Generalized Abelian Deformations: Application to Nambu Mechanics

Giuseppe Dito[1*] and Moshé Flato[2]

[1] Research Institute for Mathematical Sciences
Kyoto University
Kitashirakawa, Oiwake-cho
Sakyo-ku, Kyoto 606-01 Japan

[2] Département de Mathématiques
Université de Bourgogne
BP 138, F-21004 Dijon Cedex France

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Abstract: We study Abelian generalized deformations of the usual product of polynomials introduced in [3]. We construct an explicit example for the case of $\mathfrak{su}(2)$ which provides a tentative of a quantum-mechanical description of Nambu Mechanics on $\mathbb{R}^3$. By introducing the notions of strong and weak triviality of generalized deformations, we show that the Zariski product is never trivial in either sense, while the example constructed here in a quantum-mechanical context is only strongly non-trivial.

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1 Introduction

The generalization of Hamiltonian Mechanics introduced more than 20 years ago by Nambu [6] has been recently formulated within a geometrical framework [7]. The importance of the so-called Fundamental Identity (FI) [5, 7] has been recognized as a dynamical consistency condition for Nambu's original formulation. Somehow the FI plays the rôle of the Jacobi identity for Poisson bracket in the usual Hamiltonian Mechanics, but its consequences are by far completely different as it imposes strong constraints on the underlying geometrical structure [7]. The reader is referred to [7] for further details on Nambu structures and manifolds, we shall limit ourselves to recall the definition. Let M be a m-dimensional manifold. Denote by N the algebra of smooth real-valued functions on M. A Nambu bracket of order n on M is a n-linear map on N taking values in N, denoted by $(f_1, \ldots, f_n) \mapsto \{f_1, \ldots, f_n\}$, $f_i \in N$, such that the following properties are satisfied for any functions $f_0, \ldots, f_{2n-1} \in N$:

a) Skew-symmetry

$$\{f_1, \ldots, f_n\} = \epsilon(\sigma)\{f_{\sigma_1}, \ldots, f_{\sigma_n}\}, \quad \forall \sigma \in S_n;$$

b) Leibniz rule

$$\{f_0 f_1, f_2, \ldots, f_n\} = f_0\{f_1, f_2, \ldots, f_n\} + \{f_0, f_2, \ldots, f_n\}f_1;$$

c) Fundamental Identity

$$\begin{aligned}\{f_1, \ldots, f_{n-1}, \{f_n, \ldots, f_{2n-1}\}\} &= \{\{f_1, \ldots, f_{n-1}, f_n\}, f_{n+1}, \ldots, f_{2n-1}\} \\ &\quad + \{f_n, \{f_1, \ldots, f_{n-1}, f_{n+1}\}, f_{n+2}, \ldots, f_{2n-1}\} \\ &\quad + \cdots + \{f_n, f_{n+1}, \ldots, f_{2n-2}, \{f_1, \ldots, f_{n-1}, f_{2n-1}\}\};\end{aligned}$$

where S_n is the group of permutations of the set $\{1, \ldots, n\}$ and $\epsilon(\sigma)$ is the sign of the permutation $\sigma \in S_n$. A Nambu bracket defines a Nambu structure on M. Then M is said to be a Nambu-Poisson manifold.

The quantization of Nambu structures turns out to be a non-trivial problem, even (or especially) in the simplest cases. Usual approaches to quantization have failed to give an appropriate solution, and in a common work [3] with D. Sternheimer and L. Takhtajan, we were led to introduce a new quantization scheme (Zariski quantization) in order to give a solution to that old problem. The central idea of Zariski quantization is to look first for an Abelian deformation of the usual product of functions instead of a direct deformation of the Nambu bracket. Then the quantization of the Nambu bracket is achieved by plugging in the Abelian deformed product into the classical Nambu bracket.

Abelian algebra deformations are classified according to the Harrison cohomology, and the second Harrison cohomology group is trivial for the space of polynomials on $\mathbb{R}^n$. One way to overcome this cohomological difficulty is to consider generalized deformations not of the usual

Gerstenhaber-type. In the case of Zariski quantization, linearity with respect to the deformation parameter does not hold: The deformed product operation annihilates the deformation parameter, so the usual Harrison cohomology is not applicable. This product is obtained by factorization of real polynomials into irreducible factors and complete symmetrization by a partial Moyal product. However the deformed product is not distributive when considered as a product between polynomials (it is only Abelian and associative). The linearization of this deformed product (by extending it to a semi-group algebra), called Zariski product, provides an Abelian algebra deformation of the usual product on the semi-group algebra generated by the set of real irreducible polynomials. By an appropriate presentation of this algebra which allows a coherent notion of derivatives, we can define in a natural way deformation of the classical Nambu bracket and, hence, a quantization of the usual Nambu structure on the semi-group algebra.

The Zariski quantization looks like a field quantization where irreducible polynomials are considered as one-particle states. One may wonder if, starting from the Moyal product, a quantum-mechanical approach would have been possible if one had made a decomposition into linear factors instead of irreducible ones. The answer is negative: Complete symmetrization by Moyal product of linear factors provides no quantization at all. However, as indicated in [3], the situation is completely different if one considers an invariant star-product on $\mathfrak{su}(2)^* \sim \mathbb{R}^3$ (the one which is associated with the quantization of angular momentum, the linear factors being here the generators of the Lie algebra). In that case it is possible to find a quantization by staying in the algebra of smooth functions on $\mathbb{R}^3$.

In this paper we shall work out some remarks stated in [3] regarding a possible quantum-mechanical approach (with finite number of degrees of freedom), non-triviality of the generalized deformations (i.e. not of Gerstenhaber-type) and develop the example for $\mathfrak{su}(2)^*$. It is shown in Section 2 that some kinds of $\mathbb{R}[\nu]$-linear deformations are not interesting in our context, in the sense that they cannot be associative without coinciding with the usual product. In Section 3, we shall make precise what we call a quantum-mechanical approach to generalized Abelian deformations. The questions of non-triviality and equivalence of these deformations are also studied. We introduce the notions of A- and B-equivalences (and the corresponding notions of triviality: strong and weak) for generalized deformations, and in particular we show that the Zariski product is non-trivial in both senses (weak and strong). A detailed example is presented in Section 4 for the $\mathfrak{su}(2)^*$-case by giving an explicit form of the deformed product in terms of differential operators, providing a strongly non-trivial deformation of the usual product on $\mathbb{R}^3$. We conclude this paper by some remarks on some spectrality properties of the generalized Abelian deformations and their application to the quantization of the Nambu bracket.

2 $\mathbb{R}[\nu]$-linear products

The generalized deformation introduced in [3] is not an algebra deformation in the sense of Gerstenhaber. Also, when restricted to the space of polynomials on $\mathbb{R}^n$, it is not additive due to the factorization into irreducible polynomials involved in the definition of the product. If instead we had performed a decomposition (by addition of monomials and multiplicative factorization) into *linear* polynomials, it would have been possible to stay in the framework of Gerstenhaber-type deformations by requiring linearity with respect to the deformation parameter. However, we shall show here that this attempt is not interesting: First, we would have found nothing but a trivial deformation of the usual product (the Harrison cohomology is trivial here) and, which is worst, one cannot expect to conciliate associativity of this product with quantization. We shall show that this kind of products are associative if and only if they coincide with the usual product of functions.

Consider $\mathbb{R}^n$ with coordinates denoted by $(x_1, \ldots, x_n)$. Let P be a Poisson bracket on $\mathbb{R}^n$ and $*$ a star-product on $\mathbb{R}^n$ with deformation parameter ν such that the star-bracket $[f, g]_* \equiv (f * g - g * f)/2\nu$, $f, g \in C^\infty(\mathbb{R}^n)$, defines a Lie algebra deformation of the Lie algebra $(C^\infty(\mathbb{R}^n), P)$ (see [1, 2] for the first extensive papers on star-products). Denote by N the algebra of real polynomials on $\mathbb{R}^n$ and by $N[\nu]$ the space of polynomials in ν with coefficients in N. The symmetric tensor algebra of N (with scalars) is denoted by $\mathcal{S}$ with symmetric tensor product $\otimes$. $\mathcal{S}[\nu]$ is the space of polynomials in ν with coefficients in $\mathcal{S}$.

