f, \qquad f \in \mathrm{dom}(\sqrt{-W}). \tag{4.6}$$

In accordance with (1.1) we assume that for non-positive perturbations W on has a *stability domain* $\mathcal{D}$

$$0 \leq (-Wf, f) \leq a(A_0 f, f) + b(f, f), \qquad f \in \mathcal{D}, \quad 0 < a < 1,\, b > 0, \tag{4.7}$$

where A_0 is a non-negative closed symmetric operator for which $\mathcal{D}$ is a core, i.e. $\overline{A_0|\mathcal{D}} = A_0$. Denoting by $\hat{A}$ the Friedrichs extension of A_0 and by $\hat{\nu}$ the quadratic form which is associated with $\hat{A}$ we immediately get from (4.7) that

$$\mathrm{dom}(\hat{\nu}) \subseteq \mathrm{dom}(\sqrt{-W}) \tag{4.8}$$

and

$$0 \leq (\sqrt{-W}f, \sqrt{-W}f) \leq a\hat{\nu}(f, f) + b(f, f), \qquad f \in \mathrm{dom}(\hat{\nu}), \quad 0 < a < 1,\, b > 0, \tag{4.9}$$

which shows that $-W$ is relatively bounded with respect to $\hat{A}$ in the form sense. Then, by the KLMN-theorem [10] the form sum $\hat{H} = \hat{A} \dotplus W$ is well-defined and the operator $\hat{H}$ is semibounded from below. Using (4.9), we get

$$s-\lim_{n\to\infty}\sqrt{-W_n}(\hat{A} - z)^{-1} = \sqrt{-W}(\hat{A} - z)^{-1} \tag{4.10}$$

$$s-\lim_{n\to\infty}(\hat{A} - z)^{-1}\sqrt{-W_n} = (\hat{A} - z)^{-1}\sqrt{-W} \tag{4.11}$$

$$s-\lim_{n\to\infty}\sqrt{-W_n}(\hat{A} - z)^{-1}\sqrt{-W_n} = \sqrt{-W}(\hat{A} - z)^{-1}\sqrt{-W}, \tag{4.12}$$

for $\Im m(z) \neq 0$. As it is shown in [8] the approximating sequence $\{\hat{H}_n\}_{n=1}^{\infty}$,

$$\hat{H}_n = \hat{A} + W_n, \tag{4.13}$$

converges in the strong resolvent sense to $\hat{H}$, i.e.

$$s-\lim_{n\to\infty}(\hat{H}_n - z)^{-1} = (\hat{H} - z)^{-1}, \qquad \Im m(z) \neq 0. \tag{4.14}$$

Using (4.10) - (4.12), one proves that in addition to (4.14) we have the following limits:

$$s-\lim_{n\to\infty}\sqrt{-W_n}(\hat{H}_n-z)^{-1}=\sqrt{-W}(\hat{H}-z)^{-1}, \tag{4.15}$$

$$s-\lim_{n\to\infty}(\hat{H}_n-z)^{-1}\sqrt{-W_n}=(\hat{H}-z)^{-1}\sqrt{-W}, \tag{4.16}$$

$$s-\lim_{n\to\infty}\sqrt{-W_n}(\hat{H}_n-z)^{-1}\sqrt{-W_n}=\sqrt{-W}(\hat{H}-z)^{-1}\sqrt{-W}. \tag{4.17}$$

The purpose of this section is to find *necessary* and *sufficient* conditions ensuring that the approximating sequence $\{\tilde{H}_n\}_{n=1}^{\infty}$ defined by (1.3) converges to the form sum $\hat{H}$ (1.7) in the *generalized strong resolvent sense*, which means that in addition to the strong resolvent convergence we have

$$s-\lim_{n\to\infty}\sqrt{-W_n}(\tilde{H}_n-z)^{-1}=\sqrt{-W}(\hat{H}-z)^{-1}, \tag{4.18}$$

$$s-\lim_{n\to\infty}(\tilde{H}_n-z)^{-1}\sqrt{-W_n}=(\hat{H}-z)^{-1}\sqrt{-W}, \tag{4.19}$$

$$s-\lim_{n\to\infty}\sqrt{-W_n}(\tilde{H}_n-z)^{-1}\sqrt{-W_n}=\sqrt{-W}(\hat{H}-z)^{-1}\sqrt{-W}, \tag{4.20}$$

for some non-real z. According to this definition, the limits (4.15) - (4.17) imply that the approximating sequence $\{\hat{H}_n\}_{n=1}^{\infty}$ converges in the generalized strong resolvent sense to the form sum $\hat{H}$.

Replacing in (3.12) Γ by Γ_n, W by W_n and $\hat{H}$ by $\hat{H}_n$ one gets the representation

$$\begin{aligned}(\tilde{H}_n-\eta-z)^{-1}&=(\hat{H}_n-\eta-z)^{-1}+\\ &[I-(\hat{H}_n-\eta-z)^{-1}(W_n-z)]P\Gamma_n(z)^{-1}P[I-(W_n-z)(\hat{H}_n-\eta-z)^{-1}],\end{aligned} \tag{4.21}$$

with

$$\Gamma_n(z)=B+PW_nP-L_n(z) \tag{4.22}$$

and

$$L_n(z)=P(W_n-z)(\hat{H}_n-\eta-z)^{-1}(W_n-z)P+zI. \tag{4.23}$$

Proposition 4.2 *Let W be a non-positive self-adjoint operator and let A_0 be a non-negative closed symmetric operator. Assume that there is a core $\mathcal{D}$ of A_0 such that (4.7) is satisfied (stability domain). Further, let $\{W_n\}_{n=1}^{\infty}$ be a non-increasing regularizing sequence for W and $\tilde{A}$ be a self-adjoint extension of A_0 given by the extension parameter B. Then the approximating sequence $\{\tilde{H}_n\}_{n=1}^{\infty}$ converges in the generalized strong resolvent sense to the form sum $\hat{H}$ if and only if one has the following limits*

$$s-\lim_{n\to\infty}\Gamma_n(z)^{-1}=0, \tag{4.24}$$

$$s-\lim_{n\to\infty}\sqrt{-W_n}P\Gamma_n(z)^{-1}=0, \tag{4.25}$$

$$s-\lim_{n\to\infty}\Gamma_n(z)^{-1}P\sqrt{-W_n}=0, \tag{4.26}$$

$$s-\lim_{n\to\infty}\sqrt{-W_n}P\Gamma_n(z)^{-1}P\sqrt{-W_n}=0, \tag{4.27}$$

for some non-real z.

