# Continuity of Symplectically Adjoint Maps and the Algebraic Structure of Hadamard Vacuum Representations for Quantum Fields on Curved Spacetime

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26 September 1996

**Abstract.** We derive for a pair of operators on a symplectic space which are adjoints of each other with respect to the symplectic form (that is, they are sympletically adjoint) that, if they are bounded for some scalar product on the symplectic space dominating the symplectic form, then they are bounded with respect to a one-parametric family of scalar products canonically associated with the initially given one, among them being its "purification". As a typical example we consider a scalar field on a globally hyperbolic spacetime governed by the Klein-Gordon equation; the classical system is described by a symplectic space and the temporal evolution by symplectomorphisms (which are symplectically adjoint to their inverses). A natural scalar product is that inducing the classical energy norm, and an application of the above result yields that its "purification" induces on the one-particle space of the quantized system a topology which coincides with that given by the two-point functions of quasifree Hadamard states. These findings will be shown to lead to new results concerning the structure of the local (von Neumann) observable-algebras in representations of quasifree Hadamard states of the Klein-Gordon field in an arbitrary globally hyperbolic spacetime, such as local definiteness, local primarity and Haag-duality (and also split- and type $\mathrm{III}_1$-properties). A brief review of this circle of notions, as well as of properties of Hadamard states, forms part of the article.

*Supported by a Von Neumann Fellowship of the Operator Algebras Network, EC Human Capital and Mobility Programme.

# 1 Introduction

In the first part of this paper we shall investigate a special case of relative continuity of symplectically adjoint maps of a symplectic space. By this, we mean the following. Suppose that $(S, \sigma)$ is a symplectic space, i.e. $S$ is a real-linear vector space with an anti-symmetric, non-degenerate bilinear form $\sigma$ (the symplectic form). A pair $V, W$ of linear maps of $S$ will be called *symplectically adjoint* if $\sigma(V\phi, \psi) = \sigma(\phi, W\psi)$ for all $\phi, \psi \in S$. Let $\mu$ and $\mu'$ be two scalar products on $S$ and assume that, for each pair $V, W$ of symplectically adjoint linear maps of $(S, \sigma)$, the boundedness of both $V$ and $W$ with respect to $\mu$ implies their boundedness with respect to $\mu'$. Such a situation we refer to as *relative $\mu - \mu'$ continuity of symplectically adjoint maps* (of $(S, \sigma)$). A particular example of symplectically adjoint maps is provided by the pair $T, T^{-1}$ whenever $T$ is a symplectomorphism of $(S, \sigma)$. (Recall that a symplectomorphism of $(S, \sigma)$ is a bijective linear map $T : S \to S$ which preserves the symplectic form, $\sigma(T\phi, T\psi) = \sigma(\phi, \psi)$ for all $\phi, \psi \in S$.)

In the more specialized case to be considered in the present work, which will soon be indicated to be relevant in applications, we show that a certain distinguished relation between a scalar product $\mu$ on $S$ and a second one, $\mu'$, is sufficient for the relative $\mu - \mu'$ continuity of symplectically adjoint maps. (We give further details in Chapter 2, and in the next paragraph.) The result will be applied in Chapter 3 to answer a couple of open questions concerning the algebraic structure of the quantum theory of the free scalar field in arbitrary globally hyperbolic spacetimes: the local definiteness, local primarity and Haag-duality in representations of the local observable algebras induced by quasifree Hadamard states, as well as the determination of the type of the local von Neumann algebras in such representations. Technically, what needs to be proved in our approach to this problem is the continuity of the temporal evolution of the Cauchy-data of solutions of the scalar Klein-Gordon equation

$$(\nabla^a \nabla_a + r)\varphi = 0 \tag{1.1}$$

in a globally hyperbolic spacetime with respect to a certain topology on the Cauchy-data space. (Here, $\nabla$ is the covariant derivative of the metric $g$ on the spacetime, and $r$ an arbitrary realvalued, smooth function.) The Cauchy-data space is a symplectic space on which the said temporal evolution is realized by symplectomorphisms. It turns out that the classical "energy-norm" of solutions of (1.1), which is given by a scalar product $\mu_0$ on the Cauchy-data space, and the topology relevant for the required continuity statement (the "Hadamard one-particle space norm"), induced by a scalar product $\mu_1$ on the Cauchy-data space, are precisely in the relation for which our result on relative $\mu_0 - \mu_1$ continuity of symplectically adjoint maps applies. Since the continuity of the Cauchy-data evolution in the classical energy norm, i.e. $\mu_0$, is well-known, the desired continuity in the $\mu_1$-topology follows.

The argument just described may be viewed as the prime example of application of the relative continuity result. In fact, the relation between $\mu_0$ and $\mu_1$

is abstracted from the relation between the classical energy-norm and the one-particle space norms arising from "frequency-splitting" procedures in the canonical quantization of (linear) fields. This relation has been made precise in a recent paper by Chmielowski [11]. It provides the starting point for our investigation in Chapter 2, where we shall see in the that one can associate with a dominating scalar product $\mu \equiv \mu_0$ on $S$ in a canonical way a positive, symmetric operator $|R_\mu|$ on the $\mu$-completion of $S$, and a family of scalar products $\mu_s$, $s > 0$, on $S$, defined as $\mu$ with $|R_\mu|^s$ as an operator kernel. Using abstract interpolation, it will be shown that then relative $\mu_0 - \mu_s$ continuity of symplectically adjoint maps holds for all $0 \leq s \leq 2$. The relative $\mu_0 - \mu_1$ continuity arises as a special case. In fact, it turns out that the indicated interpolation argument may even be extended to an apparently more general situation from which the relative $\mu_0 - \mu_s$ continuity of symplectically adjoint maps derives as a corollary, see Theorem 2.2.

Chapter 3 will be concerned with the application of the result of Thm. 2.2 as indicated above. In the preparatory Section 3.1, some notions of general relativity will be summarized, along with the introduction of some notation. Section 3.2 contains a brief synopsis of the notions of local definiteness, local primarity and Haag-duality in the the context of quantum field theory in curved spacetime. In Section 3.3 we present the $C^*$-algebraic quantization of the KG-field obeying (1.1) on a globally hyperbolic spacetime, following [16]. Quasifree Hadamard states will be described in Section 3.4 according to the definition given in [45]. In the same section we briefly summarize some properties of Hadamard two-point functions, and derive, in Proposition 3.5, the result concerning the continuity of the Cauchy-data evolution maps in the topology of the Hadamard two-point functions which was mentioned above. It will be seen in the last Section 3.5 that this leads, in combination with results obtained earlier, [64,65,66], to Theorem 3.6 establishing detailed properties of the algebraic structure of the local von Neumann observable algebras in representations induced by quasifree Hadamard states of the Klein-Gordon field over an arbitrary globally hyperbolic spacetime.

# 2 Relative Continuity of Symplectically Adjoint Maps

Let $(S, \sigma)$ be a symplectic space. A (real-linear) scalar product $\mu$ on $S$ is said to *dominate* $\sigma$ if the estimate

$$|\sigma(\phi, \psi)|^2 \leq 4 \cdot \mu(\phi, \phi)\, \mu(\psi, \psi)\,, \quad \phi, \psi \in S\,, \tag{2.1}$$

holds; the set of all scalar products on $S$ which dominate $\sigma$ will be denoted by $\mathsf{q}(S, \sigma)$. Given $\mu \in \mathsf{q}(S, \sigma)$, we write $H_\mu \equiv \overline{S}^\mu$ for the completion of $S$ with respect to the topology induced by $\mu$, and denote by $\sigma_\mu$ the $\mu$-continuous extension, guaranteed to uniquely exist by (2.1), of $\sigma$ to $H_\mu$. The estimate (2.1) then extends to $\sigma_\mu$ and all $\phi, \psi \in H_\mu$. This entails that there is a uniquely determined, $\mu$-bounded linear operator $R_\mu : H_\mu \to H_\mu$ with the property

$$\sigma_\mu(x, y) = 2\, \mu(x, R_\mu y)\,, \quad x, y \in H_\mu\,. \tag{2.2}$$

The antisymmetry of $\sigma_\mu$ entails for the $\mu$-adjoint $R_\mu^*$ of $R_\mu$

$$R_\mu^* = -R_\mu \,, \tag{2.3}$$

and by (2.1) one finds that the operator norm of $R_\mu$ is bounded by 1, $|| R_\mu || \leq 1$. The operator $R_\mu$ will be called the *polarizator* of $\mu$.

In passing, two things should be noticed here:

(1) $R_\mu | S$ is injective since $\sigma$ is a non-degenerate bilinear form on $S$, but $R_\mu$ need not be injective on on all of $H_\mu$, as $\sigma_\mu$ may be degenerate.

(2) In general, it is not the case that $R_\mu(S) \subset S$.

Further properties of $R_\mu$ will be explored below. Let us first focus on two significant subsets of $\mathsf{q}(S,\sigma)$ which are intrinsically characterized by properties of the corresponding $\sigma_\mu$ or, equivalently, the $R_\mu$.

The first is $\mathsf{pr}(S,\sigma)$, called the set of *primary* scalar products on $(S,\sigma)$, where $\mu \in \mathsf{q}(S,\sigma)$ is in $\mathsf{pr}(S,\sigma)$ if $\sigma_\mu$ is a symplectic form (i.e. non-degenerate) on $H_\mu$. In view of (2.2) and (2.3), one can see that this is equivalent to either (and hence, both) of the following conditions:

(i) $R_\mu$ is injective,

(ii) $R_\mu(H_\mu)$ is dense in $H_\mu$.

The second important subset of $\mathsf{q}(S,\sigma)$ is denoted by $\mathsf{pu}(S,\sigma)$ and defined as consisting of those $\mu \in \mathsf{q}(S,\sigma)$ which satisfy the *saturation property*

$$\mu(\phi,\phi) = \sup_{\psi \in S\backslash\{0\}} \frac{|\sigma(\phi,\psi)|^2}{4\mu(\psi,\psi)} \,, \quad \psi \in S \,. \tag{2.4}$$

The set $\mathsf{pu}(S,\sigma)$ will be called the set of *pure* scalar products on $(S,\sigma)$. It is straightforward to check that $\mu \in \mathsf{pu}(S,\sigma)$ if and only if $R_\mu$ is a unitary anti-involution, or complex structure, i.e. $R_\mu^{-1} = R_\mu^*$, $R_\mu^2 = -1$. Hence $\mathsf{pu}(S,\sigma) \subset \mathsf{pr}(S,\sigma)$.

Our terminology reflects well-known relations between properties of quasifree states on the (CCR-) Weyl-algebra of a symplectic space $(S,\sigma)$ and properties of $\sigma$-dominating scalar products on $S$, which we shall briefly recapitulate. We refer to [1,3,5,45,49] and also references quoted therein for proofs and further discussion of the following statements.

Given a symplectic space $(S,\sigma)$, one can associate with it uniquely (up to $C^*$-algebraic equivalence) a $C^*$-algebra $\mathcal{A}[S,\sigma]$, which is generated by a family of unitary elements $W(\phi)$, $\phi \in S$, satisfying the canonical commutation relations (CCR) in exponentiated form,

$$W(\phi)W(\psi) = \mathrm{e}^{-i\sigma(\phi,\psi)/2}W(\phi+\psi) \,, \quad \phi,\psi \in S \,. \tag{2.5}$$

The algebra $\mathcal{A}[S,\sigma]$ is called the *Weyl-algebra*, or *CCR-algebra*, of $(S,\sigma)$. It is not difficult to see that if $\mu \in \mathsf{q}(S,\sigma)$, then one can define a state (i.e., a positive, normalized linear functional) $\omega_\mu$ on $\mathcal{A}[S,\sigma]$ by setting

$$\omega_\mu(W(\phi)) := \mathrm{e}^{-\mu(\phi,\phi)/2}\,, \quad \phi \in S\,. \tag{2.6}$$

Any state on the Weyl-algebra $\mathcal{A}[S,\sigma]$ which can be realized in this way is called a *quasifree state*. Conversely, given any quasifree state $\omega_\mu$ on $\mathcal{A}[S,\sigma]$, one can recover its $\mu \in \mathsf{q}(S,\sigma)$ as

$$\mu(\phi,\psi) = 2\mathsf{Re}\,\frac{\partial}{\partial t}\frac{\partial}{\partial \tau}\bigg|_{t=\tau=0} \omega_\mu(W(t\phi)W(\tau\psi))\,, \quad \phi,\psi \in S\,. \tag{2.7}$$

So there is a one-to-one correspondence between quasifree states on $\mathcal{A}[S,\sigma]$ and dominating scalar products on $(S,\sigma)$.

Let us now recall the subsequent terminology. To a state $\omega$ on a $C^*$-algebra $\mathcal{B}$ there corresponds (uniquely up to unitary equivalence) a triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, called the GNS-representation of $\omega$ (see e.g. [5]), characterized by the following properties: $\mathcal{H}_\omega$ is a complex Hilbertspace, $\pi_\omega$ is a representation of $\mathcal{B}$ by bounded linear operators on $\mathcal{H}_\omega$ with cyclic vector $\Omega_\omega$, and $\omega(B) = \langle \Omega_\omega, \pi_\omega(B)\Omega_\omega\rangle$ for all $B \in \mathcal{B}$. Hence one is led to associate with $\omega$ and $\mathcal{B}$ naturally the $\omega$-induced von Neumann algebra $\pi_\omega(\mathcal{B})^-$, where the bar means taking the closure with respect to the weak operator topology in the set of bounded linear operators on $\mathcal{H}_\omega$. One refers to $\omega$ (resp., $\pi_\omega$) as *primary* if $\pi_\omega(\mathcal{B})^- \cap \pi_\omega(\mathcal{B})' = \mathbf{C}\cdot 1$ (so the center of $\pi_\omega(\mathcal{B})^-$ is trivial), where the prime denotes taking the commutant, and as *pure* if $\pi_\omega(\mathcal{B})' = \mathbf{C}\cdot 1$ (i.e. $\pi_\omega$ is irreducible — this is equivalent to the statement that $\omega$ is not a (non-trivial) convex sum of different states).

In the case where $\omega_\mu$ is a quasifree state on a Weyl-algebra $\mathcal{A}[S,\sigma]$, it is known that (cf. [1,49])

(I) $\omega_\mu$ is primary if and only if $\mu \in \mathsf{pr}(S,\sigma)$,

(II) $\omega_\mu$ is pure if and only if $\mu \in \mathsf{pu}(S,\sigma)$.

We return to the investigation of the properties of the polarizator $R_\mu$ for a dominating scalar product $\mu$ on a symplectic space $(S,\sigma)$. It possesses a polar decomposition

$$R_\mu = U_\mu |R_\mu| \tag{2.8}$$

on the Hilbertspace $(H_\mu,\mu)$, where $U_\mu$ is an isometry and $|R_\mu|$ is symmetric and has non-negative spectrum. Since $R_\mu^* = -R_\mu$, $R_\mu$ is normal and thus $|R_\mu|$ and $U_\mu$ commute. Moreover, one has $|R_\mu|U_\mu^* = -U_\mu|R_\mu|$, and hence $|R_\mu|$ and $U_\mu^*$ commute as well. One readily observes that $(U_\mu^* + U_\mu)|R_\mu| = 0$. The commutativity can by the spectral calculus be generalized to the statement that, whenever $f$ is a real-valued, continuous function on the real line, then

$$[f(|R_\mu|), U_\mu] = 0 = [f(|R_\mu|), U_\mu^*]\,, \tag{2.9}$$

where the brackets denote the commutator.

In a recent work [11], Chmielowski noticed that if one defines for $\mu \in \mathsf{q}(S,\sigma)$ the bilinear form

$$\tilde{\mu}(\phi,\psi) := \mu(\phi, |R_\mu|\psi)\,, \quad \phi,\psi \in S, \tag{2.10}$$

then it holds that $\tilde{\mu} \in \mathsf{pu}(S,\sigma)$. The proof of this is straightforward. That $\tilde{\mu}$ dominates $\sigma$ will be seen in Proposition 2.1 below. To check the saturation property (2.4) for $\tilde{\mu}$, it suffices to observe that for given $\phi \in H_\mu$, the inequality in the following chain of expressions:

$$\begin{aligned}
\frac{1}{4}|\sigma_\mu(\phi,\psi)|^2 &= |\mu(\phi, U_\mu|R_\mu|\psi)|^2 \;=\; |\mu(\phi, -U_\mu^*|R_\mu|\psi)|^2 \\
&= |\mu(|R_\mu|^{1/2}U_\mu\phi, |R_\mu|^{1/2}\psi)|^2 \\
&\leq \mu(|R_\mu|^{1/2}U_\mu\phi, |R_\mu|^{1/2}U_\mu\phi)\cdot\mu(|R_\mu|^{1/2}\psi, |R_\mu|^{1/2}\psi)
\end{aligned}$$

is saturated and becomes an equality upon choosing $\psi \in H_\mu$ so that $|R_\mu|^{1/2}\psi$ is parallel to $|R_\mu|^{1/2}U_\mu\phi$. Therefore one obtains for all $\phi \in S$

$$\begin{aligned}
\sup_{\psi\in S\setminus\{0\}} \frac{|\sigma(\phi,\psi)|^2}{4\mu(\psi,|R_\mu|\psi)} &= \mu(|R_\mu|^{1/2}U_\mu\phi, |R_\mu|^{1/2}U_\mu\phi) \\
&= \mu(U_\mu|R_\mu|^{1/2}\phi, U_\mu|R_\mu|^{1/2}\phi) \\
&= \tilde{\mu}(\phi,\phi)\,,
\end{aligned}$$

which is the required saturation property.

Following Chmielowski, the scalar product $\tilde{\mu}$ on $S$ associated with $\mu \in \mathsf{q}(S,\sigma)$ will be called the *purification* of $\mu$.

It appears natural to associate with $\mu \in \mathsf{q}(S,\sigma)$ the family $\mu_s$, $s > 0$, of symmetric bilinear forms on $S$ given by

$$\mu_s(\phi,\psi) := \mu(\phi, |R_\mu|^s\psi)\,, \quad \phi,\psi \in S\,. \tag{2.11}$$

We will use the convention that $\mu_0 = \mu$. Observe that $\tilde{\mu} = \mu_1$. The subsequent proposition ensues.

