One-Dimensional Random Field Ising Model: Gibbs States and Structure of Ground States

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Abstract: We consider the random Gibbs field formalism for the ferromagnetic 1-D dichotomous random field Ising model as the simplest example of quenched disordered systems. We prove that for non–zero temperatures the Gibbs state is unique for any realization of the external field. Then we prove that as $T \to 0$, the Gibbs state converges to a limit, a ground state, for almost all realizations of the external field. The ground state turns out to be a probability measure concentrated on an infinite set of configurations, and we give a constructive description of this measure.

Key words: Gibbs states, ground states, residual entropy, random field, Ising model.

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1 Introduction

The Random Field Ising Model (RFIM) gives a challenging example of disordered system demonstrating non-triviality of the effect of randomness on

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the thermodynamics properties and the structure of grounds states, see e.g. [1–3]. Besides the physical motivations to be a model of certain classes of random diluted magnets [4] or of phase separation in some porous media [5], it provokes a serious theoretical interest as a model for fundamental concepts of quenched random systems [6].

The fundamental problem of existence of magnetization in RFIM (for low temperatures and "small" random field) was solved in [7] for $d \geq d_\ell$, where the lower critical dimension $d_\ell = 3$. Simultaneously this problem was considered for the mean–field RFIM [8]. In the latter case one can go further in the problem of the order parameter and describe explicitly the structure of Gibbs states [9]. On the other hand for the RFIM on a Bethe lattice only the ground state is found [10] and the $T \neq 0$ critical behavior is argued to be mean-field like.

The one-dimensional RFIM for dichotomous random field has a particular interest because it can be solved at $T = 0$ [11] (and partially for $T \neq 0$ [12]) and because it gives interesting effects (e.g., the residual entropy has infinite number of spikes in addition to discontinuities as a function of the field amplitude) related to the physics of frustration. As it was shown in [13] the one-dimensional RFIM is closely related to a stochastic mapping. Then the analysis of this model coincides with a standard Markov chain study of the stochastic mapping and the properties of its invariant measure [14–16]. The fractal nature of the support of the invariant measure as well as the complicated structure of ground state [17] give a new insight in understanding of the frustration in this model, while a rigorous study of Gibbs states and their limits for $T \to 0$ was (up to our knowledge) absent. The aim of the present paper is to fill this gap in the study of one-dimensional RFIM for the independent dichotomous random field.

In the next section we present main definitions, recall the Markov chain approach, and prove uniqueness of Gibbs states when $T > 0$ for *any* fixed configuration of the external field. In Section 3 we give an explicit algorithm for the construction of the whole set of ground states for *almost all* realizations of the external field. We indicate a relation between the structure of ground states and the behavior of residual entropy [11–17]. Proofs for the results of Section 3 are given in Section 4, and concluding remarks in Section 5.

2 Definitions and uniqueness of Gibbs states

In this paper we consider the ferromagnetic one-dimensional RFIM (with nearest neighbours interaction) in a *quenched dichotomous stochastic field.*

Let $\mathbb{Z}$ be a one-dimensional lattice, $X = \{-1, 1\}$ be the individual spin space and $\Omega = X^{\mathbb{Z}} = \{\sigma = \{\sigma_i\}_{i\in\mathbb{Z}} : \sigma_i \in X\}$ be the *configuration space* of the Ising model on $\mathbb{Z}$. For a subset $\Lambda \subset \mathbb{Z}$, we use $\pi_\Lambda : \Omega \to \Omega_\Lambda = X^\Lambda$ to denote the projection onto the coordinates in Λ: $\pi_\Lambda\sigma = \sigma^\Lambda = \{\sigma_i\}_{i\in\Lambda}$. If the subset Λ is finite and $B_\Lambda \subset \Omega_\Lambda$, then $C(B_\Lambda) = \pi_\Lambda^{-1}(B_\Lambda)$ is a cylindrical set with the base B_Λ. We let Σ and Σ^Λ be the σ-algebra generated by cylindrical sets in Ω and Ω_Λ.

We also introduce a sequence $h = \{h_j\}_{j\in\mathbb{Z}}$ of real-valued independent identically distributed random variables (i.i.d.r.v.) according to the probability measure $d\nu$. The probability space of h is $(\mathbb{R}^{\mathbb{Z}}, \mathcal{F}, \lambda)$, with the σ-algebra $\mathcal{F}$, generated by the cylindrical sets $C(\mathbb{R}^{\mathbb{Z}})$, and with infinite product measure $d\lambda = \prod_{j\in\mathbb{Z}} d\nu(h_j)$. We shall consider a dichotomous field taking the values α and $-\alpha$ with probability 1/2: $d\nu(h_i) = \frac{1}{2}\delta(h_i - \alpha) + \frac{1}{2}\delta(h_i + \alpha)$.

The Hamiltonian in the subset $\Lambda = [m, n]$ is given by

$$H_{\Lambda,h}(\sigma^\Lambda \mid \bar{\sigma}^{\Lambda^c}) = -\sum_{i=m}^{n-1} \sigma_i\sigma_{i+1} - \sum_{i=m}^{n} h_i\sigma_i - (\sigma_m\bar{\sigma}_{m-1} + \sigma_n\bar{\sigma}_{n+1})\ , \tag{2.1}$$

where $\Lambda^c = \mathbb{Z} \setminus \Lambda$. The *finite-volume Gibbs measure* (state) on Ω_Λ at inverse temperature $\beta = T^{-1}$ and boundary condition $\bar{\sigma}^{\Lambda^c}$, which specifies the spin configuration outside of Λ, is defined by:

$$\mathbb{P}_{\Lambda,\beta,h}(\sigma^\Lambda \mid \bar{\sigma}^{\Lambda^c}) = Z^{-1}_{\beta,h}(\Lambda \mid \bar{\sigma}^{\Lambda^c}) e^{-\beta H_\Lambda(\sigma^\Lambda|\bar{\sigma}^{\Lambda^c})}\ , \tag{2.2}$$

where $Z_{\beta,h}$ is the partition:

$$Z_{\beta,h}(\Lambda \mid \bar{\sigma}^{\Lambda^c}) = \sum_{\sigma^\Lambda\in\Omega_\Lambda} e^{-\beta H_\Lambda(\sigma^\Lambda|\bar{\sigma}^{\Lambda^c})}. \tag{2.3}$$