Following the lines of [3], we define a $\mathbb{R}[\nu]$-linear map $\lambda \colon N[\nu] \to \mathcal{S}[\nu]$ by

$$\lambda(x_1^{k_1} \cdots x_n^{k_n}) = (x_1^{\overset{k_1}{\otimes}}) \otimes \cdots \otimes (x_n^{\overset{k_n}{\otimes}}), \quad \forall k_1, \cdots, k_n \geq 0. \tag{1}$$

In particular, $\lambda(1) = I$, the identity of $\mathcal{S}[\nu]$.

Let $T \colon \mathcal{S}[\nu] \to N[\nu]$ be the unique $\mathbb{R}[\nu]$-linear map defined by $T(I) = 1$ and

$$T(F_1 \otimes \cdots \otimes F_k) = \frac{1}{k!} \sum_{\sigma \in S_k} F_{\sigma_1} * \cdots * F_{\sigma_k}, \quad \forall k \geq 1, \tag{2}$$

where $F_i \in N$, $1 \leq i \leq k$.

Definition 1 *Let $*$ be a star-product on $\mathbb{R}^n$ endowed with some Poisson bracket. Let us define a new product $\circ_\nu$ on $N[\nu]$ by the following formula:*

$$F \circ_\nu G = T(\lambda(F) \otimes \lambda(G)), \quad \forall F, G \in N[\nu]. \tag{3}$$

It is a $\mathbb{R}[\nu]$-distributive Abelian product. We shall call it the $\circ_\nu$-product associated with $*$.

In general, the product defined by Eq. (3) is not associative, and the following shows that it is associative only in the trivial case:

Proposition 1 *Let $*$ be a star-product on $\mathbb{R}^n$ such that its associated $\circ_\nu$-product is associative, then the product $\circ_\nu$ is the usual pointwise product on N.*

Proof. It is easy to see that the map λ is an algebra homomorphism: $\lambda(FG) = \lambda(F) \otimes \lambda(G)$, $\forall F, G \in N[\nu]$. Hence the product $F \circ_\nu G$, $F, G \in N[\nu]$, defined by Eq. (3) can be written in the following form:

$$F \circ_\nu G = T(\lambda(FG)) = \sum_{r \geq 0} \nu^r \rho_r(FG), \tag{4}$$

where $\rho_0(FG) = FG$ and ρ_r, $r \geq 1$, are $\mathbb{R}[\nu]$-linear maps on $N[\nu]$ (whose restrictions to N are $\mathbb{R}$-linear map on N).

Using Eq. (4), the associativity of the product $\circ_\nu$ can be formulated as

$$\sum_{r \geq 0} \nu^r \sum_{\substack{u+v=r \\ u,v \geq 0}} \rho_u(\rho_v(FG)H) = \sum_{r \geq 0} \nu^r \sum_{\substack{u+v=r \\ u,v \geq 0}} \rho_u(F\rho_v(GH)), \quad \forall F, G, H \in N[\nu]. \tag{5}$$

Consider the last equation for $F, G, H \in N$. By identifying the coefficients of the different powers of ν on both side, we find for $r = 0$: FGH=FGH, and for $r = 1$:

$$\rho_1(FG)H = F\rho_1(GH), \quad \forall F, G, H \in N. \tag{6}$$

Set $G = H = 1$ in Eq. (6), one finds $\rho_1(F) = F\rho_1(1)$, $\forall F \in N$. By definition, $T(\lambda(1)) = 1$ and Eq. (4) implies $\rho_r(1) = 0$, $\forall r \geq 1$, thus $\rho_1(F) = 0$, $\forall F \in N$.

Now suppose that for some $k \geq 2$ we have $\rho_i(F) = 0$ for all $1 \leq i \leq k$, $\forall F \in N$. By equating the coefficients of ν^{k+1} on both sides of Eq. (5), we find that

$$\rho_{k+1}(FG)H = F\rho_{k+1}(GH), \quad \forall F, G, H \in N,$$

and by the same argument used for showing that $\rho_1(F) = 0$, $\forall F \in N$, we find $\rho_{k+1}(F) = 0$, $\forall F \in N$. In conclusion, $\rho_r(F) = 0$, $\forall r \geq 1$, $\forall F \in N$, and Eq. (4) gives $F \circ_\nu G = FG$, $\forall F, G \in N$ and this shows the Proposition.

3 Non-Triviality and Equivalence

In Zariski quantization, the map $\alpha \colon N[\nu] \to \mathcal{S}$ which replaces the usual product appearing in the decomposition of some polynomial into irreducible factors by the symmetric tensor product, is not $\mathbb{R}[\nu]$-multiplicative [3]. In order to keep associativity, the map α has to annihilate non-zero powers of ν. Also the space $N[\nu]$ endowed with the Zariski product is not an Abelian algebra, it is only an Abelian semi-group. Indeed the distributivity with respect to the addition of N is lacking due to the factorization into irreducible factors. However if one replaces "factorization into irreducible factors" by "factorization of monomials into linear factors (and linear combinations)" in the definition of Zariski products, then one does get an Abelian algebra structure on $N[\nu]$. In the case of the Moyal product, for reasons explained in the Introduction, this does not lead to a new product on $N[\nu]$, and one has to go through the construction developed in [3]. However there are other star-products than Moyal and in general it is possible to find a generalized Abelian algebra deformation in a quantum-mechanical framework [4]. The example of $\mathfrak{su}(2)$ will be treated to some extend in Section 4 (in this case the linear factors are generators of the Lie algebra $\mathfrak{su}(2)$, linear functions on $\mathfrak{su}(2)^* \sim \mathbb{R}^3$; they have a quadratic expression in $\mathbb{R}^6$).

After making precise what we call a quantum-mechanical approach by defining a deformed Abelian associative product on $N[\nu]$, we shall be concerned with the possible definitions of equivalence and triviality of Abelian generalized deformations. Also we shall discuss their consequences for both $\odot_\nu$-products studied in the present paper and Zariski products introduced in [3].

Let us first define an Abelian product in a way similar to what is done in [3], with the main difference that it is $\mathbb{R}$-distributive. Let $\pi\colon N[\nu] \to N$ be the natural projection of $N[\nu]$ onto N. Let $\lambda_0 = \lambda \circ \pi\colon N[\nu] \to \mathcal{S}$, where λ is defined by Eq. (1), and let $\tilde{T}\colon \mathcal{S} \to N[\nu]$ be the restriction to $\mathcal{S}$ of the map defined by Eq. (2).

Definition 2 *Let $*$ be a star-product on $\mathbb{R}^n$. We shall call the product $\odot_\nu$ on $N[\nu]$ defined by*

$$F \odot_\nu G = \tilde{T}(\lambda_0(F) \otimes \lambda_0(G)), \quad \forall F, G \in N[\nu], \tag{7}$$

the $\odot_\nu$-product associated with $*$.

