Fractional Differentiability of Nowhere Differentiable Functions and Dimensions

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Abstract

Weierstrass's everywhere continuous but nowhere differentiable function is shown to be locally continuously fractionally differentiable everywhere for all orders below the 'critical order' $2-s$ and not so for orders between $2-s$ and 1, where s, $1<s<2$, is the box dimension of the graph of the function. This observation is consolidated in the general result showing a direct connection between local fractional differentiability and the box dimension/ local Hölder exponent. Lévy index for one dimensional Lévy flights is shown to be the critical order of its characteristic function. Local fractional derivatives of multifractal signals (non-random functions) are shown to provide the local Hölder exponent. It is argued that Local fractional derivatives provide a powerful tool to analyze pointwise behavior of irregular signals.

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1 Introduction

The importance of studying continuous but nowhere differentiable functions was emphasized a long time ago by Perrin, Poincare and others (see refs. [1, 2]). It is possible for a continuous function to be sufficiently irregular so that its graph is a fractal. This observation points out to a connection between the lack of differentiability of such a function and the dimension of its graph. Quantitatively one would like to convert the question concerning the lack of differentiability into one concerning the amount of loss of differentiability. In other words, one would look at derivatives of fractional order rather than only those of integral order and relate them to dimensions. Indeed some recent papers [3-6] indicate a connection between fractional calculus[7-9] and fractal structure [2, 10] or fractal processes [11, 12, 13]. Mandelbrot and Van Ness [11] have used fractional integrals to formulate fractal processes such as fractional Brownian motion. In refs. [4, 5] a fractional diffusion equation has been proposed for the diffusion on fractals. Also W. G. Glöckle and T. F. Nonnenmacher [12] have formulated fractional differential equations for some relaxation processes which are essentially fractal time [13] processes. Recently Zaslavasky [14] showed that the Hamiltonian chaotic dynamics of particles can be described by a fractional generalization of the Fokker-Plank-Kolmogorov equation which is defined by two fractional critical exponents (α, β) responsible for the space and time derivatives of the distribution function correspondingly. However, to our knowledge, the precise nature of the connection between the dimension of the graph of a fractal curve and fractional differentiability properties has not been established.

Irregular functions arise naturally in various branches of physics. It is well known that the graphs of projections of Brownian paths are nowhere differentiable and have dimension 3/2. A generalization of Brownian motion called fractional Brownian motion [2, 10] gives rise to graphs having dimension between 1 and 2. Typical Feynmann paths [15, 16], like the Brownian paths are continuous but nowhere differentiable. Also, passive scalars advected by a turbulent fluid [17, 18] can have isoscalar surfaces which are highly irregular, in the limit of diffusion constant going to zero. Attractors of some dynamical systems have been shown [19] to be continuous but nowhere differentiable.

All these irregular functions are characterized at every point by a local Hölder exponent typically lying between 0 and 1. In the case of functions having same Hölder exponent h at every point it is well known that box dimension of its graph is $2 - h$. Not all functions have the same exponent h at every point but have a range of Hölder exponents. A set $\{x|h(x) = h\}$ may be a fractal set. In such situations the corresponding functions are multifractal. These kind of functions also arise in various physical situations, for instance, velocity field of a turbulent fluid [20] at low viscosity.

Also there exists a class of problems where one has to solve a partial differentiable equation subject to fractal boundary conditions, e.g. Laplace equation near a fractal conducting surface. As noted in reference [21] irregular boundaries may appear, down to a certain spatial resolution, to be non-differentiable everywhere and/or may exhibit convolutions over many length scales. Keeping in view such problems there is a need to characterize pointwise behavior using something which can be readily used.

We consider the Weierstrass function as a prototype example of a function which is continuous everywhere but differentiable nowhere and has an exponent which is constant everywhere. One form of Weierstrass function is

$$W_\lambda(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} sin\lambda^k t, \qquad t \quad \text{real}. \tag{1}$$

For this form, when $\lambda > 1$ it is well known [22] that $W_\lambda(t)$ is nowhere differentiable if $1 < s < 2$. This curve has been extensively studied [1, 10, 23, 24] and its graph is known to have a box dimension s, for sufficiently large λ. Incidently, the Weierstrass functions are not just mathematical curiosities but occur at several places. For instance, the graph of this function is known [10, 19] to be a repeller or attractor of some dynamical systems. This kind of function can also be recognized as the characteristic function of a Lévy flight on a one dimensional lattice [25], which means that such a Lévy flight can be considered as a superposition of Weierstrass type functions. This function has also been used [10] to generate fractional Brownian signal by multiplying every term by a random amplitude and randomizing phases of every term.

