A Limit Evolution Problem for Time-Dependent Point Interactions

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Abstract

We study the diffusion in R^3 of a particle interacting with N fixed points through point interactions whose strength varies in time.
Under mild assumptions on the time dependence of the strengths we prove existence for all times and uniqueness of the solution, for which we provide a rather explicit expression. We also prove that, under a suitable rescaling of the interaction strengths, the solution converges, when $N \to \infty$, to the solution of a diffusion equation with a regular killing term (potential).
We use properties of the local self-adjoint extensions of the laplacian, and results from the theory of fractional integrals and derivatives.

1. Introduction and results

In recent years singular perturbations of selfadjoint differential operators have attracted increasing attention in Applied Physics as well as in Pure Mathematics. Models of "small" obstacles for scattering of waves or particles, faults in rock fracture dynamics, etc. are examples of situations where one is naturally led to consider second (or higher) order partial differential operators on some domain of R^d, perturbed on manifolds of dimension lower than d.

Perturbations of the Laplacian supported on a discrete set of points appeared first in Theoretical Physics. During the thirties, zero range potentials were introduced in the attempt to find a realistic model for the interaction of particles in nuclei ([1]). For several decades such kind of perturbations of the Laplacian have been used in the form of a computational scheme giving rules to evaluate eigenvalues and scattering data. A rigorous definition, based on the theory of selfadjoint extensions of symmetric operators in Hilbert spaces, was made available only in the sixties ([2]).

In the following we will consider a particular class of selfajoint extensions of the Laplacian restricted to $C_0^\infty(R^d \backslash Y)$, where $Y = \{y_1, y_2, ..., y_N\}$ is a set of N points of R^d. We will recall the definition of this class of extensions (often referred to as the "local" ones) and list some of their spectral properties in the case $d = 3$. For more details and relevant applications in non relativistic quantum mechanics see [3].

For any collection of N real numbers $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in R$, $i = 1, \ldots, N$ let us define the linear operator $\Delta_{\alpha,Y}$ on $L^2(R^3)$ as follows

$$D(\Delta_{\alpha,Y}) = \{u \in L^2(R^3) \mid u = \phi^\lambda + \sum_{i=1}^N q_i G^\lambda(\cdot - y_i),$$

$$\phi^\lambda \in H^2(R^3),\ \phi^\lambda(y_i) = \sum_{j=1}^N (\Gamma^\lambda_{\alpha,Y})_{ij} q_j\}\ , \tag{1.1}$$

$$(\Delta_{\alpha,Y} - \lambda)u = (\Delta - \lambda)\phi^\lambda\ . \tag{1.2}$$

We have introduced the following notation

$$G^\lambda(x - x') = (-\Delta + \lambda)^{-1}(x - x') = \frac{e^{-\sqrt{\lambda}|x-x'|}}{4\pi|x - x'|},\ \ \lambda > 0, \tag{1.3}$$

$$(\Gamma^{\lambda}_{\alpha,Y})_{ij} = \left(\alpha_i + \frac{\sqrt{\lambda}}{4\pi}\right)\delta_{ij} - \tilde{G}^{\lambda}_{ij}, \tag{1.4}$$

$$\tilde{G}^{\lambda}_{ii} = 0,\ \forall i;\ \tilde{G}^{\lambda}_{ij} = G^{\lambda}(y_i - y_j),\ \forall i \neq j. \tag{1.5}$$

$H^n(R^3)$ is the standard Sobolev space of order n and G^{λ} will denote the integral operator with kernel $G^{\lambda}(x - x')$.
Observe that a function u in the operator domain (1.1) consists of a "regular part" ϕ^{λ} plus the "potential" of the point charges q_i. The last equality in (1.1) relating q_i and ϕ^{λ} expresses the boundary condition satisfied by u at Y. The parameter λ is chosen large enough to guarantee the existence of the inverse of the matrix $\Gamma^{\lambda}_{\alpha,Y}$. In the sequel we shall drop the dependence on λ of the regular part ϕ^{λ} of u.

We briefly recall some properties of the linear operator $\Delta_{\alpha,Y}$ defined above.

$\Delta_{\alpha,Y}$ is a selfadjoint extension of Δ restricted to $C_0^{\infty}(R^3 \setminus Y)$. It is in particular a local extension, in the sense that the boundary condition satisfied by $u \in D(\Delta_{\alpha,Y})$ at each point y_i depends only on the behaviour of u in a neighborhood of y_i.

The spectrum of $-\Delta_{\alpha,Y}$ consists of an absolutely continous part, always coinciding with $[0, \infty)$, and at most N negative eigenvalues. A real negative number $-\lambda_0,\ \lambda_0 > 0$, is an eigenvalue of $-\Delta_{\alpha,Y}$ if and only if $det(\Gamma^{\lambda_0}_{\alpha,Y}) = 0$; its multiplicity is the same as the multiplicity of the eigenvalue 0 of the matrix $(\Gamma^{\lambda_0}_{\alpha,Y})$. The corresponding eigenfunctions of $-\Delta_{\alpha,Y}$ are explicitly computable in terms of the eigenvectors of $(\Gamma^{\lambda_0}_{\alpha,Y})$ corresponding to the eigenvalue 0.

The parameters α_i give a measure of the strength of the interaction (for $\alpha_i = \pm\infty$ one has the free laplacian) and are related to the scattering length associated to $-\Delta_{\alpha,Y}$. In particular for $N = 1$ the scattering length is $-1/4\pi\alpha$.

Notice that each selfadjoint extension in the class is characterized by a minimal set of parameters: the strength parameters α_i and the position y_i of the points. This particularly easy setting makes point interactions suitable for a generalization to time dependent models, where coupling constants and/or the geometry of the perturbation is changing with time. The actual time dependence might be a priori known or given implicitly, depending on the model one is interested in.

We want to mention briefly two subjects, in the framework of time dependent point interactions, which we consider worth being considered in further work:

_ Mobile points : point interactions with support changing in time have direct applications in different fields of Applied Physics such as diffusions in randomly changing environment, scattering of neutrons by liquids and many others. On a more general level, any selfconsistent description of the dynamics of a tagged particle inside a system of a great number of identical interacting particles requires, as a first step, the analysis of time dependent interactions.

_ Non linear models : here the parameters of the interaction "are coupled to the dynamics under consideration", in the sense that their value at time t depends on properties of the solution u.

We shall study the evolution problem

$$\frac{\partial u(t)}{\partial t} = \Delta_{\alpha(t),Y} u(t) \quad t \in [0,T] \tag{1.6}$$

$$u(0) = u^0 \in D(\Delta_{\alpha(0),Y}) \ , \tag{1.7}$$

where now $\alpha(t) = (\alpha_1(t), \ldots, \alpha_N(t))$ is a given function defined on $[0,T]$ with values in R^N.

General evolution problems for linear parabolic equations with a generator explicitely depending on time is a subject widely studied in Mathematics. For recent results and an extensive bibliography on the subject see ([4],[5],[6],[7]).

Depending on the assumptions on $\alpha(t)$, we shall prove two existence and uniqueness results for (1.6), (1.7). For $0 < \mu < 1$ we consider the following function spaces

$$C^\mu([0,T]) = \left\{ f \ : \ [0,T] \to R \mid \sup_{t,s \in [0,T]} \frac{|f(t) - f(s)|}{|t-s|^\mu} < c \right\}, \tag{1.8}$$

$$B^\mu([0,T]) = \left\{ f \ : \ [0,T] \to R \mid \left(\int_h^T dt |f(t-h) - f(t)|^2 \right)^{1/2} = O(h^\mu) \right\}. \tag{1.9}$$

(Here and in the following c will denote a generic positive constant).
Note that the space B^μ consists of functions satisfying a sort of integrated Hölder condition of order μ. It has been studied in [8] in connection with the theory of fractional integrals and derivatives (see section 2).
Then we have

Theorem 1. Assume that $\alpha_i \in C^\eta([0,T])$, $\eta > 1/4$, $1,\dots,N$. Then there exists a unique $u \in C^1([0,T], L^2(R^3)) \cap C([0,T], D(\Delta_{\alpha,Y}))$ which solves problem (1.6), (1.7). Moreover u is given by

$$u(t) = P_t u^0 + \sum_{i=1}^N \int_0^t ds\ P_{t-s}(\cdot - y_i) q_i(s)\ , \tag{1.10}$$

where the charges $q_i(t)$ satisfy

$$\frac{q_i(t)}{4\sqrt{\pi}} + \int_0^t ds \frac{\alpha_i(s) q_i(s)}{\sqrt{t-s}} - \sum_{j=1, i\neq j}^N \frac{1}{4\pi|y_i - y_j|} \int_0^t ds \frac{\exp\left[-\frac{|y_i-y_j|^2}{4(t-s)}\right]}{\sqrt{t-s}} q_j(s) = f_i(t) \tag{1.11}$$

$$f_i(t) = \int_0^t ds \frac{P_s \phi^0(y_i)}{\sqrt{t-s}} + \frac{q_i^0}{4\sqrt{\pi}} \exp(\lambda t)\ \mathrm{erfc}\sqrt{\lambda t} + \sum_{j=1, j\neq i}^N q_j^0 \int_0^t ds \frac{k_{ij}^\lambda(s)}{\sqrt{t-s}}\ . \tag{1.12}$$