We recall that a probability measure μ on (Ω, Σ) is called a *Gibbs state* corresponding to specifications (2.2) if for all finite $\Lambda \subset \mathbb{Z}$ and each $A \in \Sigma_\Lambda$, one has the Dobrushin-Lanford-Ruelle equation (see e.g. [18]):

$$\mu(\pi_\Lambda^{-1} A \mid \Sigma^{\Lambda^c})(\sigma) = \mathbb{P}_\Lambda(A \mid \pi_{\Lambda^c}\sigma) \quad \mu - \text{almost sure (a.s.)}, \tag{2.4}$$

or, equivalently, by the property of conditional probability, if

$$(\pi_\Lambda \mu)(A) = \int_\Omega \mu(d\sigma)\, \mathbb{P}_\Lambda(A \mid \pi_{\Lambda^c}\sigma) , \tag{2.5}$$

where $(\pi_\Lambda \mu)(A) = \mu(\pi_\Lambda^{-1} A)$. As it is well known, see e.g. [18], the set of Gibbs states coincides with the closed convex hull of the set of weak limits of finite-volume Gibbs measures (2.2).

We now turn to the study of these limits for $T > 0$. We first define an Hamiltonian in $\Lambda = [m, n]$ with generalized b.c.:

$$H_{\Lambda,h}(\sigma^\Lambda \mid a, b) = -\sum_{i=m}^{n-1} \sigma_i \sigma_{i+1} - \sum_{i=m}^{n} h_i \sigma_i - (\sigma_m a + \sigma_n b) , \tag{2.6}$$

and use respectively $\mathbb{P}_{\Lambda,\beta,h}(\sigma^\Lambda \mid a, b)$ and $Z_{\beta,h}(\Lambda \mid a, b)$ to denote the corresponding state and partition function. Let $\Delta = [k, \ell] \subset [m, n]$, be a subset of Λ. To calculate the measure (2.2) of the cylindrical set $C(B_\Delta)$ based on the space Ω_Δ, we use the identities

$$\sum_{\sigma_i = \pm 1} e^{\beta[\sigma_i \sigma_{i+1} + \sigma_i (h_i + u_i)]} = e^{\beta[\sigma_{i+1} f_\beta(h_i + u_i) + g_\beta(h_i + u_i)]}, \tag{2.7}$$

$$\sum_{\sigma_i = \pm 1} e^{\beta[\sigma_{i-1} \sigma_i + \sigma_i (h_i + v_i)]} = e^{\beta[\sigma_{i-1} f_\beta(h_i + v_i) + g_\beta(h_i + v_i)]}, \tag{2.8}$$

where

$$\begin{cases} f_\beta(x) = (1/2\beta) \ln[\cosh\beta(x+1)/\cosh\beta(x-1)], \\ g_\beta(x) = (1/2\beta) \ln[4\cosh\beta(x+1)\cosh\beta(x-1)], \end{cases} \tag{2.9}$$

to sum up over the spins in the set $\Lambda \setminus \Delta$. The step by step summing up from the (m)–th spin to the $(k-1)$–th spin, see (2.1) and (2.7), generates mapping:

$$u_i^{(m)} = f_\beta(h_{i-1} + u_{i-1}^{(m)}), \quad i = m+1, m+2, \ldots, k, \tag{2.10}$$

where $u_m^{(m)} \equiv \bar\sigma_{m-1}$. Notice that $u_i^{(m)}$ depends on $h_{i-1}, h_{i-2}, \ldots, h_m, \bar\sigma_{m-1}$: $u_i^{(m)} = u_i^{(m)}(h_{i-1}, h_{i-2}, \ldots, h_m, \bar\sigma_{m-1}), \quad m+1 \le i \le k.$

The same procedure for the spins on $[\ell+1, n]$, see (2.1) and (2.8), generates mapping

$$v_i^{(n)} = f_\beta(h_{i+1} + v_{i+1}^{(n)}), \quad i = n-1, n-2, \ldots, \ell, \tag{2.11}$$

where $v_n^{(n)} \equiv \bar\sigma_{n-1}$. Here, $v_i^{(n)}$ depends on $h_{i+1}, h_{i+2}, \ldots, h_n, \bar\sigma_{n+1}$: $v_i^{(n)} = v_i^{(n)}(h_{i+1}, h_{i+2}, \ldots, h_n, \bar\sigma_{n+1}), \quad \ell \le i \le n-1$.

Applying the mappings (2.10) and (2.11) to the calculation of the partition function (2.3), one gets

$$\begin{aligned} Z_{\beta,h}(\Lambda \mid \bar\sigma_{m-1}, \bar\sigma_{n+1}) &= \exp\left(\beta \sum_{i=m}^{k-1} g_\beta(h_i + u_i^{(m)})\right) \\ &\times Z_{\beta,h}\left(\Delta \mid u_k^{(m)}, v_\ell^{(n)}\right) \exp\left(\beta \sum_{i=\ell+1}^{n} g_\beta(h_i + v_i^{(n)})\right) . \end{aligned} \tag{2.12}$$

The same calculations for the numerator of (2.2) gives for the measure of the cylindrical set $C(B_\Delta)$ the following

$$\begin{aligned} \mathbb{P}_{\Lambda,\beta,h}\left(C(B_\Delta) \mid \bar\sigma^{\Lambda^c}\right) &= \frac{\sum\limits_{\sigma^\Delta \in B_\Delta} e^{-\beta H_{\Delta,h}(\sigma^\Delta \mid u_k^{(m)}, v_\ell^{(n)})}}{Z_{\beta,h}\left(\Delta \mid u_k^{(m)}, v_\ell^{(n)}\right)} \\ &= \mathbb{P}_{\Delta,\beta,h}\left(B_\Delta \mid u_k^{(m)}, v_\ell^{(n)}\right) . \end{aligned} \tag{2.13}$$

Theorem 2.1 *Let $h \in \mathbb{R}^{\mathbb{Z}}$ be a fixed configuration of the external random field. Then for any positive temperature $T > 0$ and any cylindrical set $C(B_\Delta)$, the limit of the measure* (2.13) *exists and is independent of the boundary conditions:*

$$\lim_{\Lambda \uparrow \mathbb{Z}} \mathbb{P}_{\Lambda,\beta,h}(C(B_\Delta) \mid \bar\sigma^{\Lambda^c}) = \mathbb{P}_{\Delta,\beta,h}(B_\Delta \mid u_k, v_\ell), \tag{2.14}$$

where

$$\begin{aligned} u_k &= \lim_{m \to -\infty} u_k^{(m)}(h_{k-1}, h_{k-2}, \ldots, h_m, \bar\sigma_{m-1}) \\ v_\ell &= \lim_{n \to \infty} v_\ell^{(n)}(h_{\ell+1}, h_{\ell+2}, \ldots, h_n, \bar\sigma_{n+1}) \end{aligned} \tag{2.15}$$

depend uniquely on the restriction of the field respectively on the intervals $(-\infty, k-1]$ and $[\ell+1, +\infty)$.