The main aim of the present paper is to explore the precise nature of the connection between fractional differentiability properties of irregular (non-differentiable) curves and dimensions/ Hölder exponents of their graphs. A second aim is to provide a possible tool to study pointwise behavior. The organization of the paper is as follows. In section II we motivate and define what we call local fractional differentiability, formally and use local fractional derivative to formulate Taylor series. Then in section III we apply this definition to a specific example, viz, Weierstrass' nowhere differentiable function and show that this function, at every point, is locally fractionally differentiable for all orders below $2-s$ and it is not so for orders between $2-s$ and 1, where s, $1 < s < 2$, is the box dimension of the graph of the function. In section IV we prove a general result showing the relation between local fractional differentiability of nowhere differentiable functions and the local Hölder exponent/ the dimension of its graph. In section V we demonstrate the use of the local fractional derivatives (LFD) in unmasking isolated singularities and in the study of pointwise behavior of multifractal functions. In section VI we conclude after pointing out a few possible consequences of our results.

2 Fractional Differentiability

We begin by recalling the Riemann-Liouville definition of fractional integral of a real function, which is given by [7, 9]

$$\frac{d^q f(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{f(y)}{(x-y)^{q+1}} dy \quad \text{for} \quad q < 0, \quad a \ \text{ real}, \tag{2}$$

and of the fractional derivative

$$\frac{d^q f(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^{q}} dy \quad \text{for} \quad 0 < q < 1. \tag{3}$$

The case of $q > 1$ is of no relevance in this paper. For future reference we note [7, 9]

$$\frac{d^q x^p}{dx^q} = \frac{\Gamma(p+1)}{\Gamma(p-q+1)} x^{p-q} \quad \text{for} \quad p > -1. \tag{4}$$

We also note that the fractional derivative has the property (see ref [7]), viz,

$$\frac{d^q f(\beta x)}{dx^q} = \beta^q \frac{d^q f(\beta x)}{d(\beta x)^q} \tag{5}$$

which makes it suitable for the study of scaling.

One may note that except in the case of positive integral q, the $q^t h$ derivative will be nonlocal through its dependence on the lower limit 'a'. On the other hand we wish to study local scaling properties and hence we need to introduce the notion of local fractional differentiability. Secondly from eqn(4) it is clear that the fractional derivative of a constant function is not zero. These two features play an important role in defining local fractional differentiability. We note that changing the lower limit or adding a constant to a function alters the value of the fractional derivative. This forces one to choose the lower limit as well as the additive constant before hand. The most natural choices are as follows. 1) We subtract, from the function, the value of the function at the point where fractional differentiability is to be checked. This makes the value of the function zero at that point, washing out the effect of any constant term. 2) Natural choice of a lower limit will be that point, where we intend to examine the fractional differentiability, itself. This has an advantage in that it preserves local nature of the differentiability property. With these motivations we now introduce:

Definition 1 *If, for a function $f : [0,1] \to I\!R$, the limit*

$$I\!D^q f(y) = \lim_{x \to y} \frac{d^q (f(x) - f(y))}{d(x-y)^q} \tag{6}$$

exists and is finite, then we say that the local fractional derivative (LFD) of order q, at $x = y$, exists.

Definition 2 *We define critical order α, at y, as*

$$\alpha(y) = Sup\{q | \text{all local fractional derivatives of order less than } q \text{ exist at } y\}.$$

Incidentally we note that Hilfer [26, 27] used a similar notion to extend Ehrenfest's classification of phase transition to continuous transitions. However in his work only the singular part of the free energy was considered. So the first of the above mentioned condition was automatically satisfied. Also no lower limit of fractional derivative was considered and by default it was taken as zero.

In order to see the information contained in the LFD we consider fractional Taylor's series with a remainder term for a real function f. Let

$$F(y, x-y; q) = \frac{d^q (f(x) - f(y))}{[d(x-y)]^q} \tag{7}$$

It is clear that

$$I\!D^q f(y) = F(y, 0; q) \tag{8}$$

Now, for $0 < q < 1$,

$$
\begin{aligned}
f(x) - f(y) &= \frac{1}{\Gamma(q)} \int_0^{x-y} \frac{F(y,t;q)}{(x-y-t)^{-q+1}} dt \qquad (9)\\
&= \frac{1}{\Gamma(q)} [F(y,t;q) \int (x-y-t)^{q-1} dt]_0^{x-y} \\
&\quad + \frac{1}{\Gamma(q)} \int_0^{x-y} \frac{dF(y,t;q)}{dt} \frac{(x-y-t)^q}{q} dt \qquad (10)
\end{aligned}
$$

provided the last term exists. Thus

$$
\begin{aligned}
f(x) - f(y) &= \frac{\mathbb{D}^q f(y)}{\Gamma(q+1)} (x-y)^q \\
&\quad + \frac{1}{\Gamma(q+1)} \int_0^{x-y} \frac{dF(y,t;q)}{dt} (x-y-t)^q dt \qquad (11)
\end{aligned}
$$

i.e.