**Proposition 2.1**

*(a) $\mu_s$ is a scalar product on $S$ for each $s \geq 0$.*

*(b) $\mu_s$ dominates $\sigma$ for $0 \leq s \leq 1$.*

*(c) Suppose that there is some $s \in (0,1)$ such that $\mu_s \in \mathsf{pu}(S,\sigma)$. Then $\mu_r = \mu_1$ for all $r > 0$. If it is in addition assumed that $\mu \in \mathsf{pr}(S,\sigma)$, then it follows that $\mu_r = \mu_1$ for all $r \geq 0$, i.e. in particular $\mu = \tilde{\mu}$.*

*(d) If $\mu_s \in \mathsf{q}(S,\sigma)$ for some $s > 1$, then $\mu_r = \mu_1$ for all $r > 0$. Assuming additionally $\mu \in \mathsf{pr}(S,\sigma)$, one obtains $\mu_r = \mu_1$ for all $r \geq 0$, entailing $\mu = \tilde{\mu}$.*

*(e) The purifications of the $\mu_s$, $0 < s < 1$, are equal to $\tilde{\mu}$: We have $\widetilde{\mu_s} = \tilde{\mu} = \mu_1$ for all $0 < s < 1$.*

*Proof.* (a) According to (b), $\mu_s$ dominates $\sigma$ for each $0 \le s \le 1$, thus it is a scalar product whenever $s$ is in that range. However, it is known that $\mu(\phi, |R_\mu|^s\phi) \ge \mu(\phi, |R_\mu|\phi)^s$ for all vectors $\phi \in H_\mu$ of unit length ($\mu(\phi,\phi) = 1$) and $1 \le s < \infty$, cf. [60 (p. 20)]. This shows that $\mu_s(\phi,\phi) \ne 0$ for all nonzero $\phi$ in $S$, $s \ge 0$.

(b) For $s$ in the indicated range there holds the following estimate:

$$\begin{aligned}
\frac{1}{4}|\sigma(\phi,\psi)|^2 &= |\mu(\phi, U_\mu|R_\mu|\psi)|^2 = |\mu(\phi, -U_\mu^*|R_\mu|\psi)|^2 \\
&= |\mu(|R_\mu|^{s/2}U_\mu\phi, |R_\mu|^{1-s/2}\psi)|^2 \\
&\le \mu(U_\mu|R_\mu|^{s/2}\phi, U_\mu|R_\mu|^{s/2}\phi)\cdot \mu(|R_\mu|^{s/2}\psi, |R_\mu|^{2(1-s)}|R_\mu|^{s/2}\psi) \\
&\le \mu_s(\phi,\phi)\cdot\mu_s(\psi,\psi)\,, \quad \phi,\psi \in S\,.
\end{aligned}$$

Here, we have used that $|R_\mu|^{2(1-s)} \le 1$.

(c) If $(\phi_n)$ is a $\mu$-Cauchy-sequence in $H_\mu$, then it is, by continuity of $|R_\mu|^{s/2}$, also a $\mu_s$-Cauchy-sequence in $H_s$, the $\mu_s$-completion of $S$. Via this identification, we obtain an embedding $j : H_\mu \to H_s$. Notice that $j(\psi) = \psi$ for all $\psi \in S$, so $j$ has dense range; however, one has

$$\mu_s(j(\phi), j(\psi)) = \mu(\phi, |R_\mu|^s\psi) \tag{2.12}$$

for all $\phi,\psi \in H_\mu$. Therefore $j$ need not be injective. Now let $R_s$ be the polarizator of $\mu_s$. Then we have

$$\begin{aligned}
2\mu_s(j(\phi), R_s j(\psi)) = \sigma_\mu(\phi,\psi) &= 2\mu(\phi, R_\mu\psi) \\
&= 2\mu(\phi, |R_\mu|^s U_\mu |R_\mu|^{1-s}\psi) \\
&= 2\mu_s(j(\phi), j(U_\mu|R_\mu|^{1-s})\psi)\,, \quad \phi,\psi \in H_\mu\,.
\end{aligned}$$

This yields

$$R_s \circ j = j \circ U_\mu |R_\mu|^{1-s} \tag{2.13}$$

on $H_\mu$. Since by assumption $\mu_s$ is pure, we have $R_s^2 = -1$ on $H_s$, and thus

$$j = -R_s j U_\mu |R_\mu|^{1-s} = -j(U_\mu|R_\mu|^{1-s})^2\,.$$

By (2.12) we may conclude

$$|R_\mu|^{2s} = -U_\mu|R_\mu|U_\mu|R_\mu| = U_\mu^* U_\mu |R_\mu|^2 = |R_\mu|^2\,,$$

which entails $|R_\mu|^s = |R_\mu|$. Since $|R_\mu| \le 1$, we see that for $s \le r \le 1$ we have

$$|R_\mu| = |R_\mu|^s \ge |R_\mu|^r \ge |R_\mu|\,,$$

hence $|R_\mu|^r = |R_\mu|$ for $s \ge r \ge 1$. Whence $|R_\mu|^r = |R_\mu|$ for all $r > 0$. This proves the first part of the statement.

For the second part we observe that $\mu \in \mathsf{pr}(S,\sigma)$ implies that $|R_\mu|$, and hence also $|R_\mu|^s$ for $0 < s < 1$, is injective. Then the equation $|R_\mu|^s = |R_\mu|$ implies that $|R_\mu|^s(|R_\mu|^{1-s} - 1) = 0$, and by the injectivity of $|R_\mu|^s$ we may conclude

$|R_\mu|^{1-s} = 1$. Since $s$ was assumed to be strictly less than 1, it follows that $|R_\mu|^r = 1$ for all $r \geq 0$; in particular, $|R_\mu| = 1$.

(d) Assume that $\mu_s$ dominates $\sigma$ for some $s > 1$, i.e. it holds that

$$4|\mu(\phi, U_\mu|R_\mu|\psi)|^2 = |\sigma_\mu(\phi,\psi)|^2 \leq 4 \cdot \mu(\phi, |R_\mu|^s\phi) \cdot \mu(\psi, |R_\mu|^s\psi)\,, \quad \phi,\psi \in H_\mu\,,$$

which implies, choosing $\phi = U_\mu\psi$, the estimate

$$\mu(\psi, |R_\mu|\psi) \leq \mu(\psi, |R_\mu|^s\psi)\,, \quad \psi \in H_\mu\,,$$

i.e. $|R_\mu| \leq |R_\mu|^s$. On the other hand, $|R_\mu| \geq |R_\mu|^r \geq |R_\mu|^s$ holds for all $1 \leq r \leq s$ since $|R_\mu| \leq 1$. This implies $|R_\mu|^r = |R_\mu|$ for all $r > 0$. For the second part of the statement one uses the same argument as given in (c).

(e) In view of (2.13) it holds that

$$\begin{aligned} |R_s|^2 j &= -R_s^2 j = -R_s j U_\mu |R_\mu|^{1-s} \\ &= -jU_\mu|R_\mu|^{1-s}U_\mu|R_\mu|^{1-s} = -jU_\mu^2(|R_\mu|^{1-s})^2\,. \end{aligned}$$

Iterating this one has for all $n \in \mathbf{N}$

$$|R_s|^{2n} j = (-1)^n j U_\mu^{2n}(|R_\mu|^{1-s})^{2n}\,.$$

Inserting this into relation (2.12) yields for all $n \in \mathbf{N}$

$$\begin{aligned} \mu_s(j(\phi), |R_s|^{2n} j(\psi)) &= \mu(\phi, |R_\mu|^s(-1)^n U_\mu^{2n}(|R_\mu|^{1-s})^{2n}\psi) \\ &= \mu(\phi, |R_\mu|^s(|R_\mu|^{1-s})^{2n}\psi)\,, \quad \phi,\psi \in H_\mu\,. \end{aligned} \tag{2.14}$$

For the last equality we used that $U_\mu$ commutes with $|R_s|^s$ and $U_\mu^2|R_\mu| = -|R_\mu|$. Now let $(P_n)$ be a sequence of polynomials on the intervall $[0,1]$ converging uniformly to the square root function on $[0,1]$. From (2.14) we infer that

$$\mu_s(j(\phi), P_n(|R_s|^2)j(\psi)) = \mu(\phi, |R_\mu|^s P_n((|R_\mu|^{1-s})^2)\psi)\,, \quad \phi,\psi \in H_\mu$$

for all $n \in \mathbf{N}$, which in the limit $n \to \infty$ gives

$$\mu_s(j(\phi), |R_s| j(\psi)) = \mu(\phi, |R_\mu|\psi)\,, \quad \phi,\psi \in H_\mu\,,$$

as desired. □

Proposition 2.1 underlines the special role of $\tilde{\mu} = \mu_1$. Clearly, one has $\tilde{\mu} = \mu$ iff $\mu \in \mathsf{pu}(S,\sigma)$. Chmielowski has proved another interesting connection between $\mu$ and $\tilde{\mu}$ which we briefly mention here. Suppose that $\{T_t\}$ is a one-parametric group of symplectomorphisms of $(S,\sigma)$, and let $\{\alpha_t\}$ be the automorphism group on $\mathcal{A}[S,\sigma]$ induced by it via $\alpha_t(W(\phi)) = W(T_t\phi)$, $\phi \in S,\ t \in \mathbf{R}$. An $\{\alpha_t\}$-invariant quasifree state $\omega_\mu$ on $\mathcal{A}[S,\sigma]$ is called *regular* if the unitary group which implements $\{\alpha_t\}$ in the GNS-representation $(\mathcal{H}_\mu, \pi_\mu, \Omega_\mu)$ of $\omega_\mu$ is strongly continuous and leaves no non-zero vector in the one-particle space of $\mathcal{H}_\mu$ invariant. Here, the one-particle space is spanned by all vectors of the form $\frac{d}{dt}\Big|_{t=0} \pi_\mu(W(t\phi))\Omega_\mu$, $\phi \in S$. It is proved in [11] that, if $\omega_\mu$ is a regular quasifree KMS-state for $\{\alpha_t\}$, then $\omega_{\tilde{\mu}}$ is the unique regular quasifree groundstate for $\{\alpha_t\}$. As explained in [11], the passage from $\mu$ to $\tilde{\mu}$ can be seen as a rigorous form of "frequency-splitting" methods employed in the canonical quantization of classical fields for which $\mu$ is induced by the classical energy norm. We shall come back to this in the concrete example of the Klein-Gordon field in Sec. 3.4.

It should be noted that the purification map $\tilde{\ } : \mathsf{q}(S,\sigma) \to \mathsf{pu}(S,\sigma)$, $\mu \mapsto \tilde{\mu}$, assigns to a quasifree state $\omega_\mu$ on $\mathcal{A}[S,\sigma]$ the pure quasifree state $\omega_{\tilde{\mu}}$ which is again a state on $\mathcal{A}[S,\sigma]$. This is different from the well-known procedure of assigning to a state $\omega$ on a $C^*$-algebra $\mathcal{A}$, whose GNS representation is primary, a pure state $\omega_0$ on $\mathcal{A}^\circ \otimes \mathcal{A}$. ($\mathcal{A}^\circ$ denotes the opposite algebra of $\mathcal{A}$, cf. [75].) That procedure was introduced by Woronowicz and is an abstract version of similar constructions for quasifree states on CCR- or CAR-algebras [45,54,75]. Whether the purification map $\omega_\mu \mapsto \omega_{\tilde{\mu}}$ can be generalized from quasifree states on CCR-algebras to a procedure of assigning to (a suitable class of) states on a generic $C^*$-algebra pure states on that same algebra, is in principle an interesting question, which however we shall not investigate here.

**Theorem 2.2**

*(a) Let $H$ be a (real or complex) Hilbertspace with scalar product $\mu(\,.\,,\,.\,)$, $R$ a (not necessarily bounded) normal operator in $H$, and $V, W$ two $\mu$-bounded linear operators on $H$ which are $R$-adjoint, i.e. they satisfy*

$$W\mathrm{dom}(R) \subset \mathrm{dom}(R) \quad \textit{and} \quad V^*R = RW \quad \text{on} \quad \mathrm{dom}(R)\,. \tag{2.15}$$

*Denote by $\mu_s$ the Hermitean form on $\mathrm{dom}(|R|^{s/2})$ given by*

$$\mu_s(x,y) := \mu(|R|^{s/2}x, |R|^{s/2}y)\,, \quad x, y \in \mathrm{dom}(|R|^{s/2}),\ 0 \leq s \leq 2\,.$$

*We write $||\,.\,||_0 := ||\,.\,||_\mu := \mu(\,.\,,\,.\,)^{1/2}$ and $||\,.\,||_s := \mu_s(\,.\,,\,.\,)^{1/2}$ for the corresponding semi-norms.*

*Then it holds for all $0 \leq s \leq 2$ that*

$$V\mathrm{dom}(|R|^{s/2}) \subset \mathrm{dom}(|R|^{s/2}) \quad \textit{and} \quad W\mathrm{dom}(|R|^{s/2}) \subset \mathrm{dom}(|R|^{s/2})\,,$$

*and $V$ and $W$ are $\mu_s$-bounded for $0 \leq s \leq 2$. More precisely, the estimates*

$$||\,Vx\,||_0 \leq v\,||\,x\,||_0 \quad \text{and} \quad ||\,Wx\,||_0 \leq w\,||\,x\,||_0\,, \quad x \in H\,, \tag{2.16}$$

*with suitable constants $v, w > 0$, imply that*

$$||\,Vx\,||_s \leq w^{s/2}v^{1-s/2}\,||\,x\,||_s \quad \text{and} \quad ||\,Wx\,||_s \leq v^{s/2}w^{1-s/2}\,||\,x\,||_s\,, \tag{2.17}$$

*for all $x \in \mathrm{dom}(|R|^{s/2})$ and $0 \leq s \leq 2$.*

*(b) (Corollary of (a)) Let $(S,\sigma)$ be a symplectic space, $\mu \in \mathsf{q}(S,\sigma)$ a dominating scalar product on $(S,\sigma)$, and $\mu_s$, $0 < s \leq 2$, the scalar products on $S$ defined in*

*(2.11). Then we have relative $\mu - \mu_s$ continuity of each pair $V, W$ of symplectically adjoint linear maps of $(S, \sigma)$ for all $0 < s \leq 2$. More precisely, for each pair $V, W$ of symplectically adjoint linear maps of $(S, \sigma)$, the estimates (2.16) for all $x \in S$ imply (2.17) for all $x \in S$.*

*Remark.* (i) In view of the fact that the operator $R$ of part (a) of the Theorem may be unbounded, part (b) can be extended to situations where it is not assumed that the scalar product $\mu$ on $S$ dominates the symplectic form $\sigma$.

(ii) When it is additionally assumed that $V = T$ and $W = T^{-1}$ with symplectomorphisms $T$ of $(S, \sigma)$, we refer in that case to the situation of relative continuity of the pairs $V, W$ as relative continuity of symplectomorphisms. In Example 2.3 after the proof of Thm. 2.2 we show that relative $\tilde{\mu} - \mu$ continuity of symplectomorphisms fails in general. Also, it is not the case that relative $\mu - \mu'$ continuity of symplectomorphisms holds if $\mu'$ is an arbitrary element in $\mathsf{pu}(S, \sigma)$ which is dominated by $\mu$ ($||\,\phi\,||_{\mu'} \leq \text{const.}||\,\phi\,||_\mu$, $\phi \in S$), see Example 2.4 below. This shows that the special relation between $\mu$ and $\tilde{\mu}$ (resp., $\mu$ and the $\mu_s$) expressed in (2.11,2.15) is important for the derivation of the Theorem.

*Proof of Theorem 2.2.* (a) In a first step, let it be supposed that $R$ is bounded. From the assumed relation (2.15) and its adjoint relation $R^*V = W^*R^*$ we obtain, for $\epsilon' > 0$ arbitrarily chosen,

$$\begin{aligned} V^*(|R|^2 + \epsilon' 1)V &= V^*RR^*V + \epsilon' V^*V = RWW^*R^* + \epsilon' V^*V \\ &\leq w^2|R|^2 + \epsilon' v^2 1 \leq w^2(|R|^2 + \epsilon 1) \end{aligned}$$

with $\epsilon := \epsilon' v^2/w^2$. This entails for the operator norms

$$||\,(|R|^2 + \epsilon 1)^{1/2} V\,|| \leq w\,||\,(|R|^2 + \epsilon 1)^{1/2}\,||\,,$$

and since $(|R|^2 + \epsilon 1)^{1/2}$ has a bounded inverse,

$$||\,(|R|^2 + \epsilon 1)^{1/2} V (|R|^2 + \epsilon 1)^{-1/2}\,|| \leq w\,.$$

On the other hand, one clearly has

$$||\,(|R|^2 + \epsilon 1)^0 V (|R|^2 + \epsilon 1)^0\,|| = ||\,V\,|| \leq v\,.$$

Now these estimates are preserved if $R$ and $V$ are replaced by their complexified versions on the complexified Hilbertspace $H \oplus iH = \mathbf{C} \otimes H$. Thus, identifying if necessary $R$ and $V$ with their complexifications, a standard interpolation argument (see Appendix A) can be applied to yield

$$||\,(|R|^2 + \epsilon 1)^\alpha V (|R|^2 + \epsilon 1)^{-\alpha}\,|| \leq w^{2\alpha} v^{1-2\alpha}$$

for all $0 \leq \alpha \leq 1/2$. Notice that this inequality holds uniformly in $\epsilon > 0$. Therefore we may conclude that

$$||\,|R|^{2\alpha} V x\,||_0 \leq w^{2\alpha} v^{1-2\alpha}\,||\,|R|^{2\alpha} x\,||_0\,, \quad x \in H\,,\ 0 \leq \alpha \leq 1/2\,,$$

which is the required estimate for $V$. The analogous bound for $W$ is obtained through replacing $V$ by $W$ in the given arguments.

