Lattice trees and super-Brownian motion*

Eric Derbez and Gordon Slade
Department of Mathematics and Statistics
McMaster University
Hamilton, ON, Canada L8S 4K1
slade@mcmaster.ca

April 25, 1997

Abstract

This article discusses our recent proof that above eight dimensions the scaling limit of sufficiently spread-out lattice trees is the variant of super-Brownian motion called integrated super-Brownian excursion (ISE), as conjectured by Aldous. The same is true for nearest-neighbour lattice trees in sufficiently high dimensions. The proof, whose details will appear elsewhere, uses the lace expansion. Here, a related but simpler analysis is applied to show that the scaling limit of a mean-field theory is ISE, in all dimensions. A connection is drawn between ISE and certain generating functions and critical exponents, which may be useful for the study of high-dimensional percolation models at the critical point.

1 Introduction

Lattice trees arise in polymer physics as a model of branched polymers and in statistical mechanics as an example exhibiting the general features of critical phenomena. A lattice tree in the d-dimensional integer lattice $\mathbb{Z}^d$ is a finite connected set of lattice bonds containing no cycles. Thus any two sites in a lattice tree are connected by a unique path in the tree. For the nearest-neighbour model, the bonds are nearest-neighbour bonds $\{x, y\}$, $x, y \in \mathbb{Z}^d$, $|x - y| = 1$ (Euclidean distance), but we will also consider "spread-out" lattice trees constructed from bonds $\{x, y\}$ with $0 < \|x - y\| \leq L$. Here L is a parameter which will later be taken large, and the norm is given by $\|x\| = \max\{x^{(1)}, \ldots, x^{(d)}\}$ for $x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{Z}^d$. We associate the uniform probability measure to the set of all n-bond lattice trees which contain the origin.

*To appear in *Canad. Math. Bull.*

Figure 1: A 2-dimensional lattice tree with 5000 vertices, created with the algorithm of [22].

We are interested in the existence of a scaling limit for lattice trees. This involves taking a continuum limit of lattice trees, in which the size of the trees increases simultaneously with a shrinking of the lattice spacing, in such a way as to produce a random fractal. The nature of the scaling limit is believed to depend in an essential way on the spatial dimension, but the existence of the limit has not been proven in low spatial dimensions. The corresponding problem for simple random walk has the well-known solution that when space is scaled down by a factor $n^{1/2}$, as the length n of the walk goes to infinity, there is convergence to Brownian motion in any dimension. For self-avoiding walks, it has been shown using the lace expansion that the scaling limit is also Brownian motion in dimensions $d \geq 5$ [8, 19, 17]. The same is believed to be true for $d = 4$ with a logarithmic adjustment to the spatial scaling, but in dimensions 2 and 3 a different limit, currently not understood, is expected.

Here we give an overview of recent work on high-dimensional lattice trees which proves that under certain assumptions the scaling limit is ISE (integrated super-Brownian excursion) for $d > 8$. To be precise about the assumptions, the scaling limit has been shown to be ISE for the spread-out model if $d > 8$ and L is sufficiently large, and for the nearest-neighbour model if d is sufficiently large. Detailed proofs will appear elsewhere [10, 11]. The hypothesis of universality implies that the scaling limit should be the same for spread-out and nearest-neighbour lattice trees, and assuming this, our results provide evidence that the scaling limit of nearest-neighbour lattice trees is ISE for $d > 8$. That the scaling limit of lattice trees should be ISE for $d > 8$ was conjectured by Aldous, who has emphasized the role of ISE as a model for the random distribution of mass [6]. In particular, Aldous has shown that ISE arises in various situations where random trees are randomly embedded into $\mathbb{R}^d$ [3, 4, 5]. ISE is super-Brownian motion (Brownian motion branching on all time scales) conditioned to have total mass 1, and is closely connected to the super-processes intensively studied in the probability literature. For our purposes, it will be most convenient to understand ISE as arising via generating functions.

It is typical of statistical mechanical models that there is an upper critical dimension above which a model's scaling properties cease to depend on the dimension and become identical with those of a simpler so-called mean-field model. For the self-avoiding walk, the mean-field model is simple random walk and the upper critical dimension is 4. For lattice trees, the fact that ISE occurs as the scaling limit for $d > 8$ adds to the already considerable evidence that the upper critical dimension is 8 [25, 7, 31, 16, 18]. The proof of convergence to ISE for $d > 8$ is based on the lace expansion, and involves the treatment of high-dimensional lattice trees as a small perturbation of a corresponding mean-field model.

This paper is organized as follows. In Section 2 we introduce a generating function approach to ISE; no previous knowledge of ISE is assumed. A connection is pointed out between ISE and the critical exponents of statistical mechanics, which may be relevant for the study of high-dimensional percolation models at the critical point. Section 3 contains precise statements of results showing that the scaling limit of high-dimensional lattice trees is ISE. Proofs of these results, deferred to [10, 11], use the lace expansion to perturb around a corresponding argument for a mean-field model. The mean-field model and its connection with ISE is discussed in Section 4.

2 Integrated super-Brownian excursion (ISE)

2.1 ISE probability densities

ISE can be considered as an abstract continuous random tree embedded in $\mathbb{R}^d$, rooted at the origin and having total mass 1 [6]. It is designed in such a way that if $0, x_1, \ldots, x_{m-1}$ are points in $\mathbb{R}^d$ contained in ISE then there is an underlying tree structure with branch points $b_1, \ldots, b_{m-2} \in \mathbb{R}^d$ and Brownian motion paths connecting the branch points and the points $0, x_1, \ldots, x_{m-1}$ according to an abstract skeleton (minimal spanning subtree); see Figure 2. There are $(2m-5)!!$ distinct "shapes" for the skeleton. See [14, (5.96)] for a proof of this elementary fact; here $N!!$ is defined recursively for $N = -1, 1, 3, 5, 7, 9, \ldots$ by $(-1)!! = 1$ and $N!! = N(N-2)!!$, $N \geq 1$. The shapes for $m = 2, 3, 4$ are illustrated in Figure 3. The joint probability density function for the skeleton shape, the durations $t_1, \ldots, t_{2m-3}$ of each of the Brownian motion paths and the positions of points and branch points is given by the explicit formula

$$\left(\sum_{i=1}^{2m-3} t_i\right) e^{-(\sum_{i=1}^{2m-3} t_i)^2/2} \prod_{i=1}^{2m-3} p_{t_i}(y_i), \tag{2.1}$$

where the y_i are the vector displacements along the skeleton paths and $p_t(y)$ is the Brownian transition function

$$p_t(y) = \frac{1}{(2\pi t)^{d/2}} e^{-y^2/2t}. \tag{2.2}$$

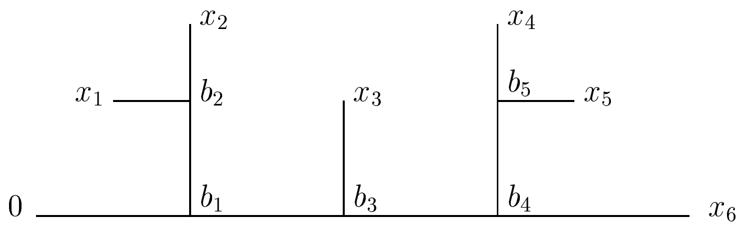

Figure 2: The branch points $b_1, \ldots, b_5$ and abstract skeleton for a realization of ISE containing the sites $0, x_1, \ldots, x_6$.

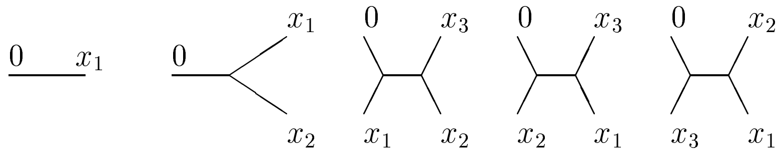

Figure 3: The unique shapes for $m = 2, 3$ and the three shapes for $m = 4$, joining points $0, x_1, \ldots, x_{m-1}$.