Proof: Due to (3.26) one gets representation

$$P\Gamma_n(z)^{-1}P = -(\hat{A}-\eta)^{-1} - (\hat{A}-\eta)^{-1}(W_n - z)(\hat{A}-\eta)^{-1} + \\ [I + (\hat{A}-\eta)^{-1}(W_n - z)](\tilde{H}_n - \eta - z)^{-1}[I + (W_n - z)(\hat{A}-\eta)^{-1}]. \tag{4.28}$$

Assume that $\{\tilde{H}_n\}_{n=1}^\infty$ converges in the generalized strong resolvent sense to the form sum $\hat{H}$. Then taking into account (4.10) - (4.12), we immediately find the relations (4.24) - (4.27). Conversely, if the relations (4.24) - (4.27) are satisfied, then by (4.21) and (4.15) - (4.17) we obtain that $\{\tilde{H}_n\}_{n=1}^\infty$ converges in the generalized strong resolvent sense to the form sum (1.7). •

Since the conditions (4.24) - (4.27) are not very practicable it is necessary to look for more comfortable ones. To this end we use the representation (3.13):

$$(\tilde{H}_n - \eta - z)^{-1} = P(B + PW_nP - z)^{-1}P + \\ [I - P(B + PW_nP - z)^{-1}P(W_n - z)](\Omega_n(z) - \eta - z)^{-1} \times \\ [I - (W_n - z)P(B + PW_nP - z)^{-1}P] \tag{4.29}$$

with

$$\Omega_n(z) = \hat{A} + W_n - (W_n - z)P(B + PW_nP - z)^{-1}P(W_n - z), \qquad n = 1, 2, \ldots, \tag{4.30}$$

which follows from (3.5) by replacing Ω by Ω_n and W by W_n.

Proposition 4.3 *Let W be a non-positive self-adjoint operator and let A_0 be a non-negative closed symmetric operator. Assume that there is a core $\mathcal{D}$ of A_0 such that (4.7) is satisfied (stability domain). Further, let $\{W_n\}_{n=1}^\infty$ be a non-increasing regularizing sequence for W and let $\tilde{A}$ be a self-adjoint extension of the A_0 which corresponds to the extension parameter B. If the approximating sequence $\{\tilde{H}_n\}_{n=1}^\infty$ converges in the generalized strong resolvent sense to the form sum $\hat{H}$, then*

$$s-\lim_{n\to\infty}(\Omega_n(z) - \eta - z)^{-1} = (\hat{H} - \eta - z)^{-1}, \tag{4.31}$$

$$s-\lim_{n\to\infty}\sqrt{-W_n}(\Omega_n(z) - \eta - z)^{-1} = \sqrt{-W}(\hat{H} - \eta - z)^{-1}, \tag{4.32}$$

$$s-\lim_{n\to\infty}(\Omega_n(z) - \eta - z)^{-1}\sqrt{-W_n} = (\hat{H} - \eta - z)^{-1}\sqrt{-W}, \tag{4.33}$$

$$s-\lim_{n\to\infty}\sqrt{-W_n}(\Omega_n(z) - \eta - z)^{-1}\sqrt{-W_n} = \sqrt{-W}(\hat{H} - \eta - z)^{-1}\sqrt{-W} \tag{4.34}$$

for some non-real z. If in addition the condition

$$\sup_n \left\|\sqrt{-W_n}P(B + PW_nP - z)^{-1}P\sqrt{-W_n}\right\| < +\infty \tag{4.35}$$

is satisfied, then

$$s-\lim_{n\to\infty}(B + PW_nP - z)^{-1} = 0, \tag{4.36}$$

$$s-\lim_{n\to\infty}\sqrt{-W_n}P(B + PW_nP - z)^{-1} = 0, \tag{4.37}$$

$$s-\lim_{n\to\infty}(B + PW_nP - z)^{-1}P\sqrt{-W_n} = 0, \tag{4.38}$$

$$s-\lim_{n\to\infty}\sqrt{-W_n}P(B+PW_nP-z)^{-1}P\sqrt{-W_n}=0 \tag{4.39}$$

for some non-real z. Conversely, if the conditions (4.31) - (4.34) and (4.36) - (4.39) are satisfied, then $\{\tilde{H}_n\}_{n=1}^{\infty}$ converges in the generalized strong resolvent sense to the form sum $\hat{H}$.

Proof: From (3.27) we find the representation

$$\begin{aligned}(\Omega_n(z)-\eta-z)^{-1} &= (\hat{A}-\eta)^{-1}-(\hat{A}-\eta)^{-1}(W_n-z)(\hat{A}-\eta)^{-1}\\ &+(\hat{A}-\eta)^{-1}(W_n-z)(\tilde{H}_n-\eta-z)^{-1}(W_n-z)(\hat{A}-\eta)^{-1}.\end{aligned} \tag{4.40}$$

Assume now that $\{\tilde{H}_n\}_{n=1}^{\infty}$ converges in the generalized strong resolvent sense to $\hat{H}$ (1.7). Then using (4.10) - (4.12) one gets directly existence of the limits (4.31) - (4.34). Taking into account the representation

$$\begin{aligned}(\hat{H}-\eta-z)^{-1} &= (\hat{A}-\eta)^{-1}-(\hat{A}-\eta)^{-1}(W-z)(\hat{A}-\eta)^{-1}\\ &+(\hat{A}-\eta)^{-1}(W-z)(\hat{H}-\eta-z)^{-1}(W-z)(\hat{A}-\eta)^{-1},\end{aligned} \tag{4.41}$$

one proves that the right-hand side of (4.40) converges to $(\hat{H}-\eta-z)^{-1}$.