Proof. From the representation (2.13) it follows that the only what we have to prove is (2.15). Let $v_i = f_\beta(h_{i+1} + v_{i+1})$ and $v_i' = f_\beta(h_{i+1} + v_{i+1}')$ for $i \ge \ell$, then:

$$v_i - v_i' = f_\beta(h_{i+1} + v_{i+1}) - f_\beta(h_{i+1} + v_{i+1}') \le f_\beta'(0)|v_{i+1} - v_{i+1}'| \tag{2.16}$$

because $f_\beta(x)$ is odd, concave for $x \geq 0$ and satisfies $0 < f'_\beta(x) \leq f'_\beta(0)$ for $\beta < \infty$. By applying (2.16) recursively we get:

$$v_i - v'_i \leq \left(f'_\beta(0)\right)^{n-i} |v_n - v'_n| \,. \tag{2.17}$$

Since $f'_{\beta<\infty}(0) < 1$, it first follows that the sequence $\{v_i(h_{i+1}, \ldots, h_n; a)\}_{n\geq i}$ is a Cauchy sequence and also that the limit $\lim_{n\to\infty} v_i(h_{i+1}, \ldots, h_n; a) = v_i(h_i, \ldots, h_n, \ldots)$ does not depend of the initial condition a, but only of the configuration $\{h_j\}_{j=i+1}^{\infty}$. This prove the second assertion of (2.15). The first one is obtained analogously. *QED*

Corollary 2.1 *For any realization of the random external field $h \in \mathbb{R}^{\mathbb{Z}}$ and any positive temperature $T > 0$, the 1-D ferromagnetic RFIM with nearest neighbour interaction has a unique limiting Gibbs state $\mu_{\beta,h}(\cdot)$ which is a weak limit of specifications $\{\mathbb{P}_{\Lambda,\beta,h}(\cdot \mid \bar{\sigma}^{\Lambda^c})\}_\Lambda$ for arbitrary boundary conditions.*

Remark 2.1 From (2.1) we get for two different boundary conditions that: $|H_{\Lambda,h}(\sigma^\Lambda \mid \bar{\sigma}_1^{\Lambda^c}) - H_{\Lambda,h}(\sigma^\Lambda \mid \bar{\sigma}_2^{\Lambda^c})| \leq 4$. Then by the general theorem proved in [19], it follows that all Gibbs states coincide.

The proof of the uniqueness theorem for Gibbs states is mainly based on the inequality $f_\beta(0) < 1$. But for $\beta \to \infty$ one gets $f_\beta(x) \to 1$ uniformly for $x \in (-1, 1)$. Therefore, the transition to $T = 0$ needs a special investigation which is the subject of the next section.

3 Ground states and residual entropy

Our aim is to prove the existence of the limit $\mu_{\infty,h} = \lim_{\beta\to\infty} \mu_{\beta,h}$ for *almost all* $h = \{h_i = \pm\alpha,\ i \in \mathbb{Z}\}$ and to describe $\mu_{\infty,h}$. In this section we formulate our main results. Their proofs are given in the next section.

Theorem 3.1 *The limit*

$$\mu_{\infty,h} = \lim_{\beta\to\infty} \mu_{\beta,h} \tag{3.1}$$

exists for almost all h.

To describe $\mu_{\infty,h}$ consider sequences $u = \{u_i,\ i \in \mathbb{Z}\}$ and $v = \{v_i,\ i \in \mathbb{Z}\}$ satisfying

$$\begin{aligned} u_i &= f_\infty(h_{i-1} + u_{i-1}), \\ v_i &= f_\infty(h_{i+1} + v_{i+1}), \qquad i \in \mathbb{Z}, \end{aligned} \tag{3.2}$$

where $f_\infty = \lim_{\beta\to\infty} f_\beta$, see (2.9), is the piecewise linear function:

$$f_\infty(x) = \begin{cases} -1 & \text{if} \quad x \le -1 \\ x & \text{if} \ -1 \le x \le 1 \\ 1 & \text{if} \quad x \ge 1 \end{cases} \tag{3.3}$$

By Lemma 4.1, see below, the sequences u and v exist for almost all h and they are unique. In addition, u_i and v_i take value only in the set $\Gamma = \Gamma_+ \cup \Gamma_-$ where

$$\begin{aligned} \Gamma_+ &= \{1, 1-\alpha, 1-2\alpha, \ldots, 1-n\alpha\}, \qquad n = \left[\frac{2}{\alpha}\right], \\ \Gamma_- &= \{-1, -1+\alpha, -1+2\alpha, \ldots, -1+n\alpha\}. \end{aligned}$$

Observe that $|u_i|, |v_i| \le 1$. Let us partition $\mathbb{Z}$ into three subsets, $\mathbb{Z} = \Lambda_+ \cup \Lambda_- \cup \Lambda$, with

$$\Lambda_\pm = \{i \in \mathbb{Z} : \quad \pm(h_i + u_i + v_i) > 0\}, \tag{3.4}$$

$$\Lambda = \{i \in \mathbb{Z} : \quad h_i + u_i + v_i = 0\}. \tag{3.5}$$

If $\alpha > 2$ then the set Λ is empty and $\sigma_i = \operatorname{sign} h_i$, so that the ground state configuration follows the field (see Statement (a) of Theorem 3.2). So we will assume $\alpha \le 2$. In this case by Lemma 4.2 below, the sets $\Lambda_\pm,\ \Lambda$ are infinite and Λ consists of a sequence of finite intervals $\Lambda_k = [i_k, j_k],\ k \in \mathbb{Z}$, $i_k \ge j_{k-1} + 2$. In addition, by Lemma 4.4 below, for $i \in \Lambda_k$ one has

$$|u_{i_k}| = |v_{j_k}| = 1. \tag{3.6}$$

and on Λ_k (3.2) reduces to the random walk,

$$\begin{aligned} u_i &= h_{i-1} + u_{i-1},\ i_k < i \le j_k, \\ v_i &= h_{i+1} + v_{i+1},\ i_k \le i < j_k. \end{aligned} \tag{3.7}$$