In Figure 2, the vector displacements (in $\mathbb{R}^d$) along the skeleton paths are $y_1 = b_1$, $y_2 = b_2 - b_1$, $y_3 = x_1 - b_2$, $y_4 = x_2 - b_2$, and so on. The ordering of the labelling of the displacements is fixed according to some convention, for each skeleton shape σ. The density (2.1) is discussed in [4, 5, 6]; see also [24].

The joint probability density function for the skeleton shape and the positions of the points and branch points, with the time variables integrated out, is given by

$$A^{(m)}(\sigma; y_1, \ldots, y_{2m-3}) = \int_0^\infty dt_1 \cdots \int_0^\infty dt_{2m-3} \left(\sum_{i=1}^{2m-3} t_i \right) e^{-(\sum_{i=1}^{2m-3} t_i)^2/2} \prod_{i=1}^{2m-3} p_{t_i}(y_i). \tag{2.3}$$

The right side is independent of the shape σ and depends only on the displacements y_i. This is indeed a probability measure, because integrating over the y_i's simply removes the product over Brownian transition functions, and the remaining integral over the t_i's equals $1/(2m-5)!!$, the reciprocal of the number of shapes. If we leave the x_i's fixed and integrate out the positions of the branch points and sum over all the $(2m-5)!!$ possible shapes, the result is a measure $P^{(m)}$ on $\mathbb{R}^{d(m-1)}$. The measures $P^{(m)}$, for $m = 2, 3, 4, \ldots$, represent the joint probability densities for ISE to contain the sites $0, x_1, \ldots, x_{m-1}$, and hence form a consistent family. For example,

$$\begin{aligned} \int_{\mathbb{R}^d} P^{(3)}(x_1, x_2) d^d x_2 &= \int_0^\infty dt_1 \int_0^\infty dt_2 \int d^d b \, p_{t_1}(b) p_{t_2}(x_1 - b) \\ &\quad \times \int_0^\infty dt_3 \, (t_1 + t_2 + t_3) e^{-(t_1+t_2+t_3)^2/2} \int d^d x_2 \, p_{t_3}(x_2 - b) \\ &= P^{(2)}(x_1), \end{aligned} \tag{2.4}$$

as can be seen by performing the integrals from right to left, using the semi-group property of $p_t(x)$ for the b-integral.

In the simplest case $m = 2$, $P^{(2)}(x)$ represents the probability density function for a point chosen randomly from the distribution of ISE. Explicitly, for $m = 2$,

$$A^{(2)}(x) = P^{(2)}(x) = (2\pi)^{-d/2} \int_0^\infty t^{1-d/2} e^{-t^2/2} e^{-x^2/2t} dt \tag{2.5}$$

and

$$\hat{A}^{(2)}(k) = \int_0^\infty t e^{-t^2/2} e^{-k^2 t/2} \, dt, \tag{2.6}$$

where our convention for the Fourier transform of a function $f : \mathbb{R}^{dn} \to \mathbb{C}$ is

$$\hat{f}(k_1, \ldots, k_n) = \int_{\mathbb{R}^{dn}} f(y_1, \ldots, y_n) e^{ik_1 \cdot y_1 + \cdots + ik_n \cdot y_n} \, d^d y_1 \cdots d^d y_n, \quad k_i \in \mathbb{R}^d. \tag{2.7}$$

The integral (2.6) can be written in terms of the parabolic cylinder function D_{-2} as $\hat{A}^{(2)}(k) = e^{k^4/16} D_{-2}(k^2/2)$ [13, 3.462.1]. For general $m \geq 2$,

$$\hat{A}^{(m)}(\sigma; k_1, \ldots, k_{2m-3}) = \int_0^\infty dt_1 \cdots \int_0^\infty dt_{2m-3} \left(\sum_{i=1}^{2m-3} t_i \right) e^{-(\sum_{i=1}^{2m-3} t_i)^2/2} e^{-\sum_{i=1}^{2m-3} k_i^2 t_i/2}. \tag{2.8}$$

2.2 ISE and generating functions

In this section, we indicate that the family of probability distributions $A^{(m)}$, $m = 2, 3, 4, \ldots$, can be encoded simply in terms of generating functions. As an analogy, consider the generating function

$$B_z(k) = \frac{1}{k^2 + 1 - z}, \quad k \in \mathbb{R}^d. \tag{2.9}$$

Expanding in a power series, we can write $B_z(k) = \sum_{n=0}^\infty b_n(k) z^n$, where $b_n(k) = (1 + k^2)^{-n-1}$. The generating function $B_z(k)$ thus gives rise to the (unit-time) Brownian transition function via

$$e^{-k^2/2} = \lim_{n \to \infty} \frac{b_n(k(2n)^{-1/2})}{b_n(0)}. \tag{2.10}$$

For ISE, beginning with $m = 2$, we define

$$C_z(k) = \frac{1}{k^2 + \sqrt{1-z}} \tag{2.11}$$

where the square root is defined to be positive for real $z < 1$ and has branch cut $[1, \infty)$ in the z-plane. This definition was motivated by the considerations of Section 2.3.1 below. Define coefficients $c_n(k)$ by

$$C_z(k) = \sum_{n=0}^\infty c_n(k) z^n, \quad |z| < 1, \tag{2.12}$$

so that

$$c_n(k) = \frac{1}{2\pi i}\oint_\Gamma C_z(k)\frac{dz}{z^{n+1}} \tag{2.13}$$

where Γ is a small circle centred at the origin. The following lemma provides a link with ISE.

Lemma 1 *For any $k \in \mathbb{R}^d$,*

$$c_n(kn^{-1/4}) \sim \frac{1}{\sqrt{\pi n}}\int_0^\infty te^{-t^2/2}e^{-\sqrt{2}k^2 t}\,dt = \frac{1}{\sqrt{\pi n}}\hat{A}^{(2)}(2^{3/4}k) \qquad \textit{as } n\to\infty. \tag{2.14}$$

In particular,

$$\lim_{n\to\infty}\frac{c_n(kn^{-1/4})}{c_n(0)} = \hat{A}^{(2)}(2^{3/4}k). \tag{2.15}$$

Proof. For $k=0$, $c_n(0)$ is given by a binomial coefficient and is asymptotic to $(\pi n)^{-1/2}$, in agreement with (2.14). Suppose henceforth that $k \neq 0$. Beginning with (2.13), we deform the contour of integration to the branch cut and make the change of variables $w = n(z-1)$. This gives

$$c_n(kn^{-1/4}) = \frac{1}{\sqrt{n}}\frac{1}{2\pi i}\int_{\Gamma'}\frac{1}{k^2+\sqrt{-w}}\frac{dw}{(1+w/n)^{n+1}}, \tag{2.16}$$

where the contour Γ' runs around the branch cut $[0,\infty)$ in the w-plane, oriented from right to left below the cut and from left to right above the cut. Then we use

$$\frac{1}{k^2+\sqrt{-w}} = \sqrt{2}\int_0^\infty dt\,\exp[-\sqrt{2}\,t(k^2+\sqrt{-w})]. \tag{2.17}$$

Taking into account the correct branches of the square root on either side of the branch cut, and applying Fubini's theorem, gives

$$c_n(kn^{-1/4}) = \frac{\sqrt{2}}{\pi\sqrt{n}}\int_0^\infty dt\,e^{-\sqrt{2}k^2 t}\int_0^\infty \frac{dw}{(1+w/n)^{n+1}}\sin(t\sqrt{2w}). \tag{2.18}$$