To show (4.36) - (4.39) we note that (4.35) yields the estimate

$$\sup_n\left\|\sqrt{-W_n}P(B+PW_nP-z)^{-1}\right\|=\sup_n\left\|(B+PW_nP-\overline{z})^{-1}P\sqrt{-W_n}\right\|<+\infty. \tag{4.42}$$

This follows from the representation

$$\begin{aligned}&\sqrt{-W_n}P\{(B+PW_nP-z)^{-1}-(B+PW_nP-\overline{z})^{-1}\}P\sqrt{-W_n}\\ &=2i\Im\mathrm{m}(z)\sqrt{-W_n}P(B+PW_nP-\overline{z})^{-1}(B+PW_nP-z)^{-1}P\sqrt{-W_n},\end{aligned} \tag{4.43}$$

which implies the estimate

$$|\Im\mathrm{m}(z)|\left\|(B+PW_nP-z)^{-1}P\sqrt{-W_n}\right\|^2\le\left\|\sqrt{-W_n}P(B+PW_nP-z)^{-1}P\sqrt{-W_n}\right\|. \tag{4.44}$$

Therefore, the condition (4.35) guarantees the condition (4.42).

Since

$$\begin{aligned}(B+PW_nP-z)^{-1} &= \Gamma_n(z)^{-1}-\\ &(B+PW_nP-z)^{-1}P(W_n-z)(\hat{H}_n-\eta-z)^{-1}(W_n-z)P\Gamma_n(z)^{-1},\end{aligned} \tag{4.45}$$

from Proposition 4.2 and relations (4.15) - (4.17), together with the estimates (4.35) and (4.42), we get limits (4.36) - (4.39).

Conversely, if (4.31) - (4.34) and (4.36) - (4.39) are valid, then the representation (4.29) shows that $\{\tilde{H}_n\}_{n=1}^{\infty}$ converges in the generalized strong resolvent sense to the form sum $\hat{H}$. •

5 Consequences

Here we study some consequences of the results obtained in the previous section.

Proposition 5.1 *Let W be a non-positive self-adjoint operator and let A_0 be a non-negative closed symmetric operator. Assume that the pair $\{A_0, W\}$ has a stability domain such that (4.7) is satisfied. Further, let $\{W_n\}_{n=1}^{\infty}$ be a non-increasing regularizing sequence for W and let $\tilde{A}$ be self-adjoint extension of A_0 given by extension parameter B. If the corresponding approximating sequence $\{\tilde{H}_n\}_{n=1}^{\infty}$ converges in the strong resolvent sense to the form sum $\hat{H}$, then*

$$dom(\sqrt{-W}) \cap dom(B) = \{0\}. \tag{5.1}$$

If the extension parameter B is non-negative, then

$$dom(\sqrt{-W}) \cap dom(\sqrt{B}) = \{0\}. \tag{5.2}$$

Proof: Assume there is a non-trivial $h \in \mathrm{dom}(\sqrt{-W}) \cap \mathrm{dom}(B)$. Then by (4.6) we have $\lim_{n\to\infty} \sqrt{-W_n + I}h = \sqrt{-W + I}h$. Taking into account (4.22), (4.23), as well as (4.14) - (4.17) and (4.24) - (4.27), one finds

$$\lim_{n\to\infty} \Gamma_n(z)^{-1}(\Gamma_n(z) + z)h = 0. \tag{5.3}$$

On the other hand, we get

$$\Gamma_n(z)^{-1}(\Gamma_n(z) + z)h = h + z \lim_{n\to\infty} \Gamma_n(z)^{-1}h = h, \qquad \Im\mathrm{m}(z) \neq 0, \tag{5.4}$$

which contradicts (5.3). Hence, $h = 0$.

If $B \geq 0$, then operator $\sqrt{B}\Gamma_n(z)^{-1}$, $\Im\mathrm{m}(z) \neq 0$, is well-defined for each $n = 1, 2, \ldots$. Furthermore, we have

$$\begin{aligned} &\|\sqrt{B}\Gamma_n(z)^{-1}h\|^2 = (h, \Gamma_n(z)^{-1}h) + \\ &\quad \|\sqrt{-W_n}P\Gamma_n(z)^{-1}h\|^2 + (P(W_n - z)(\hat{H}_n - \eta - z)^{-1}(W_n - z)P\Gamma_n(z)^{-1}h, \Gamma_n(z)^{-1}h). \end{aligned} \tag{5.5}$$

Applying again (4.14) - (4.17) and (4.24) - (4.27), we find

$$s - \lim_{n\to\infty} \sqrt{B}\Gamma_n(z)^{-1} = 0, \qquad \Im\mathrm{m}(z) \neq 0. \tag{5.6}$$

Using the representation

$$\begin{aligned} &(\Gamma_n(z)^{-1}(\Gamma_n(z) + z)h, h') = (\sqrt{B}h, \sqrt{B}\Gamma_n(\overline{z})^{-1}h') - \\ &\quad (\sqrt{-W_n}h, \sqrt{-W_n}\Gamma_n(\overline{z})^{-1}h') - ((W_n - z)(\hat{H}_n - \eta - z)^{-1}(W_n - z)h, \Gamma_n(\overline{z})^{-1}h'), \end{aligned} \tag{5.7}$$

$h \in \mathrm{dom}(\sqrt{-W}) \cap \mathrm{dom}(\sqrt{B})$, $h' \in \mathcal{N}_\eta$, $\Im\mathrm{m}(z) \neq 0$, as well as (4.14) - (5.17) and (4.24) - (4.27), one gets

$$w - \lim_{n\to\infty} \Gamma_n(z)^{-1}(\Gamma_n(z) + z)h = 0, \qquad h \in \mathrm{dom}(\sqrt{-W}) \cap \mathrm{dom}(\sqrt{B}). \tag{5.8}$$

However, this contradics (5.4).•

The conditions (5.1) and (5.2) imply that

$$\sup_n(-W_n h, h) = \infty, \tag{5.9}$$

for non-trivial elements $h \in \mathrm{dom}(B)$ or $h \in \mathrm{dom}(\sqrt{B})$. According to Definition 2.6 of [8] the property (5.9) for $B \geq 0$ means that the regularizing sequence $\{W_n\}_{n=1}^\infty$ is *$\hat{\nu}$-admissible*. Hence, Proposition 5.1 shows that admissibility is *necessary* in order to get convergence of the approximating sequence $\{\tilde{H}_n\}_{n=1}^\infty$ to the form sum $\hat{H}$.