We have denoted

$$P_t(x-x') = \exp(\Delta t)(x-x') = \frac{\exp\left(-\frac{|x-x'|^2}{4t}\right)}{(4\pi t)^{3/2}}, \tag{1.13}$$

P_t is the corresponding integral operator, $\mathrm{erfc}(\cdot)$ is the complementary error function ([9]), ϕ^0 and q_i^0 are the regular part and the charges associated to u^0 and

$$k^\lambda(x-y,t) = \int_{R^3} dz P_t(x-z) G^\lambda(z-y), \quad k_{ij}^\lambda(t) = k^\lambda(y_i - y_j, t), \quad i \neq j. \tag{1.14}$$

The function u is called a strict solution.
Under the same hypotheses on α_i existence and uniqueness of the strict solution can also be obtained applying the general criterion given in [6],[7]. Nevertheless, our method leads to a simple representation of the solution, i.e. one is reduced to study the system (1.11), (1.12) of N Volterra integral equations with weakly singular kernels, which is easily solved by iteration. Moreover, it is also clear from our analysis that the assumption on α_i is only sufficient. In particular one can easily see that Theorem 1 still holds for $\alpha_i = t^{1/4}$.
The same problem for the Schrödinger equation has been solved in [10], [11], under the assumption $\alpha \in C^2$, in the context of scattering theory for time-dependent hamiltonians.
Following the same line of the proof of Theorem 1 we also have

Theorem 1'. Assume that $\alpha_i \in B^\eta([0,T]) \cap C^0([0,T])$, $\eta > 1/4$, $i = 1,\ldots,N$. Then there exists a unique $u \in C^0([0,T], L^2(R^3)) \cap L^2([0,T], D(\Delta_{\alpha,Y}))$, with $du/dt \in L^2([0,T], L^2(R^3))$ which satisfies (1.6), (1.7). Obviously, the same representation (1.10), (1.11), (1.12) holds.

In this case the function u is called a weak solution of (1.6), (1.7).

We shall use the representation we have given for the solution to prove a reconstruction theorem for time-dependent potentials. In particular we shall prove that for N going to infinity and under the scaling $\alpha_i \to N\alpha_i$ the solution of the evolution problem (1.6), (1.7) converges to the solution of a parabolic evolution problem with a smooth generator $\Delta + V$, with V time-dependent and positive.

We introduce the following assumptions:

(a) let $Y^N = \{y_1, \ldots, y_N\}$ be a set of N independent and identically distributed random variables in R^3 with density $\rho \geq 0$, $\rho \in L^1(R^3) \cap L^2(R^3)$;

(b) let $\alpha : [0,T] \to C^0(R^3)$, with $\alpha(t) = \alpha(\cdot, t)$, $\alpha^0 = \alpha(\cdot, 0)$, such that

$$\sup_{x\in R^3} \left(\int_h^T dt |\alpha(x, t-h) - \alpha(x,t)|^2 \right)^{1/2} = O(h^\nu) \,, \quad \nu > 1/2; \tag{1.15}$$

(c) $\qquad \alpha(x,t) > \beta > 0 \quad \forall (x,t) \in R^3 \times [0,T]$.

If we denote

$$\alpha^N(t) = (\alpha_1(t), \ldots, \alpha_N(t)), \quad \alpha_i^N(t) = \alpha(y_i, t) \tag{1.16}$$

and fix

$$u^{0,N} \in D(\Delta_{N\alpha^N(0),Y^N}) \tag{1.17}$$

then there exists the weak solution $u^N(t)$ of

$$\frac{\partial u^N(t)}{\partial t} = \Delta_{N\alpha^N(t),Y^N} u^N(t) \tag{1.18}$$

$$u^N(0) = u^{0,N} \ . \tag{1.19}$$

We assume that the regular part of $u^{0,N}$ converges in probability, i.e.

(d) there exists $\phi^0 \in H^2(R^3)$ such that $\forall \epsilon > 0$

$$\lim_{N\to\infty} P^N \left(\{ Y^N \mid \|\phi^{0,N} - \phi^0\|^2_{H^2(R^3)} > \epsilon \} \right) = 0,$$

where P^N indicates the probability measure $\{\rho dx\}^N$ on $(R^3)^N$.

Our second main result is expressed by

Theorem 2. Under the assumptions (a),...,(d) and $\forall \epsilon > 0$ one has

$$\lim_{N\to\infty} P^N \left(\left\{ Y^N \mid \int_0^T dt \, \|u^N(t) - u(t)\|^2_{L^2(R^3)} > \epsilon \right\} \right) = 0, \tag{1.20}$$

where $u(t)$ solves the evolution problem

$$\frac{\partial u(t)}{\partial t} = \Delta u(t) + \frac{\rho}{\alpha(t)} u(t) \tag{1.21}$$

$$u(0) = u^0, \tag{1.22}$$

and

$$u^0 = \alpha^0 Q^0 \tag{1.23}$$

$$\alpha^0 Q^0 = \phi^0 + G^\lambda \rho Q^0 \ . \tag{1.24}$$

Theorem 2 can be considered as an extension of a previous result ([12]) on the resolvent convergence of $-\Delta_{N\alpha^N,Y^N}$ in the case of time-independent strength parameters α_i^N.
Notice that in Theorem 2 we restrict to the case α strictly positive . The case α negative seems more delicate due to the asymptotic properties for N large of the spectrum of $-\Delta_{N\alpha^N,Y^N}$. Roughly speaking for N large the eigenvalue equation $det(\Gamma^\lambda_{N\alpha^N,Y^N}) = 0$ (see (1.4)) exhibits exactly N solutions going to infinity for $N \to \infty$.
This means that, for $N \to \infty$, there are N negative eigenvalues of $-\Delta_{N\alpha^N,Y^N}$ with absolute value going to $+\infty$; the semigroup is not bounded uniformly in N and this prevents a direct proof of convergence of the solution of the evolution problem.
In the time-independent case studied in [12] the situation is different. One can exploit the fact that, for N large, one can find a gap, uniform in N, in the spectrum of $-\Delta_{N\alpha^N,Y^N}$, making it possible to have the strong resolvent convergence result also in the case of α negative.

The paper is organized as follows:
In Section 2 we recall some results on fractional integrals and derivatives which we use to solve the equation (1.11), (1.12) for the charges q_i.
In Section 3 we prove the existence and uniqueness results expressed by Theorem 1 and 1'.
In Section 4 we analyse the asymptotics of the solutions of (1.6), (1.7) for N large and we prove Theorem 2.

2. The equation for the charges

In this section we discuss the solvability of eq (1.11), (1.12) under different assumptions on $\alpha_i(t)$.
First we introduce some notation and list known results on fractional integrals and derivatives which will be used in the sequel (for a comprehensive treatment see [8],[13],[14],[15],[16]).
For a real function f defined on the interval $[0,T]$ and sufficiently well behaved, we define the fractional integral of f of order μ, $0<\mu<1$, as

$$I_\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t ds \frac{f(s)}{(t-s)^{1-\mu}} \; , \tag{2.1}$$

where Γ denotes Euler's gamma function. I_μ is also called Abel operator of order μ. For $\mu = 1$ expression (2.1) reduces to the ordinary integral of f. Moreover for $0<\mu<1$, $0<\gamma<1-\mu$

$$I_\mu(I_\gamma f) = I_{\mu+\gamma} f \tag{2.2}$$

and, for f continuous

$$\lim_{\mu\to 0} I_\mu f(t) = f(t), \quad \forall t \in [0,T]. \tag{2.3}$$

The Abel operator has nice smoothing properties. It is bounded in $L^2([0,T])$ ([15]) and its action in the spaces C^γ and B^γ can be characterized as follows ([8],[15])

Proposition 2.1. If $f \in C^\gamma([0,T])$, $f(0)=0$, and $0<\mu<1$, $0\le\gamma<1-\mu$, then $I_\mu f \in C^{\mu+\gamma}([0,T])$.