Contrary to the $\circ_\nu$-products of Section 2, the $\odot_\nu$-products annihilate all non-zero powers of the deformation parameter ν. Clearly the product $\odot_\nu$ is Abelian and distributive with respect to the addition in $N[\nu]$, but is not $\mathbb{R}[\nu]$-multiplicative, i.e., in general $(aF)\odot_\nu G \neq a(F \odot_\nu G)$ for $a \in \mathbb{R}[\nu]$. However, it is always associative as shown by:

Lemma 1 *Let $*$ be a star-product on $\mathbb{R}^n$. Its associated $\odot_\nu$-product is associative.*

Proof. Simply notice that $\lambda_0(F \odot_\nu G) = \lambda_0(F) \otimes \lambda_0(G)$, $\forall F, G \in N[\nu]$, and thus

$$\begin{aligned} F \odot_\nu (G \odot_\nu H) &= \tilde{T}(\lambda_0(F) \otimes \lambda_0(G \odot_\nu H)) = \tilde{T}(\lambda_0(F) \otimes \lambda_0(G) \otimes \lambda_0(H)) \\ &= \tilde{T}(\lambda_0(F \odot_\nu G) \otimes \lambda_0(H)) = (F \odot_\nu G) \odot_\nu H, \quad \forall F, G, H \in N[\nu]. \end{aligned}$$

Hence $N[\nu]$ endowed with a $\odot_\nu$-product defines an Abelian associative generalized deformation of the usual product on N. This makes $(N[\nu], \odot_\nu)$ into an Abelian $\mathbb{R}$-algebra. Let us mention that in general a $\odot_\nu$-product associated with some star-product differs from the usual product on N, and the star-products which have the usual product as associated $\odot_\nu$-product are all of the Moyal-type [4]. More precisely, on the dual of a Lie algebra, there exists one and only one covariant star-product whose associated $\odot_\nu$-product is the usual product of polynomials.

We shall now be concerned with the question of equivalence of $\odot_\nu$-products. Several notions of equivalence can be adapted to the kind of generalized deformations considered in the present paper. Let us present a first definition of equivalence which is similar to the usual notion of equivalence for Gerstenhaber-type deformations for associative algebras. First notice that since the map λ_0 is a homomorphism and it annihilates non-zero powers of the parameter ν, it is easy to find that any $\odot_\nu$-product can be written in the form

$$F \odot_\nu G = FG + \sum_{r \geq 0} \nu^r \rho_r(FG), \quad F, G \in N, \tag{8}$$

where $\rho_r \colon N \to N$ are linear maps (the 'cochains' of the $\odot_\nu$-product). As for associative deformations, one way to understand equivalence of $\odot_\nu$-products is given by the following:

Definition 3 *Two products $\odot_\nu$ and $\odot'_\nu$ are said to be A-equivalent, if there exists a $\mathbb{R}[\nu]$-linear (formally invertible) map $S_\nu \colon N[[\nu]] \mapsto N[[\nu]]$ of the form $S_\nu = \sum_{r \geq 0} \nu^r S_r$, where $S_r \colon N \mapsto N$, $r \geq 1$, are differential operators and $S_0 = Id$, such that*

$$S_\nu(F \odot_\nu G) = S_\mu(F) \odot'_\nu S_\mu(G)\big|_{\mu=\nu}, \quad F, G \in N,$$

which amounts to

$$\sum_{r,s \geq 0} \nu^{r+s} S_s(\rho_r(FG)) = \sum_{r,s,s' \geq 0} \nu^{r+s+s'} \rho'_r(S_s(F) S_{s'}(G)), \quad F, G \in N,$$

where ρ_r (resp. ρ'_r) are the cochains of the product $\odot_\nu$ (resp. $\odot'_\nu$).

It is straightforward to check that Definition 3 indeed defines an equivalence relation between $\odot_\nu$-products. One has the associated notion of triviality given by:

Definition 4 *A $\odot_\nu$-product associated with some star-product is said to be strongly trivial if it is A-equivalent to the usual product, i.e., there exists a $\mathbb{R}[\nu]$-linear (formally invertible) map $S_\nu \colon N[[\nu]] \mapsto N[[\nu]]$ of the form $S_\nu = \sum_{r \geq 0} \nu^r S_r$, where $S_r \colon N \mapsto N$, $r \geq 1$, are differential operators and $S_0 = Id$, such that:*

$$S_\nu(F \odot_\nu G) = S_\nu(F) \cdot S_\nu(G), \quad \forall F, G \in N, \tag{9}$$

where $\cdot$ denotes the usual product.

The following shows that in general a $\odot_\nu$-product is strongly non-trivial.

Proposition 2 *A $\odot_\nu$-product is strongly trivial if and only if it coincides with the usual product.*

Proof. Let $\odot_\nu$ be strongly trivial. Since $1\odot_\nu 1 = 1$, the map S_ν in (9) must satisfy $S_\nu(1) = S_\nu(1)\cdot S_\nu(1)$, which implies $S_\nu(1) = 1$, i.e., $S_r(1) = 0$, $\forall r \geq 1$.

The map λ_0 in (7) is a homomorphism, it implies that the product $F\odot_\nu G$, $F, G \in N$, can be expressed as

$$F\odot_\nu G = \sum_{r\geq 0} \nu^r \rho_r(FG), \quad F, G \in N,$$

where $\rho_0(FG) = FG$ and ρ_r, $r \geq 1$, are linear maps on N. By substituting this last expression for $F\odot_\nu G$ in Eq. (9), and by identifying the different powers of ν on both sides, we find:

$$\sum_{\substack{r+s=t\\ r,s\geq 0}} S_r(\rho_s(FG)) = \sum_{\substack{r+s=t\\ r,s\geq 0}} S_r(F)S_s(G), \quad \forall t \geq 0, F, G \in N. \tag{10}$$

Now set $G = 1$ in the preceding equality. Since $S_r(1) = 0$ for $r \geq 1$, Eq. (10) yields

$$S_t(F) = \sum_{\substack{r+s=t\\ r,s\geq 0}} S_r(\rho_s(F)), \quad \forall t \geq 0, F \in N. \tag{11}$$

For $t = 1$, we find that $\rho_1(F) = 0$, $\forall F \in N$. For $t \geq 2$, Eq. (11) can be written in the following form:

$$S_t(F) = S_t(F) + \sum_{\substack{r+s=t\\ r,s\geq 1}} S_r(\rho_s(F)) + \rho_t(F) \quad \forall t \geq 2, F \in N,$$

i.e, $\rho_t(F) = -\sum_{\substack{r+s=t\\ r,s\geq 1}} S_r(\rho_s(F))$, $\forall t \geq 2, F \in N$. Since $\rho_1(F) = 0$, $\forall F \in N$, it is easy to show by induction that $\rho_t(F) = 0$, $\forall t \geq 1, F \in N$. Therefore, if $\odot_\nu$ is strongly trivial, it coincides with the usual product. The converse statement being obvious, the Proposition holds.

Actually the treatment done above applies almost straightforwardly to the case of Zariski quantization yielding that the Zariski product is never strongly trivial. Let us denote by $\bullet_\nu$ the Zariski product constructed out from some star-product which is defined on the algebra $\mathcal{A}_\nu$ (the reader is referred to [3] for the definitions of $\bullet_\nu$ and $\mathcal{A}_\nu$; the deformation parameter ν was taken to be $\hbar$ in the preceding reference). The product $\bullet_\nu$ has also the form given by Eq. (8) where now the maps ρ_r are linear maps on $\mathcal{A}_0$, the classical algebra, and the "usual" product has to be interpreted as the product on the algebra $\mathcal{A}_0$. Recall that there are derivations defined on $\mathcal{A}_\nu$ [3] and hence we have a natural definition of "differential operators" acting on $\mathcal{A}_\nu$. By an obvious adaptation of Definitions 3 and 4, one gets the corresponding definitions of A-equivalence and strong triviality for Zariski products. Notice that the proof of Proposition 2 applies literally to the case of Zariski products (the differential aspect of the intertwining operator is not involved in the proof). However, by construction, a Zariski product constructed out from some star-product can never coincide with the "usual product" on $\mathcal{A}_0$ (of course, except in the trivial situation where the star-product is the usual product of polynomials on $\mathbb{R}^n$). This leads to:

Theorem 1 *Let $\bullet_\nu$ be a Zariski product associated with some star-product $*_\nu$ on $\mathbb{R}^n$, then $\bullet_\nu$ is strongly non-trivial whenever $*_\nu$ is not the usual product.*

Another natural definition of equivalence for the generalized deformations that one can consider is given by the following:

Definition 5 *Two products $\odot_\nu$ and $\odot'_\nu$ are said to be B-equivalent, if there exists a $\mathbb{R}[\nu]$-linear (formally invertible) map $S_\nu\colon N[[\nu]] \mapsto N[[\nu]]$ of the form $S_\nu = \sum_{r\geq 0} \nu^r S_r$, where $S_r\colon N \mapsto N$, $r \geq 1$, are differential operators and $S_0 = Id$, such that:*

$$S_\nu(F \odot_\nu G) = S_\nu(F) \odot'_\nu S_\nu(G), \quad F, G \in N,$$

Since the $\odot_\nu$-product annihilates the parameter ν, the preceding definition is equivalent to say that $S_\nu(F \odot_\nu G) = F \odot'_\nu G$, $F, G \in N$, holds. The difference between this definition of equivalence and that given by Definition 3 lies in the fact the latter is formulated in such a way that the terms involving the powers of the parameter ν introduced by the intertwining operator S_ν are taken into account, while the former definition allows the product operation to annihilate them. Also notice that two $\odot_\nu$-products which are B-equivalent are not necessarily A-equivalent.