Now we have to extend the argument to the case that $R$ is unbounded. Without restriction of generality we may assume that the Hilbertspace $H$ is complex, otherwise we complexify it and with it all the operators $R$,$V$,$W$, as above, thereby preserving their assumed properties. Then let $E$ be the spectral measure of $R$, and denote by $R_r$ the operator $E(B_r)RE(B_r)$ where $B_r := \{z \in \mathbf{C} : |z| \leq r\}$, $r > 0$. Similarly define $V_r$ and $W_r$. From the assumptions it is seen that $V_r^* R_r = R_r W_r$ holds for all $r > 0$. Applying the reasoning of the first step we arrive, for each $0 \leq s \leq 2$, at the bounds

$$|| V_r x ||_s \leq w^{s/2} v^{1-s/2} || x ||_s \quad \text{and} \quad || W_r x ||_s \leq w^{s/2} w^{1-s/2} || x ||_s ,$$

which hold uniformly in $r > 0$ for all $x \in \text{dom}(|R|^{s/2})$. ¿From this, the statement of the Proposition follows.

(b) This is just an application of (a), identifying $H_\mu$ with $H$, $R_\mu$ with $R$ and $V, W$ with their bounded extensions to $H_\mu$. □

**Example 2.3** We exhibit a symplectic space $(S, \sigma)$ with $\mu \in \mathsf{pr}(S, \sigma)$ and a symplectomorphism $T$ of $(S, \sigma)$ where $T$ and $T^{-1}$ are continuous with respect to $\tilde{\mu}$, but not with respect to $\mu$.

Let $S := \mathcal{S}(\mathbf{R}, \mathbf{C})$, the Schwartz space of rapidly decreasing testfunctions on $\mathbf{R}$, viewed as real-linear space. By $\langle \phi, \psi \rangle := \int \overline{\phi} \psi \, dx$ we denote the standard $L^2$ scalar product. As a symplectic form on $S$ we choose

$$\sigma(\phi, \psi) := 2\mathsf{Im}\langle \phi, \psi \rangle , \quad \phi, \psi \in S .$$

Now define on $S$ the strictly positive, essentially selfadjoint operator $A\phi := -\frac{d^2}{dx^2}\phi + \phi$, $\phi \in S$, in $L^2(\mathbf{R})$. Its closure will again be denoted by $A$; it is bounded below by 1. A real-linear scalar product $\mu$ will be defined on $S$ by

$$\mu(\phi, \psi) := \mathsf{Re}\langle A\phi, \psi \rangle , \quad \phi, \psi \in S.$$

Since $A$ has lower bound 1, clearly $\mu$ dominates $\sigma$, and one easily obtains $R_\mu = -iA^{-1}$, $|R_\mu| = A^{-1}$. Hence $\mu \in \mathsf{pr}(S, \sigma)$ and

$$\tilde{\mu}(\phi, \psi) = \mathsf{Re}\langle \phi, \psi \rangle , \quad \phi, \psi \in S .$$

Now consider the operator

$$T : S \to S , \qquad (T\phi)(x) := \mathrm{e}^{-ix^2} \phi(x) , \quad x \in \mathbf{R}, \ \phi \in S .$$

Obviously $T$ leaves the $L^2$ scalar product invariant, and hence also $\sigma$ and $\tilde{\mu}$. The inverse of $T$ is just $(T^{-1}\phi)(x) = \mathrm{e}^{ix^2}\phi(x)$, which of course leaves $\sigma$ and $\tilde{\mu}$ invariant as well. However, $T$ is not continuous with respect to $\mu$. To see this, let $\phi \in S$ be some non-vanishing smooth function with compact support, and define

$$\phi_n(x) := \phi(x - n) , \quad x \in \mathbf{R}, \ n \in \mathbf{N} .$$

Then $\mu(\phi_n, \phi_n) = \text{const.} > 0$ for all $n \in \mathbf{N}$. We will show that $\mu(T\phi_n, T\phi_n)$ diverges for $n \to \infty$. We have

$$\begin{aligned} \mu(T\phi_n, T\phi_n) &= \langle AT\phi_n, T\phi_n \rangle \geq \int \overline{(T\phi_n)'}(T\phi_n)'\, dx \\ &\geq \int (4x^2|\phi_n(x)|^2 + |\phi_n'(x)|^2)\, dx - \int 4|x\phi_n'(x)\phi_n(x)|\, dx\,, \end{aligned} \tag{2.18}$$

where the primes indicate derivatives and we have used that

$$|(T\phi_n)'(x)|^2 = 4x^2|\phi_n(x)|^2 + |\phi_n'(x)|^2 + 4 \cdot \mathsf{Im}(ix\overline{\phi_n}(x)\phi_n'(x))\,.$$

Using a substitution of variables, one can see that in the last term of (2.18) the positive integral grows like $n^2$ for large $n$, thus dominating eventually the negative integral which grows only like $n$. So $\mu(T\phi_n, T\phi_n) \to \infty$ for $n \to \infty$, showing that $T$ is not $\mu$-bounded.

**Example 2.4** We give an example of a symplectic space $(S, \sigma)$, a $\mu \in \mathsf{pr}(S, \sigma)$ and a $\mu' \in \mathsf{pu}(S, \sigma)$, where $\mu$ dominates $\mu'$ and where there is a symplectomorphism $T$ of $(S, \sigma)$ which together with its inverse is $\mu$-bounded, but not $\mu'$-bounded.

We take $(S, \sigma)$ as in the previous example and write for each $\phi \in S$, $\phi_0 := \mathsf{Re}\phi$ and $\phi_1 := \mathsf{Im}\phi$. The real scalar product $\mu$ will be defined by

$$\mu(\phi, \psi) := \langle \phi_0, A\psi_0 \rangle + \langle \phi_1, \psi_1 \rangle\,, \quad \phi, \psi \in S\,,$$

where the operator $A$ is the same as in the example before. Since its lower bound is 1, $\mu$ dominates $\sigma$, and it is not difficult to see that $\mu$ is even primary. The real-linear scalar product $\mu'$ will be taken to be

$$\mu'(\phi, \psi) = \mathsf{Re}\langle \phi, \psi \rangle\,, \quad \phi, \psi \in S\,.$$

We know from the example above that $\mu' \in \mathsf{pu}(S, \sigma)$. Also, it is clear that $\mu'$ is dominated by $\mu$. Now consider the real-linear map $T : S \to S$ given by

$$T(\phi_0 + i\phi_1) := A^{-1/2}\phi_1 - iA^{1/2}\phi_0\,, \quad \phi \in S\,.$$

One checks easily that this map is bijective with $T^{-1} = -T$, and that $T$ preserves the symplectic form $\sigma$. Also, $||\,.\,||_\mu$ is preserved by $T$ since

$$\mu(T\phi, T\phi) = \langle \phi_1, \phi_1 \rangle + \langle A^{1/2}\phi_0, A^{1/2}\phi_0 \rangle = \mu(\phi, \phi)\,, \quad \phi \in S\,.$$

On the other hand, we have for each $\phi \in S$

$$\mu'(T\phi, T\phi) = \langle \phi_1, A\phi_1 \rangle + \langle \phi_0, A^{-1}\phi_0 \rangle\,,$$

and this expression is not bounded by a ($\phi$-independent) constant times $\mu'(\phi, \phi)$, since $A$ is unbounded with respect to the $L^2$-norm.

# 3 The Algebraic Structure of Hadamard Vacuum Representations

## 3.1 Summary of Notions from Spacetime-Geometry

We recall that a spacetime manifold consists of a pair $(M, g)$, where $M$ is a smooth, paracompact, four-dimensional manifold without boundaries, and $g$ is a Lorentzian metric for $M$ with signature $(+ - - -)$. (Cf. [33,52,70], see these references also for further discussion of the notions to follow.) It will be assumed that $(M, g)$ is time-orientable, and moreover, globally hyperbolic. The latter means that $(M, g)$ possesses Cauchy-surfaces, where by a Cauchy-surface we always mean a *smooth*, spacelike hypersurface which is intersected exactly once by each inextendable causal curve in $M$. It can be shown [15,28] that this is equivalent to the statement that $M$ can be smoothly foliated in Cauchy-surfaces. Here, a foliation of $M$ in Cauchy-surfaces is a diffeomorphism $F : \mathbf{R} \times \Sigma \to M$, where $\Sigma$ is a smooth 3-manifold so that $F(\{t\} \times \Sigma)$ is, for each $t \in \mathbf{R}$, a Cauchy-surface, and the curves $t \mapsto F(t, q)$ are timelike for all $q \in \Sigma$. (One can even show that, if global hyperbolicity had been defined by requiring only the existence of a non necessarily smooth or spacelike Cauchy-surface (i.e. a topological hypersurface which is intersected exactly once by each inextendable causal curve), then it is still true that a globally hyperbolic spacetime can be smoothly foliated in Cauchy-surfaces, see [15,28].)

We shall also be interested in ultrastatic globally hyperbolic spacetimes. A globally hyperbolic spacetime is said to be *ultrastatic* if a foliation $F : \mathbf{R} \times \Sigma \to M$ in Cauchy-surfaces can be found so that $F_*g$ has the form $dt^2 \oplus (-\gamma)$ with a complete ($t$-independent) Riemannian metric $\gamma$ on $\Sigma$. This particular foliation will then be called a *natural foliation* of the ultrastatic spacetime. (An ultrastatic spacetime may posses more than one natural foliation, think e.g. of Minkowski-spacetime.)

The notation for the causal sets and domains of dependence will be recalled: Given a spacetime $(M, g)$ and $\mathcal{O} \subset M$, the set $J^{\pm}(\mathcal{O})$ (causal future/past of $\mathcal{O}$) consists of all points $p \in M$ which can be reached by future/past directed causal curves emanating from $\mathcal{O}$. The set $D^{\pm}(\mathcal{O})$ (future/past domain of dependence of $\mathcal{O}$) is defined as consisting of all $p \in J^{\pm}(\mathcal{O})$ such that every past/future inextendible causal curve starting at $p$ intersects $\mathcal{O}$. One writes $J(\mathcal{O}) := J^+(\mathcal{O}) \cup J^-(\mathcal{O})$ and $D(\mathcal{O}) := D^+(\mathcal{O}) \cup D^-(\mathcal{O})$. They are called the *causal set*, and the *domain of dependence*, respectively, of $\mathcal{O}$.

For $\mathcal{O} \subset M$, we denote by $\mathcal{O}^{\perp} := \mathrm{int}(M \backslash J(\mathcal{O}))$ the *causal complement* of $\mathcal{O}$, i.e. the largest *open* set of points which cannot be connected to $\mathcal{O}$ by any causal curve.

A set of the form $\mathcal{O}_G := \mathrm{int}\, D(G)$, where $G$ is a subset of some Cauchy-surface $\Sigma$ in $(M, g)$, will be referred to as the *diamond based on* $G$; we shall also say that $G$ is the *base* of $\mathcal{O}_G$. We note that if $\mathcal{O}_G$ is a diamond, then $\mathcal{O}_G^\perp$ is again a diamond, based on $\Sigma\backslash\overline{G}$. A diamond will be called *regular* if $G$ is an open, relatively compact subset of $\Sigma$ and if the boundary $\partial G$ of $G$ is contained in the union of finitely many smooth, two-dimensional submanifolds of $\Sigma$.

Following [45], we say that an open neighbourhood $N$ of a Cauchy-surface $\Sigma$ in $(M, g)$ is a *causal normal neighbourhood* of $\Sigma$ if (1) $\Sigma$ is a Cauchy-surface for $N$, and (2) for each pair of points $p, q \in N$ with $p \in J^+(q)$, there is a convex normal neighbourhood $\mathcal{O} \subset M$ such that $J^-(p) \cap J^+(q) \subset \mathcal{O}$. Lemma 2.2 of [45] asserts the existence of causal normal neighbourhoods for any Cauchy-surface $\Sigma$.

## 3.2 Some Structural Aspects of Quantum Field Theory in Curved Spacetime

In the present subsection, we shall address some of the problems one faces in the formulation of quantum field theory in curved spacetime, and explain the notions of local definiteness, local primarity, and Haag-duality. In doing so, we follow our presentation in [67] quite closely. Standard general references related to the subsequent discussion are [26,31,45,71].

Quantum field theory in curved spacetime (QFT in CST, for short) means that one considers quantum field theory means that one considers quantum fields propagating in a (classical) curved background spacetime manifold $(M, g)$. In general, such a spacetime need not possess any symmetries, and so one cannot tie the notion of "particles" or "vacuum" to spacetime symmetries, as one does in quantum field theory in Minkowski spacetime. Therefore, the problem of how to characterize the physical states arises. For the discussion of this problem, the setting of algebraic quantum field theory is particularly well suited. Let us thus summarize some of the relevant concepts of algebraic QFT in CST. Let a spacetime manifold $(M, g)$ be given. The observables of a quantum system (e.g. a quantum field) situated in $(M, g)$ then have the basic structure of a map $\mathcal{O} \to \mathcal{A}(\mathcal{O})$, which assigns to each open, relatively compact subset $\mathcal{O}$ of $M$ a $C^*$-algebra $\mathcal{A}(\mathcal{O})$,[1] with the properties:[2]

$$\textit{Isotony}: \qquad \mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \tag{3.1}$$

$$\textit{Locality}: \qquad \mathcal{O}_1 \subset \mathcal{O}_2^\perp \Rightarrow [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}\,. \tag{3.2}$$

A map $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ having these properties is called a *net of local observable algebras* over $(M, g)$. We recall that the conditions of locality and isotony are motivated by the idea that each $\mathcal{A}(\mathcal{O})$ is the $C^*$-algebra formed by the observables which can be measured within the spacetime region $\mathcal{O}$ on the system. We refer to [31] and references given there for further discussion.

The collection of all open, relatively compact subsets of $M$ is directed with respect to set-inclusion, and so we can, in view of (3.1), form the smallest $C^*$-algebra

[1] Throughout the paper, $C^*$-algebras are assumed to be unital, i.e. to possess a unit element, denoted by 1. It is further assumed that the unit element is the same for all the $\mathcal{A}(\mathcal{O})$.

[2] where $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{A_1A_2 - A_2A_1 : A_j \in \mathcal{A}(\mathcal{O}_j),\ j = 1, 2\}$.

$\mathcal{A} := \overline{\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})}^{||\cdot||}$ which contains all local algebras $\mathcal{A}(\mathcal{O})$. For the description of a system we need not only observables but also states. The set $\mathcal{A}_1^{*+}$ of all positive, normalized linear functionals on $\mathcal{A}$ is mathematically referred to as the set of *states* on $\mathcal{A}$, but not all elements of $\mathcal{A}_1^{*+}$ represent physically realizable states of the system. Therefore, given a local net of observable algebras $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ for a physical system over $(M, g)$, one must specify the set of physically relevant states $\mathcal{S}$, which is a suitable subset of $\mathcal{A}_1^{*+}$.

We have already mentioned in Chapter 2 that every state $\omega \in \mathcal{A}_1^{*+}$ determines canonically its GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ and thereby induces a net of von Neumann algebras (operator algebras on $\mathcal{H}_\omega$)

$$\mathcal{O} \to \mathcal{R}_\omega(\mathcal{O}) := \pi_\omega(\mathcal{O})^- \,.$$

Some of the mathematical properties of the GNS representations, and of the induced nets of von Neumann algebras, of states $\omega$ on $\mathcal{A}$ can naturally be interpreted physically. Thus one obtains constraints on the states $\omega$ which are to be viewed as physical states. Following this line of thought, Haag, Narnhofer and Stein [32] formulated what they called the "principle of local definiteness", consisting of the following three conditions to be obeyed by any collection $\mathcal{S}$ of physical states.

**Local Definiteness:** $\bigcap_{\mathcal{O} \ni p} \mathcal{R}_\omega(\mathcal{O}) = \mathbf{C} \cdot 1$ for all $\omega \in \mathcal{S}$ and all $p \in M$.

**Local Primarity:** For each $\omega \in \mathcal{S}$, $\mathcal{R}_\omega(\mathcal{O})$ is a factor.

**Local Quasiequivalence:** For each pair $\omega_1, \omega_2 \in \mathcal{S}$ and each relatively compact, open $\mathcal{O} \subset M$, the representations $\pi_{\omega_1}|\mathcal{A}(\mathcal{O})$ and $\pi_{\omega_2}|\mathcal{A}(\mathcal{O})$ of $\mathcal{A}(\mathcal{O})$ are quasiequivalent.

*Remarks.* (i) We recall (cf. the first Remark in Section 2) that $\mathcal{R}_\omega(\mathcal{O})$ is a factor if $\mathcal{R}_\omega(\mathcal{O}) \cap \mathcal{R}_\omega(\mathcal{O})' = \mathbf{C} \cdot 1$ where the prime means taking the commutant. We have not stated in the formulation of local primarity for which regions $\mathcal{O}$ the algebra $\mathcal{R}_\omega(\mathcal{O})$ is required to be a factor. The regions $\mathcal{O}$ should be taken from a class of subsets of $M$ which forms a base for the topology.

(ii) Quasiequivalence of representations means unitary equivalence up to multiplicity. Another characterization of quasiequivalence is to say that the folia of the representations coincide, where the *folium* of a representation $\pi$ is defined as the set of all $\omega \in \mathcal{A}_1^{*+}$ which can be represented as $\omega(A) = tr(\rho\,\pi(A))$ with a density matrix $\rho$ on the representation Hilbertspace of $\pi$.

(iii) Local definiteness and quasiequivalence together express that physical states have finite (spatio-temporal) energy-density with respect to each other, and local primarity and quasiequivalence rule out local macroscopic observables and local superselection rules. We refer to [31] for further discussion and background material.

A further, important property which one expects to be satisfied for physical states $\omega \in \mathcal{S}$ whose GNS representations are irreducible [3] is

[3] It is easy to see that, in the presence of local primarity, Haag-duality will be violated if $\pi_\omega$

**Haag-Duality:** $\mathcal{R}_\omega(\mathcal{O}^\perp)' = \mathcal{R}_\omega(\mathcal{O})$,
which should hold for the causally complete regions $\mathcal{O}$, i.e. those satisfying $(\mathcal{O}^\perp)^\perp = \mathcal{O}$, where $\mathcal{R}_\omega(\mathcal{O}^\perp)$ is defined as the von Neumann algebra generated by all the $\mathcal{R}_\omega(\mathcal{O}_1)$ so that $\overline{\mathcal{O}_1} \subset \mathcal{O}^\perp$.