Proposition 2.2. If $f \in B^\gamma([0,T])$, $0<\mu<1$, $0\le\gamma<1-\mu$, then $I_\mu f \in B^{\gamma+\mu}([0,T])$.

In analogy with ordinary differential calculus, one can define the fractional derivative of f of order μ, $0<\mu<1$, as

$$D^\mu f(t) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_0^t ds \frac{f(s)}{(t-s)^\mu}. \tag{2.4}$$

For a differentiable f, using the fundamental theorem of the calculus and (2.3), it is easy to see that expression (2.4) reduces to the ordinary derivative of f for $\mu \to 1$.
The space of functions having continuous (resp. square-integrable) fractional derivative will be denoted by

$$\mathcal{K}^\mu([0,T]) = \left\{f \;:\; [0,T] \to R \mid D^\mu f \in C^0([0,T])\right\}, \tag{2.5}$$

$$\mathcal{H}^\mu([0,T]) = \left\{f \;:\; [0,T] \to R \mid D^\mu f \in L^2([0,T])\right\}. \tag{2.6}$$

One of the peculiarity of fractional derivatives is that Leibnitz's rule doesn't hold. For $0 < \mu < 1$, a straightforward computation gives

$$\begin{aligned} & D^\mu(fg)(t) \\ & = g(t)\frac{1}{\Gamma(1-\mu)}\lim_{\delta\to 0}\frac{1}{\delta}\left[\int_0^{t+\delta} ds\frac{f(s)}{(t+\delta-s)^\mu} - \int_0^t ds\frac{f(s)}{(t-s)^\mu}\right] \\ & +\frac{1}{\Gamma(1-\mu)}\lim_{\delta\to 0}\frac{1}{\delta}\left[\int_0^{t+\delta} ds\frac{f(s)\left(g(s)-g(t)\right)}{(t+\delta-s)^\mu} - \int_0^t ds\frac{f(s)\left(g(s)-g(t)\right)}{(t-s)^\mu}\right] \\ & = g(t)D^\mu f(t) + \frac{\mu}{\Gamma(1-\mu)}\int_0^t ds f(s)\frac{g(t)-g(s)}{(t-s)^{1+\mu}}. \end{aligned} \tag{2.7}$$

In fact it is easy to see that formula (2.7) holds pointwise if $f \in \mathcal{K}^\mu([0,T])$ and $g \in C^\gamma([0,T])$, $\gamma > \mu$, and in the L^2 sense if $f, g \in C^0([0,T])$, $f \in \mathcal{H}^\mu([0,T])$ and $g \in B^\gamma([0,T])$, $\gamma > \mu$.
Moreover, it reduces to Leibnitz's rule for $\mu \to 1$ if f and g are differentiable.
A symmetrized version of the above formula can also be easily derived

$$\begin{aligned} D^\mu(fg)(t) = f(t)D^\mu g(t) + g(t)D^\mu f(t) - \frac{f(t)g(t)}{(1-\mu)\Gamma(1-\mu)t^\mu} \\ -\frac{\mu}{\Gamma(1-\mu)}\int_0^t ds\frac{\left(f(s)-f(t)\right)\left(g(s)-g(t)\right)}{(t-s)^{1+\mu}}. \end{aligned}$$

The following propositions give useful sufficient conditions for the existence of fractional derivatives ([8],[15])

Proposition 2.3. If $f \in C^{\gamma}([0,T])$, $0<\gamma<1$, $f(0)=0$ and $0<\mu<\gamma$, then $f \in \mathcal{K}^{\mu}([0,T])$ and $f(t) = I_{\mu}(D^{\mu} f)(t)$.

Proposition 2.4. If $f \in B^{\gamma}([0,T])$, $0<\gamma<1$, and $0<\mu<\gamma$, then $f \in \mathcal{H}^{\mu}([0,T])$ and $f(t) = I_{\mu}(D^{\mu} f)(t)$ a.e..

Finally we shall use the smoothing properties of the Abel operator in the spaces $\mathcal{K}^{\mu}$, $\mathcal{H}^{\mu}$ ([8],[15],[16])

Proposition 2.5. Fix $0 \le \mu < 1$, $0<\gamma<1-\mu$. If $f \in \mathcal{K}^{\mu}([0,T])$ then $I_{\gamma} f \in \mathcal{K}^{\mu+\gamma}([0,T])$. If $f \in \mathcal{H}^{\mu}([0,T])$ then $I_{\gamma} f \in \mathcal{H}^{\mu+\gamma}([0,T])$.

Consider now eq. (1.11) for the charges $q_i(t)$; we distinguish a diagonal term (the second term in the l.h.s.) given by an integral operator with a weakly singular kernel and a non diagonal term (the third term in the l.h.s.) given by an integral operator with a smooth integral kernel.

In general, for an integral equation with a weakly singular kernel of the type (1.11), the solution exhibits a singular behaviour in the derivative at the origin, even if the datum f_i is very smooth. This is not the case for eq. (1.11) because of a specific compensation between the diagonal term and the datum. In order to see this, it is convenient to rewrite the datum f_i as follows

$$f_i(t) = \int_0^t ds \frac{P_s \phi^0(y_i) - \phi^0(y_i)}{\sqrt{t-s}} + \frac{q_i^0}{4\sqrt{\pi}} \left[\exp(\lambda t)\ \mathrm{erfc}(\sqrt{\lambda t}) + \frac{2}{\sqrt{\pi}} \sqrt{\lambda t} \right]$$
$$+2\sqrt{t}\ \alpha_i(0) q_i^0 + \sum_{j=1, j\neq i}^{N} q_j^0 \int_0^t ds \frac{k_{ij}^{\lambda}(s) - k_{ij}^{\lambda}(0)}{\sqrt{t-s}} \ , \tag{2.8}$$

where we have used the boundary condition at Y satisfied by a function in the operator domain $D(\Delta_{\alpha(0),Y})$ (see (1.1)) and the identity

$$G^{\lambda}(y_i - y_j) = k_{ij}^{\lambda}(0)\ , \quad \forall i \neq j. \tag{2.9}$$

The equation satisfied by the difference $w_i = q_i - q_i^0$ is the same as (1.11) but with a different datum

$$\frac{w_i(t)}{4\sqrt{\pi}}+\int_0^t ds\frac{\alpha_i(s)w_i(s)}{\sqrt{t-s}}-\sum_{j=1,j\neq i}^N \frac{1}{4\pi|y_i-y_j|}\int_0^t ds\frac{\exp\left[-\frac{|y_i-y_j|^2}{4(t-s)}\right]}{\sqrt{t-s}}w_j(s)=g_i(t) \tag{2.10}$$

$$\begin{aligned} g_i(t) &= \int_0^t ds\frac{P_s\phi^0(y_i)-\phi^0(y_i)}{\sqrt{t-s}}+\frac{q_i^0}{4\sqrt{\pi}}\left[\exp(\lambda t)\ \mathrm{erfc}(\sqrt{\lambda t})-1+\frac{2}{\sqrt{\pi}}\sqrt{\lambda t}\right] \\ &+q_i^0\int_0^t ds\frac{\alpha_i(0)-\alpha_i(s)}{\sqrt{t-s}}+\sum_{j=1,j\neq i}^N q_j^0\int_0^t ds\frac{k_{ij}^\lambda(s)-k_{ij}^\lambda(0)}{\sqrt{t-s}} \\ &+\sum_{j=1,j\neq i}^N \frac{q_j^0}{4\pi|y_i-y_j|}\int_0^t ds\frac{\exp\left[-\frac{|y_i-y_j|^2}{4(t-s)}\right]}{\sqrt{t-s}}\ . \end{aligned} \tag{2.11}$$

Obviously, solving eq. (1.11), (1.12) is equivalent to solving eq. (2.10), (2.11). For α_i continuous, it is well known that eq. (2.10), (2.11) has a unique continuous solution, which can be constructed by iteration.
We are interested in solutions w_i sufficiently regular in order to construct the solution of (1.6), (1.7). In particular we shall need functions satisfying the condition

$$\sup_{t\in[0,T]}\int_0^t d\tau\frac{|\xi_i(t-\tau)-\xi_i(t)|^2}{\tau^{5/2}}<\infty \tag{2.12}$$

or

$$\int_0^T dt\int_0^t d\tau\frac{|\xi_i(t-\tau)-\xi_i(t)|^2}{\tau^{5/2}}<\infty\ . \tag{2.13}$$

Imposing additional assumptions on α_i one has

Lemma 2.1. If $\alpha_i\in C^\eta([0,T])$, $\eta>1/4$, $i=1,\dots,N$ then the solution w_i of eq. (2.10),(2.11) belongs to $C^{3/4}([0,T])\cap\mathcal{K}^{3/4}([0,T])$ and satisfies condition (2.12).
If $\alpha_i\in B^\eta([0,T])\cap C^0([0,T])$, $\eta>1/4$, $i=1,\dots,N$, then the solution w_i of eq. (2.10),(2.11) belongs to $B^{1/2+\eta}([0,T])\cap C^{1/2}([0,T])$ and satisfies condition (2.13).