The notion of triviality corresponding to B-equivalence (weaker than strong triviality, cf. Proposition 3, hence the choice of terminology) is:

Definition 6 *A $\odot_\nu$-product associated with some star-product is said to be weakly trivial if it is B-equivalent to the usual product, i.e., there exists a $\mathbb{R}[\nu]$-linear (formally invertible) map $S_\nu\colon N[[\nu]] \mapsto N[[\nu]]$ of the form $S_\nu = \sum_{r\geq 0} \nu^r S_r$, where $S_r\colon N \mapsto N$, $r \geq 1$, are differential operators and $S_0 = Id$, such that:*

$$S_\nu(F \cdot G) = F \odot_\nu G, \quad \forall F, G \in N,$$

where $\cdot$ denotes the usual product.

By adequate modifications, these definitions also apply to the case of Zariski products.

Let us study the consequences of these definitions for the $\odot_\nu$-products and then for the Zariski products. Consider a $\odot_\nu$-product whose cochains are given by differential operators (e.g. the $\odot_\nu$-product of Section 4). Then the $\odot_\nu$-product has the form $F \odot_\nu G = \rho(\pi(F \cdot G))$, $F, G \in N[\nu]$, where $\rho = Id + \sum_{r\geq 1} \nu^r \rho_r$ and the ρ_r's are differential operators acting on N, and $\pi\colon N[\nu] \mapsto N$ is the projection onto the classical part. Note that ρ is formally invertible, and we have $\rho(F \cdot G) = F \odot_\nu G$, $F, G \in N$, therefore ρ intertwines the $\odot_\nu$-product with the usual product in the sense of Definition 6. Hence we have shown the following:

Proposition 3 *A $\odot_\nu$-product is weakly trivial whenever its cochains are given by differential operators.*

For a Zariski product $\bullet_\nu$, it is true that the corresponding ρ intertwines $\bullet_\nu$ with the usual product on $\mathcal{A}_0$, but the cochains ρ_r are never given by differential operators acting on $\mathcal{A}_0$. Hence

Corollary 1 *A Zariski product is weakly non-trivial (except when the defining star-product is the usual product).*

Remark 1 One may wonder why we restrict the intertwining operators to be defined by *differentiable* cochains as in general the deformations considered are not defined by differentiable cochains. This is so because eventually the non-triviality for the Abelian generalized deformations should be linked with the usual (differentiable) Hochschild or Harrison cohomologies.

Remark 2 Of course the map $T \circ \alpha$ which defines the Zariski product satisfies

$$T \circ \alpha(F) \bullet_\nu T \circ \alpha(G) = F \bullet_\nu G = T \circ \alpha(F \cdot G), \quad \forall F, G \in \mathcal{A}_0,$$

where $\cdot$ denotes the product on the classical algebra $\mathcal{A}_0$. But the map $T \circ \alpha$ is neither invertible as a formal series nor acting by differential operators. Hence the map $T \circ \alpha$ is not an intertwiner trivializing the Zariski product in the sense of Definition 6. Even if one defines S_ν by taking the $\mathbb{R}[[\nu]]$-linear extension of the restriction of $T \circ \alpha$ to $\mathcal{A}_0$, one gets an invertible map but not an intertwining map (in the sense of B-equivalence) as S_ν is not given by differential operators on $\mathcal{A}_0$.

To sum up, the kind of Abelian deformations defined by a Zariski product considered in [3] are both strongly and weakly non-trivial. However the example developed in Section 4 is strongly non-trivial but weakly trivial. We think that the above considerations may give some hints for the definition of an appropriate cohomology for Abelian generalized deformations.

4 Example of $\mathfrak{su}(2)$

Here we shall study in some details an example of $\odot_\nu$-product associated with an invariant star-product on the dual of the $\mathfrak{su}(2)$-Lie algebra ($\sim \mathbb{R}^3$). This $\odot_\nu$-product is strongly non-trivial in the sense of Definitions 3 and 4. An explicit formula for that $\odot_\nu$-product will be given, which permits to extend the $\odot_\nu$-product to $C^\infty(\mathbb{R}^3)$ (valued in $C^\infty(\mathbb{R}^3)[[\nu]]$, whereto it extends trivially since the $\odot_\nu$-product annihilates ν). Let $*_M$ be the Moyal product on $\mathbb{R}^6$:

$$F *_M G = \sum_{r \geq 0} \nu^r P_M^{(r)}(F, G), \quad \forall F, G \in C^\infty(\mathbb{R}^6),$$

where $P_M^{(r)}$ is the r-th power of the Poisson bracket:

$$P_M(F, G) = \sum_{1 \leq i \leq 3} \Big(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \Big).$$

Consider the functions $L_i(p,q) = \sum_{1\le j,k\le 3} \varepsilon_{ijk} p_j q_k$, $1 \le i \le 3$, on $\mathbb{R}^6$, where ε_{ijk} is the totally skew-symmetric tensor with $\varepsilon_{123} = 1$. The L_i's realize the Lie algebra $\mathfrak{su}(2)$ for the Moyal bracket $[\cdot,\cdot]_M$ on $\mathbb{R}^6$, i.e.,

$$[L_i, L_j]_M \equiv \frac{L_i *_M L_j - L_j *_M L_i}{2\nu} = \sum_{1\le k\le 3} \varepsilon_{ijk} L_k, \quad 1 \le i,j \le 3.$$

One can easily show that for any polynomial $F\colon \mathbb{R}^3 \to \mathbb{R}$, the function $F(L_1, L_2, L_3)$ on $\mathbb{R}^6$ satisfies

$$L_i *_M F = L_i F + \nu \sum_{1\le j,k\le 3} \varepsilon_{ijk} L_k \frac{\partial F}{\partial L_j} + \nu^2 \Big(2 \frac{\partial F}{\partial L_i} + \sum_{1\le j\le 3} L_j \frac{\partial^2 F}{\partial L_i \partial L_j} \Big), \quad 1 \le i \le 3. \tag{12}$$

We observe that $L_i *_M F(L_1, L_2, L_3)$ is still a polynomial in the variables (L_1, L_2, L_3). By induction on k, it is readily shown that $L_{i_1} *_M \cdots *_M L_{i_k}$ is a polynomial in (L_1, L_2, L_3). Also any polynomial in (L_1, L_2, L_3) can be expressed as a $*_M$-polynomial of the L_i's, so the product $F *_M G$ of two polynomials F, G in (L_1, L_2, L_3) is a polynomial in (L_1, L_2, L_3). Hence from the Moyal product $*_M$ on $\mathbb{R}^6$, we get a star-product on $\mathbb{R}^3$ satisfying Eq. (12) for any polynomial F. This star-product, that we shall denote by $*$, is actually an invariant (and covariant) star-product on $\mathfrak{su}(2)^* \sim \mathbb{R}^3$.