The complete description of $\mu_{\infty,h}$ on Λ_k needs to distinguish the sites i, $i_k < i \le j_k$ such that $|u_i| = 1$, that we call *switches*. Consider the set M_k of all configurations $\sigma = \{\sigma_i,\ i \in \Lambda_k\}$ in Λ_k such that

(i) if i is a switch, then either $\sigma_i = \sigma_{i-1}$ or $\sigma_i = -\sigma_{i-1} = -u_i$

(ii) if $i \geq i_k + 1$ is not a switch, then $\sigma_i = \sigma_{i-1}$

In other words, for $\sigma \in M_k$, σ_i can change its value, when i runs from i_k to j_k, only at switches $i \in S_k$ and only if $\sigma_i = -u_i$ at a switch. Note that there is no restriction on the value of σ_{i_k}.

Theorem 3.2 **(a)** *$\mu_{\infty,h}$ is concentrated on configurations with $\sigma_i = +1$ for $i \in \Lambda_+$ and $\sigma_i = -1$ for $i \in \Lambda_-$.*

(b) $\pi_\Lambda \mu_{\infty,h} = \prod_{k\in\mathbb{Z}} \pi_{\Lambda_k} \mu_{\infty,h}$.

(c) *$\pi_{\Lambda_k} \mu_{\infty,h}$ is concentrated on M_k, and it is a uniform measure on M_k.*

Let us make some comments. Assume first that $\frac{2}{\alpha} \notin \mathbb{Z}$. Then the sequences Γ_+ and Γ_- do not intersect and the equations (3.6) and (3.7) imply that on each Λ_k, u_i takes values only in one of these sequences. Assume for the sake of definiteness that on a given Λ_k, u_i takes values from Γ_+. Then at all switches $i \in S_k$, $u_i = 1$. This implies that M_k consists of the following configurations: $\sigma_i \equiv +1$, $\sigma_i \equiv -1$ and all configurations such that $\sigma_j = +1$ for $j < i$ and $\sigma_j = -1$ for $j \geq i$ where i is a switch. Hence M_k consists of $2 + n_k$ configurations where n_k is the number of switches.

If $\frac{2}{\alpha}$ is integer then the sequences Γ_+ and Γ_- coincide and at switches u_i can take the both values $+1$ and -1. This increases the number of configurations in M_k and gives rise to a spike of residual entropy. Between any two neighboring spikes, when $\frac{2}{m+1} < \alpha < \frac{2}{m}$, the structure of ground state is preserved and the residual entropy is constant. This explains the behaviour of the residual entropy obtained in [11].

Theorems 3.1–3.2 are proved in the next section.

4 Proof of Theorems

Before proving Theorems 3.1–3.2 we make some observations. The finite dimensional distribution of a chain of spins $\sigma^\Delta = \sigma_k, \ldots, \sigma_\ell$ with respect to

$\mu_{\beta,h}$ is (see (2.5),(2.13)):

$$\begin{aligned}\mu_{\beta,h}(\sigma^\Delta) &= Z^{-1}\exp\left[-\beta H_\Delta(\sigma^\Delta \mid u_k, u_\ell)\right] \\ &= Z^{-1}\exp\left(\beta\sum_{i=k}^{\ell-1}\sigma_i\sigma_{i+1} + \beta\sum_{i=k}^{\ell} h_i\sigma_i + \beta u_k\sigma_k + \beta v_\ell\sigma_\ell\right), \quad (4.1)\end{aligned}$$

where hereafter we use the short hand notation $\mu_{\beta,h}(A)$ instead of $\mu_{\beta,h}(\pi_\Delta^{-1}A)$ with $\Delta = [k,\ell]$, and the numbers $u_i = u_i(\beta)$, $v_i = v_i(\beta)$, $i \in \mathbb{Z}$, are defined with the help of the recurrent equations

$$\begin{aligned} u_i &= f_\beta(h_{i-1} + u_{i-1}), \\ v_i &= f_\beta(h_{i+1} + v_{i+1}). \end{aligned} \quad (4.2)$$

As proven in Theorem 2.1, when β is finite, due to the contractive property of the map $t \to f_\beta(t)$, these equations have for *all* h a unique solution which do not depend on initial conditions. The following lemma asserts that when $\beta = \infty$, this is true for *almost all* h i.e. λ–a.s..

Lemma 4.1 *For almost all h the recursive equations* (3.2) *have unique solutions u and v.*

Proof. Let us prove the existence of a unique solution u for the first equation in (3.2). To this end let us consider a solution $u^{(N)} = \{u_i^{(-N)}\}$ of (3.2) on $\{i \geq -N\}$ with an arbitrary initial value $u_{-N}^{(-N)} = \gamma$, $|\gamma| \leq 1$, and let us prove that for almost all h, as $N \to \infty$, $u^{(N)}$ approaches a limit u. Indeed, for a given h consider the set $B(h)$ of $j \in \mathbb{Z}$ such that

$$h_j = h_{j-1} = \ldots = h_{j-n-1} = \alpha, \quad (4.3)$$

where $n = \left[\frac{2}{\alpha}\right]$. Then (3.2) implies that if $j \in B(h)$ and $j > -N+n+1$ then $u_j^{(-N)} = 1$ since

$$u_{j-n-1}^{(-N)} + (n+1)\alpha \geq -1 + \left(\left[\frac{2}{\alpha}\right] + 1\right)\alpha \geq 1. \quad (4.4)$$