Proof. First we assume $\alpha_i \in C^\eta([0,T])$, $\eta > 1/4$, $1,\ldots,N$ and establish the class of regularity of the datum g_i.
Concerning the first term in (2.11), using Schwartz's inequality and a rescaling of the integration variables, we have

$$
\begin{aligned}
&\Big|P_t\phi^0(y_i) - P_{t+h}\phi^0(y_i)\Big| \\
&\leq c\int_{R^3} dk|\hat{\phi}^0(k)|\exp(-k^2t)\left[1-\exp(-k^2h)\right] \\
&\leq c\|\Delta\phi^0\|_{L^2(R^3)}\left[\int_{R^3} dk\frac{\exp(-2k^2t)}{k^4}\left(1-\exp(-k^2h)\right)^2\right]^{1/2} \\
&\leq c\|\Delta\phi^0\|_{L^2(R^3)}h^{1/4}\left[\int_0^\infty dv\frac{\exp(-2v^2\frac{t}{h})}{v^2}\left(1-\exp(-v^2)\right)^2\right]^{1/2} \\
&\leq \|\Delta\phi^0\|_{L^2(R^3)}h^{1/4},
\end{aligned}
\tag{2.14}
$$

where we have denoted by $\hat{\xi}$ the Fourier transform of ξ.
Using the convolution property of the Laplace transform we also have

$$
\begin{aligned}
&\Big|D^{1/4}\left(P_t\phi^0(y_i) - \phi^0(y_i)\right)\Big| \\
&= \left|\frac{1}{\Gamma(3/4)}\frac{d}{dt}\int_0^t ds\frac{1}{(t-s)^{1/4}}\left(P_s\phi^0(y_i)-\phi^0(y_i)\right)\right| \\
&\leq c\int_{R^3} dk|\hat{\phi}^0(k)|k^2\int_0^t ds\frac{\exp(-k^2s)}{(t-s)^{1/4}} \\
&\leq c\|\Delta\phi^0\|_{L^2(R^3)}\left[\int_0^\infty dv v^2\left(\int_0^1 d\tau\frac{\exp(-v^2\tau)}{(1-\tau)^{1/4}}\right)^2\right]^{1/2} \\
&\leq c\|\Delta\phi^0\|_{L^2(R^3)}.
\end{aligned}
\tag{2.15}
$$

¿From (2.14),(2.15) and Propositions 2.1, 2.5 the first term in (2.11) belongs to $C^{3/4}([0,T]) \cap \mathcal{K}^{3/4}([0,T])$.
Now we show that it also satisfies condition (2.12) (and then obviously (2. 13)). A rescaling of the integration variables yields

$$
\int_0^t \frac{d\tau}{\tau^{5/2}}\left(\int_0^{t-\tau} ds\frac{P_s\phi^0(y_i)-\phi^0(y_i)}{\sqrt{t-\tau-s}} - \int_0^t ds\frac{P_s\phi^0(y_i)-\phi^0(y_i)}{\sqrt{t-s}}\right)^2
$$

$$
\leq 2\int_0^t \frac{d\tau}{\tau^{5/2}}(t-\tau)\left(\int_0^1 dz \frac{P_{(t-\tau)z}\phi^0(y_i) - P_{tz}\phi^0(y_i)}{\sqrt{1-z}}\right)^2
$$

$$
+2\int_0^t \frac{d\tau}{\tau^{5/2}}\left(\sqrt{t}-\sqrt{t-\tau}\right)^2\left(\int_0^1 dz \frac{P_{tz}\phi^0(y_i) - \phi^0(y_i)}{\sqrt{1-z}}\right)^2. \tag{2.16}
$$

Proceeding as in (2.14), the first term in the r.h.s. of (2.16) can be estimated by

$$
c\|\Delta\phi^0\|_{L^2(R^3)} \int_0^\infty dz \frac{(1-\exp(-z^2))}{z^2}\int_0^t d\tau \frac{t-\tau}{\tau^2}\exp(-2z^2\frac{t}{\tau})
$$

$$
= c\|\Delta\phi^0\|^2_{L^2(R^3)}\int_0^\infty dz \frac{(1-\exp(-z^2))^2}{z^4}\int_{z^2}^\infty dv\left(1-\frac{z^2}{v}\right)\exp(-2v)
$$

$$
\leq c\|\Delta\phi^0\|^2_{L^2(R^3)}; \tag{2.17}
$$

using (2.14), the second term in the r.h.s. of (2.16) is estimated by

$$
c\|\Delta\phi^0\|^2_{L^2(R^3)}\sqrt{t}\int_0^t \frac{d\tau}{\tau^{5/2}}\left(\sqrt{t}-\sqrt{t-\tau}\right)^2
$$

$$
c\|\Delta\phi^0\|^2_{L^2(R^3)}\int_0^1 \frac{d\sigma}{\sigma^{5/2}}\left(1-\sqrt{1-\sigma}\right)^2
$$

$$
\leq c\|\Delta\phi^0\|^2_{L^2(R^3)}. \tag{2.18}
$$

We can conclude that the first term in (2.11) satisfies condition (2.12).
Due to Propositions 2.1, 2.3, the term containing α_i in (2.11) belongs to $C^{1/2+\eta}([0,T]) \cap \mathcal{K}^{1/2+\nu}([0,T])$, for any $\nu < \eta$, and satisfies condition (2.12). The same is obviously true for the remaining terms in (2.11), which are in fact differentiable in $[0,T]$.
Then we have shown that the datum g_i belongs to $C^{3/4}([0,T]) \cap \mathcal{K}^{3/4}([0,T])$ and satisfies (2.12).
Now consider eq. (2.10). Using Propositions 2.1, 2.3 and the smoothing property of the non diagonal term we easily find that w_i belongs to $C^{3/4}([0,T]) \cap \mathcal{K}^{3/4}([0,T])$ and satisfies (2.12).

Let us suppose $\alpha_i \in B^\eta([0,T]) \cap C^0([0,T])$. By Propositions 2.1, 2.3 the datum g_i belongs to $B^{1/2+\eta}([0,T]) \cap C^{1/2}([0,T])$ and satisfies condition (2.13). ¿From eq. (2.10) it is easily seen that the same is true for the solution w_i, concluding the proof of the lemma. □

3. Solution of the evolution problem

We turn next to the evolution problem (1.6), (1.7). Given q_i solution of eq. (1.11), (1.12) we shall prove that

$$u(t) = P_t u^0 + \sum_{i=1}^{N} \int_0^t ds P_{t-s}(\cdot - y_i) q_i(s) \tag{3.1}$$

solves problem (1.6), (1.7), in the strict or in the weak sense depending on the assumptions on α_i.
This goal will be obtained through some lemmas. The aim is to emphasize the specific assumptions that we need for each step of the proof. First we have

Lemma 3.1. Assume that $q_i \in C^0([0,T])$. Then $u \in C^0([0,T], L^2(R^3))$, $u(0) = u^0$ and $\lim_{x\to y_i} 4\pi|x-y_i|u(x,t) = q_i(t)$.

Proof. A change of variable in the integral in (3.1) yields

$$\int_0^t ds P_{t-s}(x-y_i) q_i(s) = \frac{1}{2\pi^{3/2}|x-y_i|} \int_{\frac{|x-y_i|}{2\sqrt{t}}}^{\infty} d\sigma\, e^{-\sigma^2} q_i\left(t - \frac{|x-y_i|}{4\sigma^2}\right). \tag{3.2}$$

Using (3.2) and the dominated convergence theorem, one immediately obtains the proof of the lemma. □

Define the regular part $\phi(t)$ of $u(t)$ as

$$\phi(t) = u(t) - \sum_{i=1}^{N} q_i(t) G^\lambda(\cdot - y_i) \ , \tag{3.3}$$

and denote $w_i = q_i - q_i^0$ as in the previous section. Then we have

Lemma 3.2. Assume that $w_i \in C^{3/4}([0,T])$ and satisfies condition (2.12). Then $\phi \in C^0([0,T], H^2(R^3))$.