We comment that Haag-duality means that the von Neumann algebra $\mathcal{R}_\omega(\mathcal{O})$ of local observables is maximal in the sense that no further observables can be added without violating the condition of locality. It is worth mentioning here that the condition of Haag-duality plays an important role in the theory of superselection sectors in algebraic quantum field theory in Minkowski spacetime [31,59]. For local nets of observables generated by Wightman fields on Minkowski spacetime it follows from the results of Bisognano and Wichmann [4] that a weaker condition of "wedge-duality" is always fulfilled, which allows one to pass to a new, potentially larger local net (the "dual net") which satisfies Haag-duality.

In quantum field theory in Minkowski-spacetime where one is given a vacuum state $\omega_0$, one can define the set of physical states $\mathcal{S}$ simply as the set of all states on $\mathcal{A}$ which are locally quasiequivalent (i.e., the GNS representations of the states are locally quasiequivalent to the vacuum-representation) to $\omega_0$. It is obvious that local quasiequivalence then holds for $\mathcal{S}$. Also, local definiteness holds in this case, as was proved by Wightman [72]. If Haag-duality holds in the vacuum representation (which, as indicated above, can be assumed to hold quite generally), then it does not follow automatically that all pure states locally quasiequivalent to $\omega_0$ will also have GNS representations fulfilling Haag-duality; however, it follows once some regularity conditions are satisfied which have been checked in certain quantum field models [19,61]. So far there seems to be no general physically motivated criterion enforcing local primarity of a quantum field theory in algebraic formulation in Minkowski spacetime. But it is known that many quantum field theoretical models satisfy local primarity.

For QFT in CST we do in general not know what a vacuum state is and so $\mathcal{S}$ cannot be defined in the same way as just described. Yet in some cases (for some quantum field models) there may be a set $\mathcal{S}_0 \subset \mathcal{A}_1^{*+}$ of distinguished states, and if this class of states satisfies the four conditions listed above, then the set $\mathcal{S}$, defined as consisting of all states $\omega_1 \in \mathcal{A}_1^{*+}$ which are locally quasiequivalent to any (and hence all) $\omega \in \mathcal{S}_0$, is a good candidate for the set of physical states.

For the free scalar Klein-Gordon field (KG-field) on a globally hyperbolic spacetime, the following classes of states have been suggested as distinguished, physically reasonable states [4]

(1) (quasifree) states fulfilling local stability [3,22,31,32]

(2) (quasifree) states fulfilling the wave front set (or microlocal) spectrum condition [6,47,55]

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is not irreducible.

[4]The following list is not meant to be complete, it comprises some prominent families of states of the KG-field over a generic class of spacetimes for which mathematically sound results are known. Likewise, the indicated references are by no means exhaustive.

(3) quasifree Hadamard states [12,68,45]

(4) adiabatic vacua [38,48,53]

The list is ordered in such a way that the less restrictive condition preceeds the stronger one. There are a couple of comments to be made here. First of all, the specifications (3) and (4) make use of the information that one deals with the KG-field (or at any rate, a free field obeying a linear equation of motion of hyperbolic character), while the conditions (1) and (2) do not require such input and are applicable to general – possibly interacting – quantum fields over curved spacetimes. The conditions imposed on the classes of states (1), (2) and (3) are related in that they are ultralocal remnants of the spectrum condition requiring a certain regularity of the short distance behaviour of the respective states which can be formulated in generic spacetimes. The class of states (4) is more special and can only be defined for the KG-field (or other linear fields) propagating in Robertson-Walker-type spacetimes. Here a distinguished choice of a time-variable can be made, and the restriction imposed on adiabatic vacua is a regularity condition on their spectral behaviour with respect to that special choice of time. (A somewhat stronger formulation of local stability has been proposed in [34].)

It has been found by Radzikowski [55] that for quasifree states of the KG-field over generic globally hyperbolic spacetimes the classes (2) and (3) coincide. The microlocal spectrum condition is further refined and applied in [6,47]. Recently it was proved by Junker [38] that adiabatic vacua of the KG-field in Robertson-Walker spacetimes fulfill the microlocal spectrum condition and thus are, in fact, quasifree Hadamard states. The notion of the microlocal spectrum condition and the just mentioned results related to it draw on pseudodifferential operator techniques, particularly the notion of the wave front set, see [20,36,37].

Quasifree Hadamard states of the KG-field (see definition in Sec. 3.4 below) have been investigated for quite some time. One of the early studies of these states is [12]. The importance of these states, especially in the context of the semiclassical Einstein equation, is stressed in [68]. Other significant references include [24,25] and, in particular, [45] where, apparently for the first time, a satisfactory definition of the notion of a globally Hadamard state is given, cf. Section 3.4 for more details. In [66] it is proved that the class of quasifree Hadamard states of the KG-field fulfills local quasiequivalence in generic globally hyperbolic spacetimes and local definiteness, local primarity and Haag-duality for the case of ultrastatic globally hyperbolic spacetimes. As was outlined in the beginning, the purpose of the present chapter is to obtain these latter results also for arbitrary globally hyperbolic spacetimes which are not necessarily ultrastatic. It turns out that some of our previous results can be sharpened, e.g. the local quasiequivalence specializes in most cases to local unitary equivalence, cf. Thm. 3.6. For a couple of other results about the algebraic structure of the KG-field as well as other fields over curved spacetimes we refer to [2,6,15,16,17,40,41,46,63,64,65,66,74].

### 3.3 The Klein-Gordon Field

In the present section we summarize the quantization of the classical KG-field over a globally hyperbolic spacetime in the $C^*$-algebraic formalism. This follows in major parts the the work of Dimock [16], cf. also references given there. Let $(M,g)$ be a globally hyperbolic spacetime. The KG-equation with potential term $r$ is

$$(\nabla^a\nabla_a + r)\varphi = 0 \tag{3.3}$$

where $\nabla$ is the Levi-Civita derivative of the metric $g$, the potential function $r \in C^\infty(M,\mathbf{R})$ is arbitrary but fixed, and the sought for solutions $\varphi$ are smooth and real-valued. Making use of the fact that $(M,g)$ is globally hyperbolic and drawing on earlier results by Leray, it is shown in [16] that there are two uniquely determined, continuous [5] linear maps $E^\pm : C_0^\infty(M,\mathbf{R}) \to C^\infty(M,\mathbf{R})$ with the properties

$$(\nabla^a\nabla_a + r)E^\pm f = f = E^\pm(\nabla^a\nabla_a + r)f\,, \quad f \in C_0^\infty(M,\mathbf{R})\,,$$

and

$$\mathrm{supp}(E^\pm f) \subset J^\pm(\mathrm{supp}(f))\,, \quad f \in C_0^\infty(M,\mathbf{R})\,.$$

The maps $E^\pm$ are called the advanced(+)/retarded(–) fundamental solutions of the KG-equation with potential term $r$ in $(M,g)$, and their difference $E := E^+ - E^-$ is referred to as the *propagator* of the KG-equation.

One can moreover show that the Cauchy-problem for the KG-equation is well-posed. That is to say, if $\Sigma$ is any Cauchy-surface in $(M,g)$, and $u_0 \oplus u_1 \in C_0^\infty(M,\mathbf{R}) \oplus C_0^\infty(M,\mathbf{R})$ is any pair of Cauchy-data on $\Sigma$, then there exists precisely one smooth solution $\varphi$ of the KG-equation (3.3) having the property that

$$P_\Sigma(\varphi) := \varphi|\Sigma \oplus n^a\nabla_a\varphi|\Sigma = u_0 \oplus u_1\,. \tag{3.4}$$

The vectorfield $n^a$ in (3.4) is the future-pointing unit normalfield of $\Sigma$. Furthermore, one has "finite propagation speed", i.e. when the supports of $u_0$ and $u_1$ are contained in a subset $G$ of $\Sigma$, then $\mathrm{supp}(\varphi) \subset J(G)$. Notice that compactness of $G$ implies that $J(G) \cap \Sigma'$ is compact for any Cauchy-surface $\Sigma'$.

The well-posedness of the Cauchy-problem is a consequence of the classical energy-estimate for solutions of second order hyperbolic partial differential equations, cf. e.g. [33]. To formulate it, we introduce further notation. Let $\Sigma$ be a Cauchy-surface for $(M,g)$, and $\gamma_\Sigma$ the Riemannian metric, induced by the ambient Lorentzian metric, on $\Sigma$. Then denote the Laplacian operator on $C_0^\infty(\Sigma,\mathbf{R})$ corresponding to $\gamma_\Sigma$ by $\Delta_{\gamma_\Sigma}$, and define the *classical energy scalar product* on $C_0^\infty(\Sigma,\mathbf{R}) \oplus C_0^\infty(\Sigma,\mathbf{R})$ by

$$\mu_\Sigma^E(u_0 \oplus u_1, v_0 \oplus v_1) := \int_\Sigma (u_0(-\Delta_{\gamma_\Sigma} + 1)v_0 + u_1 v_1)\, d\eta_\Sigma\,, \tag{3.5}$$

where $d\eta_\Sigma$ is the metric-induced volume measure on $\Sigma$. As a special case of the energy estimate presented in [33] one then obtains

[5] With respect to the usual locally convex topologies on $C_0^\infty(M,\mathbf{R})$ and $C^\infty(M,\mathbf{R})$, cf. [13].

**Lemma 3.1** *(Classical energy estimate for the KG-field.) Let $\Sigma_1$ and $\Sigma_2$ be a pair of Cauchy-surfaces in $(M,g)$ and $G$ a compact subset of $\Sigma_1$. Then there are two positive constants $c_1$ and $c_2$ so that there holds the estimate*

$$c_1\,\mu^E_{\Sigma_1}(P_{\Sigma_1}(\varphi),P_{\Sigma_1}(\varphi)) \leq \mu^E_{\Sigma_2}(P_{\Sigma_2}(\varphi),P_{\Sigma_2}(\varphi)) \leq c_2\,\mu^E_{\Sigma_1}(P_{\Sigma_1}(\varphi),P_{\Sigma_1}(\varphi)) \quad (3.6)$$

*for all solutions $\varphi$ of the KG-equation (3.3) which have the property that the supports of the Cauchy-data $P_{\Sigma_1}(\varphi)$ are contained in $G$.* [6]

We shall now indicate that the space of smooth solutions of the KG-equation (3.3) has the structure of a symplectic space, locally as well as globally, which comes in several equivalent versions. To be more specific, observe first that the Cauchy-data space

$$\mathcal{D}_\Sigma := C_0^\infty(\Sigma,\mathbf{R}) \oplus C_0^\infty(\Sigma,\mathbf{R})$$

of an arbitrary given Cauchy-surface $\Sigma$ in $(M,g)$ carries a symplectic form

$$\delta_\Sigma(u_0 \oplus u_1, v_0 \oplus v_1) := \int_\Sigma (u_0 v_1 - v_0 u_1)\, d\eta_\Sigma\,.$$

It will also be observed that this symplectic form is dominated by the classical energy scalar product $\mu^E_\Sigma$.

Another symplectic space is $S$, the set of all real-valued $C^\infty$-solutions $\varphi$ of the KG-equation (3.3) with the property that, given any Cauchy-surface $\Sigma$ in $(M,g)$, their Cauchy-data $P_\Sigma(\varphi)$ have compact support on $\Sigma$. The symplectic form on $S$ is given by

$$\sigma(\varphi,\psi) := \int_\Sigma (\varphi n^a \nabla_a \psi - \psi n^a \nabla_a \varphi)\, d\eta_\Sigma$$

which is independent of the choice of the Cauchy-surface $\Sigma$ on the right hand side over which the integral is formed; $n^a$ is again the future-pointing unit normalfield of $\Sigma$. One clearly finds that for each Cauchy-surface $\Sigma$ the map $P_\Sigma : S \to \mathcal{D}_\Sigma$ establishes a symplectomorphism between the symplectic spaces $(S,\sigma)$ and $(\mathcal{D}_\Sigma,\delta_\Sigma)$.

A third symplectic space equivalent to the previous ones is obtained as the quotient $K := C_0^\infty(M,\mathbf{R})/\ker(E)$ with symplectic form

$$\kappa([f],[h]) := \int_M f(Eh)\, d\eta\,, \quad f,h \in C_0^\infty(M,\mathbf{R})\,,$$

where $[\,.\,]$ is the quotient map $C_0^\infty(M,\mathbf{R}) \to K$ and $d\eta$ is the metric-induced volume measure on $M$.

Then define for any open subset $\mathcal{O} \subset M$ with compact closure the set $K(\mathcal{O}) := [C_0^\infty(\mathcal{O},\mathbf{R})]$. One can see that the space $K$ has naturally the structure of an isotonous, local net $\mathcal{O} \to K(\mathcal{O})$ of subspaces, where locality means that the

[6] The formulation given here is to some extend more general than the one appearing in [33] where it is assumed that $\Sigma_1$ and $\Sigma_2$ are members of a foliation. However, the more general formulation can be reduced to that case.

symplectic form $\kappa([f],[h])$ vanishes for $[f] \in K(\mathcal{O})$ and $[h] \in K(\mathcal{O}_1)$ whenever $\mathcal{O}_1 \subset \mathcal{O}^{\perp}$. Dimock has proved in [16 (Lemma A.3)] that moreover there holds

$$K(\mathcal{O}_G) \subset K(N) \tag{3.7}$$

for all open neighbourhoods $N$ (in $M$) of $G$, whenever $\mathcal{O}_G$ is a diamond. Using this, one obtains that the map $(K,\kappa) \to (S,\sigma)$ given by $[f] \mapsto Ef$ is surjective, and by Lemma A.1 in [16], it is even a symplectomorphism. Clearly, $(K(\mathcal{O}_G), \kappa|K(\mathcal{O}_G))$ is a symplectic subspace of $(K,\kappa)$ for each diamond $\mathcal{O}_G$ in $(M,g)$. For any such diamond one then obtains, upon viewing it (or its connected components separately), equipped with the appropriate restriction of the spacetime metric $g$, as a globally hyperbolic spacetime in its own right, local versions of the just introduced symplectic spaces and the symplectomorphisms between them. More precisely, if we denote by $S(\mathcal{O}_G)$ the set of all smooth solutions of the KG-equation (3.3) with the property that their Cauchy-data on $\Sigma$ are compactly supported in $G$, then the map $P_\Sigma$ restricts to a symplectomorphism $(S(\mathcal{O}_G), \sigma|S(\mathcal{O}_G)) \to (\mathcal{D}_G, \delta_G)$, $\varphi \mapsto P_\Sigma(\varphi)$. Likewise, the symplectomorphism $[f] \mapsto Ef$ restricts to a symplectomorphism $(K(\mathcal{O}_G), \kappa|K(\mathcal{O}_G)) \to (S(\mathcal{O}_G), \sigma|S(\mathcal{O}_G))$.

To the symplectic space $(K,\kappa)$ we can now associate its Weyl-algebra $\mathcal{A}[K,\kappa]$, cf. Chapter 2. Using the afforementioned local net-structure of the symplectic space $(K,\kappa)$, one arrives at the following result.

**Proposition 3.2** [16]. *Let $(M,g)$ be a globally hyperbolic spacetime, and $(K,\kappa)$ the symplectic space, constructed as above, for the KG-eqn. with smooth potential term $r$ on $(M,g)$. Its Weyl-algebra $\mathcal{A}[K,\kappa]$ will be called the* Weyl-algebra of the KG-field with potential term $r$ over $(M,g)$. *Define for each open, relatively compact $\mathcal{O} \subset M$, the set $\mathcal{A}(\mathcal{O})$ as the $C^*$-subalgebra of $\mathcal{A}[K,\kappa]$ generated by all the Weyl-operators $W([f])$, $[f] \in K(\mathcal{O})$. Then $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ is a net of $C^*$-algebras fulfilling isotony (3.1) and locality (3.2), and moreover* primitive causality, *i.e.*

$$\mathcal{A}(\mathcal{O}_G) \subset \mathcal{A}(N) \tag{3.8}$$

*for all neighbourhoods $N$ (in $M$) of $G$, whenever $\mathcal{O}_G$ is a (relatively compact) diamond.*

It is worth recalling (cf. [5]) that the Weyl-algebras corresponding to symplectically equivalent spaces are canonically isomorphic in the following way: Let $W(x)$, $x \in K$ denote the Weyl-generators of $\mathcal{A}[K,\kappa]$ and $W_S(\varphi)$, $\varphi \in S$, the Weyl-generators of $\mathcal{A}[S,\sigma]$. Furthermore, let $T$ be a symplectomorphism between $(K,\kappa)$ and $(S,\sigma)$. Then there is a uniquely determined $C^*$-algebraic isomorphism $\alpha_T : \mathcal{A}[K,\kappa] \to \mathcal{A}[S,\sigma]$ given by $\alpha_T(W(x)) = W_S(Tx)$, $x \in K$. This shows that if we had associated e.g. with $(S,\sigma)$ the Weyl-algebra $\mathcal{A}[S,\sigma]$ as the algebra of quantum observables of the KG-field over $(M,g)$, we would have obtained an equivalent net of observable algebras (connected to the previous one by a net isomorphism, see [3]), rendering the same physical information.

### 3.4 Hadamard States

We have indicated above that quasifree Hadamard states are distinguished by their short-distance behaviour which allows the definition of expectation values of energy-momentum observables with reasonable properties [26,68,69,71]. If $\omega_\mu$ is a quasifree state on the Weyl-algebra $\mathcal{A}[K,\kappa]$, then we call

$$\lambda(x,y) := \mu(x,y) + \frac{i}{2}\kappa(x,y)\,, \quad x,y \in K\,,$$

its *two-point function* and

$$\Lambda(f,h) := \lambda([f],[h])\,, \quad f,h \in C_0^\infty(M,\mathbf{R})\,,$$

its *spatio-temporal* two-point function. In Chapter 2 we have seen that a quasifree state is entirely determined through specifying $\mu \in \mathsf{q}(K,\kappa)$, which is equivalent to the specification of the two-point function $\lambda$. Sometimes the notation $\lambda_\omega$ or $\lambda_\mu$ will be used to indicate the quasifree state $\omega$ or the dominating scalar product $\mu$ which is determined by $\lambda$.