Hence

$$\lim_{N\to\infty} u_j^{(-N)} = 1, \qquad \forall\, j \in B(h). \quad (4.5)$$

If $i > j$ where $j \in B(h)$ then by (3.2),

$$\lim_{N\to\infty} u_i^{(-N)} = f_\infty(h_{i-1} + f_\infty(h_{i-2} + \ldots f_\infty(h_j + 1)\ldots)) \in \Gamma. \tag{4.6}$$

For almost all h the set $B(h)$ is unbounded in the sense that $\forall\, M > 0$,

$$B(h) \cap \{j \le -M\} \ne \emptyset, \qquad B(h) \cap \{j \ge M\} \ne \emptyset. \tag{4.7}$$

Indeed, if we partition $\mathbb{Z}$ into blocks $A_k = \{(n+1)k \le j < (n+1)(k+1)\}$, then for a given $L > 0$ the probability that the event $\{h_j = \alpha,\ \forall j \in A_k\}$ does not hold for all $k \le -L$ is equal to

$$\prod_{k \le -L} (1 - 2^{-n-1}) = 0. \tag{4.8}$$

This proves the existence of a unique solution u for almost all h. Similarly we prove the existence of a unique solution v for almost all h. *QED*

Lemma 4.2 *If $\alpha \le 2$ then for almost all h, the sets $\Lambda_\pm$ and Λ are unbounded in the sense that for $A = \Lambda_\pm, \Lambda$ and $\forall\, M > 0$,*

$$A \cap \{j \le -M\} \ne \emptyset, \qquad A \cap \{j \ge M\} \ne \emptyset. \tag{4.9}$$

Corollary 4.1 *The set Λ consists of an infinite sequence of finite intervals $\Lambda_k = [i_k, j_k]$, $k \in \mathbb{Z}$, $i_k \ge j_{k-1} + 2$.*

Proof of Lemma 4.2. For a given h consider the set $B(h)$ of $j \in \mathbb{Z}$ such that (4.3) holds. Then $u_j = 1$ (see the proof of Lemma 4.1), so

$$h_j + u_j + v_j = \alpha + 1 + v_j \ge \alpha > 0, \qquad \forall\, j \in B(h). \tag{4.10}$$

Thus $B(h) \subset \Lambda_+$, and hence by (3.6) the set Λ_+ is unbounded for almost all h. Similarly we prove that Λ_- is unbounded. It remains to prove that Λ is unbounded. For a given h consider the set $C(h)$ of $j \in \mathbb{Z}$ such that

$$\begin{aligned} h_{j-1} &= \ldots = h_{j-n-1} = \alpha, \qquad h_j = -\alpha; \\ h_{j+2} &= \ldots = h_{j+n+2} = -\alpha, \qquad h_{j+1} = \alpha. \end{aligned} \tag{4.11}$$

Then by (3.2), $u_j = 1$ and $v_{j+1} = -1$, so that $v_j = -1 + \alpha$ and

$$h_j + u_j + v_j = -\alpha + 1 + (-1 + \alpha) = 0, \qquad \forall\, j \in C(h). \tag{4.12}$$

Hence $C(h) \subset \Lambda$. Since the probability that $C(h)$ is unbounded is equal to 1, the set Λ is unbounded with probability 1 as well. *QED*

The following lemma implies (3.6) and (3.7).

Lemma 4.3 *The sequences $\{u_i\}$ and $\{v_i\}$ satisfy $|u_{i_k-1} + h_{i_k-1}| > 1$ and $|v_{j_k+1} + h_{j_k+1}| > 1$, while for $i \in \Lambda_k$ one has $|u_i + h_i| \le 1$, $|v_i + h_i| \le 1$.*

Proof. Observe the following: if $u_i + v_i + h_i = 0$ and

$$\begin{cases} u_{i+1} = u_i + h_i, \\ \quad v_i \ = v_{i+1} + h_{i+1}, \end{cases} \tag{4.13}$$

then $u_{i+1} + v_{i+1} + h_{i+1} = 0$. This means that i cannot be the right endpoint of Λ_k. Therefore, at $i = j_k$ at least one of the equations (4.13) is violated. As it follows from the recurrent equation (3.2), the only way to violate the first equation in (4.13) is to have

$$|u_i| = 1, \qquad u_i h_i > 0.$$

But then $|u_i + h_i| > 1$ and $u_i + v_i + h_i = 0$ which is impossible. So we conclude that at $i = j_k$ the second equation in (4.13) is violated, which is possible only when

$$|v_{j_k}| = 1, \qquad v_{j_k+1} h_{j_k+1} > 0.$$

Hence

$$|v_{j_k+1} + h_{j_k+1}| > 1,$$

and one proves analogously $|u_{i_k-1} + h_{i_k-1}| > 1$. To prove the other inequalities assume the contrary, i.e., that $|v_i + h_i| > 1$ for some $i \in \Lambda_k$. But then the equality $u_i + v_i + h_i = 0$ is impossible so i cannot belong to Λ_k. This contradiction proves $|v_i + h_i| \le 1$. The inequality $|u_i + h_i| \le 1$ is proven analogously. *QED*

The following Lemma gives the asymptotics of $u_i(\beta)$ and $v_i(\beta)$ as $\beta \to \infty$ for almost all h.

Lemma 4.4 *Let $N > 0$ be an arbitrary number. Then for almost all h,*

$$\begin{aligned} |u_i(\beta) - u_i(\infty) - c_i\beta^{-1}| \le C_i\beta^{-N}, \\ |v_i(\beta) - v_i(\infty) - d_i\beta^{-1}| \le D_i\beta^{-N}, \end{aligned} \tag{4.14}$$

with some coefficients $c_i = c_i(h)$, $d_i = d_i(h)$ and some constants $C_i = C_i(h) > 0$, $D_i = D_i(h) > 0$.

The proof of Lemma 4.4 utilizes the following asymptotics of the function $f_\beta(t)$ as $\beta \to \infty$.