Proof. The regular part of u can be written as follows

$$\phi(x,t) = (P_t\phi^0)(x) + \sum_{i=1}^N \int_0^t ds P_{t-s}(x-y_i) w_i(s) - \sum_{i=1}^N w_i(t) G^\lambda(x-y_i)$$
$$+ \sum_{i=1}^N q_i^0 \left[k^\lambda(x-y_i,t) + \int_0^t ds P_{t-s}(x-y_i) - G^\lambda(x-y_i) \right] . \tag{3.4}$$

In the r.h.s. of (3.4) the first term is clearly in $C^0([0,T], H^2(R^3))$. The second term can be analyzed using Fourier transform. If we denote

$$\zeta_i(x,t) = \int_0^t ds P_{t-s}(x-y_i) w_i(s) - w_i(t) G^\lambda(x-y_i) \tag{3.5}$$

then we have

$$k^2 \hat{\zeta}_i(k,t) = k^2 \int_0^t ds \exp\left[-k^2(t-s) + ik\cdot y_i\right] w_i(s) - \frac{k^2 \exp(ik\cdot y_i)}{k^2+\lambda} w_i(t)$$
$$= k^2 \exp(ik\cdot y_i) \int_0^t ds \exp\left[-k^2(t-s)\right] (w_i(s) - w_i(t))$$
$$- \exp(-k^2 t + ik\cdot y_i) w_i(t) + \frac{\lambda \exp(ik\cdot y_i)}{k^2+\lambda} w_i(t) \ . \tag{3.6}$$

Using Fubini's theorem, a rescaling of the integration variable and Schur's test for integral operators in L^2-spaces ([17]) we have

$$\int_{R^3} dk \left| k^2 \exp(ik\cdot y_i) \int_0^t ds \exp\left[-k^2(t-s)\right] (w_i(s) - w_i(t)) \right|^2$$
$$\leq c \int_0^t ds \int_0^t ds' |w_i(s) - w_i(t)| |w_i(s') - w_i(t)| \frac{1}{(2t-s-s')^{7/2}}$$
$$\leq c \left(\sup_{z\in[0,t]} z^{5/4} \int_0^t dz' \frac{z'^{5/4}}{(z+z')^{7/2}} \right) \int_0^t dz \frac{|w_i(t-z) - w_i(t)|^2}{z^{5/2}}$$
$$= c \left(\sup_{z\in[0,t]} \int_0^{t/z} d\tau \frac{\tau^{5/4}}{(1+\tau)^{7/2}} \right) \int_0^t dz \frac{|w_i(t-z) - w_i(t)|^2}{z^{5/2}}$$
$$\leq c \int_0^t dz \frac{|w_i(t-z) - w_i(t)|^2}{z^{5/2}} \ . \tag{3.7}$$

Moreover, explicit computations yield

$$\int_{R^3} dk \left|\exp(-k^2 t + ik \cdot y_i) w_i(t)\right|^2 \leq \frac{c}{t^{3/2}} |w_i(t)|^2 \, , \tag{3.8}$$

$$\int_{R^3} dk \left|\lambda \frac{\exp(ik \cdot y_i)}{k^2 + \lambda} w_i(t)\right|^2 \leq c \; \lambda^{3/2} |w_i(t)|^2 \, . \tag{3.9}$$

¿From (3.7), (3.8), (3.9) and (3.6) we conclude

$$\begin{aligned} &\int_{R^3} dk |k^2 \zeta_i(k,t)|^2 \\ &\leq c \left[\left(\lambda^{3/2} + \frac{1}{t^{3/2}} \right) |w_i(t)|^2 + \int_0^t dz \frac{|w_i(t-z) - w_i(t)|^2}{z^{5/2}} \right] . \end{aligned} \tag{3.10}$$

Finally we consider the last term in the r.h.s. of (3.4). Applying the laplacian and taking the Fourier transform for each term of the sum one has

$$\begin{aligned} &k^2 \int_0^t ds \exp\left[-k^2(t-s) + ik \cdot y_i\right] - \frac{k^2 \exp(ik \cdot y_i)}{k^2 + \lambda} + \frac{k^2 \exp(-k^2 t + ik \cdot y_i)}{k^2 + \lambda} \\ &= \frac{\lambda \exp(ik \cdot y_i)}{k^2 + \lambda} \left[1 - \exp(-k^2 t)\right] \, , \end{aligned} \tag{3.11}$$

which obviously belongs to $C^0([0,T], L^2(R^3))$. From (3.10), (3.11), we conclude the proof of the lemma. □

The following lemma shows that eq. (2.10), (2.11) for the charges implies the boundary condition at Y for $u(t)$ (see (1.1)).

Lemma 3.3. Assume that the solution w_i of eq. (2.10), (2.11) satisfies condition (2.12) and $w_i \in C^{3/4}([0,T])$. Then $u \in C^0([0,T], D(\Delta_{\alpha,Y}))$.

Proof. First we show that the boundary condition at Y is satisfied. Taking the fractional derivative of order 1/2 of eq. (2.10) and using the convolution property of the Laplace transform one easily gets

$$\frac{D^{1/2} w_i(t)}{4\pi} + \alpha_i(t) w_i(t) - \sum_{j=1, j \neq i}^{N} \int_0^t ds P_{t-s}(y_i - y_j) w_j(s) = \frac{D^{1/2} g_i(t)}{\sqrt{\pi}} . \tag{3.12}$$

The r.h.s. of (3.12) can also be explicitely computed using Laplace transform (see e.g. [18]). Taking into account the boundary condition at Y for $t=0$ and the identity $k^{\lambda}_{ij}(0)=G^{\lambda}(y_i-y_j)$, eq. (3.12) can be rewritten as follows

$$\frac{D^{1/2}w_i(t)}{4\pi}+\alpha_i(t)w_i(t)-\sum_{j=1,j\neq i}^{N}\int_0^t dsP_{t-s}(y_i-y_j)w_j(s)$$
$$=P_t\phi^0(y_i)-q_i^0\frac{\sqrt{\lambda}}{4\pi}\exp(\lambda t)\ \mathrm{erfc}(\sqrt{\lambda t})+\sum_{j=1,j\neq i}^{N}q_j^0\ k^{\lambda}_{ij}(t)\ . \tag{3.13}$$

Using eq. (3.4), we can write $P_t\phi^0$ in terms of the charges w_i and of the regular part ϕ of u.
If we write $w_i(t)=I_{1/2}(D^{1/2}w_i)(t)$, we have

$$\int_0^t dsP_{t-s}(x-y_i)w_i(s)-w_i(t)G^{\lambda}(x-y_i)$$
$$=\frac{1}{4\pi^{3/2}|x-y_i|}\int_0^t ds\frac{\exp\left[-\frac{|x-y_i|^2}{4(t-s)}\right]-1}{\sqrt{t-s}}D^{1/2}w_i(s)$$
$$+\,w_i(t)\frac{1-\exp(-\sqrt{\lambda}|x-y_i|)}{4\pi|x-y_i|}$$
$$=\frac{1}{4\pi^{3/2}}\int_0^{\frac{t}{|x-y_i|^2}}d\tau\frac{\exp(-\frac{1}{4\tau})-1}{\sqrt{\tau}}D^{1/2}w_i(t-|x-y_i|^2\tau)$$
$$+\,w_i(t)\frac{1-\exp(-\sqrt{\lambda}|x-y_i|)}{4\pi|x-y_i|}. \tag{3.14}$$

¿From (3.14), an explicit integration (see e.g [9]) and the dominated convergence theorem we conclude

$$\left(\int_0^t dsP_{t-s}(x-y_i)w_i(s)-w_i(t)G^{\lambda}(x-y_i)\right)_{x=y_i}=-\frac{D^{1/2}w_i(t)}{4\pi}+\frac{\sqrt{\lambda}}{4\pi}w_i(t)\ . \tag{3.15}$$

Analogous computation gives

$$\left(\int_0^t dsP_{t-s}(x-y_i)-G^{\lambda}(x-y_i)+k^{\lambda}(x-y_i,t)\right)_{x=y_i}$$

$$= \frac{\sqrt{\lambda}}{4\pi}\left[1 - \exp(\lambda t)\ \mathrm{erfc}(\sqrt{\lambda t})\right] . \qquad (3.16)$$

¿From (3.15), (3.16), (3.4), we have

$$\begin{aligned}
&P_t\phi^0(y_i) = \phi^0(y_i,t) + \frac{D^{1/2}w_i(t)}{4\pi} - \frac{\sqrt{\lambda}}{4\pi}q_i(t)\\
&- \sum_{j=1, j\neq i}^{N}\left(\int_0^t dsP_{t-s}(y_i - y_j)q_j(s) - q_j(t)G^{\lambda}(y_i - y_j)\right)\\
&+ q_i^0\ \frac{\sqrt{\lambda}}{4\pi}\exp(\lambda t)\ \mathrm{erfc}(\sqrt{\lambda t}) - \sum_{j=1, j\neq i}^{N} q_j^0\ k_{ij}^{\lambda}(t)\ . \qquad (3.17)
\end{aligned}$$

Substituting (3.17) in (3.13) we finally conclude that the required boundary condition at Y

$$\left(\frac{\sqrt{\lambda}}{4\pi} + \alpha_i(t)\right)q_i(t) - \sum_{j=1}^{N} q_j(t)\tilde{G}_{ij}^{\lambda} = \phi(y_i,t), \qquad (3.18)$$

is satisfied $\forall t \in [0,T]$. Taking into account Lemmas 3.1, 3.2, the proof is easily completed. $\Box$

Collecting the preceedings results, we can give the proof of Theorem 1.