Next we are interested in the question what is going on if $\mathrm{dom}(\sqrt{-W}) \cap \mathcal{N}_\eta \neq \{0\}$. It turns out that in this case not the all approximating sequences $\{\tilde{H}_n\}_{n=1}^\infty$ converge to the form sum $\hat{H}$.

Proposition 5.2 *Let W be a non-positive self-adjoint operator and let A_0 be a non-negative closed symmetric operator. Assume that the pair $\{A_0, W\}$ has a stability domain $\mathcal{D}$ such that condition (4.7) is satisfied. If $dom(\sqrt{-W}) \cap \mathcal{N}_\eta \neq \{0\}$, $\mathcal{N}_\eta = \ker(A_0^* - \eta I)$, for some $\eta < 0$, then there is a continuum of different semibounded self-adjoint extensions $\tilde{A}$ of A_0 such that*

$$dom(\tilde{\nu}) \subseteq dom(\sqrt{-W}), \tag{5.10}$$

$$dom(\tilde{\nu}) \cap \mathcal{N}_\eta = dom(\sqrt{-W}) \cap \mathcal{N}_\eta, \qquad \forall \eta < 0, \tag{5.11}$$

and

$$(\sqrt{-W}f, \sqrt{-W}f) \leq \tilde{a}\tilde{\nu}(f,f) + \tilde{b}(f,f), \quad f \in dom(\tilde{\nu}), \quad 0 < \tilde{a} < 1, \quad \tilde{b} > 0 \tag{5.12}$$

where $\tilde{\nu}$ is the closed quadratic form associated with $\tilde{A}$.

Proof: First we remark that if $\mathrm{dom}(\sqrt{-W}) \cap \mathcal{N}_\eta = \{0\}$ for some $\eta < 0$, then this relation holds for any $\eta < 0$. To see this we have to use the representation [2]

$$\mathrm{dom}(A_0^*) = \mathrm{dom}(\hat{A}) \dotplus \mathcal{N}_\eta \tag{5.13}$$

which holds for each $\eta < 0$. Assume that $\mathrm{dom}(\sqrt{-W}) \cap \mathcal{N}_\eta = \{0\}$ for some $\eta < 0$. Since $\mathrm{dom}(\sqrt{-W}) \cap \mathrm{dom}(A_0^*) = \mathrm{dom}(\sqrt{-W}) \cap \mathrm{dom}(\hat{A})$ we have $\mathrm{dom}(\sqrt{-W}) \cap \mathcal{N}_{\eta'} = \{0\}$ for any $\eta' < 0$.

Let $\eta < -\frac{b}{a}$ where the constants a and b are taken from (4.7). We set

$$\begin{aligned} \tilde{q}_\rho(h,h) &= \rho(\sqrt{-W}h, \sqrt{-W}h), \qquad 0 \leq \rho < \infty, \\ h &\in \mathrm{dom}(\tilde{q}_\rho) = \mathrm{dom}(\sqrt{-W}) \cap \mathcal{N}_\eta. \end{aligned} \tag{5.14}$$

Since W is a non-positive self-adjoint operator, the quadratic form $\tilde{q}_\rho$ is non-negative and closed. In accordance with (2.4) we associate with $\tilde{q}_\rho$ a semi-bounded self-adjoint extension $\tilde{A}_\rho$ of A_0. By (2.3) and (2.4) we have

$$\mathrm{dom}(\tilde{\nu}_\rho) = \mathrm{dom}(\hat{\nu}) \dotplus \mathrm{dom}(\tilde{q}_\rho) \subseteq \mathrm{dom}(\sqrt{-W}), \tag{5.15}$$

$$\tilde{\nu}_\rho(g+h, g+h) - \eta\|g+h\|^2 = \hat{\nu}(g,g) - \eta\|g\|^2 + \tilde{q}_\rho(h,h),\ g \in \mathrm{dom}(\hat{\nu}), h \in \mathrm{dom}(\tilde{q}). \tag{5.16}$$

Using the estimate

$$(\sqrt{-W}(g+h),\sqrt{-W}(g+h)) \le \qquad (5.17)$$
$$(1+\epsilon)(\sqrt{-W}g,\sqrt{-W}g)+(1+\frac{1}{\epsilon})(\sqrt{-W}h,\sqrt{-W}h), \qquad \epsilon>0,$$

and (4.9) we find

$$(\sqrt{-W}(g+h),\sqrt{-W}(g+h)) \le (1+\epsilon)\left\{a\hat{\nu}(g,g)+b\|g\|^2\right\}+\frac{1+\epsilon}{\epsilon}(\sqrt{-W}h,\sqrt{-W}h), \quad (5.18)$$

which implies

$$(\sqrt{-W}(g+h),\sqrt{-W}(g+h)) \le \qquad (5.19)$$
$$(1+\epsilon)a\left\{\hat{\nu}(g,g)+\frac{b}{(1+\epsilon)a}\|g\|^2+\frac{1}{a\epsilon}(\sqrt{-W}h,\sqrt{-W}h)\right\}.$$

Since $\frac{b}{(1+\epsilon)a}\le\frac{b}{a}<-\eta$, one gets the estimate

$$(\sqrt{-W}(g+h),\sqrt{-W}(g+h)) \le (1+\epsilon)a\left\{\hat{\nu}(g,g)-\eta\|g\|^2+\tilde{q}_\rho(h,h)\right\} \qquad (5.20)$$

for each $\rho\ge\frac{1}{a\epsilon}$. Therefore, we have

$$(\sqrt{-W}(g+h),\sqrt{-W}(g+h)) \le (1+\epsilon)a\tilde{\nu}_\rho(g+h,g+h)-(1+\epsilon)a\eta\|g+h\|^2. \qquad (5.21)$$

Choosing ϵ small enough, we satisfy the condition $\tilde{a}=(1+\epsilon)a<1$. Setting $\tilde{b}=-(1+\epsilon)\eta$, we prove (5.12).