Lemma 4.5 *Let $N > 0$ and $1 > \tau > 0$ be arbitrary numbers. Then*

$$\begin{aligned} &\sup_{t:|t|\le 1-\tau} |f_\beta(t) - t| = O(\beta^{-N}); \\ &\sup_{t:t\ge 1+\tau} |f_\beta(t) - 1| = O(\beta^{-N}); \\ &\sup_{s:|s|\le \tau\beta} \left| f_\beta(1+\beta^{-1}s) - 1 + \beta^{-1}\frac{\ln(1+e^{-2s})}{2} \right| = O(\beta^{-N}). \end{aligned} \tag{4.15}$$

and

$$f_\beta(t) = f_\infty(t) + O(\beta^{-1}). \tag{4.16}$$

We shall prove Lemmas 4.4 and 4.5 later, and now we prove Theorems 3.1–3.2.

Proof of Theorem 3.1. Let $M_{k\ell}$ be the set of ground state configurations of the Hamiltonian

$$H(\sigma_k,\dots,\sigma_\ell) = -\sum_{i=k}^{\ell-1} \sigma_i\sigma_{i+1} - \sum_{i=k}^{\ell} h_i\sigma_i - u_k(\infty)\sigma_k - v_\ell(\infty)\sigma_\ell\ ,$$

Then by (4.1) and (4.14), $\lim_{\beta\to\infty} \mu_{\beta,h}(\sigma_k,\dots,\sigma_\ell) = 0$, if $(\sigma_k,\dots,\sigma_\ell) \notin M_{k\ell}$. Let E be the energy of a ground state configuration. Then by (4.14), if $(\sigma_k,\dots,\sigma_\ell) \in M_{k\ell}$, then

$$\begin{aligned} \beta\sum_{i=k}^{\ell-1} \sigma_i\sigma_{i+1} + \beta\sum_{i=k}^{\ell} h_i\sigma_i + \beta u_k(\beta)\sigma_k + \beta v_\ell(\beta)\sigma_\ell \\ = -\beta E + \beta[u_k(\beta) - u_k(\infty)]\sigma_k + \beta[v_\ell(\beta) - v_\ell(\infty)]\sigma_\ell \\ = -\beta E + c_k\sigma_k + d_\ell\sigma_\ell + O(\beta^{-N}), \end{aligned}$$

hence

$$\lim_{\beta\to\infty} \mu_{\beta,h}(\sigma_k,\dots,\sigma_\ell) = Z^{-1}\exp(c_k\sigma_k + d_\ell\sigma_\ell), \tag{4.17}$$

where, from now onwards, we denote by Z^{-1} the corresponding normalizing factors, cf. (4.1) and (4.18). This proves the existence of limit,

$$\lim_{\beta\to\infty} \mu_{\beta,h}(\sigma_k,\dots,\sigma_\ell) = \mu_{\infty,h}(\sigma_k,\dots,\sigma_\ell)$$

for any sequence of spins $(\sigma_k,\dots,\sigma_\ell)$. *QED*

Proof of Theorem 3.2. When $k=\ell$, (4.1) reduces to

$$\mu_{\beta,h}(\sigma_k) = Z^{-1}\exp\{\beta[h_k + u_k(\beta) + v_k(\beta)]\sigma_k\}. \tag{4.18}$$

This implies that if $h_k + u_k(\infty) + v_k(\infty) > 0$, then

$$\lim_{\beta\to\infty} \beta[h_k + u_k(\beta) + v_k(\beta)] = \lim_{\beta\to\infty} \beta[h_k + u_k(\infty) + v_k(\infty)] + c_k + d_k = \infty,$$

hence $\sigma_k = 1$ in the limit when $\beta\to\infty$. Similarly, if $h_k+u_k(\infty)+v_k(\infty) < 0$ then $\sigma_k = -1$ in the limit when $\beta\to\infty$. This proves Statement (a).

The Gibbs measure $\mu_{\beta,h}$ (see Corollary 2.1) possesses the Markov property. Hence $\mu_{\infty,h}$, as a limit measure of $\mu_{\beta,h}$, possesses the Markov property as well. Consider any finite-dimensional distribution $\mu_{\infty,h}(\sigma_m,\dots,\sigma_n)$ such that $m,n\in\Lambda_+\cup\Lambda_-$. Let $\Lambda_k,\dots,\Lambda_\ell$ be all the connected components of Λ in $[m,n]$. Then by Statement (a) of Theorem 3.2 for $i\in[m,n]\setminus\Lambda$ the spin σ_i takes a deterministic value, hence the Markov property of $\mu_{\infty,h}$ implies that

$$\pi_{\Lambda\cap[m,n]}\mu_{\infty,h} = \pi_{\Lambda_k}\mu_{\infty,h}\dots\pi_{\Lambda_\ell}\mu_{\infty,h}.$$

So, Statement (b) is proved.

By (4.17)

$$\mu_{\infty,h}(\sigma_{i_k-1},\sigma_{i_k},\dots,\sigma_{j_k},\sigma_{j_k+1}) = Z^{-1}\exp\left(c_{i_k-1}\sigma_{i_k-1} + d_{j_k+1}\sigma_{j_k+1}\right), \tag{4.19}$$

for any ground state configuration of the Hamiltonian $H(\sigma_{i_k-1},\dots,\sigma_{j_k+1})$. In addition, the probability of any configuration which is not a ground state configuration is 0. Notice that $i_k-1, j_k+1\in\Lambda_+\cup\Lambda_-$, hence by Statement (a) of Theorem 3.2, σ_{i_k-1} and σ_{j_k+1} take deterministic values. Hence (4.19) implies that $\mu_{\infty,h}$ is a uniform measure on the set of ground state configurations. Let us describe now all ground state configurations of the Hamiltonian $H(\sigma_{i_k-1},\ldots,\sigma_{j_k+1})$. A characteristic property of ground state configuration is

$$\mu_{\infty,h}(\sigma_{i_k-1},\ldots,\sigma_{j_k+1})>0.$$

Notice that $\sigma_{i_k},\ldots,\sigma_{j_k}$ form a finite Markov chain, with values in Γ. Consider one-point and two-point distributions of this Markov chain. By (4.17), if $i_k\le i\le j_k$ then

$$\mu_{\infty,h}(\sigma_i)=Z^{-1}\exp[(c_i+d_i)\sigma_i],$$

which shows that σ_i takes the values both $+1$ and -1 with positive probability. Assume now that $i_k\le i-1<i\le j_k$. Then by(4.1),

$$\begin{aligned}
&\mu_{\beta,h}(\sigma_{i-1},\sigma_i)\\
&\quad=\frac{1}{Z}\exp\left\{\beta\left[\sigma_{i-1}\sigma_i+h_{i-1}\sigma_{i-1}+h_i\sigma_i+u_{i-1}(\beta)\sigma_{i-1}+v_i(\beta)\sigma_i\right]\right\}\\
&\quad=\frac{1}{Z}\exp\Big\{\beta\left[\sigma_{i-1}\sigma_i+h_{i-1}\sigma_{i-1}+h_i\sigma_i+u_{i-1}(\infty)\sigma_{i-1}+v_i(\infty)\sigma_i\right]\\
&\qquad\qquad+c_{i-1}\sigma_{i-1}+d_i\sigma_i+O(\beta^{-N})\Big\}\,.
\end{aligned}$$