Since $(1+\frac{w}{n})^{n+1} \geq 1 + \frac{(n+1)n}{2}(\frac{w}{n})^2 \geq 1 + \frac{w^2}{2}$ for all $n \geq 1$, the dominated convergence theorem can be applied to give

$$c_n(kn^{-1/4}) \sim \frac{\sqrt{2}}{\pi\sqrt{n}}\int_0^\infty dt\,e^{-\sqrt{2}k^2 t}\int_0^\infty dw\,e^{-w}\sin(t\sqrt{2w}). \tag{2.19}$$

The desired result then follows, since $\int_0^\infty dw\,e^{-w}\sin(t\sqrt{2w}) = (\pi/2)^{1/2}te^{-t^2/2}$. $\square$

For any shape σ and any $m \geq 3$, recalling the definition of $C_z(k)$ in (2.11), let

$$C_z^{(m)}(\sigma; k_1, \ldots, k_{2m-3}) = \prod_{j=1}^{2m-3} C_z(k_j). \tag{2.20}$$

We write the Maclaurin series of (2.20) as

$$C_z^{(m)}(\sigma; k_1, \ldots, k_{2m-3}) = \sum_{n=0}^{\infty} c_n^{(m)}(\sigma; k_1, \ldots, k_{2m-3}) z^n, \quad |z| < 1. \tag{2.21}$$

A calculation similar to that used in the proof of Lemma 1, using (2.17) for each of the $2m-3$ factors in (2.20) and a limiting argument if any $k_j = 0$, then gives

$$c_n^{(m)}(\sigma; k_1 n^{-1/4}, \ldots, k_{2m-3} n^{-1/4}) \sim \frac{2^{m-2} n^{m-5/2}}{\sqrt{\pi}} \hat{A}^{(m)}(\sigma; 2^{3/4} k_1, \ldots, 2^{3/4} k_{2m-3}). \tag{2.22}$$

Since $\hat{A}^{(m)}(\sigma; 0, \ldots, 0) = 1/(2m-5)!!$ is the reciprocal of the number of shapes, this gives

$$\lim_{n \to \infty} \frac{c_n^{(m)}(\sigma; k_1 n^{-1/4}, \ldots, k_{2m-3} n^{-1/4})}{\sum_\sigma c_n^{(m)}(\sigma; 0, \ldots, 0)} = \hat{A}^{(m)}(\sigma; 2^{3/4} k_1, \ldots, 2^{3/4} k_{2m-3}). \tag{2.23}$$

Thus the distributions $A^{(m)}$ arise as the scaling limits of the coefficients of the generating functions $C_z^{(m)}$, $m \geq 2$. In particular, this essential aspect of ISE follows solely from (2.11) and (2.20). Moreover, small perturbations of (2.11) and (2.20) will not affect the scaling limit; see Section 4.2. In Sections 3 and 4 below, we indicate how generating functions can be related directly to (2.1) itself, rather than to its integral (2.3).

2.3 ISE and critical exponents

This section shows that the generating function approach to ISE outlined in Section 2.2 provides a link between ISE and the critical exponents of statistical mechanics. For lattice trees, it is the exponents η and γ which are relevant, while for percolation it is η and δ.

2.3.1 Lattice trees

A lattice tree containing the points $0, x_1, \ldots, x_{m-1}$ has a unique skeleton (the minimal spanning subtree for $0, x_1, \ldots, x_{m-1}$), with $m-2$ branch points $b_1, \ldots, b_{m-2}$ and $2m-3$ paths. Let $y_1, \ldots, y_{2m-3}$ denote the vector displacements of the skeleton paths, as in Figure 4, and let $t_n^{(m)}(\sigma; y_1, \ldots, y_{2m-3})$ denote the number of n-bond trees having skeleton of shape σ and skeleton path displacements $y_1, \ldots, y_{2m-3}$. Equivalently, $t_n^{(m)}(\sigma; y_1, \ldots, y_{2m-3})$ is the number of n-bond lattice trees containing the branch points $b_1, \ldots, b_{m-2}$ and sites $0, x_1, \ldots, x_{m-1}$ consistent with the displacements $y_1, \ldots, y_{2m-3}$ and joined together by a skeleton of shape σ. Define

$$G_z^{(m)}(\sigma; y_1, \ldots, y_{2m-3}) = \sum_{n=0}^{\infty} t_n^{(m)}(\sigma; y_1, \ldots, y_{2m-3}) z^n. \tag{2.24}$$

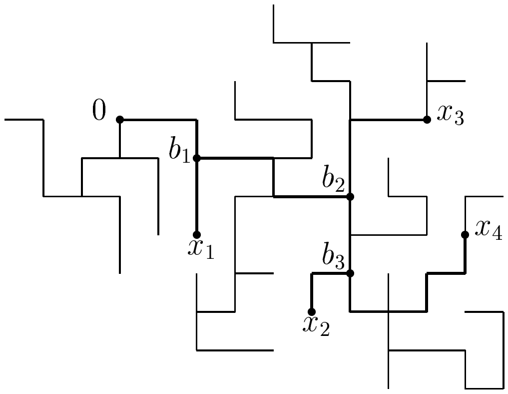

Figure 4: A lattice tree containing the sites $0, x_1, x_2, x_3, x_4$ with its corresponding skeleton and branch points b_1, b_2, b_3. The vector displacements along the skeleton paths are $y_1 = b_1$, $y_2 = x_1 - b_1$, $y_3 = b_2 - b_1$, $y_4 = x_3 - b_2$, and so on. The ordering of the labelling of the displacements is fixed according to some convention, for each skeleton shape σ.

It can be shown via a subadditivity argument that summing the above expression over $y_1, \ldots, y_{2m-3}$ results in a power series having a radius of convergence $z_c \in (0, \infty)$, independent of m. The two principal ingredients involved in the proof of convergence of lattice trees to ISE in high dimensions are to show that the functions $G_z^{(m)}$ obey, to leading order, (2.11) and (2.20).

In terms of critical exponents, the Fourier transform of the two-point function is believed to behave asymptotically as

$$\hat{G}_{z_c}^{(2)}(k) \sim \frac{c_1}{k^{2-\eta}} \quad \text{as } k \to 0, \qquad \hat{G}_z^{(2)}(0) \sim \frac{c_2}{(1 - z/z_c)^{\gamma}} \quad \text{as } z \to z_c, \tag{2.25}$$

with the mean-field values $\eta = 0$ and $\gamma = \frac{1}{2}$ for $d > 8$. Here the Fourier transform is the discrete one, given for $f : \mathbb{Z}^d \to \mathbb{C}$ by

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) e^{ik \cdot x}, \quad k \in [-\pi, \pi]^d. \tag{2.26}$$

For $d > 8$, the simplest possible combination of the two asymptotic relations in (2.25) is

$$\hat{G}_z^{(2)}(k) = \frac{C_1}{D_1^2 k^2 + 2^{3/2}(1 - z/z_c)^{1/2}} + \text{error term}, \tag{2.27}$$

where C_1 and D_1 are positive constants depending on d and L, and the factor $2^{3/2}$ has been inserted for later convenience. The error term is meant to be of lower order than the main term, in some suitable sense, as $k \to 0$ and $z \to z_c$. The first step in the proof of convergence to ISE is to show that (2.27) does hold in high dimensions, with a controlled error, so that much as (2.11) leads to (2.15),

$$\lim_{n\to\infty} \frac{\hat{t}_n^{(2)}(kD_1^{-1}n^{-1/4})}{\hat{t}_n^{(2)}(0)} = \hat{A}^{(2)}(k). \tag{2.28}$$

This can be interpreted as asserting that in the scaling limit the distribution of a site $xD_1n^{1/4}$ in an n-bond lattice tree is the distribution of a point from ISE. It was the anticipation of this conclusion which motivated Section 2.2.