For a quasifree Hadamard state, the spatio-temporal two-point function is of a special form, called Hadamard form. The definition of Hadamard form which we give here follows that due to Kay and Wald [45]. Let $N$ is a causal normal neighbourhood of a Cauchy-surface $\Sigma$ in $(M,g)$. Then a smooth function $\chi : N \times N \to [0,1]$ is called *N-regularizing* if it has the following property: There is an open neighbourhood, $\Omega_*$, in $N \times N$ of the set of pairs of causally related points in $N$ such that $\overline{\Omega_*}$ is contained in a set $\Omega$ to be described presently, and $\chi \equiv 1$ on $\Omega_*$ while $\chi \equiv 0$ outside of $\overline{\Omega}$. Here, $\Omega$ is an open neighbourhood in $M \times M$ of the set of those $(p,q) \in M \times M$ which are causally related and have the property that (1) $J^+(p) \cap J^-(q)$ and $J^+(q) \cap J^-(p)$ are contained within a convex normal neighbourhood, and (2) $s(p,q)$, the square of the geodesic distance between $p$ and $q$, is a well-defined, smooth function on $\Omega$. (One observes that there are always sets $\Omega$ of this type which contain a neighbourhood of the diagonal in $M \times M$, and that an $N$-regularizing function depends on the choice of the pair of sets $\Omega_*, \Omega$ with the stated properties.) It is not difficult to check that $N$-regularizing functions always exist for any causal normal neighbourhood; a proof of that is e.g. given in [55]. Then denote by $U$ the square root of the VanVleck-Morette determinant, and by $v_m$, $m \in \mathbf{N}_0$ the sequence determined by the Hadamard recursion relations for the KG-equation (3.3), see [23,27] and also [30] for their definition. They are all smooth functions on $\Omega$.[7] Now set for $n \in \mathbf{N}$,

$$V^{(n)}(p,q) := \sum_{m=0}^{n} v_m(p,q)(s(p,q))^m\,, \quad (p,q) \in \Omega\,,$$

and, given a smooth time-function $T : M \to \mathbf{R}$ increasing towards the future, define for all $\epsilon > 0$ and $(p,q) \in \Omega$,

$$Q_T(p,q;\epsilon) := s(p,q) - 2i\epsilon(T(p) - T(q)) - \epsilon^2\,,$$

[7] For any choice of $\Omega$ with the properties just described.

and

$$G_\epsilon^{T,n}(p,q) := \frac{1}{4\pi^2}\left(\frac{U(p,q)}{Q_T(p,q;\epsilon)} + V^{(n)}(p,q)ln(Q_T(p,q;\epsilon))\right),$$

where $ln$ is the principal branch of the logarithm. With this notation, one can give the

**Definition 3.3** [45]. *A* $\mathbf{C}$*-valued bilinear form* $\Lambda$ *on* $C_0^\infty(M,\mathbf{R})$ *is called an* Hadamard form *if, for a suitable choice of a causal normal neighbourhood* $N$ *of some Cauchy-surface* $\Sigma$*, and for suitable choices of an* $N$*-regularizing function* $\chi$ *and a future-increasing time-function* $T$ *on* $M$*, there exists a sequence* $H^{(n)} \in C^n(N \times N)$*, so that*

$$\Lambda(f,h) = \lim_{\epsilon\to 0+}\int_{M\times M} \Lambda_\epsilon^{T,n}(p,q)f(p)h(q)\,d\eta(p)\,d\eta(q) \tag{3.9}$$

*for all* $f,h \in C_0^\infty(N,\mathbf{R})$*, where* [8]

$$\Lambda_\epsilon^{T,n}(p,q) := \chi(p,q)G_\epsilon^{T,n}(p,q) + H^{(n)}(p,q)\,. \tag{3.10}$$

Based on results of [24,25], it is shown in [45] that this is a reasonable definition. The findings of these works will be collected in the following

**Proposition 3.4**

*(a) If* $\Lambda$ *is of Hadamard form on a causal normal neighbourhood* $N$ *of a Cauchy-surface* $\Sigma$ *for some choice of a time-function* $T$ *and some* $\mathcal{N}$*-regularizing function* $\chi$ *(i.e. that (3.9),(3.10) hold with suitable* $H^{(n)} \in C^n(N\times N)$*), then so it is for any other time-function* $T'$ *and* $\mathcal{N}$*-regularizing* $\chi'$*. (This means that these changes can be compensated by choosing another sequence* $H'^{(n)} \in C^n(N\times N)$*.)*

*(b) (Causal Propagation Property of the Hadamard Form)*
*If* $\Lambda$ *is of Hadamard form on a causal normal neighbourhood* $N$ *of some Cauchy-surface* $\Sigma$*, then it is of Hadamard form in any causal normal neighbourhood* $N'$ *of any other Cauchy-surface* $\Sigma'$*.*

*(c) Any* $\Lambda$ *of Hadamard form is a regular kernel distribution on* $C_0^\infty(M\times M)$*.*

*(d) There exist pure, quasifree Hadamard states (these will be referred to as* Hadamard vacua*) on the Weyl-algebra* $\mathcal{A}[K,\kappa]$ *of the KG-field in any globally hyperbolic spacetime. The family of quasifree Hadamard states on* $\mathcal{A}[K,\kappa]$ *spans an infinite-dimensional subspace of the continuous dual space of* $\mathcal{A}[K,\kappa]$*.*

*(e) The dominating scalar products* $\mu$ *on* $K$ *arising from quasifree Hadamard states* $\omega_\mu$ *induce locally the same topology, i.e. if* $\mu$ *and* $\mu'$ *are arbitrary such scalar products and* $\mathcal{O}\subset M$ *is open and relatively compact, then there are two positive constants* $a, a'$ *such that*

$$a\,\mu([f],[f]) \le \mu'([f],[f]) \le a'\,\mu([f],[f])\,, \quad [f]\in K(\mathcal{O})\,.$$

[8] The set $\Omega$ on which the functions forming $G_\epsilon^{T,n}$ are defined and smooth is here to coincide with the $\Omega$ with respect to which $\chi$ is defined.

*Remark.* Observe that this definition of Hadamard form rules out the occurence of spacelike singularities, meaning that the Hadamard $\Lambda$ is, when tested on functions $f, h$ in (3.9) whose supports are acausally separated, given by a $C^\infty$-kernel. For that reason, the definition of Hadamard form as stated above is also called *global* Hadamard form (cf. [45]). A weaker definition of Hadamard form would be to prescribe (3.9),(3.10) only for sets $N$ which, e.g., are members of an open covering of $M$ by convex normal neighbourhoods, and thereby to require the Hadamard form locally. In the case that $\Lambda$ is the spatio-temporal two-point function of a state on $\mathcal{A}[K,\kappa]$ and thus dominates the symplectic form $\kappa$ ($|\kappa([f],[h])|^2 \leq 4\,\Lambda(f,f)\Lambda(h,h)$), it was recently proved by Radzikowski that if $\Lambda$ is locally of Hadamard form, then it is already globally of Hadamard form [56]. However, if $\Lambda$ doesn't dominate $\kappa$, this need not hold [29,51,56]. Radzikowski's proof makes use of a characterization of Hadamard forms in terms of their wave front sets which was mentioned above. A definition of Hadamard form which is less technical in appearence has recently been given in [44]. We should add that the usual Minkowski-vacuum of the free scalar field with constant potential term is, of course, an Hadamard vacuum. This holds, more generally, also for ultrastatic spacetimes, see below.

*Notes on the proof of Proposition 3.4.* The property (a) is proved in [45]. The argument for (b) is essentially contained in [25] and in the generality stated here it is completed in [45]. An alternative proof using the "propagation of singularities theorem" for hyperbolic differential equations is presented in [55]. Also property (c) is proved in [45 (Appendix B)] (cf. [66 (Prop. C.2)]). The existence of Hadamard vacua (d) is proved in [24] (cf. also [45]); the stated Corollary has been observed in [66] (and, in slightly different formulation, already in [24]). Statement (e) has been shown to hold in [66 (Prop. 3.8)].

In order to prepare the formulation of the next result, in which we will apply our result of Chapter 2, we need to collect some more notation. Suppose that we are given a quasifree state $\omega_\mu$ on the Weyl-algebra $\mathcal{A}[K,\kappa]$ of the KG-field over some globally hyperbolic spacetime $(M,g)$, and that $\Sigma$ is a Cauchy-surface in that spacetime. Then we denote by $\mu_\Sigma$ the dominating scalar product on $(\mathcal{D}_\Sigma, \delta_\Sigma)$ which is, using the symplectomorphism between $(K,\kappa)$ and $(\mathcal{D}_\Sigma,\delta_\Sigma)$, induced by the dominating scalar product $\mu$ on $(K,\kappa)$, i.e.

$$\mu_\Sigma(P_\Sigma E f, P_\Sigma E h) = \mu([f],[h])\,, \quad [f],[h] \in K\,. \tag{3.11}$$

Conversely, to any $\mu_\Sigma \in \mathsf{q}(\mathcal{D}_\Sigma,\delta_\Sigma)$ there corresponds via (3.11) a $\mu \in \mathsf{q}(K,\kappa)$.

Next, consider a complete Riemannian manifold $(\Sigma,\gamma)$, with corresponding Laplacian $\Delta_\gamma$, and as before, consider the operator $-\Delta_\gamma + 1$ on $C_0^\infty(\Sigma,\mathbf{R})$. Owing to the completeness of $(\Sigma,\gamma)$ this operator is, together with all its powers, essentially selfadjoint in $L^2_{\mathbf{R}}(\Sigma, d\eta_\gamma)$ [10], and we denote its selfadjoint extension by $A_\gamma$. Then one can introduce the Sobolev scalar products of $m$-th order,

$$\langle u,v\rangle_{\gamma,m} := \langle u, A_\gamma^m v\rangle\,, \quad u,v \in C_0^\infty(\Sigma,\mathbf{R}),\ m \in \mathbf{R}\,,$$

where on the right hand side is the scalar product of $L^2_{\mathbf{R}}(\Sigma, d\eta_\gamma)$. The completion of $C_0^\infty(\Sigma, \mathbf{R})$ in the topology of $\langle\, .\, ,\, .\,\rangle_{\gamma,m}$ will be denoted by $H_m(\Sigma,\gamma)$. It turns out that the topology of $H_m(\Sigma,\gamma)$ is locally independent of the complete Riemannian metric $\gamma$, and that composition with diffeomorphisms and multiplication with smooth, compactly supported functions are continuous operations on these Sobolev spaces. (See Appendix B for precise formulations of these statements.) Therefore, whenever $G \subset \Sigma$ is open and relatively compact, the topology which $\langle\, .\, ,\, .\,\rangle_{m,\gamma}$ induces on $C_0^\infty(G, \mathbf{R})$ is independent of the particular complete Riemannian metric $\gamma$, and we shall refer to the topology which is thus locally induced on $C_0^\infty(\Sigma, \mathbf{R})$ simply as the (local) *$H_m$-topology.*

Let us now suppose that we have an ultrastatic spacetime $(\tilde{M}, \tilde{\gamma})$, given in a natural foliation as $(\mathbf{R} \times \tilde{\Sigma}, dt^2 \otimes (-\gamma))$ where $(\tilde{\Sigma}, \gamma)$ is a complete Riemannian manifold. We shall identify $\tilde{\Sigma}$ and $\{0\} \times \tilde{\Sigma}$. Consider again $A_\gamma$ = selfadjoint extension of $-\Delta_\gamma + 1$ on $C_0^\infty(\tilde{\Sigma}, \mathbf{R})$ in $L^2_{\mathbf{R}}(\tilde{\Sigma}, d\eta_\gamma)$ with $\Delta_\gamma$ = Laplacian of $(\tilde{\Sigma}, \gamma)$, and the scalar product $\mu^\circ_{\tilde{\Sigma}}$ on $\mathcal{D}_{\tilde{\Sigma}}$ given by

$$\begin{aligned} \mu^\circ_{\tilde{\Sigma}}(u_0 \oplus u_1, v_0 \oplus v_1) &:= \frac{1}{2}\Big(\langle u_0, A_\gamma^{1/2} v_0\rangle + \langle u_1, A_\gamma^{-1/2} v_1\rangle\Big) \\ &= \frac{1}{2}\Big(\langle u_0, v_0\rangle_{\gamma,1/2} + \langle u_1, v_1\rangle_{\gamma,-1/2}\Big) \end{aligned} \qquad (3.12)$$

for all $u_0 \oplus u_1, v_0 \oplus v_1 \in \mathcal{D}_{\tilde{\Sigma}}$. It is now straightforward to check that $\mu^\circ_{\tilde{\Sigma}} \in \mathsf{pu}(\mathcal{D}_{\tilde{\Sigma}}, \delta_{\tilde{\Sigma}})$, in fact, $\mu^\circ_{\tilde{\Sigma}}$ is the purification of the classical energy scalar product $\mu^E_{\tilde{\Sigma}}$ defined in eqn. (3.5). (We refer to [11] for discussion, and also the treatment of more general situations along similar lines.) What is furthermore central for the derivation of the next result is that $\mu^\circ_{\tilde{\Sigma}}$ corresponds (via (3.11)) to an Hadamard vacuum $\omega^\circ$ on the Weyl-algebra of the KG-field with potential term $r \equiv 1$ over the ultrastatic spacetime $(\mathbf{R} \times \tilde{\Sigma}, dt^2 \oplus (-\gamma))$. This has been proved in [24]. The state $\omega^\circ$ is called the *ultrastatic vacuum* for the said KG-field over $(\mathbf{R} \times \tilde{\Sigma}, dt^2 \oplus (-\gamma))$; it is the unique pure, quasifree ground state on the corresponding Weyl-algebra for the time-translations $(t, q) \mapsto (t + t', q)$ on that ultrastatic spacetime with respect to the chosen natural foliation (cf. [40,42]).

*Remark.* The passage form $\mu^E_{\tilde{\Sigma}}$ to $\mu^\circ_{\tilde{\Sigma}}$, where $\mu^\circ_{\tilde{\Sigma}}$ is the purification of the classical energy scalar product, may be viewed as a refined form of "frequency-splitting" procedures (or Hamiltonian diagonalization), in order to obtain pure dominating scalar products and hence, pure states of the KG-field in curved spacetimes, see [11]. However, in the case that $\tilde{\Sigma}$ is not a Cauchy-surface lying in the natural foliation of an ultrastatic spacetime, but an arbitrary Cauchy-surface in an arbitrary globally hyperbolic spacetime, the $\mu^\circ_{\tilde{\Sigma}}$ may fail to correspond to a quasifree Hadamard state — even though, as the following Proposition demonstrates, $\mu^\circ_{\tilde{\Sigma}}$ gives locally on the Cauchy-data space $\mathcal{D}_{\tilde{\Sigma}}$ the same topology as the dominating scalar products induced on it by any quasifree Hadamard state. More seriously, $\mu^\circ_{\tilde{\Sigma}}$ may even correspond to a state which is no longer locally quasiequivalent to any quasifree Hadamard state. For an explicit example demonstrating this in a closed Robertson-Walker universe, and for additional discussion, we refer to Sec. 3.6 in [38].

We shall say that a map $T : \mathcal{D}_\Sigma \to \mathcal{D}_{\Sigma'}$, with $\Sigma, \Sigma'$ Cauchy-surfaces, is *locally continuous* if, for any open, locally compact $G \subset \Sigma$, the restriction of $T$ to $C_0^\infty(G, \mathbf{R}) \oplus C_0^\infty(G, \mathbf{R})$ is continuous (with respect to the topologies under consideration).

**Proposition 3.5** *Let $\omega_\mu$ be a quasifree Hadamard state on the Weyl-algebra $\mathcal{A}[K,\kappa]$ of the KG-field with smooth potential term $r$ over the globally hyperbolic spacetime $(M,g)$, and $\Sigma, \Sigma'$ two Cauchy-surfaces in $(M,g)$.*

*Then the Cauchy-data evolution map*

$$T_{\Sigma',\Sigma} := P_{\Sigma'} \circ P_\Sigma^{-1} : \mathcal{D}_\Sigma \to \mathcal{D}_{\Sigma'} \tag{3.13}$$

*is locally continuous in the $H_\tau \oplus H_{\tau-1}$-topology, $0 \le \tau \le 1$, on the Cauchy-data spaces, and the topology induced by $\mu_\Sigma$ on $\mathcal{D}_\Sigma$ coincides locally (i.e. on each $C_0^\infty(G, \mathbf{R}) \oplus C_0^\infty(G, \mathbf{R})$ for $G \subset \Sigma$ open and relatively compact) with the $H_{1/2} \oplus H_{-1/2}$-topology.*

*Remarks.* (i) Observe that the continuity statement is reasonably formulated since, as a consequence of the support properties of solutions of the KG-equation with Cauchy-data of compact support ("finite propagation speed") it holds that for each open, relatively compact $G \subset \Sigma$ there is an open, relatively compact $G' \subset \Sigma'$ with $T_{\Sigma',\Sigma}(C_0^\infty(G, \mathbf{R}) \oplus C_0^\infty(G, \mathbf{R})) \subset C_0^\infty(G', \mathbf{R}) \oplus C_0^\infty(G', \mathbf{R})$.

(ii) For $\tau = 1$, the continuity statement is just the classical energy estimate. It should be mentioned here that the claimed continuity can also be obtained by other methods. For instance, Moreno [50] proves, under more restrictive assumptions on $\Sigma$ and $\Sigma'$ (among which is their compactness), the continuity of $T_{\Sigma',\Sigma}$ in the topology of $H_\tau \oplus H_{\tau-1}$ for all $\tau \in \mathbf{R}$, by employing an abstract energy estimate for first order hyperbolic equations (under suitable circumstances, the KG-equation can be brought into this form). We feel, however, that our method, using the results of Chapter 2, is physically more appealing and emphasizes much better the "invariant" structures involved, quite in keeping with the general approach to quantum field theory.