Proof of Theorem 1. Fix $\psi \in C_0^{\infty}(R^3)$ and $t \in (0,T]$. Then for u given by (3.1) we have

$$\left(\psi, \frac{\partial u(t)}{\partial t}\right) = \left(\psi, \frac{\partial P_t u^0}{\partial t}\right) + \sum_{i=1}^{N}\left(\psi, \frac{\partial}{\partial t}\int_0^t dsP_{t-s}(\cdot - y_i)q_i(s)\right) . \qquad (3.19)$$

$P_t u^0$ solves the heat equation with initial datum u^0; moreover $\int_0^t dsP_{t-s}(\cdot - y_i)q_i(s)$ is the unique solution (in the distributional sense) of the problem

$$\frac{\partial \chi_i(t)}{\partial t} = \Delta\chi_i(t) + q_i(t)\delta_{y_i} \qquad (3.20)$$

$$\chi_i(0) = 0\ , \qquad (3.21)$$

where δ_{y_i} is the Dirac measure concentrated at y_i. Then, from eq. (3.19) we have

$$\begin{aligned}
&\left(\psi, \frac{\partial u(t)}{\partial t}\right) \\
&= \left(\psi, \Delta P_t u^0\right) + \sum_{i=1}^{N} \left(\psi, \Delta \int_0^t ds P_{t-s}(\cdot - y_i) q_i(s)\right) + \sum_{i=1}^{N} q_i(t)(\psi, \delta_{y_i}) \\
&= (\psi, \Delta\phi(t)) + \lambda \sum_{i=1}^{N} q_i(t)(\psi, G^\lambda(\cdot - y_i)) = (\psi, \Delta_{\alpha(t),Y} u(t)) \ . \qquad (3.22)
\end{aligned}$$

¿From Lemmas 3.1, 3.2 ,3.3 and eq. (3.22) we conclude that $u(t)$ is a strict solution of problem (1.6), (1.7).
It remains to prove uniqueness. Suppose that u_1, u_2 are two strict solutions with the same initial datum. Then $v = u_1 - u_2$ is a strict solution with initial datum zero. Since $v(t) \in D(\Delta_{\alpha(t),Y})$ we know that $v(t)$ has a regular part $\bar{\phi}(t) \in H^2(R^3)$ and continuous charges $\bar{q}_i(t)$, which are obviously zero for $t = 0$. Moreover

$$\begin{aligned}
\Delta_{\alpha(t),Y} v(t) &= (\Delta - \lambda)\bar{\phi}(t) + \lambda v(t) \\
&= \Delta \left(v(t) - \sum_{i=1}^{N} \bar{q}_i(t) G^\lambda(\cdot - y_i)\right) + \lambda \sum_{i=1}^{N} \bar{q}_i(t) G^\lambda(\cdot - y_i) \ . \qquad (3.23)
\end{aligned}$$

Then $v(t)$ is the solution of a problem of the type (3.20), (3.21) i.e.

$$v(t) = \sum_{i=1}^{N} \int_0^t ds P_{t-s}(\cdot - y_i)\bar{q}_i(s) \ . \qquad (3.24)$$

Now we impose the boundary condition at Y. Applying the Abel operator of order 1/2 to the boundary condition (3.18), one easily finds that $\bar{q}_i$ satisfies the usual equation for the charges (1.11) with datum f_i equal to zero. This means that $\bar{q}_i = 0$ and then also $v = 0$, concluding the proof of Theorem 1. $\square$

The proof of Theorem 1' can be easily obtained following exactly the same line of the proof of Theorem 1, with only minor changes; we omit the details.

4. The limit $N \to \infty$

Here we analyse the asymptotics of the solution of our evolution problem when the number of points and their strength increase to $+\infty$.
¿From now on we use the notation introduced in assumptions (a),...,(d) in section 1.
¿From (1.15) and Proposition 2.4 we know that α is jointly continuous in (x,t) (see e.g. [16]). The hypotheses of Theorem 1' are then satisfied and we can represent the (weak) solution of (1.6), (1.7) as

$$u^N(t) = P_t u^{0,N} + \sum_{i=1}^{N} \int_0^t ds P_{t-s}(\cdot - y_i) q_i(s), \tag{4.1}$$

where $w_i = q_i - q_i^0$ satisfies eq. (2.10), (2.11) with α_i replaced by $N\alpha_i$.
Due to the assumptions on ρ and α one has

$$D\left(\Delta + \frac{\rho}{\alpha(t)}\right) = D(\Delta) \tag{4.2}$$

and it is well known that the weak solution of (1.21), (1.22) can be represented by

$$u(t) = P_t u^0 + \int_0^t ds \int_{R^3} dy P_{t-s}(\cdot - y)\rho(y) Q(y,s), \tag{4.3}$$

where the charge distribution $Q(t) = Q(\cdot, t)$ satisfies

$$\alpha(t) Q(t) = P_t u^0 + \int_0^t ds \int_{R^3} dy P_{t-s}(\cdot - y)\rho(y) Q(y,s). \tag{4.4}$$

Obviously one has $u(t) = \alpha(t)Q(t)$. Since $u(x,t)$ is jointly continuous in (x,t), the same is true for $Q(x,t)$. In the sequel we shall also need the following estimate

Lemma 4.1

$$\sup_{x \in R^3} \int_0^T dt \left| D^{1/2}\left(Q(x,t) - Q^0(x)\right)\right|^2 < c. \tag{4.5}$$

Proof. If we define $\tilde{\alpha}(t) = \alpha(t) - \alpha^0$ and $\tilde{Q}(t) = Q(t) - Q^0$ then from eq. (4.4) we have

$$\tilde{\alpha}(t)\tilde{Q}(t)+\alpha^0\tilde{Q}(t)+Q^0\tilde{\alpha}(t)=P_t u^0-u^0$$
$$+\int_0^t ds\int_{R^3}dyP_{t-s}(\cdot-y)\rho(y)\tilde{Q}(y,s)+\int_0^t ds\int_{R^3}dyP_{t-s}(\cdot-y)\rho(y)Q^0(y)\ . \tag{4.6}$$

Taking the derivative of order 1/2 in eq. (4.6) and using formula (2.7) we obtain the equation

$$\begin{aligned}
&\alpha(t)D^{1/2}\tilde{Q}(t)-D^{1/2}\left(\int_0^t ds\int_{R^3}dyP_{t-s}(\cdot-y)\rho(y)\tilde{Q}(y,s)\right)\\
&=D^{1/2}(P_t u^0-u^0)-\frac{1}{2\sqrt{\pi}}\int_0^t ds\ \tilde{Q}(s)\frac{\tilde{\alpha}(t)-\tilde{\alpha}(s)}{(t-s)^{3/2}}\ -\ Q^0D^{1/2}\tilde{\alpha}(t)\\
&+D^{1/2}\left(\int_0^t ds\int_{R^3}dyP_{t-s}(\cdot-y)\rho(y)Q^0(y)\right)\ .
\end{aligned} \tag{4.7}$$

The first term in the r.h.s. of (4.7) can be estimated as in the proof of Lemma 2.1. One obtains

$$\sup_{x\in R^3}\int_0^T dt\left|D^{1/2}(P_t u^0(x)-u^0(x))\right|^2\le\ c\ \|\Delta u^0\|_{L^2(R^3)}T^{1/2}. \tag{4.8}$$

By our assumptions on α it is easy to see that

$$\sup_{x\in R^3}\int_0^T dt\left(\int_0^t ds\tilde{Q}(x,s)\frac{\tilde{\alpha}(x,t)-\tilde{\alpha}(x,s)}{(t-s)^{3/2}}\right)^2<c, \tag{4.9}$$

$$\sup_{x\in R^3}|Q^0(x)|\int_0^T ds|D^{1/2}\tilde{\alpha}(x,s)|^2<c\ . \tag{4.10}$$