$$
f(x) = f(y) + \frac{\mathbb{D}^q f(y)}{\Gamma(q+1)} (x-y)^q + R_1(x,y) \qquad (12)
$$

where $R_1(x,y)$ is a remainder given by

$$
R_1(x,y) = \frac{1}{\Gamma(q+1)} \int_0^{x-y} \frac{dF(y,t;q)}{dt} (x-y-t)^q dt \qquad (13)
$$

Equation (12) is a fractional Taylor expansion of $f(x)$ involving only the lowest and the second leading terms. This expansion can be carried to higher orders provided the corresponding remainder term is well defined.

We note that the local fractional derivative as defined above (not just fractional derivative) provides the coefficient A in the approximation of $f(x)$ by the function $f(y) + A(x-y)^q/\Gamma(q+1)$, for $0 < q < 1$, in the vicinity of y. We further note that the terms on the RHS of eqn(11) are nontrivial and finite only in the case $q = \alpha$. Osler in ref.[28] has constructed fractional Taylor series using usual (not local in the present sense) fractional derivatives. His results are, however, applicable to analytic functions and cannot be used for non-differentiable scaling functions directly. Further Osler's formulation involves terms with negative q also and hence is not suitable for approximating schemes.

One may further notice that when q is set equal to one in the above approximation one gets the equation of the tangent. It may be recalled that all the curves passing through a point y and having same tangent form an equivalence class (which is modeled by a linear behavior). Analogously all the functions (curves) with the same critical order α and the same $\mathbb{D}^\alpha$ will form an equivalence class modeled by x^α (If f differs from x^α by a logarithmic correction then terms on RHS of eqn(11) do not make sense precisely as in the case of ordinary calculus). This is how one may generalize the geometric interpretation of derivatives in terms of tangents. This observation is useful when one wants to approximate an irregular function by a piecewise smooth (scaling) function.

To illustrate the definitions of local fractional differentiability and critical order consider an example of a polynomial of degree n with its graph passing through the origin and for which first derivative at origin is not zero. Then all the local fractional derivatives of order less than or equal to one exist at the origin. Also all derivatives of integer order greater than one exist, as expected. But local derivatives of any other order, e.g. between 1 and 2, (see equations (4) and (6)) do not exist. Therefore critical order for this function at $x = 0$ is one. In fact, except at finite number of points where the function has vanishing first derivative, critical order of a polynomial function will be one, since the linear term is expected to dominate near these points.

Remark: We would like to point out that there is a multiplicity of definitions of fractional derivative. The use of Riemann-Liouville definition, and other equivalent definitions such as Grunwald's definition are suitable for our purpose. The other definitions of fractional derivatives which do not allow control over both the limits, such as Wyel's definition or definition using Fourier transforms, are not suitable since it would not be possible to retrieve local nature of differentiability property which is essential for study of local behavior. Also, the important difference between our work and the work of [4, 12] is that while we are trying to study the local scaling behavior these works apply to asymptotic scaling properties.

3 Fractional Differentiability of Weierstrass Function

Consider a form of Weierstrass function as given above, viz,

$$W_\lambda(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} sin\lambda^k t, \quad \lambda > 1. \tag{14}$$

Note that $W_\lambda(0) = 0$. Now

$$\begin{aligned} \frac{d^q W_\lambda(t)}{dt^q} &= \sum_{k=1}^{\infty} \lambda^{(s-2)k} \frac{d^q sin(\lambda^k t)}{dt^q} \\ &= \sum_{k=1}^{\infty} \lambda^{(s-2+q)k} \frac{d^q sin(\lambda^k t)}{d(\lambda^k t)^q} \end{aligned}$$

provided the right hand side converges uniformly. Using, for $0 < q < 1$,

$$\frac{d^q sin(x)}{dx^q} = \frac{d^{q-1} cos(x)}{dx^{q-1}}$$

we get

$$\frac{d^q W_\lambda(t)}{dt^q} = \sum_{k=1}^{\infty} \lambda^{(s-2+q)k} \frac{d^{q-1} cos(\lambda^k t)}{d(\lambda^k t)^{q-1}} \tag{15}$$

From second mean value theorem it follows that the fractional integral of $cos(\lambda^k t)$ of order $q-1$ is bounded uniformly for all values of $\lambda^k t$. This implies that the series on the right hand side will converge uniformly for $q < 2 - s$, justifying our action of taking the fractional derivative operator inside the sum.