The second step in the proof of convergence to ISE involves showing that in high dimensions there is an approximate independence of the form

$$\hat{G}_z^{(m)}(\sigma; k_1, \ldots, k_{2m-3}) = v^{m-2} \prod_{j=1}^{2m-3} \hat{G}_z(k_j) + \text{error term}, \tag{2.29}$$

where v is a finite positive constant which translates the self-avoidance interactions of lattice trees into a renormalized vertex factor. Then, with sufficient control on the error term in (2.29), the finite-dimensional distributions can be shown to have scaling limit $\hat{A}^{(m)}(\sigma; k_1, \ldots, k_{2m-3})$, just as (2.20) leads to (2.23).

We believe that the above discussion should apply also to lattice animals for $d > 8$, yielding ISE for their scaling limit for $d > 8$ and consistent with the general belief that lattice trees and lattice animals have the same scaling properties in all dimensions.

2.3.2 Percolation

Reasoning of the above type has led Hara and Slade to conjecture that for $d > 6$ the scaling limit of large percolation clusters at the critical point is ISE. The remainder of this section discusses the basis for the conjecture. Further discussion of the scaling limit can be found in [1].

Consider independent Bernoulli bond percolation on $\mathbb{Z}^d$ with p fixed and equal to its critical value p_c [14]. Let $C(0)$ denote the random set of sites connected to 0, let

$$\tau_n^{(2)}(x) = P_{p_c}\{C(0) \ni x, |C(0)| = n\} \tag{2.30}$$

denote the probability at the critical point that the origin is connected to x via a cluster containing n sites, and let

$$T_z^{(2)}(x) = \sum_{n=1}^{\infty} \tau_n^{(2)}(x) z^n, \quad |z| \leq 1. \tag{2.31}$$

The generating function (2.31) converges absolutely if $|z| \leq 1$. Let $\tau(p; 0, x)$ denote the probability that 0 is connected to x. Then $T_1^{(2)}(x) = \tau(p_c; 0, x)$ (assuming no infinite cluster at p_c). The conventional definitions [14, Section 7.1] of the critical exponents η and δ lead to

$$\hat{T}_1^{(2)}(k) \sim \frac{c_1}{k^{2-\eta}}, \quad \text{as } k \to 0, \quad \hat{T}_z^{(2)}(0) \sim \frac{c_2}{(1-z)^{1-1/\delta}}, \quad \text{as } z \to 1. \tag{2.32}$$

Using the mean-field values $\eta = 0$ and $\delta = 2$ above six dimensions, the simplest combination of the above asymptotic relations for $d > 6$, analogous to (2.27), is

$$\hat{T}_z^{(2)}(k) = \frac{C_2}{D_2^2 k^2 + 2^{3/2}(1-z)^{1/2}} + \text{error term}, \tag{2.33}$$

for some constants C_2, D_2.

Proving (2.33) would provide an analogue of (2.11). With sufficient control of the error in (2.33), contour integration with respect to z may then lead to

$$\lim_{n\to\infty} \frac{\hat{\tau}_n^{(2)}(kD_2^{-1}n^{-1/4})}{\hat{\tau}_n^{(2)}(0)} = \hat{A}^{(2)}(k). \tag{2.34}$$

The above equation can be interpreted as asserting that in the scaling limit the distribution of a site $xD_2n^{1/4}$ in the cluster of the origin, conditional on the cluster being of size n, is the distribution of a point in ISE.

The study of percolation clusters containing $m \geq 3$ sites is more difficult than for lattice trees because for percolation there is not a unique skeleton nor therefore unique branch points and corresponding displacements for a cluster containing m specified points (the same is true for lattice animals). Nevertheless, for $d > 6$ this lack of uniqueness should be a "local" effect whose role is unimportant in the scaling limit, and we expect that a relation of the form

$$\hat{T}_z^{(m)}(\sigma; k_1, \ldots, k_{2m-3}) = v^{m-2} \prod_{i=1}^{2m-3} \hat{T}_z^{(2)}(k_i) + \text{error term} \tag{2.35}$$

should hold for a suitably defined generating function $\hat{T}_z^{(m)}(\sigma; k_1, \ldots, k_{2m-3})$. Such a statement would provide a relation analogous to (2.29), and could lead to the ISE correlation $\hat{A}^{(m)}(\sigma; k_1, \ldots, k_{2m-3})$ in the scaling limit. An asymptotic relation in the spirit of (2.35) was conjectured for $d > 6$ already in [2]. There it was argued that the sum, over sites $x_1, \ldots, x_{m-1}$ in the lattice, of the probability that the cluster of the origin contains $x_1, \ldots, x_{m-1}$, should behave asymptotically as $v^{m-2}\hat{\tau}(p;0)^{2m-3}$ in the limit $p \to p_c$, where v is a positive constant.

Hara and Slade are currently investigating whether the method of [11] can be combined with the method of [15] to prove the conjecture. The methods of [29, 30] could possibly serve as a starting point to study related questions for oriented percolation.

3 Lattice trees in high dimensions

In this section, we state precise results for the scaling limit of high-dimensional lattice trees. We begin by introducing some notation and recalling some previous results.

Let $t_n^{(1)}$ denote the number of n-bond lattice trees containing the origin, with $t_0^{(1)} = 1$. By a subadditivity argument [23], the limit $z_c^{-1} = \lim_{n\to\infty}(t_n^{(1)})^{1/n}$ exists and is positive and finite. For $m \geq 2$, let $t_n^{(m)}(\sigma; \vec{y}, \vec{s})$ be the number of n-bond lattice trees with skeleton shape σ and skeleton displacements $y_1, \ldots, y_{2m-3}$ as in Figure 4, with the skeleton path corresponding to y_i consisting of s_i steps ($i = 1, \ldots, 2m-3$). We also define

$$t_n^{(m)}(\sigma; \vec{y}) = \sum_{\vec{s}} t_n^{(m)}(\sigma; \vec{y}, \vec{s}), \tag{3.1}$$

$$t_n^{(m)}(\vec{y}) = \sum_{\sigma} t_n^{(m)}(\sigma; \vec{y}). \tag{3.2}$$

We will make use of Fourier transforms with respect to the $\vec{y}$ variables, for example,

$$\hat{t}_n^{(m)}(\sigma; \vec{k}) = \sum_{\vec{y}} t_n^{(m)}(\sigma; \vec{y}) e^{i(k_1 \cdot y_1 + \cdots + k_{2m-3} \cdot y_{2m-3})}, \quad k_i \in [-\pi, \pi]^d. \tag{3.3}$$

Note that for $m \geq 2$,

$$\hat{t}_n^{(m)}(\vec{0}) = \sum_{\sigma} \sum_{\vec{y}} t_n^{(m)}(\sigma; \vec{y}) = (n+1)^{m-1} t_n^{(1)}. \tag{3.4}$$

To see this, perform the sums over σ and $\vec{y}$ by first fixing the values of $x_1, \ldots, x_{m-1}$ and then summing over all shapes and branch points compatible with $x_1, \ldots, x_{m-1}$ as in Figure 4. This leaves the sum over $x_1, \ldots, x_{m-1}$ of the number of n-bond lattice trees containing the origin and $x_1, \ldots, x_{m-1}$. Then (3.4) follows from the fact that an n-bond lattice tree contains $n+1$ sites.

In [16, 18], some critical exponents for lattice trees were proven to exist and to assume their mean-field values when $d > 8$. More precisely, the results were obtained for the nearest-neighbour model when $d \geq d_0$ for some undetermined dimension $d_0 > 8$, and for spread-out trees when $d > 8$ and L is sufficiently large depending on d. We will refer to either of these restrictions on the dimension and L as the "high-dimension condition."