The last term in the r.h.s. of (4.7) can be explicitely computed using Laplace transform (see e.g. [18]). Then

$$\begin{aligned}
&\int_0^T dt\left|D^{1/2}\left(\int_0^t ds\int_{R^3}dyP_{t-s}(x-y)\rho(y)Q^0(y)\right)\right|^2\\
&=c\int_0^T dt\left|\int_{R^3}dy\rho(y)Q^0(y)\frac{\exp\left(-\frac{|x-y|^2}{4t}\right)}{\sqrt{t}|x-y|}\right|^2
\end{aligned}$$

$$
\begin{aligned}
&\le c \sup_{x\in R^3} |Q^0(x)|^2 \, \|\rho\|^2_{L^2(R^3)} \int_0^T dt \frac{1}{t} \int_{R^3} dy \frac{\exp\left(-\frac{|x-y|^2}{2t}\right)}{|x-y|^2} \\
&\le c \sup_{x\in R^3} |Q^0(x)|^2 \, \|\rho\|^2_{L^2(R^3)} T^{1/2}. \qquad (4.11)
\end{aligned}
$$

Now consider the second term in the l.h.s. of (4.7). Using the convolution property of the Laplace transform and Schwartz's inequality one has

$$
\begin{aligned}
&\int_0^T dt \left| D^{1/2} \left(\int_0^t ds \int_{R^3} dy P_{t-s}(x-y) \rho(y) \tilde{Q}(y,s) \right) \right|^2 \\
&= \int_0^T dt \left| \int_{R^3} dy \rho(y) \int_0^t ds P_{t-s}(x-y) D^{1/2} \tilde{Q}(y,s) \right|^2 \\
&\le \|\rho\|^2_{L^2(R^3)} \int_{R^3} dy \int_0^T dt \left(\int_0^t ds P_{t-s}(x-y) D^{1/2} \tilde{Q}(y,s) \right)^2 . \qquad (4.12)
\end{aligned}
$$

Using Schur's test ([17]) and an explicit integration (see e.g. [9]) we can estimate the norm in $L^2([0,T])$ of the integral operator with kernel $P_{t-s}(x-y)$. We obtains

$$
\begin{aligned}
&\int_0^T dt \left| D^{1/2} \left(\int_0^t ds \int_{R^3} dy P_{t-s}(x-y) \rho(y) \tilde{Q}(y,s) \right) \right|^2 \\
&\le c \|\rho\|^2_{L^2(R^3)} \int_{R^3} dy \frac{1}{|x-y|^2} \mathrm{erfc}^2 \left(\frac{|x-y|}{2\sqrt{T}} \right) \int_0^T dt \left| D^{1/2} \tilde{Q}(y,t) \right|^2 \\
&\le c \|\rho\|^2_{L^2(R^3)} T^{1/2} \sup_{x\in R^3} \int_0^T dt \left| D^{1/2} \tilde{Q}(x,t) \right|^2 . \qquad (4.13)
\end{aligned}
$$

By the assumptions on α and (4.7)...(4.13), we first get the estimate (4.5) for T small and then, using the evolution equation (1.21), (1.22), for any T. $\square$

Now we study the difference $u - u^N$ for $N \to \infty$.
Exploiting the fact that $Q(x,t)$ is jointly continuous in (x,t), we can write

$$
u(t) - u^N(t) = v_1^N(t) + v_2^N(t) + v_3^N(t), \qquad (4.14)
$$
$$
v_1^N(t) = P_t(u^0 - u^{0,N}), \qquad (4.15)
$$

$$v_2^N(t) = \frac{1}{N}\sum_{i=1}^N \int_0^t ds P_{t-s}(\cdot - y_i)\left(Q(y_i,s) - Nq_i(s)\right), \tag{4.16}$$

$$v_3^N(t) = \int_0^t ds \int_{R^3} dy P_{t-s}(\cdot - y)\rho(y)Q(y,s) - \frac{1}{N}\sum_{i=1}^N \int_0^t ds P_{t-s}(\cdot - y_i)Q(y_i,s). \tag{4.17}$$

In the next three lemmas we prove the convergence of $v_j^N, \;\; j = 1,2,3$.

Lemma 4.2. Under assumptions (a),...,(d) and $\forall \epsilon > 0$ one has

$$\lim_{N\to\infty} P^N\left(\left\{Y^N \mid \sup_{t\in[0,T]} \|v_1^N(t)\|_{L^2(R^3)} > \epsilon\right\}\right) = 0 \tag{4.18}$$

Proof. First we observe

$$\|v_1^N(t)\|_{L^2(R^3)} \leq \|u^0 - u^{0,N}\|_{L^2(R^3)}. \tag{4.19}$$

Since $u^{0,N} \in D(\Delta_{N\alpha(0),Y^N})$, we have $u^{0,N} = (-\Delta_{N\alpha(0),Y^N} + \lambda)^{-1}h^{0,N}$, where $h^{0,N} = (-\Delta + \lambda)\phi^{0,N}$.
In the same way we can write $u^0 = (-\Delta - \frac{\rho}{\alpha^0} + \lambda)^{-1}h^0$, with $h^0 = (-\Delta + \lambda)\phi^0$. Taking into account assumption (d) and the result proved in [12] we obtain the proof of the lemma. □

A direct comparison between the two equations for the charges (1.11), (4.4), gives the estimate of v_2^N.

Lemma 4.3. Under assumption (a),...,(d) there exists $T_1 > 0$, sufficiently small and independent of N, such that $\forall \epsilon > 0$ one has

$$\lim_{N\to\infty} P^N\left(\left\{Y^N \mid \int_0^{T_1} dt\, \|v_2^N(t)\|^2_{L^2(R^3)} > \epsilon\right\}\right) = 0. \tag{4.20}$$

Proof. For $\zeta \in L^2(R^3)$ we have

$$\begin{aligned} |(\zeta, v_2^N(t))| &\leq \frac{1}{N}\sum_{i=1}^N \int_0^t ds |P_{t-s}\zeta(y_i)||Q(y_i,s) - Nq_i(s)| \\ &\leq \frac{c}{N}\|\zeta\|_{L^2(R^3)} \sum_{i=1}^N \int_0^t ds \frac{|Q(y_i,s) - Nq_i(s)|}{(t-s)^{3/4}}, \end{aligned} \tag{4.21}$$

where we have used Schwartz's inequality. The boundedness in $L^2([0,T])$ of the Abel operator of order $1/4$ implies

$$\int_0^T dt \|v_2^N(t)\|^2_{L^2(R^3)} \leq \frac{c}{N}\sum_{i=1}^N \int_0^T dt |Q(y_i,s) - Nq_i(s)|^2. \tag{4.22}$$

The proof is then reduced to the estimate of the difference of the charges. Denoting $\xi_i(t) = Q(y_i,t) - Q^0(y_i) - N(q_i(t) - q_i^0)$, subtracting eq. (1.11) from eq. (4.4) and taking the derivative of order $1/2$, we find

$$\frac{D^{1/2}\xi_i(t)}{4\pi N} + \alpha_i(t)\xi_i(t) - \frac{1}{N}\sum_{j=1, j\neq i}^N \int_0^t ds P_{t-s}(y_i - y_j)(Q(y_j,s) - Nq_j(s)) = F_i(t) \tag{4.23}$$

$$\begin{aligned} F_i(t) &= P_t(\phi^0 - \phi^{0,N})(y_i) + P_t\left(G^\lambda \rho Q^0 - \sum_{j=1,j\neq i}^N q_j^0 G^\lambda(\cdot - y_j)\right)(y_i) \\ &+ q_i^0 \frac{\sqrt{\lambda}}{4\pi}\exp(\lambda t)\,\mathrm{erfc}(\sqrt{\lambda t}) - \alpha_i(t)(Q^0(y_i) - Nq_i^0) + \frac{1}{4\pi N}D^{1/2}(Q(y_i,t) - Q^0(y_i)), \end{aligned} \tag{4.24}$$

where we have used the decomposition $u^0 = \phi^0 + G^\lambda \rho Q^0$ (see (1.23), (1.24)). A straightforward computation shows that the fractional differential operator $(4\pi N)^{-1}D^{1/2} + \alpha_i$ is bounded from below by a positive constant. In fact, if $\xi_i = I_{1/2}\sqrt{\alpha_i}\eta_i$, we have

$$\begin{aligned} &\int_0^T dt \left| \frac{D^{1/2}\xi_i(t)}{4\pi N} + \alpha_i(t)\xi_i(t)\right|^2 \geq \beta \int_0^T dt \left|\frac{\eta_i(t)}{4\pi N} + (\sqrt{\alpha_i}I_{1/2}\sqrt{\alpha_i}\eta_i)(t)\right|^2 \\ &= \beta\left(\frac{1}{4\pi N}\int_0^T dt|\eta_i(t)|^2 + \int_0^T dt |(\sqrt{\alpha_i}I_{1/2}\sqrt{\alpha_i}\eta_i)(t)|^2 \right. \\ &\left. + \frac{1}{4\pi N}\int_0^T dt \int_0^T ds\ \eta_i(t)\sqrt{\alpha_i(t)}\eta_i(s)\sqrt{\alpha_i(s)}\frac{1}{\sqrt{|t-s|}}\right) \\ &\geq \beta \int_0^T dt |(\sqrt{\alpha_i}I_{1/2}\sqrt{\alpha_i}\eta_i)(t)|^2 \geq \beta^2 \int_0^T dt |\xi_i(t)|^2. \end{aligned} \tag{4.25}$$