Also as $t \to 0$ for any k the fractional integral in the summation of equation (15) goes to zero. Therefore it is easy to see from this that

$$I\!\!D^q W_\lambda(0) = \lim_{t\to 0} \frac{d^q W_\lambda(t)}{dt^q} = 0 \quad \text{for} \quad q < 2 - s. \tag{16}$$

This shows that q^{th} local derivative of the Weierstrass function exists and is continuous, at $t = 0$, for $q < 2 - s$.

To check the fractional differentiability at any other point, say τ, we use $t' = t - \tau$ and $\widetilde{W}(t') = W(t' + \tau) - W(\tau)$ so that $\widetilde{W}(0) = 0$. We have

$$\begin{aligned}\widetilde{W}_\lambda(t') &= \sum_{k=1}^{\infty} \lambda^{(s-2)k} sin\lambda^k (t' + \tau) - \sum_{k=1}^{\infty} \lambda^{(s-2)k} sin\lambda^k \tau \\ &= \sum_{k=1}^{\infty} \lambda^{(s-2)k} (cos\lambda^k \tau sin\lambda^k t' + sin\lambda^k \tau (cos\lambda^k t' - 1))\end{aligned} \tag{17}$$

Taking fractional derivative of this with respect to t' and following the same procedure we can show that the fractional derivative of the Weierstrass function of order $q < 2 - s$ exists at all points.

For $q > 2 - s$, right hand side of the equation (15) seems to diverge. We now prove that the LFD of order $q > 2 - s$ in fact does not exist. We do this by showing that there exists a sequence of points approaching 0 along which the limit of the fractional derivative of order $2 - s < q < 1$ does not exist. We use the property of the Weierstrass function [10], viz, for each $t' \in [0, 1]$ and $0 < \delta \leq \delta_0$ there exists t such that $|t - t'| \leq \delta$ and

$$c\delta^\alpha \leq |W(t) - W(t')|, \tag{18}$$

where $c > 0$ and $\alpha = 2 - s$, provided λ is sufficiently large. We consider the case of $t' = 0$ and $t > 0$. Define $g(t) = W(t) - ct^\alpha$.

Now the above mentioned property, along with continuity, of the Weierstrass function assures us a sequence of points $t_1 > t_2 > ... > t_n > ... \geq 0$ such that $t_n \to 0$ as $n \to \infty$ and $g(t_n) = 0$ and $g(t) > 0$ on (t_n, ϵ) for some $\epsilon > 0$, for all n (It is not ruled out that t_n may be zero for finite n). Define

$$\begin{aligned} g_n(t) &= 0 \quad \text{if} \quad t \leq t_n \\ &= g(t) \quad \text{otherwise.}\end{aligned}$$

Now we have, for $0 < \alpha < q < 1$,

$$\frac{d^q g_n(t)}{d(t - t_n)^q} = \frac{1}{\Gamma(1 - q)} \frac{d}{dt} \int_{t_n}^{t} \frac{g(y)}{(t - y)^q} dy,$$

where $t_n \leq t \leq t_{n-1}$. We assume that left hand side of the above equation exists for if it does not then we have nothing to prove. Let

$$h(t) = \int_{t_n}^{t} \frac{g(y)}{(t - y)^q} dy.$$

Now $h(t_n) = 0$ and $h(t_n + \epsilon) > 0$, for a suitable ϵ, as the integrand is positive. Due to continuity there must exist an $\epsilon' > 0$ and $\epsilon' < \epsilon$ such that $h(t)$ is increasing on (t_n, ϵ'). Therefore

$$0 \leq \frac{d^q g_n(t)}{d(t - t_n)^q}|_{t=t_n} \quad n = 1, 2, 3.... \tag{19}$$

This implies that

$$c\frac{d^q t^\alpha}{d(t-t_n)^q}|_{t=t_n} \leq \frac{d^q W(t)}{d(t-t_n)^q}|_{t=t_n} \quad n = 1, 2, 3... \tag{20}$$

But we know from eqn(4) that, when $0 < \alpha < q < 1$, the left hand side in the above inequality approaches infinity as $t \to 0$. This implies that the right hand side of the above inequality does not exist as $t \to 0$. This argument can be generalized for all nonzero t' by changing the variable $t'' = t - t'$. This concludes the proof.

Therefore the critical order of the Weierstrass function will be $2 - s$ at all points.