Now we can verify (5.11). For the chosen $\eta<-\frac{b}{a}$ the condition (5.11) is satisfied by the construction (5.15). In order to show the condition (5.11) for any $\eta<0$ we note that

$$\mathrm{dom}(\tilde{\nu}_\rho)\cap\mathrm{dom}(A_0^*)=\mathrm{dom}(\hat{\nu})\dotplus\{\mathrm{dom}(\tilde{q}_\rho)\cap\mathcal{N}_\eta\}=\mathrm{dom}(\hat{\nu})\dotplus\{\mathrm{dom}(\sqrt{-W})\cap\mathcal{N}_\eta\}. \quad (5.22)$$

This yields $\mathrm{dom}(\tilde{\nu}_\rho)\cap\mathcal{N}_\eta=\mathrm{dom}(\sqrt{-W})\cap\mathcal{N}_\eta$ for any $\eta<0$.

Since for different $\rho\ge\frac{1}{a\epsilon}$ quadratic forms $\tilde{q}_\rho$ are different, the corresponding self-adjoint extensions $\tilde{A}_\rho$ are different too. Hence, there is a continuum family of self-adjoint extensions of A_0 satisfying (5.10) - (5.12).•

Replacing $\hat{A}$ by $\tilde{A}$ one easily gets that (5.10) and (5.12) are equivalent to (4.8) and (4.9). Again by the KLMN-Theorem the form sum $\tilde{H}=\tilde{A}\dotplus W$ is well-defined and semi-bounded from below. Further, setting $\tilde{H}_n=\tilde{A}+W_n$ and replacing $\hat{A}$, $\hat{H}$ and $\hat{H}_n$ by $\tilde{A}$, $\tilde{H}$ and $\tilde{H}_n$, respectively, we obtain relations similar to (4.10) - (4.12) and (4.14) - (4.17). Hence, the expectation that each approximating sequence $\{\tilde{H}_n\}_{n=1}^\infty$ converges to the form sum $\hat{H}$ is *false* if $\mathrm{dom}(\sqrt{-W})\cap\mathcal{N}_\eta\neq\{0\}$.

Condition $\mathrm{dom}(\sqrt{-W})\cap\mathcal{N}_\eta=\{0\}$ means that the domain $\mathrm{dom}(\hat{\nu})$ has to be *maximal* with respect to W, i.e., if $\tilde{A}$ is any semibounded self-adjoint extension of A_0 with corresponding quadratic form $\tilde{\nu}$ such that

$$\mathrm{dom}(\hat{\nu})\subseteq\mathrm{dom}(\tilde{\nu})\subseteq\mathrm{dom}(\sqrt{-W}), \qquad (5.23)$$

then $\mathrm{dom}(\tilde{\nu})=\mathrm{dom}(\hat{\nu})$, which yields $\tilde{A}=\hat{A}$.

Obviously the last remark implies the following question. Assume that $\hat{\nu}$ is *maximal* with respect to W and there is approximating sequence $\{\tilde{H}_n\}_{n=1}^\infty$ which converges to some self-adjoint extension $\tilde{H}$ of $\dot{H}$. Does $\tilde{H}$ coincide necessarily with the form sum $\hat{H}$?

Proposition 5.3 *Let W be a non-positive self-adjoint operator and let A_0 be a non-negative closed symmetric operator such that dom($\hat{\nu}$) is maximal with respect to W. Assume that (4.7) is satisfied for a core $\mathcal{D}$ of A_0 and $\tilde{A}$ is a self-adjoint extension of A_0. Further, let $\{W_n\}_{n=1}^{\infty}$ be a regularizing sequence for W such that*

$$\lim_{n\to\infty} \sqrt{|W_n|+I}f = \sqrt{-W+I}f, \qquad f \in dom(\sqrt{-W}). \tag{5.24}$$

If the corresponding approximating sequence $\{\tilde{H}_n\}_{n=1}^{\infty}$ converges in the strong resolvent sense to a semibounded self-adjoint extension $\tilde{H}$ of $\dot{H}$ and

$$\sup_n \|\sqrt{|W_n|}(\tilde{H}_n - z)^{-1}\sqrt{|W_n|}\| < +\infty \tag{5.25}$$

for some non-real z, then $\tilde{H}$ coincides with the form sum $\hat{H}$.

Proof: Let us introduce the sequence $\{S_n(z)\}_{n=1}^{\infty}$ of bounded operators

$$S_n(z) = \sqrt{|W_n|+I}(\tilde{H}_n - z)^{-1}\sqrt{|W_n|+I}, \tag{5.26}$$

which is uniformly bounded by (5.25), i.e. $\sup_n \|S_n(z)\| < +\infty$. This yields the existence of a bounded operator $S(z)$ such that for a subsequence $\{n_k\}_{k=1}^{\infty}$ we have the weak limit

$$w - \lim_{k\to\infty} S_{n_k}(z) = S(z). \tag{5.27}$$

Let $f, h \in \mathrm{dom}(\sqrt{-W})$. Then taking into account (5.24) we find

$$\begin{aligned} \lim_{n_k\to\infty}(S_{n_k}(z)f,h) &= ((\tilde{H}_{n_k} - z)^{-1}\sqrt{-W_{n_k}+I}f, \sqrt{-W_{n_k}+I}h) = \\ &((\tilde{H} - z)^{-1}\sqrt{-W+I}f, \sqrt{-W+I}h) = (S(z)f,h), \end{aligned} \tag{5.28}$$

which shows that

$$\mathrm{dom}(\tilde{H}) \subseteq \mathrm{dom}(\sqrt{-W}) \tag{5.29}$$

and

$$S(z)f = \sqrt{-W+I}(\tilde{H} - z)^{-1}\sqrt{-W+I}f, \qquad f \in \mathrm{dom}(\sqrt{-W}). \tag{5.30}$$