Let $\odot_\nu$ be the $\odot_\nu$-product associated with $*$. Remark that

$$L_i * L_j = L_i L_j + \nu \sum_{1\le k\le 3} \varepsilon_{ijk} L_k + 2\nu^2 \delta_{ij}, \quad 1 \le i,j \le 3,$$

and $L_i \odot_\nu L_j = L_i L_j + 2\nu^2 \delta_{ij}$, so this $*$-product does provide quantum terms. In the following we shall derive an explicit expression for $\odot_\nu$. Before stating the main result of this section, we need some combinatorial preliminaries. We shall denote the cochains of the star-product $*$ on $\mathbb{R}^3$ by C_r, i.e., $F * G = \sum_{r\ge 0} \nu^r C_r(F,G)$, where $C_1(F,G) = P(F,G) \equiv \sum_{1\le i,j,k\le 3} \varepsilon_{ijk} L_k \frac{\partial F}{\partial L_i} \frac{\partial G}{\partial L_j}$ is the standard Poisson bracket on $\mathfrak{su}(2)^* \sim \mathbb{R}^3$. We denote by Δ the Laplacian operator: $\Delta = \sum_{1\le k\le 3} \frac{\partial^2}{\partial L_k^2}$.

Lemma 2 *For any $n \ge 1$ and any indices $i_1, \ldots, i_n \in \{1,2,3\}$, we have:*

a) $\displaystyle\sum_{(i_1,\ldots,i_n)} \frac{\partial}{\partial L_{i_1}} (L_{i_2} \cdots L_{i_n}) = \Delta(L_{i_1} \cdots L_{i_n});$

b) $\displaystyle\sum_{(i_1,\ldots,i_n)} L_{i_1} \Delta^m (L_{i_2} \cdots L_{i_n}) = (n - 2m) \Delta^m (L_{i_1} \cdots L_{i_n}), \quad \forall m \ge 0;$

c) $\displaystyle\sum_{(i_1,\ldots,i_n)} P(L_{i_1}, \Delta^m (L_{i_2} \cdots L_{i_n})) = 0, \quad \forall m \ge 0;$

d) $\displaystyle\sum_{(i_1,\ldots,i_n)} C_2(L_{i_1}, \Delta^m (L_{i_2} \cdots L_{i_n})) = (n - 2m) \Delta^{m+1} (L_{i_1} \cdots L_{i_n}), \quad \forall m \ge 0;$

where $\sum\limits_{(i_1,\ldots,i_n)}$ *stands for summation over cyclic permutations of* $(i_1,\ldots,i_n)$*, and we set* $L_{i_2}\cdots L_{i_n}=1$ *when* $n=1$*.*

Proof. Statement a) is proved by performing the derivations with respect to the L_i's on both side. We have that

$$\frac{\partial}{\partial L_{i_1}}(L_{i_2}\cdots L_{i_n})=\sum_{2\le k\le n}\delta_{i_1 i_k}L_{i_2}\cdots\hat{L}_{i_k}\cdots L_{i_n},$$

where the symbol $\hat{}$ stands for omission. The sum on the right-hand side can be written as $\sum_{\substack{1\le k\le n\\ k\ne 1}}\delta_{i_1 i_k}\hat{L}_{i_1}L_{i_2}\cdots\hat{L}_{i_k}\cdots L_{i_n}$. The sum over cyclic permutations gives

$$\begin{aligned}\sum_{(i_1,\ldots,i_n)}\frac{\partial}{\partial L_{i_1}}(L_{i_2}\cdots L_{i_n}) &= \sum_{(i_1,\ldots,i_n)}\sum_{\substack{1\le k\le n\\ k\ne 1}}\delta_{i_1 i_k}\hat{L}_{i_1}L_{i_2}\cdots\hat{L}_{i_k}\cdots L_{i_n}\\ &= \sum_{1\le s\le n}\sum_{\substack{1\le k\le n\\ k\ne s}}\delta_{i_s i_k}L_{i_1}\cdots\hat{L}_{i_s}\cdots\hat{L}_{i_k}\cdots L_{i_n}.\end{aligned}$$

A straightforward calculation shows that the right-hand side of the last equality is equal to $\Delta(L_{i_1}\cdots L_{i_n})$ and therefore statement a) is true.

The following identity is easily established by induction on k:

$$\Delta^k(L_iF)=L_i\Delta^k(F)+2k\Delta^{k-1}\frac{\partial F}{\partial L_i},\quad \forall F\in N, k\ge 1, 1\le i\le 3. \tag{13}$$

Statement b) is clearly true for $m=0$. Suppose that it is valid for $m=k-1$, $k\ge 1$, then by using identity (13), we find that

$$\Delta^k(L_{i_1}\cdots L_{i_n})=L_{i_1}\Delta^k(L_{i_2}\cdots L_{i_n})+2k\Delta^{k-1}\frac{\partial}{\partial L_{i_1}}(L_{i_2}\cdots L_{i_n}).$$

Summing over cyclic permutations on both sides of the last equation and using statement a) of the Lemma, gives

$$n\Delta^k(L_{i_1}\cdots L_{i_n})=\sum_{(i_1,\ldots,i_n)}L_{i_1}\Delta^k(L_{i_2}\cdots L_{i_n})+2k\Delta^k(L_{i_1}\cdots L_{i_n}),$$

which shows that statement b) is true for $m=k$, hence statement b) holds.

In order to show c), we apply identity (13) to the Poisson bracket $P(L_i,\Delta^m F)=\sum_{1\le j,k\le 3}\varepsilon_{ijk}L_k\Delta^m\frac{\partial F}{\partial L_j}$ and find that

$$\Delta^m(P(L_i,F))=\sum_{1\le j,k\le 3}\varepsilon_{ijk}L_k\frac{\partial}{\partial L_j}\Delta^m F+2m\sum_{1\le j,k\le 3}\varepsilon_{ijk}\Delta^m\frac{\partial^2 F}{\partial L_j\partial L_k}.$$

The first term on the right-hand side is simply $P(L_i, \Delta^m F)$, while the second term vanishes due to the skew-symmetry of ε_{ijk}, hence we have found that $\Delta^m(P(L_i, F)) = P(L_i, \Delta^m F)$, $\forall F \in N, m \geq 0$. Setting $F = L_{i_2} \cdots L_{i_n}$ in this relation and summing over cyclic permutations yield

$$\sum_{(i_1,\ldots,i_n)} P(L_{i_1}, \Delta^m(L_{i_2} \cdots L_{i_n})) = \Delta^m(\sum_{(i_1,\ldots,i_n)} P(L_{i_1}, L_{i_2} \cdots L_{i_n})),$$

whose right-hand side vanishes due to Leibniz property and skew-symmetry of the Poisson bracket. This shows statement c).

The differential operator $\mathcal{D} = \sum_{1\leq k\leq 3} L_k \frac{\partial}{\partial L_k}$ acting on a homogeneous polynomial F of degree $\deg(F)$ gives $\mathcal{D}(F) = \deg(F)F$. The second cochain C_2 in Eq. (12) can be written as

$$C_2(L_i, F) = (2 + \mathcal{D})(\frac{\partial F}{\partial L_i}).$$

Let $F = \Delta^m(L_{i_2} \cdots L_{i_n})$ in the preceding equation and sum over cyclic permutations; this gives

$$\sum_{(i_1,\ldots,i_n)} C_2(L_{i_1}, \Delta^m(L_{i_2} \cdots L_{i_n})) = (2 + \mathcal{D})\Delta^m\Big(\sum_{(i_1,\ldots,i_n)} \frac{\partial}{\partial L_{i_1}}(L_{i_2} \cdots L_{i_n})\Big).$$

The right-hand side of this equation is equal to $(2+\mathcal{D})\Delta^{m+1}(L_{i_1} \cdots L_{i_n})$ by statement a) of the Lemma; since $\Delta^{m+1}(L_{i_1} \cdots L_{i_n})$ is a homogeneous polynomial of degree $n - 2m + 2$, we have $(2 + \mathcal{D})\Delta^{m+1}(L_{i_1} \cdots L_{i_n}) = (n - 2m)\Delta^{m+1}(L_{i_1} \cdots L_{i_n})$, and this shows statement d).