In (4.25) we have used the positivity of the integral operator in $L^2([0,T])$ with kernel $(\sqrt{|t-s|})^{-1}$.
The non diagonal term in eq. (4.23) can be made small, uniformly in N, if we restrict the time interval. In fact for any $T_1 \leq T$

$$
\begin{aligned}
&\frac{1}{N^3}\sum_{i=1}^{N}\int_0^{T_1} dt \left| \sum_{j=1,j\neq i}^{N} \int_0^t ds (Q(y_j,s) - N q_j(s)) P_{t-s}(y_i - y_j) \right|^2 \\
&\leq \frac{1}{N^3}\sum_{i=1}^{N}\left[\sum_{j=1,j\neq i}^{N} \left(\int_0^{T_1} dt \left| \int_0^t ds (Q(y_j,s) - N q_j(s)) P_{t-s}(y_i - y_j) \right|^2 \right)^{1/2} \right]^2 \\
&\leq \frac{4\pi}{N^3}\sum_{i=1}^{N}\left[\sum_{j=1,j\neq i}^{N} \left(\int_0^{T_1} dt (Q(y_j,t) - N q_j(t))^2 \right)^{1/2} \frac{\operatorname{erfc}\left(\frac{|y_i-y_j|}{2\sqrt{T_1}}\right)}{|y_i-y_j|} \right]^2 \\
&\leq \left(\frac{1}{N}\sum_{j=1} N \int_0^{T_1} dt (Q(y_j,t) - N q_j(t))^2 \right) \frac{1}{N^2} \sum_{i,j=1,i\neq j}^{N} \frac{\operatorname{erfc}^2\left(\frac{|y_i-y_j|}{2\sqrt{T_1}}\right)}{|y_i-y_j|^2} \\
&\leq c(T_1) \left(\frac{1}{N}\sum_{j=1}^{N} \int_0^{T_1} dt (Q(y_j,t) - N q_j(t))^2 \right). \qquad (4.26)
\end{aligned}
$$

In (4.26) we have used again Schur's test to estimate the integral operator in $L^2([0,T])$ with kernel $P_{t-s}(y_i - y_j)$ and Schwatz's inequality. The last inequality in (4.26), where $c(T_1)$ is independent of N, holds with probability one as a consequence of the law of large numbers.
Moreover by the dominated convergence theorem we have

$$
\lim_{T_1 \to 0} c(T_1) = 0. \qquad (4.27)
$$

Using (4.25), (4.26) and choosing T_1 such that $4c(T_1) < \beta^2$, we obtain the estimate

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\int_0^{T_1} dt (Q(y_i,t) - N q_i(t))^2 \\
&\leq \frac{2\beta^2 T_1}{(\beta^2 - 4c(T_1))N}\sum_{i=1}^{N}(Q^0(y_i) - N q_i^0)^2 + \frac{4}{(\beta^2 - 4c(T_1))N}\sum_{i=1}^{N}\int_0^{T_1} dt\, F_i^2(t). \qquad (4.28)
\end{aligned}
$$

The first term in (4.28) goes to zero in probability for $N \to \infty$ (see [12]). The final step of the proof is the estimate of the datum F_i. Direct computation gives

$$\frac{1}{N}\sum_{i=1}^{N}\int_0^{T_1} dt |P_t(\phi^0 - \phi^{0,N})(y_i)|^2 \leq c\, T_1 \|\phi^0 - \phi^{0,N}\|_{H^2(R^3)}. \tag{4.29}$$

For the second term in (4.24) we have

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\int_0^{T_1} dt \left| P_t\left(G^\lambda \rho Q^0 - \sum_{j=1, j\neq i}^{N} q_j^0 G^\lambda(\cdot - y_j)\right)(y_j)\right|^2 \\
&\leq \frac{2}{N}\sum_{i=1}^{N}\int_0^{T_1} dt \left| \frac{1}{N}\sum_{j=1,j\neq i}^{N} (Q^0(y_j) - N q_y^0) k_{ij}^\lambda(t)\right|^2 \\
&+ \frac{2}{N}\sum_{i=1}^{N}\int_0^{T_1} dt \left| \int_{R^3} dy \rho(y) Q^0(y) k^\lambda(y - y_i, t) - \frac{1}{N}\sum_{j=1,j\neq i}^{N} Q^0(y_j) k_{ij}^\lambda(t)\right|^2 \\
&\equiv J_1^N + J_2^N.
\end{aligned} \tag{4.30}$$

Explicit differentiation gives

$$\dot{k}^\lambda(x-y,t) = \lambda k^\lambda(x-y,t) - P_t(x-y), \quad x \neq y. \tag{4.31}$$

Then using (2.9), (3.2) and Gronwall's lemma

$$|k^\lambda(x-y,t)| \leq c\, \exp(\lambda T_1) G^0(x-y), \quad x\neq y. \tag{4.32}$$

By Schwartz's inequality and the estimate (4.32)

$$J_1^N \leq c\, \exp(2\lambda T_1)\, T_1 \left(\frac{1}{N^2}\sum_{i,j=1,i\neq j}^{N} \frac{1}{|y_i - y_j|^2}\right)\left[\frac{1}{N}\sum_{j=1}^{N}(Q^0 - N q_j^0)^2\right]. \tag{4.33}$$

By the law of large numbers and the result proved in [12] we conclude that J_1^N goes to zero in probability for $N \to \infty$.
Taking the expectation E^N (with respect to P^N) of the last term in (4.30) we get

$$
\begin{aligned}
E^N(J_2^N) &= \frac{1}{N^2}\int_0^{T_1} dt \int_{R^3} dx\rho(x)\left(\int_{R^3} dy\rho(y)Q^0(y)k^\lambda(x-y,t)\right)^2 \\
&\leq \frac{1}{N^2}\int_0^{T_1} dt \int_{R^3} dx\rho(x)\left(\int_{R^3} dz P_t(x-z)\sup_{\xi\in R^3}\int_{R^3} dy\rho(y)|Q^0(y)|G^\lambda(y-\xi)\right)^2 \\
&\leq \frac{c}{N^2}T_1\|\rho\|^2_{L^2(R^3)}\left(\sup_{y\in R^3}|Q^0(y)|\right)^2 . \qquad (4.34)
\end{aligned}
$$

¿From (4.33), (4.34) we conclude that the second term in (4.24) goes to zero in probability for $N\to\infty$.
Due the result in [12] and the estimate (4.5), the remaining terms in (4.24) are also going to zero in probability for $N\to\infty$. This concludes the proof of the lemma. □

A straightforward computation of the expectation E^N gives the estimate of v_3^N.

Lemma 4.4. Under assumptions (a),...,(d) one has

$$
E^N\left(\int_0^T dt\ \|v_3^N(t)\|^2_{L^2(R^3)}\right) \leq \frac{c}{N}T^{3/2}\left(\sup_{x,t}|Q(x,t)|\right)^2 . \qquad (4.35)
$$

Proof. Explicit computation gives

$$
\begin{aligned}
&E^N\left(\int_0^T dt\|v_3^N(t)\|^2_{L^2(R^3)}\right) \\
&= \frac{1}{N}\int_0^T dt\int_{R^3} dx\int_0^t ds\int_0^t d\sigma\int_{R^3} dy\rho(y)Q(y,s)Q(y,\sigma)P_{t-s}(x-y)P_{t-\sigma}(x-y) \\
&-\frac{1}{N}\int_0^T dt\int_{R^3} dx\left(\int_0^t ds\int_{R^3} dy\rho(y)Q(y,s)P_{t-s}(x-y)\right)^2 . \qquad (4.36)
\end{aligned}
$$

Using the continuity of Q, (3.2) and (for the last integral in (4.36)) Schwartz's inequality one easily obtains the proof of the lemma. □

Proof of Theorem 2. Lemmas 4.2, 4.3, 4.4 allow us to prove Theorem 2 for $T\leq T_1$, with T_1 sufficiently small and independent of N. Using the evolution equations for u and u^N one easily extends the proof of Theorem 2 to any $T > 0$. $\square$

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