Due to the equations $u_i(\infty)=u_{i-1}(\infty)+h_{i-1}$ and $h_i+v_i(\infty)=-u_i(\infty)$, the expression in the brackets on the right can be reduced to

$$\sigma_{i-1}\sigma_i+h_{i-1}\sigma_{i-1}+h_i\sigma_i+u_{i-1}(\infty)\sigma_{i-1}+v_i(\infty)\sigma_i=\sigma_{i-1}\sigma_i+u_i(\infty)(\sigma_{i-1}-\sigma_i).$$

This gives 1 for both $++$ and $--$ configurations, $-1+2u_i$ for $+-$ and $-1-2u_i$ for $-+$, here $u_i\equiv u_i(\infty)$. Since $|u_i|\le 1$ this implies that $++$ and $--$ are ground states and

$$\mu_{\infty,h}(++)>0,\qquad \mu_{\infty,h}(--)>0.$$

In addition, the configuration $+-$ is ground state configuration and

$$\mu_{\infty,h}(+-)>0\quad\text{if and only if}\quad u_i=1.$$

Respectively,

$$\mu_{\infty,h}(-+)>0\quad\text{if and only if}\quad u_i=-1.$$

This means that in the Markov chain $\sigma_{i_k},\ldots,\sigma_{j_k}$, allowed transitions are $+\to+$ and $-\to-$ and also $+\to-$ when $u_i=1$ and $-\to+$ when $u_i=-1$. Thus all configurations from the set M_k and only these configurations are allowed, i.e., have positive probability with respect to $\mu_{\infty,h}$. As it was noticed before, all allowed configurations have equal probabilities. This proves Statement (c) and finishes the proof of Theorem 3.2. *QED*

It remains to prove Lemma 4.4. First we prove Lemma 4.5.

Proof of Lemma 4.5. Observe that for large x,

$$\tanh x = 1 - 2e^{-2x} + O(e^{-4x}), \qquad x \to \infty,$$

and for small $x > 0$,

$$\operatorname{artanh}(1-x) = -\frac{\ln(x/2)}{2} + O(x), \qquad x \to +0.$$

Hence, when $|t| \le 1-\tau$,

$$\begin{aligned} f_\beta(t) &= \beta^{-1} \operatorname{artanh}[\tanh\beta\, \tanh(\beta t)] \\ &= \beta^{-1} \operatorname{artanh}[\tanh(\beta t)] + O(e^{-2\tau\beta}) = t + O(e^{-2\tau\beta}), \end{aligned}$$

which proves the first line in (4.15). Next, when $s \ge -\tau\beta$,

$$\begin{aligned} f_\beta(1+\beta^{-1}s) &= \beta^{-1} \operatorname{artanh}[\tanh\beta\, \tanh(\beta+s)] \\ &= \beta^{-1}\left[-\frac{\ln\left(e^{-2\beta}+e^{-2\beta-2s}\right)}{2} + O(e^{-2(1-\tau)\beta})\right] \\ &= 1 - \beta^{-1}\frac{\ln(1+e^{-2s})}{2} + O(e^{-2(1-\tau)\beta}), \end{aligned}$$

which proves the last two lines in (4.15). Finally, since

$$\ln(1+e^{-2t}) = \begin{cases} O(1), & \text{if } t > 0, \\ -2t + O(1), & \text{if } t < 0, \end{cases}$$

(4.16) follows from (4.15). Lemma 4.5 is proven. *QED*

Proof of Lemma 4.4. Consider the following sequences: for a given $0 < \beta < \infty$, the sequence $u(\beta) = \{u_i(\beta),\ i \in \mathbb{Z}\}$, which is the unique solution of (4.2), then the sequence $u(\infty) = \{u_i(\infty),\ i \in \mathbb{Z}\}$, which is the unique solution of (3.2) (it is defined for almost all h), and finally the sequence $\{u_i,\ i \ge 0\}$, defined by the recurrent equation $u_{i+1} = f_\infty(h_i + u_i)$, $i \geq 0$, with the initial value $u_0 = u_0(\beta)$. Let L be an arbitrary number greater than $n = \left[\frac{2}{\alpha}\right]$. Define the set A_L of h such that the sequence $u(\infty)$ is uniquely defined and

$$h_k = h_{k-1} = \ldots = h_{k-n-1} = \alpha \tag{4.20}$$

for some k in the interval $n+1 < k < L$. Then $u_k = u_k(\infty) = 1$ (see the proof of Lemma 4.1) and hence by (4.16),

$$|u_k(\beta) - 1| \leq q_k(h)\beta^{-1}, \tag{4.21}$$

with some $q_k(h) > 0$. By the second line in (4.15) we get now that if β is large enough then

$$|u_{k+1}(\beta) - u_{k+1}(\infty)| = |f_\beta(u_k(\beta) + \alpha) - 1| \leq C_k(h)\beta^{-N}. \tag{4.22}$$