*Proof of Proposition 3.5.* We note that there is a diffeomorphism $\Psi : \Sigma \to \Sigma'$. To see this, observe that we may pick a foliation $F : \mathbf{R} \times \tilde{\Sigma} \to M$ of $M$ in Cauchy-surfaces. Then for each $q \in \tilde{\Sigma}$, the curves $t \mapsto F(t,q)$ are inextendible, timelike curves in $(M,g)$. Each such curve intersects $\Sigma$ exactly once, at the parameter value $t = \tau(q)$. Hence $\Sigma$ is the set $\{F(\tau(q),q) : q \in \tilde{\Sigma}\}$. As $F$ is a diffeomorphism and $\tau : \tilde{\Sigma} \to \mathbf{R}$ must be $C^\infty$ since, by assumption, $\Sigma$ is a smooth hypersurface in $M$, one can see that $\Sigma$ and $\tilde{\Sigma}$ are diffeomorphic. The same argument shows that $\Sigma'$ and $\tilde{\Sigma}$ and therefore, $\Sigma$ and $\Sigma'$, are diffeomorphic.

Now let us first assume that the $g$-induced Riemannian metrics $\gamma_\Sigma$ and $\gamma_{\Sigma'}$ on $\Sigma$, resp. $\Sigma'$, are complete. Let $d\eta$ and $d\eta'$ be the induced volume measures on $\Sigma$ and $\Sigma'$, respectively. The $\Psi$-transformed measure of $d\eta$ on $\Sigma'$, $\Psi^* d\eta$, is given through

$$\int_\Sigma (u \circ \Psi)\, d\eta = \int_{\Sigma'} u\,(\Psi^* d\eta)\,, \quad u \in C_0^\infty(\Sigma')\,. \tag{3.14}$$

Then the Radon-Nikodym derivative $(\rho(q))^2 := (\Psi^* d\eta/d\eta')(q)$, $q \in \Sigma'$, is a smooth, strictly positive function on $\Sigma'$, and it is now easy to check that the linear map

$$\vartheta : (\mathcal{D}_\Sigma, \delta_\Sigma) \to (\mathcal{D}_{\Sigma'}, \delta_{\Sigma'}) , \quad u_0 \oplus u_1 \mapsto \rho \cdot (u_0 \circ \Psi^{-1}) \oplus \rho \cdot (u_1 \circ \Psi^{-1}) ,$$

is a symplectomorphism. Moreover, by the result given in Appendix B, $\vartheta$ and its inverse are locally continuous maps in the $H_s \oplus H_t$-topologies on both Cauchy-data spaces, for all $s, t \in \mathbf{R}$.

By the energy estimate, $T_{\Sigma',\Sigma}$ is locally continuous with respect to the $H_1 \oplus H_0$-topology on the Cauchy-data spaces, and the same holds for the inverse $(T_{\Sigma',\Sigma})^{-1} = T_{\Sigma,\Sigma'}$. Hence, the map $\Theta := \vartheta^{-1} \circ T_{\Sigma',\Sigma}$ is a symplectomorphism of $(\mathcal{D}_\Sigma, \delta_\Sigma)$, and $\Theta$ together with its inverse is locally continuous in the $H_1 \oplus H_0$-topology on $\mathcal{D}_\Sigma$. Here we made use of Remark (i) above. Now pick two sets $G$ and $G'$ as in Remark (i), then there is some open, relatively compact neighbourhood $\tilde{G}$ of $\Psi^{-1}(G') \cup G$ in $\Sigma$. We can choose a smooth, real-valued function $\chi$ compactly supported on $\Sigma$ with $\chi \equiv 1$ on $\tilde{G}$. It is then straightforward to check that the maps $\chi \circ \Theta \circ \chi$ and $\chi \circ \Theta^{-1} \circ \chi$ ($\chi$ to be interpreted as multiplication with $\chi$) is a pair of symplectically adjoint maps on $(\mathcal{D}_\Sigma, \delta_\Sigma)$ which are bounded with respect to the $H_1 \oplus H_0$-topology, i.e. with respect to the norm of $\mu^E_\Sigma$. At this point we use Theorem 2.2(b) and consequently $\chi \circ \Theta \circ \chi$ and $\chi \circ \Theta^{-1} \circ \chi$ are continuous with respect to the norms of the $(\mu^E_\Sigma)_s$, $0 \leq s \leq 2$. Inspection shows that

$$(\mu^E_\Sigma)_s(u_0 \oplus u_1, v_0 \oplus v_1) = \frac{1}{2}\left(\langle u_0, A^{1-s/2}_{\gamma_\Sigma} v_0\rangle + \langle u_1, A^{-s/2}_{\gamma_\Sigma} v_1\rangle\right)$$

for $0 \leq s \leq 2$. From this it is now easy to see that $\Theta$ restricted to $C_0^\infty(G, \mathbf{R}) \oplus C_0^\infty(G, \mathbf{R})$ is continuous in the topology of $H_\tau \oplus H_{\tau-1}$, $0 \leq \tau \leq 1$, since $\chi \circ \Theta \circ \chi(u_0 \oplus u_1) = \Theta(u_0 \oplus u_1)$ for all $u_0 \oplus u_1 \in C_0^\infty(G, \mathbf{R}) \oplus C_0^\infty(G, \mathbf{R})$ by the choice of $\chi$. Using that $\Theta = \vartheta^{-1} \circ T_{\Sigma',\Sigma}$ and that $\vartheta$ is locally continuous with respect to all the $H_s \oplus H_t$-topologies, $s, t \in \mathbf{R}$, on the Cauchy-data spaces, we deduce that that $T_{\Sigma',\Sigma}$ is locally continuous in the $H_\tau \oplus H_{\tau-1}$-topology, $0 \leq \tau \leq 1$, as claimed.

If the $g$-induced Riemannian metrics $\gamma_\Sigma$, $\gamma_{\Sigma'}$ are not complete, one can make them into complete ones $\hat{\gamma}_\Sigma := f \cdot \gamma_\Sigma$, $\hat{\gamma}_{\Sigma'} := h \cdot \gamma_{\Sigma'}$ by multiplying them with suitable smooth, strictly positive functions $f$ on $\Sigma$ and $h$ on $\Sigma'$ [14]. Let $d\hat{\eta}$ and $d\hat{\eta}'$ be the volume measures corresponding to the new metrics. Consider then the density functions $\phi_1 := (d\eta/d\hat{\eta})$, $\phi_2 := (d\hat{\eta}'/d\eta')$, which are $C^\infty$ and strictly positive, and define $(\mathcal{D}_\Sigma, \hat{\delta}_\Sigma)$, $(\mathcal{D}_{\Sigma'}, \hat{\delta}_{\Sigma'})$ and $\hat{\vartheta}$ like their unhatted counterparts but with $d\hat{\eta}$ and $d\hat{\eta}'$ in place of $d\eta$ and $d\eta'$. Likewise define $\hat{\mu}^E_\Sigma$ with respect to $\hat{\gamma}_\Sigma$. Then $\hat{T}_{\Sigma',\Sigma} := \phi_2 \circ T_{\Sigma',\Sigma} \circ \phi_1$ (understanding that $\phi_1, \phi_2$ act as multiplication operators) and its inverse are symplectomorphisms between $(\mathcal{D}_\Sigma, \hat{\delta}_\Sigma)$ and $(\mathcal{D}_{\Sigma'}, \hat{\delta}_{\Sigma'})$ which are locally continuous in the $H_1 \oplus H_0$-topology. Now we can apply the argument above showing that $\hat{\Theta} = \hat{\vartheta}^{-1} \circ \hat{T}_{\Sigma',\Sigma}$ and, hence, $\hat{T}_{\Sigma',\Sigma}$ is locally continuous in the $H_\tau \oplus H_{\tau-1}$-topology for $0 \leq \tau \leq 1$. The same follows then for $T_{\Sigma',\Sigma} = \phi_2^{-1} \circ \hat{T}_{\Sigma',\Sigma} \circ \phi_1^{-1}$.

For the proof of the second part of the statement, we note first that in [24] it is shown that there exists another globally hyperbolic spacetime $(\hat{M}, \hat{g})$ of the form $\hat{M} = \mathbf{R} \times \Sigma$ with the following properties:

(1) $\Sigma_0 := \{0\} \times \Sigma$ is a Cauchy-surface in $(\hat{M}, \hat{g})$, and a causal normal neighbourhood $N$ of $\Sigma$ in $M$ coincides with a causal normal neigbourhood $\hat{N}$ of $\Sigma_0$ in $\hat{M}$, in such a way that $\Sigma = \Sigma_0$ and $g = \hat{g}$ on $N$.

(2) For some $t_0 < 0$, the $(-\infty, t_0) \times \Sigma$-part of $\hat{M}$ lies properly to the past of $\hat{N}$, and on that part, $\hat{g}$ takes the form $dt^2 \oplus (-\gamma)$ where $\gamma$ is a complete Riemannian metric on $\Sigma$.

This means that $(\hat{M}, \hat{g})$ is a globally hyperbolic spacetime which equals $(M, g)$ on a causal normal neighbourhood of $\Sigma$ and becomes ultrastatic to the past of it.

Then consider the Weyl-algebra $\mathcal{A}[\hat{K}, \hat{\kappa}]$ of the KG-field with potential term $\hat{r}$ over $(\hat{M}, \hat{g})$, where $\hat{r} \in C_0^\infty(\hat{M}, \mathbf{R})$ agrees with $r$ on the neighbourhood $\hat{N} = N$ and is identically equal to 1 on the $(-\infty, t_0) \times \Sigma$-part of $\hat{M}$. Now observe that the propagators $E$ and $\hat{E}$ of the respective KG-equations on $(M, g)$ and $(\hat{M}, \hat{g})$ coincide when restricted to $C_0^\infty(N, \mathbf{R})$. Therefore one obtains an identification map

$$[f] = f + \ker \mapsto [f]\hat{} = f + \ker(\hat{E}) \,, \quad f \in C_0^\infty(N, \mathbf{R}) \,,$$

between $K(N)$ and $\hat{K}(\hat{N})$ which preserves the symplectic forms $\kappa$ and $\hat{\kappa}$. Without danger we may write this identification as an equality, $K(N) = \hat{K}(\hat{N})$. This identification map between $(K(N), \kappa|K(N))$ and $(\hat{K}(\hat{N}), \hat{\kappa}|\hat{K}(\hat{N}))$ lifts to a $C^*$-algebraic isomorphism between the corresponding Weyl-algebras

$$\begin{aligned} \mathcal{A}[K(N), \kappa|K(N)] &= \mathcal{A}[\hat{K}(\hat{N}), \hat{\kappa}|\hat{K}(\hat{N})] \,, \\ W([f]) &= \hat{W}([f]\hat{}) \,, \quad f \in C_0^\infty(N, \mathbf{R}) \,. \end{aligned} \tag{3.15}$$

Here we followed our just indicated convention to abbreviate this identification as an equality. Now we have $D(N) = M$ in $(M, g)$ and $D(\hat{N}) = \hat{M}$ in $(\hat{M}, \hat{g})$, implying that $K(N) = K$ and $\hat{K}(\hat{N}) = \hat{K}$. Hence $\mathcal{A}[K(N), \kappa|K(N)] = \mathcal{A}[K, \kappa]$ and the same for the “hatted” objects. Thus (3.15) gives rise to an identification between $\mathcal{A}[K, \kappa]$ and $\mathcal{A}[\hat{K}, \hat{\kappa}]$, and so the quasifree Hadamard state $\omega_\mu$ induces a quasifree state $\omega_{\hat{\mu}}$ on $\mathcal{A}[\hat{K}, \hat{\kappa}]$ with

$$\hat{\mu}([f]\hat{}, [h]\hat{}) = \mu([f], [h]) \,, \quad f, h \in C_0^\infty(N, \mathbf{R}) \,. \tag{3.16}$$

This state is also an Hadamard state since we have

$$\begin{aligned} \Lambda(f, h) &= \mu([f], [h]) + \frac{i}{2}\kappa([f], [h]) \\ &= \hat{\mu}([f]\hat{}, [h]\hat{}) + \frac{i}{2}\hat{\kappa}([f]\hat{}, [h]\hat{}) \,, \quad f, h \in C_0^\infty(N, \mathbf{R}) \,, \end{aligned}$$

and $\Lambda$ is, by assumption, of Hadamard form. However, due to the causal propagation property of the Hadamard form this means that $\hat{\mu}$ is the dominating scalar product on $(\hat{K}, \hat{\kappa})$ of a quasifree Hadamard state on $\mathcal{A}[\hat{K}, \hat{\kappa}]$. Now choose some $t < t_0$, and let $\Sigma_t = \{t\} \times \Sigma$ be the Cauchy-surface in the ultrastatic part of

$(\hat{M}, \hat{g})$ corresponding to this value of the time-parameter of the natural foliation. As remarked above, the scalar product

$$\mu^{\circ}_{\Sigma_t}(u_0 \oplus u_1, v_0 \oplus v_1) = \frac{1}{2}\left(\langle u_0, v_0\rangle_{\gamma,1/2} + \langle u_1, v_1\rangle_{\gamma,-1/2}\right) , \quad u_0 \oplus u_1, v_0 \oplus v_1 \in \mathcal{D}_{\Sigma_t} , \tag{3.17}$$

is the dominating scalar product on $(\mathcal{D}_{\Sigma_t}, \delta_{\Sigma_t})$ corresponding to the ultrastatic vacuum state $\omega^{\circ}$ over the ultrastatic part of of $(\hat{M}, \hat{g})$, which is an Hadamard vacuum. Since the dominating scalar products of all quasifree Hadamard states yield locally the same topology (Prop. 3.4(e)), it follows that the dominating scalar product $\hat{\mu}_{\Sigma_t}$ on $(\mathcal{D}_{\Sigma_t}, \delta_{\Sigma_t})$, which is induced (cf. (3.11)) by the the dominating scalar product of $\hat{\mu}$ of the quasifree Hadamard state $\omega_{\hat{\mu}}$, endows $\mathcal{D}_{\Sigma_t}$ locally with the same topology as does $\mu^{\circ}_{\Sigma_t}$. As can be read off from (3.17), this is the local $H_{1/2} \oplus H_{-1/2}$-topology.

To complete the argument, we note that (cf. (3.11,3.13))

$$\hat{\mu}_{\Sigma_0}(u_0 \oplus u_1, v_0 \oplus v_1) = \hat{\mu}_{\Sigma_t}(T_{\Sigma_t,\Sigma_0}(u_0 \oplus u_1), T_{\Sigma_t,\Sigma_0}(v_0 \oplus v_1)) , \quad u_0 \oplus u_1, v_0 \oplus v_1 \in \mathcal{D}_{\Sigma_0} .$$

But since $\hat{\mu}_{\Sigma_t}$ induces locally the $H_{1/2} \oplus H_{-1/2}$-topology and since the symplectomorphism $T_{\Sigma_t,\Sigma_0}$ as well as its inverse are locally continuous on the Cauchy-data spaces in the $H_{1/2} \oplus H_{-1/2}$-topology, the last inequality entails that $\hat{\mu}_{\Sigma_0}$ induces the local $H_{1/2} \oplus H_{-1/2}$-topology on $\mathcal{D}_{\Sigma_0}$. In view of (3.16), the Proposition is now proved. $\Box$

### 3.5 Local Definiteness, Local Primarity, Haag-Duality, etc.

In this section we prove Theorem 3.6 below on the algebraic structure of the GNS-representations associated with quasifree Hadamard states on the CCR-algebra of the KG-field on an arbitrary globally hyperbolic spacetime $(M, g)$. The results appearing therein extend our previous work [64,65,66].

Let $(M, g)$ be a globally hyperbolic spacetime. We recall that a subset $\mathcal{O}$ of $M$ is called a regular diamond if it is of the form $\mathcal{O} = \mathcal{O}_G = \text{int}\, D(G)$ where $G$ is an open, relatively compact subset of some Cauchy-surface $\Sigma$ in $(M, g)$ having the property that the boundary $\partial G$ of $G$ is contained in the union of finitely many smooth, closed, two-dimensional submanifolds of $\Sigma$. We also recall the notation $\mathcal{R}_\omega(\mathcal{O}) = \pi_\omega(\mathcal{A}(\mathcal{O}))^-$ for the local von Neumann algebras in the GNS-representation of a state $\omega$. The $C^*$-algebraic net of observable algebras $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ will be understood as being that associated with the KG-field in Prop. 3.2.