In particular, it was shown in [18] that under the high-dimension condition there is a positive constant A (depending on d and L) such that

$$t_n^{(1)} \sim A z_c^{-n} n^{-3/2}, \quad \text{as } n \to \infty. \tag{3.5}$$

In terms of the critical exponent θ occurring in the conjectured relation $t_n^{(1)} \sim A z_c^{-n} n^{1-\theta}$, this says that $\theta = \frac{5}{2}$ under the high-dimension condition. The bounds $c_1 n^{-c_2 \log n} z_c^{-n} \leq t_n^{(1)} \leq c_3 n^{1/d} z_c^{-n}$, proved respectively in [21] and [26] and believed not to be sharp, are the best general bounds known at present for $t_n^{(1)}$. The critical exponent θ is formally related to the exponent γ, discussed in Section 2.3.1 and defined by $\hat{G}_z^{(2)}(0) \sim \text{const.}(1 - z/z_c)^{-\gamma}$ as $z \to z_c$, by $\theta = 3 - \gamma$. It had been proved earlier, in [16], that $\gamma = \frac{1}{2}$ under the high-dimension condition. With (3.5), (3.4) gives

$$\hat{t}_n^{(m)}(\vec{0}) \sim A z_c^{-n} n^{m-5/2}. \tag{3.6}$$

Another critical exponent involves R_n, the average radius of gyration of n-bond trees. The squared average radius of gyration is defined by

$$R_n^2 = \frac{1}{t_n^{(1)}} \sum_{T: |T| = n, T \ni 0} R(T)^2, \tag{3.7}$$

where

$$R(T)^2 = \frac{1}{|T|+1} \sum_{x \in T} |x - \bar{x}_T|^2 \tag{3.8}$$

is the squared radius of gyration of T. Here we write $|T|$ to denote the number of bonds in a lattice tree T, $\bar{x}_T = (|T|+1)^{-1} \sum_{x \in T} x$ to denote the centre of mass of T (considered as a set of unit masses at the *sites* of T), and we say that $x \in T$ if x is an element of a bond in T. Equivalently,

$$R_n^2 = \frac{1}{2\hat{t}_n^{(2)}(0)} \sum_x |x|^2 t_n^{(2)}(x). \tag{3.9}$$

It is believed that there is a critical exponent ν such that $R_n \sim Dn^\nu$ as $n \to \infty$, but very little has been proved rigorously about this in general dimensions.

Under the high-dimension condition, it is proved in [18] that

$$R_n \sim Dn^{1/4}, \tag{3.10}$$

so that $\nu = \frac{1}{4}$. The amplitude D of (3.10) is a positive constant which depends on d, and for the spread-out model, also on L. Asymptotically, for fixed d, D behaves like a multiple of L as $L \to \infty$. For later use, we define

$$D_1 = 2^{3/4} d^{-1/2} \pi^{-1/4} D. \tag{3.11}$$

The fact that $\nu = \frac{1}{4}$ under the high-dimension condition can be interpreted as saying that the mass n of a tree grows on average like the fourth power of its radius, suggesting a 4-dimensional nature for lattice trees in high dimensions. This compares well with the fact that ISE has Hausdorff dimension 4 (as can be shown to follow from [9, Theorem 1.4]), and also permits the upper critical dimension 8 to be interpreted as the dimension above which two 4-dimensional objects generically do not intersect.

Define

$$p_n^{(m)}(\sigma; \vec{y}) = \frac{t_n^{(m)}(\sigma; \vec{y})}{\hat{t}_n^{(m)}(\vec{0})}, \tag{3.12}$$

which is the probability that an n-bond lattice tree containing the origin has a skeleton of shape σ mediating displacements $y_1, \ldots, y_{2m-3}$. The following theorem [10, 11] shows that this distribution has the corresponding ISE distribution as its scaling limit, under the high-dimension condition.

Theorem 1 *Let $m \geq 2$ and $k_i \in \mathbb{R}^d$ $(i = 1, \ldots, 2m-3)$. For nearest-neighbour trees in sufficiently high dimensions $d \geq d_0$, and for sufficiently spread-out trees above eight dimensions,*

$$\lim_{n \to \infty} \hat{p}_n^{(m)}(\sigma; \vec{k} D_1^{-1} n^{-1/4}) = \hat{A}^{(m)}(\sigma; \vec{k}),$$

where D_1 is given by (3.11).

For a more refined statement than Theorem 1, we wish to see the integrand

$$\hat{a}^{(m)}(\sigma;\vec{k},\vec{t}) \equiv \left(\sum_{i=1}^{2m-3} t_i\right) e^{-(\sum_{i=1}^{2m-3} t_i)^2/2} e^{-\sum_{i=1}^{2m-3} k_i^2 t_i/2} \tag{3.13}$$

of the integral representation (2.8) of $\hat{A}^{(m)}(\sigma;\vec{k})$ as corresponding to Brownian motion paths arising from the scaling limit of the skeleton. For this, we denote by

$$p_n^{(m)}(\sigma;\vec{y},\vec{s}) = \frac{t_n^{(m)}(\sigma;\vec{y},\vec{s})}{\hat{t}_n^{(m)}(\vec{0})} \tag{3.14}$$

the probability that an n-bond lattice tree containing the origin has a skeleton of shape σ mediating displacements $y_1,\dots,y_{2m-3}$ with skeleton paths of respective lengths $s_1,\dots,s_{2m-3}$. The following theorem [11] shows that, for $m=2,3$, the skeleton paths converge to Brownian motions, with the weight factor appearing in (3.13).

Theorem 2 *Let $m=2$ or $m=3$, $k_i \in \mathbb{R}^d$ and $t_i \in [0,\infty)$ $(i=1,\dots,2m-3)$. For nearest-neighbour trees in sufficiently high dimensions $d \geq d_0$, and for sufficiently spread-out trees above eight dimensions, there is a constant T_1 depending on d and L such that*

$$\lim_{n\to\infty} (T_1 n^{1/2})^{2m-3}\, \hat{p}_n^{(m)}(\sigma;\vec{k}D_1^{-1}n^{-1/4},\vec{t}T_1 n^{1/2}) = \hat{a}^{(m)}(\sigma;\vec{k},\vec{t}). \tag{3.15}$$

(As an argument of $\hat{p}_n^{(m)}$, $t_i T_1 n^{1/2}$ is to be interpreted as its integer part $\lfloor t_i T_1 n^{1/2} \rfloor$.)

We believe that Theorem 2 is valid for also for $m \geq 4$, but we encounter technical difficulties in attempting a proof. It would be of interest to extend Theorem 2 to general m, and also to investigate tightness with the aim of obtaining a stronger statement of convergence to ISE.

The factor $(T_1 n^{1/2})^{2m-3}$ on the left side of (3.15) has a natural interpretation. In fact, writing $t_i = s_i/(T_1 n^{1/2})$ in the right side of (3.15), and then multiplying by $(T_1 n^{1/2})^{-(2m-3)}$ and summing over the s_i, gives a Riemann sum approximation to (2.8). Theorem 2 indicates that skeleton paths with length of order $\sqrt{n}$ are typical, and that the skeleton paths converge to Brownian motion paths in the scaling limit.

The proofs of Theorems 1 and 2 are given in [10, 11]. The proofs use generating functions and contour integration, with the generating functions controlled using the lace expansion. To define the generating functions, we begin with $m=1$ and define

$$g(z) = \sum_{n=0}^{\infty} t_n^{(1)} z^n = \sum_{T:T\ni 0} z^{|T|}. \tag{3.16}$$

For $m \geq 2$, let

$$G_z^{(m)}(\sigma;\vec{y}) = \sum_{n=0}^{\infty} t_n^{(m)}(\sigma;\vec{y}) z^n. \tag{3.17}$$

The series in (3.17) and (3.16) converge if $|z| < z_c$. When the high-dimension condition is satisfied, it was shown in [16] that $g(z_c) < \infty$.