A straightforward computation shows that if $S(z)$ makes sense and if it is bounded for some non-real z, then the same holds for each $z \in \rho(\tilde{H})$, in particular, for *real* points of the resolvent set $\rho(\tilde{H})$. They exist by the assumption that $\tilde{H}$ is semibounded. Hence, there is a $\tau \le 0$ such that $\tilde{H} - \tau I \ge 0$ and the inverse operator $(\tilde{H} - \tau I)^{-1}$ exists and is bounded. Since $S(\tau) = \sqrt{-W+I}(\tilde{H} - \tau)^{-1}\sqrt{-W+I}$ is bounded, we obtain that $\mathrm{dom}((\tilde{H} - \tau)^{1/2}) \subseteq \mathrm{dom}(\sqrt{-W})$. Let $\tilde{\mu}$ be the quadratic form which is associated with $\tilde{H}$. Then we have

$$\dim(\tilde{\mu}) \subseteq \mathrm{dom}(\sqrt{-W}). \tag{5.31}$$

Let us introduce closed quadratic form $\tilde{\nu}$

$$\tilde{\nu}(f,f) = \tilde{\mu}(f,f) + (\sqrt{-W}f, \sqrt{-W}f), \qquad f \in \mathrm{dom}(\tilde{\nu}) = \mathrm{dom}(\tilde{\mu}). \tag{5.32}$$

The form $\tilde{\nu}$ is obviously semibounded. Moreover, if $f \in \mathcal{D} \subseteq \mathrm{dom}(A_0) \cap \mathrm{dom}(W)$, then

$$\tilde{\nu}(f,f) = (\dot{H}f,f) - (Wf,f) = (A_0 f,f). \tag{5.33}$$

Hence, the closed quadratic form $\tilde{\nu}$ defines a semibounded self-adjoint extension $\tilde{A}$ of A_0. Since $\mathrm{dom}(\hat{\nu})$ is maximal with respect to W and

$$\mathrm{dom}(\hat{\nu}) \subseteq \mathrm{dom}(\tilde{\nu}) \subseteq \mathrm{dom}(\sqrt{-W}), \tag{5.34}$$

we get $\mathrm{dom}(\tilde{\nu}) = \mathrm{dom}(\hat{\nu})$, which yields $\tilde{\nu} = \hat{\nu}$ and $\tilde{A} = \hat{A}$. Therefore, $\mathrm{dom}(\tilde{\mu}) = \mathrm{dom}(\hat{\nu})$ and

$$\tilde{\mu}(f,f) = \tilde{\nu}(f,f) - (\sqrt{-W}f, \sqrt{-W}f) = \hat{\nu}(f,f) - (\sqrt{-W}f, \sqrt{-W}f) = \hat{\nu}(f,f)$$

$f \in \mathrm{dom}(\tilde{\mu}) = \mathrm{dom}(\hat{\mu}) = \mathrm{dom}(\hat{\nu})$, where $\hat{\mu}$ is the quadratic form which is associated with $\hat{H}$. Consequently, we obtain $\tilde{H} = \hat{H}$. •

If the $\mathrm{dom}(\hat{\nu})$ is *maximal* with respect to the perturbation W and the *additional* condition (5.25) holds, then Proposition 5.3 tells us that the approximating sequence $\{\tilde{H}_n\}_{n=1}^{\infty}$ *either* converges to the form sum $\hat{H}$ *or* it does not converge at all.

Finally let us show that for each semibounded self-adjoint extension $\tilde{A}$ of A_0 there is *always* a regularizing sequence $\{W_n\}_{n=1}^{\infty}$ for $W \leq 0$ such that the corresponding approximating sequence $\{\tilde{H}_n\}_{n=1}^{\infty}$ tends to the form sum $\hat{H}$ in the *generalized strong resolvent sense.*

Proposition 5.4 *Let W be a non-positive self-adjoint operator and let A_0 be a non-negative closed symmetric operator such that $\mathrm{dom}(\hat{\nu})$ is maximal with respect to W. Let the pair $\{A_0, W\}$ has a stability domain and condition (4.7) is satisfied for a core $\mathcal{D}$ of A_0. Then for each semibounded self-adjoint extension $\tilde{A}$ of A_0 there is a regularizing sequence $\{W_n\}_{n=1}^{\infty}$ for W such that*

$$|(W_n f,f)| \leq a(A_0 f,f) + (b+2)(f,f), \qquad f \in \mathcal{D}, \tag{5.35}$$

where a and b are taken from (4.7), and the corresponding approximating sequence $\{\tilde{H}_n\}_{n=1}^{\infty}$ given by (1.3) obeys

$$s - \lim_{n\to\infty} (\tilde{H}_n - z)^{-1} = (\hat{H} - z)^{-1}, \tag{5.36}$$

$$s - \lim_{n\to\infty} \sqrt{|W_n|}(\tilde{H}_n - z)^{-1} = \sqrt{-W}(\hat{H} - z)^{-1}, \tag{5.37}$$

$$s - \lim_{n\to\infty} (\tilde{H}_n - z)^{-1}\sqrt{|W_n|} = (\hat{H} - z)^{-1}\sqrt{-W}, \tag{5.38}$$

$$s - \lim_{n\to\infty} \sqrt{|W_n|}(\tilde{H} - z)^{-1}\sqrt{|W_n|} = \sqrt{-W}(\hat{H} - z)^{-1}\sqrt{-W} \tag{5.39}$$

for non-real z.