**Theorem 3.6** *Let $(M, g)$ be a globally hyperbolic spacetime and $\mathcal{A}[K, \kappa]$ the Weyl-algebra of the KG-field with smooth, real-valued potential function $r$ over $(M, g)$. Suppose that $\omega$ and $\omega_1$ are two quasifree Hadamard states on $\mathcal{A}[K, \kappa]$. Then the following statements hold.*

*(a) The GNS-Hilbertspace $\mathcal{H}_\omega$ of $\omega$ is infinite dimensional and separable.*

*(b) The restrictions of the GNS-representations $\pi_\omega|\mathcal{A}(\mathcal{O})$ and $\pi_{\omega_1}|\mathcal{A}(\mathcal{O})$ of any open, relatively compact $\mathcal{O} \subset M$ are quasiequivalent. They are even unitarily equivalent when $\mathcal{O}^\perp$ is non-void.*

*(c) For each $p \in M$ we have local definiteness,*

$$\bigcap_{\mathcal{O} \ni p} \mathcal{R}_\omega(\mathcal{O}) = \mathbf{C} \cdot 1 .$$

*More generally, whenever $C \subset M$ is the subset of a compact set which is contained in the union of finitely many smooth, closed, two-dimensional submanifolds of an arbitrary Cauchy-surface $\Sigma$ in $M$, then*

$$\bigcap_{\mathcal{O} \supset C} \mathcal{R}_\omega(\mathcal{O}) = \mathbf{C} \cdot 1 . \tag{3.18}$$

*(d) Let $\mathcal{O}$ and $\mathcal{O}_1$ be two relatively compact diamonds, based on Cauchy-surfaces $\Sigma$ and $\Sigma_1$, respectively, such that $\overline{\mathcal{O}} \subset \mathcal{O}_1$. Then the split-property holds for the pair $\mathcal{R}_\omega(\mathcal{O})$ and $\mathcal{R}_\omega(\mathcal{O}_1)$, i.e. there exists a type $\mathrm{I}_\infty$ factor $\mathcal{N}$ such that one has the inclusion*

$$\mathcal{R}_\omega(\mathcal{O}) \subset \mathcal{N} \subset \mathcal{R}_\omega(\mathcal{O}_1) .$$

*(e) Inner and outer regularity*

$$\mathcal{R}_\omega(\mathcal{O}) = \Bigg( \bigcup_{\overline{\mathcal{O}_I} \subset \mathcal{O}} \mathcal{R}_\omega(\mathcal{O}_I) \Bigg)'' = \bigcap_{\mathcal{O}_1 \supset \overline{\mathcal{O}}} \mathcal{R}_\omega(\mathcal{O}_1) \tag{3.19}$$

*holds for all regular diamonds $\mathcal{O}$.*

*(f) If $\omega$ is pure (an Hadamard vacuum), then we have Haag-Duality*

$$\mathcal{R}_\omega(\mathcal{O})' = \mathcal{R}_\omega(\mathcal{O}^\perp)$$

*for all regular diamonds $\mathcal{O}$. (By the same arguments as in* [65 (Prop. 6)]*, Haag-Duality extends to all pure (but not necessarily quasifree or Hadamard) states $\omega$ which are locally normal (hence, by (d), locally quasiequivalent) to any Hadamard vacuum.)*

*(g) Local primarity holds for all regular diamonds, that is, for each regular diamond $\mathcal{O}$, $\mathcal{R}_\omega(\mathcal{O})$ is a factor. Moreover, $\mathcal{R}_\omega(\mathcal{O})$ is isomorphic to the unique hyperfinite type* $\mathrm{III}_1$ *factor if $\mathcal{O}^\perp$ is non-void. In this case, $\mathcal{R}_\omega(\mathcal{O}^\perp)$ is also hyperfinite and of type* $\mathrm{III}_1$*, and if $\omega$ is pure, $\mathcal{R}_\omega(\mathcal{O}^\perp)$ is again a factor. Otherwise, if $\mathcal{O}^\perp = \emptyset$, then $\mathcal{R}_\omega(\mathcal{O})$ is a type* $\mathrm{I}_\infty$ *factor.*

*Proof.* The key point in the proof is that, by results which for the cases relevant here are to large extend due to Araki [1], the above statement can be equivalently translated into statements about the structure of the one-particle space, i.e. essentially the symplectic space $(K, \kappa)$ equipped with the scalar product $\lambda_\omega$ We shall use, however, the formalism of [40,45]. Following that, given a symplectic space $(K,\kappa)$ and $\mu \in \mathsf{q}(K,\kappa)$ one calls a real linear map $\mathbf{k} : K \to H$ a *one-particle Hilbertspace structure* for $\mu$ if (1) H is a complex Hilbertspace, (2) the complex linear span of $\mathbf{k}(K)$ is dense in $H$ and (3)

$$\langle \mathbf{k}(x), \mathbf{k}(y)\rangle = \lambda_\mu(x,y) = \mu(x,y) + \frac{i}{2}\kappa(x,y)$$

for all $x, y \in K$. It can then be shown (cf. [45,(Appendix A)]) that the GNS-representation of the quasifree state $\omega_\mu$ on $\mathcal{A}[K,\kappa]$ may be realized in the following way: $\mathcal{H}_{\omega_\mu} = F_s(H)$, the Bosonic Fock-space over the one-particle space $H$, $\Omega_{\omega_\mu}$ = the Fock-vacuum, and

$$\pi_{\omega_\mu}(W(x)) = \mathrm{e}^{i(a(\mathbf{k}(x))+a^+(\mathbf{k}(x)))^-}\,, \quad x \in K\,,$$

where $a(\,.\,)$ and $a^+(\,.\,)$ are the Bosonic annihilation and creation operators, respectively.

Now it is useful to define the symplectic complement $F^{\mathrm{v}} := \{\chi \in H : \mathsf{Im}\,\langle\chi,\phi\rangle = 0 \ \ \forall\phi \in F\}$ for $F \subset H$, since it is known that

(i) $\mathcal{R}_{\omega_\mu}(\mathcal{O})$ is a factor iff $\mathbf{k}(K(\mathcal{O}))^- \cap \mathbf{k}(K(\mathcal{O}))^{\mathrm{v}} = \{0\}$,

(ii) $\mathcal{R}_{\omega_\mu}(\mathcal{O})' = \mathcal{R}_{\omega_\mu}(\mathcal{O}^\perp)$ iff $\mathbf{k}(K(\mathcal{O}))^{\mathrm{v}} = \mathbf{k}(K(\mathcal{O}^\perp))^-$,

(iii) $\bigcap_{\mathcal{O}\supset C}\mathcal{R}_{\omega_\mu}(\mathcal{O}) = \mathbf{C}\cdot 1$ iff $\bigcap_{\mathcal{O}\supset C}\mathbf{k}(K(\mathcal{O}))^- = \{0\}$,

cf. [1,21,35,49,58].

After these preparations we can commence with the proof of the various statements of our Theorem.

(a) Let $\mathbf{k} : K \to H$ be the one-particle Hilbertspace structure of $\omega$. The local one-particle spaces $\mathbf{k}(K(\mathcal{O}_G))^-$ of regular diamonds $\mathcal{O}_G$ based on $G \subset \Sigma$ are topologically isomorphic to the completions of $C_0^\infty(G,\mathbf{R}) \oplus C_0^\infty(G,\mathbf{R})$ in the $H_{1/2} \oplus H_{-1/2}$-topology and these are separable. Hence $\mathbf{k}(K)^-$, which is generated by a countable set $\mathbf{k}(K(\mathcal{O}_{G_n}))$, for $G_n$ a sequence of locally compact subsets of $\Sigma$ eventually exhausting $\Sigma$, is also separable. The same holds then for the one-particle Hilbertspace $H$ in which the complex span of $\mathbf{k}(K)$ is dense, and thus separability is implied for $\mathcal{H}_\omega = F_s(H)$. The infinite-dimensionality is clear.

(b) The local quasiequivalence has been proved in [66] and we refer to that reference for further details. We just indicate that the proof makes use of the fact that the difference $\Lambda - \Lambda_1$ of the spatio-temporal two-point functions of any pair of quasifree Hadamard states is on each causal normal neighbourhood of any Cauchy-surface given by a smooth integral kernel — as can be directly read off from the Hadamard form — and this turns out to be sufficient for local quasiequivalence. The statement about the unitary equivalence can be inferred from (g) below, since it is known that every $*$-preserving isomorphism between von Neumann algebras of type III acting on separable Hilbertspaces is given by the adjoint action of a unitary operator which maps the Hilbertspaces onto each other. See e.g. Thm. 7.2.9 and Prop. 9.1.6 in [45].

(c) Here one uses that there exist Hadamard vacua, i.e. pure quasifree Hadamard states $\omega_\mu$. Since by Prop. 3.4 the topology of $\mu_\Sigma$ in $\mathcal{D}_\Sigma$ is locally that of $H_{1/2} \oplus H_{-1/2}$, one can show as in [66 (Chp. 4 and Appendix)] that under the stated hypotheses about $C$ it holds that $\bigcap_{\mathcal{O}\supset C} \mathbf{k}(K(\mathcal{O}))^- = \{0\}$ for the one-particle Hilbertspace structures of Hadamard vacua. From the local equivalence of the topologies induced by the dominating scalar products of all quasifree Hadamard states (Prop. 3.4(e)), this extends to the one-particle structures of all quasifree Hadamard states. By (iii), this yields the statement (c).

(d) This is proved in [65] under the additional assumption that the potential term $r$ is a positive constant. (The result was formulated in [65] under the hypothesis that $\Sigma = \Sigma_1$, but it is clear that the present statement without this hypothesis is an immediate generalization.) To obtain the general case one needs in the spacetime deformation argument of [65] the modification that the potential term $\hat{r}$ of the KG-field on the new spacetime $(\hat{M}, \hat{g})$ is equal to a positive constant on its ultrastatic part while being equal to $r$ in a neighbourhood of $\Sigma$. We have used that procedure already in the proof of Prop. 3.5, see also the proof of (f) below where precisely the said modification will be carried out in more detail.

(e) Inner regularity follows simply from the definition of the $\mathcal{A}(\mathcal{O})$; one deduces that for each $A \in \mathcal{A}(\mathcal{O})$ and each $\epsilon > 0$ there exists some $\overline{\mathcal{O}_I} \subset \mathcal{O}$ and $A_\epsilon \in \mathcal{A}(\mathcal{O}_I)$ so that $||\, A - A_\epsilon \,|| < \epsilon$. It is easy to see that inner regularity is a consequence of this property.

So we focus now on the outer regularity. Let $\mathcal{O} = \mathcal{O}_G$ be based on the subset $G$ of the Cauchy-surface $\Sigma$. Consider the symplectic space $(\mathcal{D}_\Sigma, \delta_\Sigma)$ and the dominating scalar product $\mu_\Sigma$ induced by $\mu \in \mathsf{q}(\mathcal{D}_\Sigma, \delta_\Sigma)$, where $\omega_\mu = \omega$; the corresponding one-particle Hilbertspace structure we denote by $\mathbf{k}_\Sigma : \mathcal{D}_\Sigma \to H_\Sigma$. Then we denote by $\mathcal{W}(\mathbf{k}_\Sigma(\mathcal{D}_G))$ the von Neumann algebra in $B(F_s(H_\Sigma))$ generated by the unitary groups of the operators $(a(\mathbf{k}_\Sigma(u_0 \oplus u_1)) + a^+(\mathbf{k}_\Sigma(u_0 \oplus u_1)))^-$ where $u_0 \oplus u_1$ ranges over $\mathcal{D}_G := C_0^\infty(G, \mathbf{R}) \oplus C_0^\infty(G, \mathbf{R})$. So $\mathcal{W}(\mathbf{k}_\Sigma(\mathcal{D}_G)) = \mathcal{R}_\omega(\mathcal{O}_G)$. It holds generally that $\bigcap_{G_1 \supset \overline{G}} \mathcal{W}(\mathbf{k}_\Sigma(\mathcal{D}_{G_1})) = \mathcal{W}(\bigcap_{G_1\supset\overline{G}} \mathbf{k}_\Sigma(\mathcal{D}_{G_1})^-)$ [1], hence, to establish outer regularity, we must show that

$$\bigcap_{G_1 \supset \overline{G}} \mathbf{k}_\Sigma(\mathcal{D}_{G_1})^- = \mathbf{k}_\Sigma(\mathcal{D}_G)^- \,. \tag{3.20}$$

In [65] we have proved that the ultrastatic vacuum $\omega^\circ$ of the KG-field with potential term $\equiv 1$ over the ultrastatic spacetime $(M^\circ, g^\circ) = (\mathbf{R} \times \Sigma, dt^2 \oplus (-\gamma))$ (where $\gamma$ is any complete Riemannian metric on $\Sigma$) satisfies Haag-duality. That means, we have

$$\mathcal{R}^\circ_{\omega^\circ}(\mathcal{O}_\circ)' = \mathcal{R}^\circ_{\omega^\circ}(\mathcal{O}_\circ^\perp) \tag{3.21}$$

for any regular diamond $\mathcal{O}_\circ$ in $(M^\circ, g^\circ)$ which is based on any of the Cauchy-surfaces $\{t\} \times \Sigma$ in the natural foliation, and we have put a "$\circ$" on the local von Neumann algebras to indicate that they refer to a KG-field over $(M^\circ, g^\circ)$. But since we have inner regularity for $\mathcal{R}^\circ_{\omega^\circ}(\mathcal{O}_\circ^\perp)$ — by the very definition — the outer regularity of $\mathcal{R}^\circ_{\omega^\circ}(\mathcal{O}_\circ)$ follows from the Haag-duality (3.21). Translated into conditions on the one-particle Hilbertspace structure $\mathbf{k}^\circ_\Sigma : \mathcal{D}_\Sigma \to H^\circ_\Sigma$ of $\omega^\circ$, this means that the equality

$$\bigcap_{G_1 \supset \overline{G}} \mathbf{k}^\circ_\Sigma(\mathcal{D}_{G_1})^- = \mathbf{k}^\circ_\Sigma(\mathcal{D}_G)^- \tag{3.22}$$

holds. Now we know from Prop. 3.5 that $\mu_\Sigma$ induces locally the $H_{1/2} \oplus H_{-1/2}$-topology on $\mathcal{D}_\Sigma$. However, this coincides with the topology locally induced by $\mu^\circ_\Sigma$ on $\mathcal{D}_\Sigma$ (cf. (3.11)) — even though $\mu^\circ_\Sigma$ may, in general, not be viewed as corresponding to an Hadamard vacuum of the KG-field over $(M,g)$. Thus the required relation (3.20) is implied by (3.22).

(f) In view of outer regularity it is enough to show that, given any $\mathcal{O}_1 \supset \overline{\mathcal{O}}$, it holds that

$$\mathcal{R}_\omega(\mathcal{O}^\perp)' \subset \mathcal{R}_\omega(\mathcal{O}_1)\,. \tag{3.23}$$

The demonstration of this property relies on a spacetime deformation argument similar to that used in the proof of Prop. 3.5. Let $G$ be the base of $\mathcal{O}$ on the Cauchy-surface $\Sigma$ in $(M,g)$. Then, given any other open, relatively compact subset $G_1$ of $\Sigma$ with $\overline{G} \subset G_1$, we have shown in [65] that there exists an ultrastatic spacetime $(\hat{M},\hat{g})$ with the properties (1) and (2) in the proof of Prop. 3.5, and with the additional property that there is some $t < t_0$ such that

$$\left(\mathrm{int}\,\hat{J}(G) \cap \Sigma_t\right)^- \subset \mathrm{int}\,\hat{D}(G_1) \cap \Sigma_t\,.$$

Here, $\Sigma_t = \{0\} \times \Sigma$ are the Cauchy-surfaces in the natural foliation of the ultrastatic part of $(\hat{M},\hat{g})$. The hats indicate that the causal set and the domain of dependence are to be taken in $(\hat{M},\hat{g})$. This implies that we can find some regular diamond $\mathcal{O}^t := \mathrm{int}\hat{D}(S^t)$ in $(\hat{M},\hat{g})$ based on a subset $S^t$ of $\Sigma_t$ which satisfies

$$\left(\mathrm{int}\,\hat{J}(G) \cap \Sigma_t\right)^- \subset S^t \subset \mathrm{int}\,\hat{D}(G_1) \cap \Sigma_t\,. \tag{3.24}$$

Setting $\hat{\mathcal{O}} := \mathrm{int}\,\hat{D}(G)$ and $\hat{\mathcal{O}}_1 := \mathrm{int}\,\hat{D}(G_1)$, one derives from (3.24) the relations

$$\hat{\mathcal{O}} \subset \mathcal{O}^t \subset \hat{\mathcal{O}}_1\,. \tag{3.25}$$

These are equivalent to

$$\hat{\mathcal{O}}_1^\perp \subset (\mathcal{O}^t)^\perp \subset \hat{\mathcal{O}}^\perp \tag{3.26}$$

where $\perp$ is the causal complementation in $(\hat{M},\hat{g})$.

Now as in the proof of Prop. 3.5, the given Hadamard vacuum $\omega$ on the Weyl-algebra $\mathcal{A}[K,\kappa]$ of the KG-field over $(M,g)$ induces an Hadamard vacuum $\hat{\omega}$ on the Weyl-algebra $\mathcal{A}[\hat{K},\hat{\kappa}]$ of the KG-field over $(\hat{M},\hat{g})$ whose potential term $\hat{r}$ is 1 on the ultrastatic part of $(\hat{M},\hat{g})$. Then by Prop. 6 in [65] we have Haag-duality

$$\hat{\mathcal{R}}_{\hat{\omega}}(\hat{\mathcal{O}}_t{}^\perp)' = \hat{\mathcal{R}}_{\hat{\omega}}(\hat{\mathcal{O}}_t) \tag{3.27}$$

for all regular diamonds $\hat{\mathcal{O}}_t$ with base on $\Sigma_t$; we have put hats on the von Neumann algebras to indicate that they refer to $\mathcal{A}[\hat{K},\hat{\kappa}]$. (This was proved in [65] assuming that $(\hat{M},\hat{g})$ is globally ultrastatic. However, with the same argument, based on primitive causality, as we use it next to pass from (3.28) to (3.30), one can easily establish that (3.27) holds if only $\Sigma_t$ is, as here, a member in the natural foliation of the ultrastatic part of $(\hat{M},\hat{g})$.) Since $\mathcal{O}^t$ is a regular diamond based on $\Sigma_t$, we obtain

$$\hat{\mathcal{R}}_{\hat{\omega}}((\mathcal{O}^t)^{\perp})' = \hat{\mathcal{R}}_{\hat{\omega}}(\mathcal{O}^t)$$

and thus, in view of (3.25) and (3.26),

$$\hat{\mathcal{R}}_{\hat{\omega}}(\hat{\mathcal{O}}^{\perp})' \subset \hat{\mathcal{R}}_{\hat{\omega}}((\mathcal{O}^t)^{\perp})' = \hat{\mathcal{R}}_{\hat{\omega}}(\mathcal{O}^t) \subset \hat{\mathcal{R}}_{\hat{\omega}}(\hat{\mathcal{O}}_1)\,. \tag{3.28}$$

Now recall (see proof of Prop. 3.5) that $(\hat{M},\hat{g})$ coincides with $(M,g)$ on a causal normal neighbourhood $N$ of $\Sigma$. Primitive causality (Prop. 3.2) then entails

$$\hat{\mathcal{R}}_{\hat{\omega}}(\hat{\mathcal{O}}^{\perp}\cap N)' \subset \hat{\mathcal{R}}_{\hat{\omega}}(\hat{\mathcal{O}}_1 \cap N)\,. \tag{3.29}$$

On the other hand, $\hat{\mathcal{O}}^{\perp} = \mathrm{int}\hat{D}(\Sigma\backslash G)$ and $\hat{\mathcal{O}}_1$ are diamonds in $(\hat{M},\hat{g})$ based on $\Sigma$. Since $(M,g)$ and $(\hat{M},\hat{g})$ coincide on the causal normal neighbourhood $N$ of $\Sigma$, one obtains that $\mathrm{int}\,D(\tilde{G})\cap N = \mathrm{int}\,\hat{D}(\tilde{G})\cap N$ for all $\tilde{G}\in\Sigma$. Hence, with $\mathcal{O} = \mathrm{int}\,D(G)$, $\mathcal{O}_1 = \mathrm{int}\,D(G_1)$ (in $(M,g)$), we have that (3.23) entails

$$\mathcal{R}_{\omega}(\mathcal{O}^{\perp}\cap N)' \subset \mathcal{R}_{\omega}(\mathcal{O}_1\cap N)$$

(cf. the proof of Prop. 3.5) where the causal complement $\perp$ is now taken in $(M,g)$. Using primitive causality once more, we deduce that

$$\mathcal{R}_{\omega}(\mathcal{O}^{\perp})' \subset \mathcal{R}_{\omega}(\mathcal{O}_1)\,. \tag{3.30}$$

The open, relatively compact subset $G_1$ of $\Sigma$ was arbitrary up to the constraint $\overline{G}\subset G_1$. Therefore, we arrive at the conclusion that the required inclusion (3.23) holds of all $\overline{\mathcal{O}}_1 \supset \mathcal{O}$.