For $m = 2$, there is only one shape, and the two-point function $G_z^{(2)}(x)$ is the generating function for lattice trees containing the sites 0 and x. A lattice tree T containing these two sites contains a unique self-avoiding walk ω connecting them (the skeleton), and removing the bonds of ω from T leaves behind trees $R_0, R_1, \ldots, R_{|\omega|}$ attached along the skeleton sites, with the restriction that as sets of sites $R_i \cap R_j = \emptyset$ if $i \neq j$. Explicitly,

$$G_z^{(2)}(x) = \sum_{\omega:0\to x} z^{|\omega|} \prod_{i=0}^{|\omega|} \sum_{R_i:R_i \ni \omega(i)} z^{|R_i|} I[R_j \cap R_k = \emptyset \text{ if } j \neq k], \tag{3.18}$$

where the sum is over self-avoiding walks ω from 0 to x (spread-out or nearest-neighbour as appropriate) and $|\omega|$ denotes the number of steps of ω. There is a differential equation relating $\hat{G}_z^{(2)}(0)$ and g_z, since by (3.4)

$$\hat{G}_z^{(2)}(0) = \sum_{n=0}^{\infty} \hat{t}_n^{(2)}(0) z^n = \sum_{n=0}^{\infty} (n+1) t_n^{(1)} z^n = \frac{d}{dz}\left(zg(z)\right). \tag{3.19}$$

The lace expansion was used in [16] to show, under the high-dimension condition, that

$$\hat{G}_z^{(2)}(k) = \frac{g_z + \hat{\Pi}_z(k)}{1 - z\Omega \hat{D}(k)(g_z + \hat{\Pi}_z(k))}, \tag{3.20}$$

where Ω is the number of bonds emanating from the origin ($2d$ for the nearest-neighbour model and $(2L+1)^d - 1$ for the spread-out model), and

$$\hat{D}(k) = \frac{1}{\Omega} \sum_{x \in \Omega} e^{ik \cdot x} \tag{3.21}$$

(letting Ω denote also the set of sites which together with the origin form a bond). The function $\hat{\Pi}_z(k)$ is complicated but explicit and is well-understood under the high-dimension condition, where it can be regarded as a small perturbation of $g(z)$. It can be shown [10, 11] that under the high dimension condition

$$\hat{G}_z^{(2)}(k) = \frac{C_1}{D_1^2 k^2 + 2^{3/2}\sqrt{1 - z/z_c}} + \text{error term}, \tag{3.22}$$

with appropriate control on the error term. This shows that the critical exponent η takes on its mean-field value $\eta = 0$. The bounds on the error term are sufficient that, using Lemma 1, and [19, Lemma 3.3] or [27, Lemma 6.3.3] for the error term, the result of Theorem 1 for the two-point function can be concluded. For the m-point function, the basic step involves showing that

$$\hat{G}_z^{(m)}(\sigma; k_1, \ldots, k_{2m-3}) = v^{m-2} \prod_{j=1}^{2m-3} \hat{G}_z^{(2)}(k_j) + \text{error term}, \tag{3.23}$$

with v a positive constant. With sufficient control on the error term in (3.23), this is enough to prove Theorem 1.

To prove Theorem 2, more refined generating functions are used. For the two-point function, this involves the insertion of a factor $\zeta^{|\omega|}$ on the right side of (3.18), where ζ is a complex fugacity (in the unit disk) for the length of the skeleton. This modifies (3.20) to

$$\hat{G}^{(2)}_{z,\zeta}(k) = \frac{g_z + \hat{\Pi}_{z,\zeta}(k)}{1 - \zeta z \Omega \hat{D}(k)(g_z + \hat{\Pi}_{z,\zeta}(k))}, \tag{3.24}$$

where now the interaction term $\hat{\Pi}_{z,\zeta}(k)$ depends also on ζ. It is shown in [11] that

$$\hat{G}^{(2)}_{z,\zeta}(k) = \frac{C_1}{D_1^2 k^2 + 2^{3/2}\sqrt{1 - z/z_c} + 2T_1(1-\zeta)} + \text{error term}, \tag{3.25}$$

with appropriate control on the error term. This permits us to perform a contour integration in the ζ-plane to extract the coefficient of ζ^s, corresponding to trees with an s-step skeleton, and then perform a contour integration in the z-plane to extract the coefficient of z^n, corresponding to n-bond trees with an s-step skeleton. This aspect is discussed in more detail in Section 4.2, in the context of the mean-field model. The situation is similar for the three-point function, using also a ζ-dependent analogue of (3.23).

4 Mean-field theory

The proofs of Theorems 1 and 2 can be interpreted as perturbations of corresponding calculations for a mean-field theory. This section sketches the main ideas of an analysis of the mean-field theory, valid in all dimensions, which serves as a basis for the proofs of Theorems 1 and 2 in [10, 11]. The mean-field model studied here can be regarded as a model of non-self-interacting lattice trees with no self-avoidance constraint, and is closely related to the model studied in [7]. Its scaling limit is ISE in all dimensions. For convenience, we will consider throughout this section the case where z is a site fugacity rather than a bond fugacity, which has the effect of replacing factors $z^{|T|}$ by $z^{|T|+1}$, for example in (3.16). This is not a substantive change. We also restrict attention to the nearest-neighbour model. At the end of this section, we make some remarks of a combinatorial nature.

4.1 Definition

Switching to a site fugacity, if we remove the indicator function containing the interaction in the two-point function (3.18), we obtain $\sum_{\omega:0\to x}(g(z))^{|\omega|+1}$. This prompts us to define the two-point function of the mean-field theory by

$$F^{(2)}_z(x) = \sum_{\omega:0\to x} (f(z))^{|\omega|+1}, \tag{4.1}$$

where the function $f(z)$, specified below, can be regarded as the generating function for mean-field trees containing the origin. The sum in (4.1) is taken over simple random walks, and by analogy with (3.19) (taking into account the switch to site fugacity), $f(z)$ is required to satisfy the differential equation

$$\hat{F}_z^{(2)}(0) = z\frac{df(z)}{dz}. \tag{4.2}$$

We demand, by analogy with (3.16), that $f(z) \sim z$ as $z \to 0$. In the next paragraph, we show that this uniquely defines $f(z)$ and hence $\hat{F}_z^{(2)}(k)$. For the m-point function of the mean-field model, we define

$$\hat{F}_z^{(m)}(\sigma; k_1, \ldots, k_{2m-3}) = \prod_{i=1}^{2m-3} \hat{F}_z^{(2)}(k_i), \tag{4.3}$$

an exact analogue of (2.20). This definition of the m-point function as a product of 2-point functions is consistent with a lack of self-interaction for the mean-field model.

Taking the Fourier transform of (4.1) gives

$$\hat{F}_z^{(2)}(k) = \frac{f(z)}{1 - 2df(z)\hat{D}(k)}, \tag{4.4}$$

where, as in (3.21), $\hat{D}(k) = d^{-1}\sum_{j=1}^{d}\cos k^{(j)}$. Combining (4.2) and (4.4) gives

$$\frac{f(z)}{1 - 2df(z)} = z\frac{df(z)}{dz}. \tag{4.5}$$

Let z_0 be defined by $2df(z_0) = 1$. Solving the separable equation (4.5) gives

$$f(z)e^{-2df(z)} = \frac{z}{2dez_0}. \tag{4.6}$$

Comparing the asymptotic behaviour of both sides as $z \to 0$, using $f(z) \sim z$, gives $z_0 = 1/(2de)$ and hence

$$f(z)e^{-2df(z)} = z. \tag{4.7}$$

4.2 Analysis

This section sketches proofs of analogues of Theorems 1 and 2 for the mean-field model.