Proof: Let $\{W_n\}_{n=1}^{\infty}$ be a non-increasing regularizing sequence for $W \leq 0$ (4.1), (4.2). Further, let B be the extension parameter which corresponds to $\tilde{A}$. Choosing η smaller than $\inf \sigma(\tilde{A})$ we can assume that $B \geq 0$. We set $X_n = PW_nP \leq 0$, $n = 1, 2, \ldots$. There is an isometry $\Gamma_n : \tilde{Q}_\eta \longrightarrow \mathcal{H}$ such that

$$\sqrt{-W_n + IP} = \Gamma_n \sqrt{-X_n + IP}, \qquad n = 1, 2, \ldots \quad . \tag{5.40}$$

Introducing the operator

$$J_n = I - 2\Gamma_n\Gamma_n^*, \qquad n = 1, 2, \ldots, \tag{5.41}$$

we define a sequence $\{V_n\}_{n=1}^{\infty}$ of bounded operators by

$$V_n = -(\sqrt{-W_n + I}J_n\sqrt{-W_n + I} - I). \tag{5.42}$$

Let us show that $\{V_n\}_{n=1}^{\infty}$ is a regularizing sequence for W. By monotonicity, for each $f \in \mathrm{dom}(\sqrt{-W})$ we have $s - \lim_{n\to\infty}\sqrt{-W_n + I}f = \sqrt{-W + I}f$, see (4.6). Using (5.40) we get representation

$$\Gamma_n^* f = \frac{1}{\sqrt{-X_n + I}}P\sqrt{-W_n + I}f, \qquad f \in \mathrm{dom}(\sqrt{-W}),\, n = 1, 2 \ldots \quad . \tag{5.43}$$

Since $\mathrm{dom}(\hat{\nu})$ is maximal with respect to W, we have $s - \lim_{n\to\infty}(-X_n + I)^{-1/2} = 0$, see Lemma 2.10 of [8]. This immediately implies

$$\lim_{n\to\infty} \Gamma_n^* f = 0, \qquad f \in \mathrm{dom}(\sqrt{-W}). \tag{5.44}$$

Since $\{\Gamma_{n=1}^*\}_{n=1}^{\infty}$ is a sequence of partial isometries this yields $s - \lim_{n\to\infty}\Gamma_n^* = 0$. Hence one gets $s - \lim_{n\to\infty}\Gamma_n\Gamma_n^* = 0$. Therefore $s - \lim_{n\to\infty} J_n = I$. If $\tau \in \mathbf{R}$ is sufficiently small, then the operator $(I + \tau(I - W_n)^{-1}J_n)^{-1}$ has a bounded inverse such that

$$\|(I + \tau(I - W_n)^{-1})^{-1}J_n\| \leq \frac{1}{1 - |\tau|}. \tag{5.45}$$

Notice that $\|J_n\| = 1$. Therefore, one gets the following representation

$$(\sqrt{-W_n + I}J_n\sqrt{-W_n + I} + \tau)^{-1} = \frac{1}{\sqrt{-W_n + I}}J_n(I + \tau(I - W_n)^{-1}J_n)^{-1}\frac{1}{\sqrt{-W_n + I}}. \tag{5.46}$$

Since the sequence $\{W_n\}_{n=1}^{\infty}$ tends in the strong resolvent sense to W and $s - \lim_{n\to\infty} J_n = I$, we find

$$s - \lim_{n\to\infty}(\sqrt{-W_n + I}J_n\sqrt{-W_n + I} + \tau)^{-1} = (-W + I + \tau)^{-1}. \tag{5.47}$$

Hence, $\{\sqrt{-W_n + I}J_n\sqrt{-W_n + I}\}_{n=1}^{\infty}$ tends in the strong resolvent sense to $-W + I$, which shows, see (5.42), that $\{V_n\}_{n=1}^{\infty}$ tends in the strong resolvent sense to W. Hence $\{V_n\}_{n=1}^{\infty}$ is a regularizing sequence for W in spite of it is not, in general, a non-increasing one.

To prove (5.35) we note that by (5.41) and (5.42) one has

$$V_n = \tag{5.48}$$
$$-\left(\sqrt{-W_n + I}\{I - 2\Gamma_n\Gamma_n^*\}\sqrt{-W_n + I} - I\right) = W_n + 2\sqrt{-W_n + I}\Gamma_n\Gamma_n^*\sqrt{-W_n + I},$$

which yields

$$W_n \leq V_n, \qquad n = 1, 2, \ldots \quad . \tag{5.49}$$

Moreover, since $\|\Gamma_n\| \leq 1$, we find

$$V_n \leq -W_n + 2I, \qquad n = 1, 2, \ldots \quad , \tag{5.50}$$

which proves the estimate

$$|(V_n f,f)| \leq (-W_n f,f) + 2(f,f), \qquad n=1,2,\ldots \quad f \in \mathcal{H}. \tag{5.51}$$

Since $-W_n \leq -V$, $n = 1,2,\ldots$, we obtain (5.35) by (4.7).

To verify the convergence of the approximating sequence $\{\tilde{H}_n\}_{n=1}^{\infty}$ to the form sum $\hat{H}$ we use the representation (4.29). Then, by (5.40) one finds

$$PV_nP = -P\sqrt{-W_n+I}J_n\sqrt{-W_n+I}P + P = -\sqrt{-X_n+I}\Gamma_n^* J_n \Gamma_n \sqrt{-X_n+I} + P. \tag{5.52}$$

Since $\Gamma_n^* J_n \Gamma_n = -I$, one finally gets

$$PV_nP = -X_n + 2P, \qquad n = 1,2\ldots. \tag{5.53}$$

This immediately implies

$$B + PV_nP - I = B - X_n + I \geq I, \qquad n = 1,2\ldots. \tag{5.54}$$

Now we can calculate the operator-valued function $\Omega_n(z)$ for $z = 1$. Using (4.30) and (5.54) we find

$$\begin{aligned}\Omega_n(1) = \hat{A} + I - \sqrt{-W_n+I}J_n\sqrt{-W_n+I} - \\ \sqrt{-W_n+I}J_n\sqrt{-W_n+I}P(B-X_n+I)^{-1}P\sqrt{-W_n+I}J_n\sqrt{-W_n+I}.\end{aligned} \tag{5.55}$$

Taking into account (5.41), one gets

$$\begin{aligned}\Omega_n(1) = \hat{A} + I - \\ \sqrt{-W_n+I}\left\{J_n + \Gamma_n\sqrt{-X_n+I}(B-X_n+I)^{-1}\sqrt{-X_n+I}\Gamma_n^*\right\}\sqrt{-W_n+I}.\end{aligned} \tag{5.56}$$

Inserting (5.41) into (5.56), we finally obtain

$$\Omega_n(1) = \hat{A} + W_n + \Gamma_n\{2I - \sqrt{-X_n+I}(B-X_n+I)^{-1}\sqrt{-X_n+I}\}\Gamma_n^*. \tag{5.57}$$