Using this technical lemma, we are now in position to prove the central result of this section.

Proposition 4 *Let H_n be the subspace of N consisting of homogeneous polynomials of degree n. There exist constants $a(n, r)$, $n, r \geq 0$, such that the product $\odot_\nu$ can be written*

$$F \odot_\nu G = \sum_{r\geq 0} \nu^{2r} \eta_r(FG), \quad \forall F, G \in N,$$

where η_0 is the identity and $\eta_r : N \to N$, $r \geq 1$, are linear maps whose restrictions to H_n are given by

$$\eta_r|_{H_n} = a(n, r)\Delta^r, \quad \forall n, r \geq 0.$$

Proof. In Section 2, we saw that the product $\odot_\nu$ can be written in the form

$$F \odot_\nu G = \sum_{r\geq 0} \nu^r \rho_r(FG), \quad \forall F, G \in N,$$

where $\rho_0 = Id$ and for $r \geq 1$, ρ_r is a linear map on N. Consider the product $L_{i_1} \odot_\nu \cdots \odot_\nu L_{i_n}$ for any $i_k \in \{1, 2, 3\}$, $\forall 1 \leq k \leq n$. By definition, it is given by

$$L_{i_1} \odot_\nu \cdots \odot_\nu L_{i_n} = \frac{1}{n!} \sum_{\sigma\in S_n} L_{i_{\sigma_1}} * \cdots * L_{i_{\sigma_n}},$$

which can be expressed as

$$L_{i_1}\odot_\nu\cdots\odot_\nu L_{i_n}=\frac{1}{n}\sum_{(i_1,\dots,i_n)}L_{i_1}*(L_{i_2}\odot_\nu\cdots\odot_\nu L_{i_n}), \tag{14}$$

where $\sum_{(i_1,\dots,i_n)}$ denotes sum over cyclic permutations. Using the form (14) for the product $\odot_\nu$ and taking into account that $L_i*F=L_iF+\nu P(L_i,F)+\nu^2C_2(L_i,F)$, $\forall F\in N$, $1\le i\le 3$, we find that the coefficient of the r-th power of ν in $L_{i_1}\odot_\nu\cdots\odot_\nu L_{i_n}$ satisfies the following induction relation for $k\ge 2$ and $n\ge 1$ (when $n=1$, we set $L_{i_2}\cdots L_{i_n}=1$):

$$\rho_k(L_{i_1}\cdots L_{i_n})= \tag{15}$$

$$\frac{1}{n}\sum_{(i_1,\dots,i_n)}\Big(L_{i_1}\rho_k(L_{i_2}\cdots L_{i_n})+P(L_{i_1},\rho_{k-1}(L_{i_2}\cdots L_{i_n}))+C_2(L_{i_1},\rho_{k-2}(L_{i_2}\cdots L_{i_n}))\Big),$$

and for $k=1$ and $n\ge 1$:

$$\rho_1(L_{i_1}\cdots L_{i_n})=\frac{1}{n}\sum_{(i_1,\dots,i_n)}\Big(L_{i_1}\rho_1(L_{i_2}\cdots L_{i_n})+P(L_{i_1},L_{i_2}\cdots L_{i_n})\Big). \tag{16}$$

The second term on the right-hand side of Eq. (16) vanishes by skew-symmetry of the Poisson bracket. From $1\odot_\nu 1=1$, we have $\rho_1(1)=0$. By induction on n, we easily find that $\rho_1(L_{i_1}\cdots L_{i_n})=0$, $\forall n\ge 0$, i.e., $\rho_1=0$.

Let Z_k, $k\ge 0$, be the following property:

$$\rho_{2k}|_{H_n}=a(n,k)\Delta^k,\quad \forall n\ge 0,\ \text{and}\ \rho_{2k+1}=0, \tag{Z_k}$$

where $a(n,k)$, $n,k\ge 0$, is a constant (notice that for $n<2k$, Δ^k vanishes on H_n, and in that case the constant $a(n,k)$ can be arbitrary). The result will be proved by induction on k.

First, remark that Z_0 is true with $a(n,0)=1$, $\forall n\ge 0$. Suppose that Z_k is true for $0\le k\le r-1$, for some $r\ge 1$. By hypothesis $\rho_{2r-1}=0$ and $\rho_{2r-2}|_{H_n}=a(n,r-1)\Delta^{r-1}$, so the induction relation (15) takes the form

$$\rho_{2r}(L_{i_1}\cdots L_{i_n})=$$

$$\frac{1}{n}\sum_{(i_1,\dots,i_n)}\Big(L_{i_1}\rho_{2r}(L_{i_2}\cdots L_{i_n})+a(n-1,r-1)C_2(L_{i_1},\Delta^{r-1}(L_{i_2}\cdots L_{i_n}))\Big),\quad \forall n\ge 1,$$

and by application of statement d) of Lemma 2 to the second term on the right-hand side, we find that

$$\rho_{2r}(L_{i_1}\cdots L_{i_n})= \tag{17}$$

$$\frac{1}{n}\Big(\sum_{(i_1,\dots,i_n)}L_{i_1}\rho_{2r}(L_{i_2}\cdots L_{i_n})\Big)+\frac{a(n-1,r-1)(n-2r+2)}{n}\Delta^r(L_{i_1}\cdots L_{i_n}),\quad \forall n\ge 1,$$

the case $n = 0$ is trivially verified by observing that $\rho_{2r}(1) = 0$ for $r \geq 1$.

Now we will show by induction on n that any linear map γ on N which satisfies along with $\gamma(1) = 0$, the following relation for some given $r \geq 1$:

$$\gamma(L_{i_1} \cdots L_{i_n}) = \frac{1}{n}\Big(\sum_{(i_1,\ldots,i_n)} L_{i_1}\gamma(L_{i_2} \cdots L_{i_n})\Big) + \alpha_n \Delta^r(L_{i_1} \cdots L_{i_n}), \quad \forall n \geq 1, \tag{18}$$

where the α_n's are constants, must be of the form $\gamma(L_{i_1} \cdots L_{i_n}) = \beta_n \Delta^r(L_{i_1} \cdots L_{i_n})$, i.e. $\gamma|_{H_n} = \beta_n \Delta^r$ for some constants β_n. Suppose $\gamma|_{H_m} = \beta_m \Delta^r$ for $0 \leq m \leq n-1$, then the induction relation (18) can be written as

$$\gamma(L_{i_1} \cdots L_{i_n}) = \frac{\beta_{n-1}}{n}\Big(\sum_{(i_1,\ldots,i_n)} L_{i_1}\Delta^r(L_{i_2} \cdots L_{i_n})\Big) + \alpha_n \Delta^r(L_{i_1} \cdots L_{i_n}), \quad \forall n \geq 1,$$

which by statement b) of Lemma 2 yields

$$\gamma(L_{i_1} \cdots L_{i_n}) = \Big((n-2r)\frac{\beta_{n-1}}{n} + \alpha_n\Big)\Delta^r(L_{i_1} \cdots L_{i_n}), \quad \forall n \geq 1,$$

and this shows that γ must be proportional to Δ^r on every H_n, $n \geq 0$, (the case $n = 0$ being trivially verified when $r \geq 1$, since $\Delta^r(1) = 0$).