By (4.7), f can be written as $f(z) = -(2d)^{-1}W(-2dz)$, where W is the principal branch of the Lambert W function, defined by $W(w)e^{W(w)} = w$. The principal branch of $W(w)$ is analytic on the w-plane with branch cut $(-\infty, -e^{-1}]$, corresponding to a branch cut $[z_0, \infty)$ for $f(z)$. Properties of f can be derived from known properties of the Lambert function, or derived directly from (4.7). In particular, f has a square root singularity at z_0, and

$$f(z) = f(z_0) - \frac{1}{d\sqrt{2}}(1 - z/z_0)^{1/2} + O(|1 - z/z_0|), \tag{4.8}$$

with the absolute value of the error term bounded by a constant multiple of $|1 - z/z_0|$ uniformly in the cut plane. Using also the fact that $\hat{D}(k) = 1 - (2d)^{-1}k^2 + O(k^4)$, this gives

$$\hat{F}_z^{(2)}(k) = \frac{1 + E_1}{k^2 + d2^{3/2}\sqrt{1 - z/z_0} + E_2}, \tag{4.9}$$

where $E_1 = -\sqrt{2}(1 - z/z_0)^{1/2} + O(|1 - z/z_0|)$ and $E_2 = O(k^4) + O(k^2|1 - z/z_0|^{1/2}) + O(|1 - z/z_0|)$. Writing $\hat{F}_z^{(m)}(k) = \sum_{n=0}^{\infty} \hat{\phi}_n^{(m)}(k) z^n$, the asymptotic form of the coefficient $\hat{\phi}_n^{(2)}(kn^{-1/4})$ can then be obtained using contour integration as in the proof of Lemma 1. The result is

$$\hat{\phi}_n^{(2)}(d^{1/2}kn^{-1/4}) \sim \frac{1}{2d\sqrt{2n\pi}} \frac{1}{z_0^n} \hat{A}^{(2)}(k), \tag{4.10}$$

and hence, as in Theorem 1,

$$\lim_{n\to\infty} \frac{\hat{\phi}_n^{(2)}(d^{1/2}kn^{-1/4})}{\hat{\phi}_n^{(2)}(0)} = \hat{A}^{(2)}(k). \tag{4.11}$$

For the general m-point function, the procedure is similar. By (4.3) and (4.9), $\hat{F}_z^{(m)}(\sigma; \vec{k})$ is approximately equal to

$$\prod_{i=1}^{2m-3} \frac{1}{k_i^2 + d2^{3/2}\sqrt{1 - z/z_0}}. \tag{4.12}$$

As in (2.22), this leads to

$$\hat{\phi}_n^{(m)}(\sigma; d^{1/2}\vec{k}n^{-1/4}) \sim \frac{n^{m-5/2}}{(2d)^{2m-3}\sqrt{2\pi}} \frac{1}{z_0^n} \hat{A}^{(m)}(\sigma; \vec{k}) \quad \text{as } n \to \infty. \tag{4.13}$$

Since $\hat{A}^{(m)}(\sigma; \vec{0}) = 1/(2m-5)!!$ is the reciprocal of the number of shapes, this gives a mean-field version of Theorem 1:

$$\lim_{n\to\infty} \frac{\hat{\phi}_n^{(m)}(\sigma; d^{1/2}\vec{k}n^{-1/4})}{\sum_\sigma \hat{\phi}_n^{(m)}(\sigma; \vec{0})} = \hat{A}^{(m)}(\sigma; \vec{k}). \tag{4.14}$$

To study the scaling behaviour of the skeleton, we introduce fugacities ζ_i ($|\zeta_i| \leq 1$) for the bonds in the skeleton paths. For the two-point function, we define

$$F_{z,\zeta}^{(2)}(x) = \sum_{\omega:0\to x} (f(z))^{|\omega|+1} \zeta^{|\omega|}, \tag{4.15}$$

so that

$$\hat{F}_{z,\zeta}^{(2)}(k) = \frac{f(z)}{1 - 2d\zeta f(z)\hat{D}(k)}. \tag{4.16}$$

Define, for $|\zeta_i| \le 1$ and $|z| < z_0$,

$$\hat{F}^{(m)}_{z,\vec{\zeta}}(\sigma;\vec{k}) = \prod_{i=1}^{2m-3} \hat{F}^{(2)}_{z,\zeta_i}(k_i) = \sum_{n=0}^{\infty}\sum_{\vec{s}} \hat{\phi}^{(m)}_n(\sigma;\vec{k},\vec{s})\zeta_1^{s_1}\cdots\zeta_{2m-3}^{s_{2m-3}} z^n, \tag{4.17}$$

where the last equality defines $\hat{\phi}^{(m)}_n(\sigma;\vec{k},\vec{s})$. To prove a mean-field version of Theorem 2, for all $m \ge 2$, we wish to argue that

$$\lim_{n\to\infty} n^{m-3/2} \frac{\hat{\phi}^{(m)}_n(\sigma; d^{1/2}\vec{k}n^{-1/4}, \vec{t}\sqrt{n})}{\sum_\sigma \hat{\phi}^{(m)}_n(\sigma;\vec{0})} = \hat{a}^{(m)}(\sigma;\vec{k},\vec{t}). \tag{4.18}$$

Since the right side of (4.16) is the sum of a geometric series (in ζ),

$$\sum_{n=0}^{\infty} \hat{\phi}^{(m)}_n(\sigma;\vec{k},\vec{s})z^n = \prod_{i=1}^{2m-3}\left[(2d\hat{D}(k_i))^{s_i}(f(z))^{s_i+1}\right], \tag{4.19}$$

and thus

$$\hat{\phi}^{(m)}_n(\sigma;\vec{k},\vec{s}) = \left(\prod_{i=1}^{2m-3}\hat{D}(k_i)^{s_i}\right)\frac{1}{(2d)^{2m-3}}\frac{1}{2\pi i}\oint_\Gamma (2df(z))^{\sum_{i=1}^{2m-3}(s_i+1)}\frac{dz}{z^{n+1}} \tag{4.20}$$

where Γ is a small circle centred at the origin. Writing $s_i = t_i\sqrt{n}$ and replacing k_i by $d^{1/2}k_i n^{-1/4}$, the Gaussian factor emerges in the limit $n \to \infty$ from

$$\prod_{i=1}^{2m-3}\hat{D}(d^{1/2}k_i n^{-1/4})^{t_i\sqrt{n}} = \prod_{i=1}^{2m-3}\left(1 - \frac{k_i^2}{2\sqrt{n}} + O\left(\frac{k_i^4}{n}\right)\right)^{t_i\sqrt{n}} \to e^{-\sum_{i=1}^{2m-3}k_i^2 t_i/2}. \tag{4.21}$$

Letting $t = \sum_i t_i$ and $M = 2m-3$, it remains to determine the asymptotic form of the integral

$$\frac{1}{2\pi i}\oint_\Gamma (2df(z))^{t\sqrt{n}+M}\frac{dz}{z^{n+1}}. \tag{4.22}$$

This can be done by deforming the contour to the branch cut $[z_0,\infty)$, making the change of variables $w = n(z/z_0 - 1)$, using the fact that $f(z)$ has a square root singularity at z_0 by (4.8), and finally applying the dominated convergence theorem, with the result

$$\frac{1}{nz_0^n}\frac{1}{\pi}\int_0^\infty e^{-w}\sin(t\sqrt{2w})dw = \frac{1}{nz_0^n}\frac{1}{\sqrt{2\pi}}te^{-t^2/2}. \tag{4.23}$$

The fact that (4.22) is asymptotic to (4.23) is a special case of a more general fact; see [28, (4.2)], with, in their notation, $\alpha = 0$ and $\lambda = t$. Combining (4.20)–(4.23), and recalling from (4.13) that

$$\sum_\sigma \hat{\phi}^{(m)}_n(\sigma;\vec{0}) \sim \frac{n^{m-5/2}}{(2d)^{2m-3}\sqrt{2\pi}}\frac{1}{z_0^n}, \tag{4.24}$$

gives the desired result (4.18) that for the mean-field model typical skeletons have path lengths proportional to $\sqrt{n}$, and have a Brownian scaling limit with the ISE weight factor $te^{-t^2/2}$.