Now let us fix some $h \in A_L$ and prove that for any j in the interval $k+1 \leq j \leq L$,

$$|u_j(\beta) - u_j(\infty) - c_j\beta^{-1}| \leq C_j(h)\beta^{-N}, \tag{4.23}$$

with some coefficient $c_j = c_j(h)$. We prove (4.23) by induction in j. For $j = k+1$, (4.23) follows from (4.22) with $c_j = 0$. Assume that (4.23) holds for some $j \geq k+1$ and prove that it holds for $j+1$. We have three cases: (i) $|h_j + u_j(\infty)| < 1$; (ii) $|h_j + u_j(\infty)| > 1$; and (iii) $|h_j + u_j(\infty)| = 1$. In the case (i) we obtain using the first line in (4.15) and the induction hypothesis (4.23) for j that

$$\begin{aligned} u_{j+1}(\beta) &= f_\beta(h_j + u_j(\beta)) = h_j + u_j(\beta) + O(\beta^{-N}) \\ &= h_j + u_j(\infty) + c_j\beta^{-1} + O(\beta^{-N}) \\ &= u_{j+1}(\infty) + c_j\beta^{-1} + O(\beta^{-N}). \end{aligned}$$

This gives (4.23) for $j+1$ with $c_{j+1} = c_j$. Similarly, in the case (ii) we apply the second line in (4.15) and we obtain (4.23) for $j+1$ with $c_{j+1} = 0$. Finally, in the case (iii) we apply the third line in (4.15) and we obtain (4.23) for $j+1$ with

$$c_{j+1} = \begin{cases} -\dfrac{\ln(1+e^{-2c_j})}{2} & \text{if} \quad h_j + u_j(\infty) = 1; \\ \dfrac{\ln(1+e^{2c_j})}{2} & \text{if} \quad h_j + u_j(\infty) = -1. \end{cases} \tag{4.24}$$

This proves (4.23). Hence (4.23) holds for $u_L(\beta)$ for any $h \in A_L$, so that it holds with probability $1 - \varepsilon_L$, where $\varepsilon_L \to 0$ as $L \to \infty$. Due to the translation invariance it holds then for any fixed j with probability $1 - \varepsilon_L$. Hence it holds for any fixed j with probability 1. The same calculations are obviously valid for $d_j = d_j(h)$. Lemma 4.4 is proven. *QED*

5 Concluding Remarks

In this paper we study the one of the simplest random spin model, i.e., $1-D$ ferromagnetic RFIM for dichotomous field, within the Gibbs field formalism.

For $T \neq 0$ we get that for *any realization* of the external field $h \in \mathbb{R}^{\mathbb{Z}}$ the limiting Gibbs measure $\mu_{\beta,h}$ is *unique*, i.e., independent of the boundary conditions (see Theorem 2.1 and Corollary 2.1). This result is well fitted with the common wisdom about 1-D systems with short–range interactions. The only difference with respect to non–random systems is in the statement *for any realization* of the random parameter (quenched randomness).

For $T = 0$ we obtain the same type of uniqueness theorem (see Theorem 3.1) but now for *almost all* (with respect to the Bernoulli measure λ) field configurations h. Moreover, we give a constructive description of the limiting ground state structure (Theorems 3.1–3.2), which gives more light on the residual entropy problem [11–17] as well as on recent calculations of the quenched two–point correlation function in 1-D RFIM for $T = 0$ [20].

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References

[1] T. Nattermann and P. Rujan, *Random field and other systems dominated by disorder fluctuations*, Int. J. Mod. Phys. B **3**, 1597–1654 (1989).

[2] J.M. Luck, *Systèmes Désordonnés Unidimensionnels*, Collection Aléa Saclay, 1992.

[3] M.R. Swift, A. Maritan, M. Cieplack, and J.R. Banavar, *Phase diagrams of random–field Ising systems*, J. Phys. A **27**, 1525–1532 (1994).

[4] S. Fishman and A. Aharony, *Random field effects in disordered anisotropic antiferromagnets*, J. Phys. C **12**, L729–733 (1979).

[5] P.G. de Gennes, *Liquid-liquid demixing inside a rigid network. Qualitative features*, J. Phys. Chem. **88**, 6469–6472 (1984).

[6] D.S. Fisher, J. Fröhlich, and T. Spencer, *The Ising model in a random magnetic field*, J. Stat. Phys. **34**, 863–870 (1984).

[7] J.Z. Imbrie, *The ground state of the three-dimensional random–field Ising model*, Commun. Math Phys. **98**, 145–176 (1985).

[8] S.R. Salinas and W.F. Wreszinski, *On the mean–field Ising model in a random external field*, J. Stat. Phys. **41**, 299–313 (1985).

[9] J.M.G. Amaros de Matos, A.E. Patrick, and V.A. Zagrebnov, *Random infinite–volume Gibbs states for the Curie–Weiss random field Ising model*, J. Stat. Phys. **66**, 139–164 (1992).

[10] R. Bruinsma, *Random-field Ising model on a Bethe lattice*, Phys. Rev. B **30**, 289–299 (1984).

[11] B. Derrida, J. Vannimenus, and Y. Pommeau, *Simple frustated systems: chains, strips and squares*, J. Phys. C: Solid State Phys. **11**, 4749–4765 (1978).

[12] R. Bruinsma and G. Aeppli, *One–dimensional Ising model in a random field*, Phys. Rev. Lett. **50**, 1494–1497 (1983).

[13] G. Györgyi and P. Rujan, *Strange attractors in disordered systems*, J. Phys. C: Solid State Phys. **17**, 4207–4212 (1984).

[14] J.M. Norman, N.L. Mehta, and H. Orland, *One dimensional random Ising model*, J. Phys. A: Math. Gen. **18**, 621–639 (1985).

[15] U. Behn and V.A. Zagrebnov, *One–dimensional random field Ising model and discrete stochastic mappings*, J. Stat. Phys. **47**, 939–946 (1987).

[16] U. Behn and V.A. Zagrebnov, *One–dimensional Markovian–field Ising model: physical properties and characteristic of the discrete stochastic mapping*, J. Phys. A: Math. Gen.**21**, 2151–2165 (1988).

[17] U. Behn, V.B. Priezzhev, and V.A. Zagrebnov, *One–dimensional random field Ising model: residual entropy, magnetization, and the "Perestroyka" of the ground state*, Physica A **167**, 481–493 (1990).

[18] H.O. Georgii, *Gibbs Measures and Phase Transitions*, de Gruyter Studies in Mathematics 9, Berlin New York 1988.

[19] J. Bricmont, J.L. Lebowitz, and C.-E. Pfister, *On the equivalence of boundary conditions*, J. Stat. Phys. **21**, 573–582 (1979).

[20] F. Iglói, *Correlations in random Ising chains at zero temperatures*, Preprint 1994.