Let us apply the preceding result to the induction relation (17); we readily find that $\rho_{2r}(L_{i_1} \cdots L_{i_n}) = a(n,r)\Delta^r(L_{i_1} \cdots L_{i_n})$ and the constant $a(n,r)$ is given by

$$a(n,r) = \frac{1}{n}\Big((n-2r)a(n-1,r) + (n-2r+2)a(n-1,r-1)\Big), \quad \forall n \geq 1, \tag{19}$$

Hence $\rho_{2r}|_{H_n} = a(n,r)\Delta^r$ and in order to complete the proof we only need to show that $\rho_{2r+1} = 0$. Under the induction hypothesis we have $\rho_{2r-1} = 0$, and we just have shown that $\rho_{2r}|_{H_n} = a(n,r)\Delta^r$, hence the induction relation (15) for ρ_{2r} yields

$$\rho_{2r+1}(L_{i_1} \cdots L_{i_n}) =$$

$$\frac{1}{n} \sum_{(i_1,\ldots,i_n)} \Big(L_{i_1}\rho_{2r+1}(L_{i_2} \cdots L_{i_n}) + a(n-1,r)P(L_{i_1}, \Delta^r(L_{i_2} \cdots L_{i_n}))\Big), \quad \forall n \geq 1.$$

By statement c) of Lemma 2, the second term on the right-hand side vanishes and we are left with

$$\rho_{2r+1}(L_{i_1} \cdots L_{i_n}) = \frac{1}{n} \sum_{(i_1,\ldots,i_n)} L_{i_1}\rho_{2r+1}(L_{i_2} \cdots L_{i_n}), \quad \forall n \geq 1.$$

We have $\rho_{2r+1}(1) = 0$ and by induction on n we easily find that $\rho_{2r+1}(L_{i_1} \cdots L_{i_n}) = 0$, $\forall n \geq 0$, i.e, $\rho_{2r+1} = 0$. This shows that the property Z_r is true and completes the proof.

The expression of the constants $a(n,r)$ appearing in the statement of Proposition 4 (also cf. Eq. (19)) can be written explicitly with the help of a generating function. Moreover, for $n \geq 2r$, the $a(n,r)$'s are polynomials of degree r in n. This fact will allow to find an expression for the product $\odot_\nu$ in terms of differential operators.

Lemma 3 *For $n \geq 2r$, the constants $a(n,r)$ of Proposition 4 are given by:*

$$a(n,r) = \frac{1}{n!}\left(\frac{\partial}{\partial\alpha}\right)^n \left((\cos(\alpha))^{-2}(\tan(\alpha))^{n-2r}\right)\Big|_{\alpha=0}, \quad n \geq 2r.$$

Moreover there exist polynomials p_r, $r \geq 0$, of degree r such that for $n \geq 2r$ we have $a(n,r) = p_r(n)$.

Proof. Let X be any generator of $\mathfrak{su}(2)$. The $*$-powers and the $\odot_\nu$-powers of X coincide and the $*$-exponential [1, 2] and the $\odot_\nu$-exponential [3] of X are identical. Here we fix the deformation parameter ν to be $i\hbar/2$, where $\hbar$ is real. The $*$-exponential of X has been explicitly computed in [2]:

$$\exp_*\left(\frac{tX}{i\hbar}\right) \equiv \sum_{n\geq 0}\frac{1}{n!}(\frac{t}{i\hbar})^n X^{\overset{n}{*}} = \cos^{-2}(t/2)\exp(\frac{2X}{i\hbar}\tan(t/2)), \tag{20}$$

for t in a neighborhood of the origin. By Proposition 4 we have (we denote $\odot_{i\hbar/2}$ by $\odot$)

$$X^{\overset{n}{\odot}} = \sum_{r=0}^{[n/2]}(\frac{i\hbar}{2})^{2r}a(n,r)\Delta^r X^n = \sum_{r=0}^{[n/2]}(\frac{i\hbar}{2})^{2r}a(n,r)\frac{n!}{(n-2r)!}X^{n-2r}, \tag{21}$$

where $[n/2]$ denotes the integer part of $n/2$. The function $\phi(\alpha,\beta) = \cos^{-2}(\alpha)\exp(\beta\tan(\alpha))$ is analytic in a neighborhood of the origin of $\mathbb{C}^2$ and notice that $\exp_*\left(\frac{tX}{i\hbar}\right) = \exp_\odot\left(\frac{tX}{i\hbar}\right) = \phi(\frac{t}{2},\frac{2X}{i\hbar})$. By Eq. (21), we have for t in a neighborhood of 0,

$$\exp_\odot\left(\frac{tX}{i\hbar}\right) \equiv \sum_{n\geq 0}\frac{1}{n!}(\frac{t}{i\hbar})^n X^{\overset{n}{\odot}} = \sum_{n\geq 0}\sum_{r=0}^{[n/2]}(\frac{t}{i\hbar})^n(\frac{i\hbar}{2})^{2r}\frac{a(n,r)}{(n-2r)!}X^{n-2r}$$

or

$$\phi(\frac{t}{2},\frac{2X}{i\hbar}) = \sum_{n\geq 0}\sum_{r=0}^{[n/2]}\frac{a(n,r)}{(n-2r)!}(\frac{t}{2})^n(\frac{2X}{i\hbar})^{n-2r},$$

so $\frac{a(n,r)}{(n-2r)!}$, for $n \geq 2r$, appears to be the coefficient of $\alpha^n\beta^{n-2r}$ in the Taylor series around the origin of $\phi(\alpha,\beta)$, hence

$$a(n,r) = \frac{1}{n!}\left(\frac{\partial}{\partial\alpha}\right)^n\left(\frac{\partial}{\partial\beta}\right)^{n-2r}\left((\cos(\alpha))^{-2}(\exp\beta\tan(\alpha))\right)\Big|_{\alpha=\beta=0}, \quad n \geq 2r,$$

or

$$a(n,r) = \frac{1}{n!}\left(\frac{\partial}{\partial\alpha}\right)^n \left((\cos(\alpha))^{-2}(\tan(\alpha))^{n-2r}\right)\Big|_{\alpha=0}, \quad n \geq 2r.$$

From the last equality, we see also that $a(n,r)$ is the coefficient of α^n in the Taylor series of $\cos^{-2}(\alpha)(\tan(\alpha))^{n-2r}$. Using the Taylor series expansions for $\cos^{-1}(\alpha)$ and $\tan(\alpha)$:

$$\cos^{-1}(\alpha) = \sum_{n\geq 0}\gamma_n\alpha^{2n}, \quad \tan(\alpha) = \sum_{n\geq 0}\tau_n\alpha^{2n+1},$$

where $\gamma_n = E_{2n}/(2n)!$ and $\tau_n = 2^{2n+2}(2^{2n+2}-1)B_{2n+2}/(2n+2)!$, the constants E_k (resp. B_k) being the Euler (resp. Bernoulli) numbers, we find that:

$$a(n,r) = \sum_{\substack{j_1+\cdots+j_{n-2r+2}=r\\ j_1,\ldots,j_{n-2r+2}\geq 0}} \gamma_{j_1}\gamma_{j_2}\tau_{j_3}\cdots\tau_{j_{n-2r+2}}, \quad n \geq 2r. \tag{22}$$

Let $b(s,r) = a(s+2r,r)$, $s,r \geq 0$, $A_k = \sum_{\substack{i+j=k\\ i,j\geq 0}}\gamma_i\gamma_j$, $k \geq 0$, and

$$c(s,k) = \sum_{\substack{j_1+\cdots+j_s=k\\ j_1,\ldots,j_s\geq 0}} \tau_{j_1}\cdots\tau_{j_s} \quad s \geq 1, k \geq 0, \tag{23}$$

and $c(0,k) = \delta_{0k}$, $k \geq 0$. Notice that $\tau_0 = 1$, so in particular we have $c(s,0) = 1, s \geq 0$. From Eq. (22), we find that $b(s,r) = \sum_{k=0}^r A_{r-k}c(s,k) = A_k + \sum_{k=1}^r A_{r-k}c(s,k)$. We will show that the $c(s,k)$'s are polynomials of degree k in s.