4.3 Combinatorics

We close with some considerations of a combinatorial nature, relating the analysis of the mean-field model to the scaling limit of lattice embeddings of abstract trees.

The Taylor series at the origin of the Lambert function is $W(w) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} w^n$, and hence

$$f(z) = -\frac{1}{2d} W(-2dz) = \sum_{n=1}^{\infty} \frac{(2dn)^{n-1}}{n!} z^n. \tag{4.25}$$

The funtion $f(z)$ is thus essentially the exponential generating function of rooted labelled trees [20, (1.7.6)], and we would like to argue that the analysis of the mean-field model captures an essential feature relevant to the scaling limit of embeddings of abstract trees into the lattice.

This claim is based on the fact that the coefficient of z^n in (4.25) is given asymptotically by

$$\frac{(2dn)^{n-1}}{n!} \sim \frac{e}{\sqrt{2\pi}} (2de)^{n-1} \frac{1}{n^{3/2}}. \tag{4.26}$$

The power law $n^{-3/2}$ corresponds to the square root singularity of $f(z)$, which in turn led to (4.9) and hence to ISE. In terms of critical exponents, the power law $n^{-3/2}$ corresponds to the critical exponent value $\theta = \frac{5}{2}$, which in turn corresponds to $\gamma = \frac{1}{2}$. This reiterates the theme of Section 2.3, relating critical exponents and ISE.

Lattice embeddings of abstract trees typically also give rise to this power law $n^{-3/2}$. For example, the number of rooted unlabelled trees with n vertices is given [20, (9.5.29)] asymptotically by $c_1 c_2^n n^{-3/2}$, where c_1 and c_2 are positive constants. Since each edge of an abstract tree can be embedded in $2d$ ways, the number of lattice embeddings is therefore asymptotic to

$$c_1 (2d)^{n-1} c_2^n \frac{1}{n^{3/2}}. \tag{4.27}$$

As another example of lattice embeddings of abstract trees that gives rise to the power law $n^{-3/2}$, consider the number of embeddings of planted planar trees with $2n$ vertices, each of degree 1 or 3 (these trees are discussed in [12, Section 2.7.2]). The number of such trees is given by the Catalan number $(n+1)^{-1}\binom{2n}{n}$, and hence the number of lattice embeddings is asymptotic to

$$\frac{1}{\sqrt{\pi}} 2^{2n} (2d)^{2n-1} \frac{1}{n^{3/2}}. \tag{4.28}$$

The difference in the rates of exponential growth of (4.26), (4.27) and (4.28) serves only to give different values of the critical point z_0 and does not play an important role. However, the common power law $n^{-3/2}$, which is well-known to combinatorialists [28], corresponds to the square-root nature of the singularity of $f(z)$ which plays an essential role in the identification of the limit as ISE.

Acknowledgements

G.S. thanks the Canadian Mathematical Society for its invitation to present the 1995 Coxeter–James Lecture and to write this article. We are grateful to Takashi Hara for many conversations and for his contribution to Section 2.3.2. We thank David Aldous, Jean-François Le Gall and Ed Perkins for enlightening correspondence concerning ISE, and Buks van Rensburg for providing Figure 1. E.D. thanks Sichun Wang for several helpful discussions. This work was supported in part by NSERC grant OGP0009351.

References

[1] M. Aizenman. On the number of incipient infinite spanning clusters. Preprint, (1996).

[2] M. Aizenman and C.M. Newman. Tree graph inequalities and critical behaviour in percolation models. *J. Stat. Phys.*, **36**:107–143, (1984).

[3] D. Aldous. The continuum random tree. I. *Ann. Probab.*, **19**:1–28, (1991).

[4] D. Aldous. The continuum random tree II: an overview. In M.T. Barlow and N.H. Bingham, editors, *Stochastic Analysis*, pages 23–70. Cambridge University Press, Cambridge, (1991).

[5] D. Aldous. The continuum random tree III. *Ann. Probab.*, **21**:248–289, (1993).

[6] D. Aldous. Tree-based models for random distribution of mass. *J. Stat. Phys.*, **73**:625–641, (1993).

[7] A. Bovier, J. Fröhlich, and U. Glaus. Branched polymers and dimensional reduction. In K. Osterwalder and R. Stora, editors, *Critical Phenomena, Random Systems, Gauge Theories*, Amsterdam, (1986). North-Holland. Les Houches 1984.

[8] D.C. Brydges and T. Spencer. Self-avoiding walk in 5 or more dimensions. *Commun. Math. Phys.*, **97**:125–148, (1985).

[9] D.A. Dawson, I. Iscoe, and E.A. Perkins. Super-Brownian motion: path properties and hitting probabilities. *Probab. Th. Rel. Fields*, **83**:135–205, (1989).

[10] E. Derbez. *The scaling limit of lattice trees above eight dimensions.* PhD thesis, McMaster University, (1996).

[11] E. Derbez and G. Slade. The scaling limit of lattice trees in high dimensions. Preprint.

[12] I.P. Goulden and D.M. Jackson. *Combinatorial Enumeration.* John Wiley and Sons, New York, (1983).

[13] I.S. Gradshteyn and I.M. Ryzhik. *Table of Integrals, Series and Products.* Academic Press, New York, 4th edition, (1965).

[14] G. Grimmett. *Percolation.* Springer, Berlin, (1989).

[15] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. *Commun. Math. Phys.*, **128**:333–391, (1990).

[16] T. Hara and G. Slade. On the upper critical dimension of lattice trees and lattice animals. *J. Stat. Phys.*, **59**:1469–1510, (1990).

[17] T. Hara and G. Slade. The lace expansion for self-avoiding walk in five or more dimensions. *Reviews in Math. Phys.*, **4**:235–327, (1992).

[18] T. Hara and G. Slade. The number and size of branched polymers in high dimensions. *J. Stat. Phys.*, **67**:1009–1038, (1992).

[19] T. Hara and G. Slade. Self-avoiding walk in five or more dimensions. I. The critical behaviour. *Commun. Math. Phys.*, **147**:101–136, (1992).

[20] F. Harary and E.M. Palmer. *Graphical Enumeration.* Academic Press, New York, (1973).

[21] E. J. Janse van Rensburg. On the number of trees in $\mathbf{Z}^d$. *J. Phys. A: Math. Gen.*, **25**:3523–3528, (1992).

[22] E. J. Janse van Rensburg and N. Madras. A non-local Monte–Carlo algorithm for lattice trees. *J. Phys. A: Math. Gen.*, **25**:303–333, (1992).

[23] D.J. Klein. Rigorous results for branched polymer models with excluded volume. *J. Chem. Phys.*, **75**:5186–5189, (1981).

[24] J.-F. Le Gall. The uniform random tree in a Brownian excursion. *Probab. Theory Relat. Fields*, **96**:369–383, (1993).

[25] T.C. Lubensky and J. Isaacson. Statistics of lattice animals and dilute branched polymers. *Phys. Rev.*, **A20**:2130–2146, (1979).

[26] N. Madras. A rigorous bound on the critical exponent for the number of lattice trees, animals and polygons. *J. Stat. Phys.*, **78**:681–699, (1995).

[27] N. Madras and G. Slade. *The Self-Avoiding Walk.* Birkhäuser, Boston, (1993).

[28] A. Meir and J.W. Moon. The asymptotic behaviour of coefficients of powers of certain generating functions. *Europ. J. Combinatorics*, **11**:581–587, (1990).

[29] B.G. Nguyen and W-S. Yang. Triangle condition for oriented percolation in high dimensions. *Ann. Probab.*, **21**:1809–1844, (1993).

[30] B.G. Nguyen and W-S. Yang. Gaussian limit for critical oriented percolation in high dimensions. *J. Stat. Phys.*, **78**:841–876, (1995).

[31] H. Tasaki and T. Hara. Critical behaviour in a system of branched polymers. *Prog. Theor. Phys. Suppl.*, **92**:14–25, (1987).