(g) Let $\Sigma$ be the Cauchy-surface on which $\mathcal{O}$ is based. For the local primarity one uses, as in (c), the existence of Hadamard vacua $\omega_\mu$ and the fact (Prop. 3.5) that $\mu_\Sigma$ induces locally the $H_{1/2}\oplus H_{-1/2}$-topology; then one may use the arguments of [66 (Chp. 4 and Appendix)] to show that due to the regularity of the boundary $\partial G$ of the base $G$ of $\mathcal{O}$ there holds

$$\mathbf{k}(K(\mathcal{O}))^{-}\cap\mathbf{k}(K(\mathcal{O}))^{\mathrm{v}} = \{0\}$$

for the one-particle Hilbertspace structures of Hadamard vacua. As in the proof of (c), this can be carried over to the one-particle structures of all quasifree Hadamard states since they induce locally on the one-particle spaces the same topology, see [66 (Chp. 4)]. We note that for Hadamard vacua the local primarity can also be established using (3.18) together with Haag-duality and primitive causality purely at the algebraic level, without having to appeal to the one-particle structures.

The type $\mathrm{III}_1$-property of $\mathcal{R}_\omega(\mathcal{O})$ is then derived using Thm. 16.2.18 in [3] (see also [73]). We note that for some points $p$ in the boundary $\partial G$ of $G$, $\mathcal{O}$ admits domains which are what is in Sect. 16.2.4 of [3] called "$\beta_p$-causal sets", as a consequence of the regularity of $\partial G$ and the assumption $\mathcal{O}^\perp \neq \emptyset$. We further note that is is straightforward to prove that the quasifree Hadamard states of the KG-field over $(M, g)$ possess at each point in $M$ scaling limits (in the sense of Sect. 16.2.4 in [3], see also [22,32]) which are equal to the theory of the massless KG-field in Minkowski-spacetime. Together with (a) and (c) of the present Theorem this shows that the the assumptions of Thm. 16.2.18 in [3] are fulfilled, and the $\mathcal{R}_\omega(\mathcal{O})$ are type $\mathrm{III}_1$-factors for all regular diamonds $\mathcal{O}$ with $\mathcal{O}^\perp \neq \emptyset$. The hyperfiniteness follows from the split-property (d) and the regularity (e), cf. Prop. 17.2.1 in [3]. The same arguments may be applied to $\mathcal{R}_\omega(\mathcal{O}^\perp)$, yielding its type $\mathrm{III}_1$-property (meaning that in its central decomposition only type $\mathrm{III}_1$-factors occur) and hyperfiniteness. If $\omega$ is an Hadamard vacuum, then $\mathcal{R}_\omega(\mathcal{O}^\perp) = \mathcal{R}_\omega(\mathcal{O})'$ is a factor unitarily equivalent to $\mathcal{R}_\omega(\mathcal{O})$. For the last statement note that $\mathcal{O}^\perp = \emptyset$ implies that the spacetime has a compact Cauchy-surface on which $\mathcal{O}$ is based. In this case $\mathcal{R}_\omega(\mathcal{O}) = \pi_\omega(\mathcal{A}[K,\kappa])''$ (use the regularity of $\partial G$, and (c), (e) and primitive causality). But since $\omega$ is quasiequivalent to any Hadamard vacuum by the relative compactness of $\mathcal{O}$, $\mathcal{R}_\omega(\mathcal{O}) = \pi_\omega(\mathcal{A}[K,\kappa])''$ is a type $\mathrm{I}_\infty$-factor. $\square$

We end this section and therefore, this work, with a few concluding remarks.

First we note that the split-property signifies a strong notion of statistical independence. It can be deduced from constraints on the phase-space behaviour ("nuclearity") of the considered quantum field theory. We refer to [9,31] for further information and also to [62] for a review, as a discussion of these issues lies beyond the scope of of this article. The same applies to a discussion of the property of the local von Neumann algbras $\mathcal{R}_\omega(\mathcal{O})$ to be hyperfinite and of type $\mathrm{III}_1$. We only mention that for quantum field theories on Minkowski spacetime it can be established under very general (model-independent) conditions that the local (von Neumann) observable algebras are hyperfinite and of type $\mathrm{III}_1$, and refer the reader to [7] and references cited therein. However, the property of the local von Neumann algebras to be of type $\mathrm{III}_1$, together with the separability of the GNS-Hilbertspace $\mathcal{H}_\omega$, has an important consequence which we would like to point out (we have used it implicitly already in the proof of Thm. 3.6(b)): $\mathcal{H}_\omega$ contains a dense subset $\mathsf{ts}(\mathcal{H}_\omega)$ of vectors which are cyclic and separating for all $\mathcal{R}_\omega(\mathcal{O})$ whenever $\mathcal{O}$ is a diamond with $\mathcal{O}^\perp \neq \emptyset$. But so far it has only been established in special cases that $\Omega_\omega \in \mathsf{ts}(\mathcal{H}_\omega)$, see [64]. At any rate, when $\Omega \in \mathsf{ts}(\mathcal{H}_\omega)$ one may consider for a pair of regular diamonds $\mathcal{O}_1, \mathcal{O}_2$ with $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$ and $\mathcal{O}_2^\perp$ nonvoid the modular operator $\Delta_2^{1/2}$ of $\mathcal{R}_\omega(\mathcal{O}_2)$,$\Omega$ (cf. [39]). The split property and the factoriality of $\mathcal{R}_\omega(\mathcal{O}_1)$ and $\mathcal{R}_\omega(\mathcal{O}_2)$ imply the that the map

$$\Xi_{1,2} : A \mapsto \Delta_2^{1/4} A\Omega\,, \quad A \in \mathcal{R}_\omega(\mathcal{O}_1)\,, \tag{3.31}$$

is compact [8]. As explained in [8], "modular compactness" or "modular nuclearity" may be viewed as suitable generalizations of "energy compactness" or "energy nuclearity" to curved spacetimes as notions to measure the phase-space behaviour of a quantum field theory (see also [65]). Thus an interesting question would be if the maps (3.31) are even nuclear.

Summarizing we may say that Thm. 3.6 shows that the nets of von Neumann observable algebras of the KG-field over a globally hyperbolic spacetime in the representations of quasifree Hadamard states have all the properties one would expect for physically reasonable representations. This supports the point of view that quasifree Hadamard states may be taken as a good choice for physical states of the KG-field over a globally hyperbolic spacetime. Similar results may be expected for other linear fields.

Finally, the reader may have noticed that we have been considering exclusively the quantum theory of a KG-field on a *globally hyperbolic* spacetime. For recent developments concerning quantum fields in the background of non-globally hyperbolic spacetimes, we refer to [44] and references cited there.

**Acknowledgements.** I would like to thank D. Buchholz for valueable comments on a very early draft of Chapter 2. Moreover, I would like to thank C. D'Antoni, R. Longo, J. Roberts and L. Zsido for their hospitiality, and their interest in quantum field theory in curved spacetimes. I also appreciated conversations with R. Conti, D. Guido and L. Tuset on various parts of the material of the present work.

# Appendix

### Appendix A

For the sake of completeness, we include here the interpolation argument in the form we use it in the proof of Theorem 2.2 and in Appendix B below. It is a standard argument based on Hadamard's three-line-theorem, cf. Chapter IX in [57].

**Lemma A.1** *Let $\mathcal{F}, \mathcal{H}$ be complex Hilbertspaces, $X$ and $Y$ two non-negative, injective, selfadjoint operators in $\mathcal{F}$ and $\mathcal{H}$, respectively, and $Q$ a bounded linear operator $\mathcal{H} \to \mathcal{F}$ such that $Q\mathrm{Ran}(Y^\tau) \subset \mathrm{dom}(X^\tau)$ for all $0 \leq \tau \leq 1$. Suppose that the operator $XQY$ admits a bounded extension $T : \mathcal{H} \to \mathcal{F}$. Then for all $0 \leq \tau \leq 1$, the operators $X^\tau Q Y^\tau$ are bounded by $||\, T\,||^\tau ||\, Q\,||^{1-\tau}$.*

*Proof.* The operators $\ln(X)$ and $\ln(Y)$ are (densely defined) selfadjoint operators. Let the vectors $x$ and $y$ belong to the spectral subspaces of $\ln(X)$ and $\ln(Y)$, respectively, corresponding to an arbitrary finite intervall. Then the functions $\mathbf{C} \ni z \mapsto \mathrm{e}^{z\ln(X)}x$ and $\mathbf{C} \ni z \mapsto \mathrm{e}^{z\ln(Y)}y$ are holomorphic. Moreover, $\mathrm{e}^{\tau\ln(X)}x = X^\tau x$ and $\mathrm{e}^{\tau\ln(Y)}y = Y^\tau y$ for all real $\tau$. Consider the function

$$F(z) := \langle \mathrm{e}^{\overline{z}\ln(X)}x, Q\mathrm{e}^{z\ln(Y)}y\rangle_{\mathcal{F}}\,.$$

It is easy to see that this function is holomorphic on $\mathbf{C}$, and also that the function is uniformly bounded for $z$ in the strip $\{z : 0 \leq \mathsf{Re}\, z \leq 1\}$. For $z = 1 + it$, $t \in \mathbf{R}$, one has

$$|F(z)| = |\langle \mathrm{e}^{-it\ln(X)}x, XQY\mathrm{e}^{it\ln(Y)}y\rangle_{\mathcal{F}}| \leq ||\,T\,||\,||\,x\,||_{\mathcal{F}}||\,y\,||_{\mathcal{H}}\,,$$

and for $z = it$, $t \in \mathbf{R}$,

$$|F(z)| = |\langle \mathrm{e}^{-it\ln(X)}x, Q\mathrm{e}^{it\ln(Y)}y\rangle_{\mathcal{F}}| \leq ||\,Q\,||\,||\,x\,||_{\mathcal{F}}||\,y\,||_{\mathcal{H}}\,.$$

By Hadamard's three-line-theorem, it follows that for all $z = \tau + it$ in the said strip there holds the bound

$$|F(\tau + it)| \leq ||\,T\,||^{\tau}||\,Q\,||^{1-\tau}||\,x\,||_{\mathcal{F}}||\,y\,||_{\mathcal{H}}\,.$$

As $x$ and $y$ were arbitrary members of the finite spectral intervall subspaces, the last estimate extends to all $x$ and $y$ lying in cores for the operators $X^{\tau}$ and $Y^{\tau}$, from which the the claimed statement follows. $\Box$

## Appendix B

For the convenience of the reader we collect here two well-known results about Sobolev norms on manifolds which are used in the proof of Proposition 3.5. The notation is as follows. $\Sigma$ and $\Sigma'$ will denote smooth, finite dimensional manifolds (connected, paracompact, Hausdorff); $\gamma$ and $\gamma'$ are complete Riemannian metrics on $\Sigma$ and $\Sigma'$, respectively. Their induced volume measures are denoted by $d\eta$ and $d\eta'$. We abbreviate by $A_{\gamma}$ the selfadjoint extension in $L^2(\Sigma, d\eta)$ of the operator $-\Delta_{\gamma} + 1$ on $C_0^{\infty}(\Sigma)$, where $\Delta_{\gamma}$ is the Laplace-Beltrami operator on $(\Sigma, \gamma)$; note that [10] contains a proof that $(-\Delta_{\gamma} + 1)^k$ is essentially selfadjoint on $C_0^{\infty}(\Sigma)$ for all $k \in \mathbf{N}$. $A'$ will be defined similarly with respect to the corresponding objects of $(\Sigma', \gamma')$. As in the main text, the $m$-th Sobolev scalar product is $\langle u, v\rangle_{\gamma,m} = \langle u, A_{\gamma}^m v\rangle$ for $u, v \in C_0^{\infty}(\Sigma)$ and $m \in \mathbf{R}$, where $\langle\,.\,,\,.\,\rangle$ is the scalar product of $L^2(\Sigma, d\eta)$. Anagolously we define $\langle\,.\,,\,.\,\rangle_{\gamma',m}$. For the corresponding norms we write $||\,.\,||_{\gamma,m}$, resp., $||\,.\,||_{\gamma',m}$.

**Lemma B.1** *(a) Let $\chi \in C_0^{\infty}(\Sigma)$. Then there is for each $m \in \mathbf{R}$ a constant $c_m$ so that*

$$||\,\chi u\,||_{\gamma,m} \leq c_m||\,u\,||_{\gamma,m}\,, \quad u \in C_0^{\infty}(\Sigma)\,.$$

*(b) Let $\phi \in C^{\infty}(\Sigma)$ be strictly positive and $G \subset \Sigma$ open and relatively compact. Then there are for each $m \in \mathbf{R}$ two positive constants $\beta_1, \beta_2$ so that*

$$\beta_1||\,\phi u\,||_{\gamma,m} \leq ||\,u\,||_{\gamma,m} \leq \beta_2||\,\phi u\,||_{\gamma,m}\,, \quad u \in C_0^{\infty}(G)\,.$$

*Proof.* (a) We may suppose that $\chi$ is real-valued (otherwise we treat real and imaginary parts separately). A tedious but straightforward calculation shows that the claimed estimate is fulfilled for all $m = 2k$, $k \in \mathbf{N}_0$. Hence $A^k\chi A^{-k}$ extends to a bounded operator on $L^2(\Sigma, d\eta)$, and the same is true of the adjoint $A^{-k}\chi A^k$. Thus by the interpolation argument, cf. Lemma A.1, $A^{\tau m}\chi A^{-\tau m}$ is bounded for all $-1 \leq \tau \leq 1$. This yields the stated estimate.

(b) This is a simple corollary of (a). For the first estimate, note that we may replace $\phi$ by a smooth function with compact support. Then note that the second estimate is equivalent to $|| \phi^{-1}v ||_{\gamma,m} \leq \beta_2 || v ||_{\gamma,m}$, $v \in C_0^\infty(G)$, and again we use that instead of $\phi^{-1}$ we may take a smooth function of compact support. $\square$

**Lemma B.2** *Let $(\Sigma, \gamma)$ and $(\Sigma', \gamma')$ be two complete Riemannian manifolds, $N$ and $N'$ two open subsets of $\Sigma$ and $\Sigma'$, respectively, and $\Psi : N \to N'$ a diffeomorphism. Given $m \in \mathbf{R}$ and some open, relatively compact subset $G$ of $\Sigma$ with $\overline{G} \subset N$, there are two positive constants $b_1, b_2$ such that*

$$b_1 || u ||_{\gamma,m} \leq || \Psi^* u ||_{\gamma',m} \leq b_2 || u ||_{\gamma,m} \,, \quad u \in C_0^\infty(G) \,,$$

*where $\Psi^* u := u \circ \Psi^{-1}$.*

*Proof.* Again it is elementary to check that such a result is true for $m = 2k$ with $k \in \mathbf{N}_0$. One infers that, choosing $\chi \in C_0^\infty(N)$ with $\chi|G \equiv 1$ and setting $\chi' := \Psi^*\chi$, there is for each $k \in \mathbf{N}_0$ a positive constant $b$ fulfilling

$$|| A^k \chi \Psi_* \chi' v ||_{\gamma,0} \leq b \, || (A')^k v ||_{\gamma',0} \,, \quad v \in C_0^\infty(\Sigma') \,;$$

here $\Psi_* v := v \circ \Psi$. Therefore,

$$A^k \circ \chi \circ \Psi_* \circ \chi' \circ (A')^{-k}$$

extends to a bounded operator $L^2(\Sigma', d\eta') \to L^2(\Sigma, d\eta)$ for each $k \in \mathbf{N}_0$. Interchanging the roles of $A$ and $A'$, one obtains that also

$$(A')^k \circ \chi' \circ \Psi^* \circ \chi \circ A^k$$

extends, for each $k \in \mathbf{N}_0$, to a bounded operator $L^2(\Sigma, d\eta) \to L^2(\Sigma', d\eta')$. The boundedness transfers to the adjoints of these two operators. Observe then that for $(\Psi_*)^\dagger$, the adjoint of $\Psi_*$, we have $(\Psi_*)^\dagger = \rho^2 \circ (\Psi^*)$ on $C_0^\infty(N)$, and similarly, for the adjoint $(\Psi^*)^\dagger$ of $\Psi^*$ we have $(\Psi^*)^\dagger = \Psi_* \circ \rho^{-2}$ on $C_0^\infty(N')$, where $\rho^2 = \Psi^* d\eta / d\eta'$ is a smooth density function on $N'$, cf. eqn. (3.14). It can now easily be worked out that the interpolation argument of Lemma A.1 yields again the claimed result.

$\square$

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