For any integer $k \geq 1$, we denote by $d(k)$ the number of partitions of k, i.e, the number of ways to write k as a sum of strictly positive integers: $k = n_1 + \cdots + n_p$, with $n_1 \geq \cdots \geq n_p$. We call p the length of the partition $(n_1, \ldots, n_p)$. Obviously we have $1 \leq p \leq k$. Let $B(k,p)$ be the set of partitions of k of length p. For a partition $(n_1,\ldots,n_p)$ of $B(k,p)$, let m_i be the multiplicity with which a given integer $a_i > 0$ appears in $(n_1,\ldots,n_p)$, so that $k = m_1a_1 + \cdots + m_ra_r$ and $a_1 > \cdots > a_r$. To that partition $(n_1,\ldots,n_p)$ we associate a symmetry factor given by

$$m(n_1,\ldots,n_p) = \frac{1}{m_1!\cdots m_r!}.$$

Let $s \geq 1$ be an integer. Let $j_1,\ldots,j_s \geq 0$ be integers such that $j_1+\cdots+j_s = k$. We can associate to the integers $j_1,\ldots,j_s$, by ordering them, a partition $(n_1,\ldots,n_p)$ of k for some p. Conversely, for a partition $(n_1,\ldots,n_p)$ of length p of k there are $\frac{s!}{(s-p)!}m(n_1,\ldots,n_p)$ ways to associate a set of non-negative integers satisfying $j_1+\cdots+j_s = k$ when $s \geq p$. Hence the sum in Eq. (23) for $s \geq 1, k \geq 1$ can be written as:

$$c(s,k) = \sum_{p=1}^k s(s-1)\cdots(s-p+1) \sum_{(n_1,\ldots,n_p)\in B(k,p)} m(n_1,\ldots,n_p)\tau_{n_1}\cdots\tau_{n_p},$$

with an obvious interpretation when $s < k$. Since $c(0,k) = 0$ for $k \geq 1$, the preceding formula also covers the case $s = 0$. Also we saw that we have $c(s,0) = 1$, $s \geq 0$, hence $c(s,k)$, $k \geq 0$, is a polynomial in s. The term $p = k$ in the preceding sum corresponds to a polynomial of degree k and since it is the polynomial with maximal degree in that sum, we conclude that $c(s,k)$ is a polynomial of degree k in s. By retracing the preceding steps, we conclude that there exist polynomials p_r, $r \geq 0$, of degree r such that for $n \geq 2r$ we have $a(n,r) = p_r(n)$. With the previous notations, these polynomials are given by $p_0(n) = 1$ and for $r \geq 1$:

$$p_r(n) = A_r + \sum_{p=1}^{r} z_{p,r}(n-2r)\cdots(n-2r-p+1), \tag{24}$$

where $z_{p,r} = \sum_{k=p}^{r} A_{r-k} \sum_{(n_1,\ldots,n_p)\in B(k,p)} m(n_1,\ldots,n_p)\tau_{n_1}\cdots\tau_{n_p}$.

This lemma allows to express the cochains of the $\odot_\nu$-product on $\mathfrak{su}(2)^*$ by differential operators and we conclude this section by a theorem which summarizes the construction developed here.

Theorem 2 *The $\odot_\nu$-product associated with the invariant star-product on $\mathfrak{su}(2)^*$ (defined by Eq. (12)) admits the following form:*

$$F \odot_\nu G = FG + \sum_{r\geq 1} \nu^{2r} \eta_r(FG), \quad F, G \in N,$$

where $\eta_r \colon N \to N$, $r \geq 1$, are differential operators given by

$$\eta_r(F) = \left(A_r + \sum_{p=1}^{r} z_{p,r}\mathcal{D}(\mathcal{D}-1)\cdots(\mathcal{D}-p+1) \right) \Delta^r(F), \quad F \in N, \tag{25}$$

where $\mathcal{D} = \sum_{1\leq k\leq 3} L_k \frac{\partial}{\partial L_k}$ and where the constants A_r and $z_{p,r}$ are given in Lemma 3. This $\odot_\nu$-product is strongly (but not weakly) non-trivial and has a unique extension to $F, G \in C^\infty(\mathbb{R}^3)$.

Proof. From Proposition 4, we have $\eta_r|_{H_n} = a(n,r)\Delta^r$ for $n, r \geq 0$. Notice that Δ^r vanishes on H_n for $n < 2r$ and by Lemma 3 we have for $n \geq 2r$, $a(n,r) = p_r(n)$, where the p_r's are the polynomials defined by Eq. (24). Hence $\eta_r|_{H_n} = p_r(n)\Delta^r$. Let $\mathcal{D} = \sum_{1\leq k\leq 3} L_k \frac{\partial}{\partial L_k}$. Since $\mathcal{D}|_{H_n} = n$ and Δ^r maps H_n to H_{n-2r} for $n \geq 2r$ and is 0 otherwise, it follows that the restriction of the differential operator $p_r(\mathcal{D}-2r)\Delta^{2r}$ to H_n coincides with $\eta_r|_{H_n}$, $n \geq 0$, therefore $\eta_r = p_r(\mathcal{D}-2r)\Delta^{2r}$ on N. By using the explicit expression for $p_r(n)$ given by Eq. (24), relation (25) follows. The last statement of the Theorem follows from the fact that $\pi(F \odot_\nu G) = FG$, $F, G \in C^\infty(\mathbb{R}^3)$, which implies, as for the polynomial case, that the $\odot_\nu$-product is associative.

5 Concluding Remarks

Spectrality. As indicated in [3], in the framework of generalized deformations, the spectrum of an observable is obtained through its corresponding $\odot$-exponential (cf. proof of Lemma 3). Consider the invariant star-product on $\mathfrak{su}(2)^*$ (with $\nu = i\hbar/2$), Eq. (20) gives explicitly the $*$-exponential of a linear element X, and from that one finds that the $*$-spectrum of X is discrete [2]. For X linear, we have noticed that $\exp_*(\frac{tX}{i\hbar}) = \exp_\odot(\frac{tX}{i\hbar})$, thus the $\odot$-spectrum and the $*$-spectrum of X coincide. The $\odot$-spectrum of X is discrete, hence non-trivial; although the $\odot_\nu$-product of Section 4 is weakly trivial. The situation changes drastically if one considers, e.g., the square of the total angular momentum $H = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2)$. Then it is no longer true that the corresponding spectra coincide.

Consider the Zariski product $\bullet$ associated with the invariant star-product $\mathfrak{su}(2)^*$. The observable H is an irreducible polynomial (over the reals), hence the $*$-powers and $\bullet$-powers of H are identical and the $\bullet$-spectrum of H is nothing but its $*$-spectrum. The situation is similar for the case of a linear polynomial X on $\mathfrak{su}(2)^*$. Hence while the $\odot_\nu$-product does not give the usual spectrum for the square of the total angular momentum (or the rotational kinetic energy), the Zariski product does.

Quantized Nambu Bracket. A quantization of the classical Nambu bracket is achieved by replacing the usual product by the $\odot_\nu$-product of Section 4. Due to the properties of the $\odot_\nu$-product, it is easy to see that actually the quantized Nambu bracket is given by:

$$[F, G, H]_{\odot_\nu} = \tilde{T} \circ \lambda_0(\{F, G, H\}), \quad F, G, H \in C^\infty(\mathbb{R}^3),$$

where $\{F, G, H\}$ denotes the classical Nambu bracket on $\mathbb{R}^3$, i.e., the Jacobian. Though the Leibniz rule is not satisfied for the $\odot_\nu$-product, this quantized Nambu bracket does satisfy the Fundamental Identity. Indeed only the weaker form of the Leibniz rule

$$F\odot_\nu(\frac{\partial}{\partial L_i}(G\odot_\nu H) - G\odot_\nu\frac{\partial H}{\partial L_i} - \frac{\partial G}{\partial L_i}\odot_\nu H) = \tilde{T} \circ \lambda_0(F(\frac{\partial}{\partial L_i}(GH) - G\frac{\partial H}{\partial L_i} - \frac{\partial G}{\partial L_i}H)) = 0,$$

is required to ensure that the Fundamental Identity is verified by the quantized Nambu bracket.

Finally let us mention that, as for the $\odot_\nu$-product case, the quantized Nambu bracket is strongly (but not weakly) non-trivial.

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