Notice that

$$0 \leq \Gamma_n\left\{2I - \sqrt{-X_n+I}(B-X_n+I)^{-1}\sqrt{-X_n+I})^{-1}\right\}\Gamma_n^* \leq 2I. \tag{5.58}$$

Therefore, by (5.57) one gets

$$\Omega_n(1) \geq \hat{A} + W_n \geq \hat{H}. \tag{5.59}$$

If $\eta + 2 < min\{\inf\sigma(\hat{H}), \inf\sigma(\tilde{A})\}$, then

$$\Omega_n(1) - (\eta+1)I \geq \hat{H}_n - (\eta+1)I \geq \hat{H} - (\eta+1)I \geq I, \tag{5.60}$$

where $\hat{H}_n = \hat{A} + V_n$, $n = 1,2,\ldots$. Hence, the inverse operator $(\Omega_n(1) - \eta - 1)^{-1}$ exists and we have

$$0 \leq (\Omega_n(1) - \eta - 1)^{-1} \leq I. \tag{5.61}$$

Let us show that $s-\lim_{n\to\infty}(\Omega_n(1)-\eta-1)^{-1}=(\hat{H}-\eta-1)^{-1}$. By (5.57) the operator $(\Omega(1)-\eta-1)^{-1}$ admits the representation

$$(\Omega_n(1)-\eta-1)^{-1}=\frac{1}{\sqrt{\hat{H}_n-\eta-1}}(I+Z_n)^{-1}\frac{1}{\sqrt{\hat{H}_n-\eta-1}}, \tag{5.62}$$

where

$$Z_n=\frac{1}{\sqrt{\hat{H}_n-\eta-1}}\Gamma_n\left\{2I-\sqrt{-X_n+I}(B-X_n+I)^{-1}\sqrt{-X_n+I}\right\}\Gamma_n^*\frac{1}{\sqrt{\hat{H}_n-\eta-1}}. \tag{5.63}$$

By the strong resolvent convergence of $\{\hat{H}_n\}_{n=1}^\infty$ to $\hat{H}$ and by (5.44) one gets

$$I=s-\lim_{n\to\infty}(I+Z_n)^{-1}, \tag{5.64}$$

which yields

$$s-\lim_{n\to\infty}(\Omega_n(1)-\eta-1)^{-1}=(\hat{H}-\eta-1)^{-1}. \tag{5.65}$$

Since $\mathrm{dom}(\hat{\nu})$ is maximal with respect to W, we have

$$s-\lim_{n\to\infty}(B-X_n+I)^{-1}=0. \tag{5.66}$$

Therefore, (5.66) yields the existence of the limit $s-\lim_{n\to\infty}(B-X_n+I)^{-1/2}$. So, the sequence of operators $\{\sqrt{-W_n+I}P(B-X_n+I)^{-1/2}\}_{n=1}^\infty$ is uniformly bounded. Then we get

$$s-\lim_{n\to\infty}\sqrt{-W_n+I}P(B-X_n+I)^{-1}=0. \tag{5.67}$$

Using (5.40) and (5.44), we obtain

$$s-\lim_{n\to\infty}(B-X_n+I)^{-1}P\sqrt{-W_n+I}=0. \tag{5.68}$$

The estimate (5.50) gives

$$|V_n|\le -W_n+2I, \tag{5.69}$$

which implies

$$s-\lim_{n\to\infty}\sqrt{|V_n|}P(B-X_n+I)^{-1}=0. \tag{5.70}$$

Since $\{V_n\}_{n=1}^\infty$ converges in the strong resolvent sense to W, one has

$$s-\lim_{n\to\infty}(\sqrt{|V_n|}+I)^{-1}=(\sqrt{-W}+I)^{-1}. \tag{5.71}$$

Hence, we obtain

$$s-\lim_{n\to\infty}(B-X_n+I)^{-1}P\sqrt{|V_n|}\frac{1}{\sqrt{|V_n|}+I}=0 \tag{5.72}$$

However, the sequence $\{(B-X_n+I)^{-1/2}P\sqrt{|V_n|}\}_{n=1}^\infty$ is uniformly bounded. Therefore, we get

$$s-\lim_{n\to\infty}(B-X_n+I)^{-1/2}P\sqrt{|V_n|}=0, \tag{5.73}$$

which yields

$$s - \lim_{n\to\infty} (B - X_n + I)^{-1} P \sqrt{|V_n|} = 0 \tag{5.74}$$

and

$$s - \lim_{n\to\infty} \sqrt{|V_n|} P (B - X_n + I)^{-1} P \sqrt{|V_n|} = 0. \tag{5.75}$$

Taking into account (4.15) - (4.17), (5.42), the representation (5.62) as well as the convergences (5.64), (5.70), (5.74), (5.75) and $s - \lim_{n\to\infty} J_n = I$, we finally obtain

$$s - \lim_{n\to\infty} \sqrt{|V_n|} P (B - X_n + I)^{-1} P V_n (\Omega_n(1) - \eta - 1)^{-1} = 0, \tag{5.76}$$

$$s - \lim_{n\to\infty} (\Omega_n(1) - \eta - 1)^{-1} V_n P (B - X_n + I)^{-1} P \sqrt{|V_n|} = 0, \tag{5.77}$$

$$s - \lim_{n\to\infty} \sqrt{|V_n|} P (B - X_n + 1)^{-1} P V_n (\Omega_n(1) - \eta - 1)^{-1} V_n P (B - X_n + 1)^{-1} P \sqrt{|V_n|} = 0. \tag{5.78}$$

Then, using the representation (4.29) and convergences (5.65), (5.66), (5.70), (5.74), (5.75), (5.76) - (5.78), we get (5.36) - (5.39). •

Remark 5.5 Notice that the sequence $\{V_n\}_{n=1}^{\infty}$ defined by (5.42) depends on the subspace $\tilde{Q}_\eta$ but not on the chosen semi-bounded self-adjoint extension $\tilde{A}$ of A_0. Moreover, the above construction holds for any non-increasing regularizing sequence $\{W_n\}_{n=1}^{\infty}$ of W.

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