Remark: Schlesinger et al [25] have considered a Lévy flight on a one dimensional periodic lattice where a particle jumps from one lattice site to other with the probability given by

$$P(x) = \frac{\omega - 1}{2\omega} \sum_{j=0}^{\infty} \omega^{-j} [\delta(x, +b^j) + \delta(x, -b^j)] \tag{21}$$

where x is magnitude of the jump, b is a lattice spacing and $b > \omega > 1$. $\delta(x, y)$ is a Kronecker delta. The characteristic function for $P(x)$ is given by

$$\tilde{P}(k) = \frac{\omega - 1}{2\omega} \sum_{j=0}^{\infty} \omega^{-j} cos(b^j k). \tag{22}$$

which is nothing but the Weierstrass cosine function. For this distribution the Lévy index is $\log\omega/\log b$, which can be identified as critical order of $\tilde{P}(k)$.

More generally for the Lévy distribution with index μ the characteristic function is given by

$$\tilde{P}(k) = A \exp c|k|^\mu. \tag{23}$$

Critical order of this function at $k = 0$ also turns out to be same as μ. Thus the Lévy index can be identified as the critical order of the characteristic function at $k = 0$.

4 Connection between critical order and the box dimension of the curve

Theorem 1 *Let $f : [0, 1] \to I\!R$ be a continuous function.*

a)If

$$\lim_{x\to y} \frac{d^q(f(x) - f(y))}{[d(x-y)]^q} = 0 \quad \text{for} \quad q < \alpha \quad ,$$

where $q, \alpha \in (0, 1)$, for all $y \in (0, 1)$, then $dim_B f(x) \leq 2 - \alpha$.

b)If there exists a sequence $x_n \to y$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{d^q(f(x_n) - f(y))}{[d(x_n - y)]^q} = \pm\infty \quad \text{for} \quad q > \alpha \quad ,$$

for all y, then $dim_B f \geq 2 - \alpha$.

Proof: (a)Without loss of generality assume $y = 0$ and $f(0) = 0$. We consider the case of $q < \alpha$.

As $0 < q < 1$ and $f(0) = 0$ we can write [7]

$$\begin{aligned} f(x) &= \frac{d^{-q}}{dx^{-q}} \frac{d^q f(x)}{dx^q} \\ &= \frac{1}{\Gamma(q)} \int_0^x \frac{\frac{d^q f(y)}{dy^q}}{(x-y)^{-q+1}} dy \end{aligned} \tag{24}$$

Now

$$|f(x)| \leq \frac{1}{\Gamma(q)} \int_0^x \frac{|\frac{d^q f(y)}{dy^q}|}{(x-y)^{-q+1}} dy$$

As, by assumption, for $q < \alpha$,

$$\lim_{x \to 0} \frac{d^q f(x)}{dx^q} = 0$$

we have, for any $\epsilon > 0$, a $\delta > 0$ such that $|\frac{d^q f(x)}{dx^q}| < \epsilon$ for all $x < \delta$

$$\begin{aligned} |f(x)| &\leq \frac{\epsilon}{\Gamma(q)} \int_0^x \frac{dy}{(x-y)^{-q+1}} \\ &= \frac{\epsilon}{\Gamma(q+1)} x^q \end{aligned}$$

As a result we have

$$|f(x)| \leq K|x|^q \quad \text{for} \quad x < \delta.$$

Now this argument can be extended for general y simply by considering $x - y$ instead of x and $f(x) - f(y)$ instead of $f(x)$. So finally we get for $q < \alpha$

$$|f(x) - f(y)| \leq K|x-y|^q \quad \text{for} \ |x-y| < \delta, \tag{25}$$

for all $y \in (0, 1)$. Hence we have [10]

$$dim_B f(x) \leq 2 - \alpha.$$

b)Now we consider the case $q > \alpha$. If we have

$$\lim_{x_n \to 0} \frac{d^q f(x_n)}{dx_n^q} = \infty \tag{26}$$

then for given $M_1 > 0$ and $\delta > 0$ we can find positive integer N such that $|x_n| < \delta$ and $d^q f(x_n)/dx_n^q \geq M_1$ for all $n > N$. Therefore by eqn(24)

$$\begin{aligned} f(x_n) &\geq \frac{M_1}{\Gamma(q)} \int_0^{x_n} \frac{dy}{(x_n-y)^{-q+1}} \\ &= \frac{M_1}{\Gamma(q+1)} x_n^q \end{aligned}$$

If we choose $\delta = x_N$ then we can say that there exists $x < \delta$ such that

$$f(x) \geq k_1 \delta^q \tag{27}$$

If we have

$$\lim_{x_n \to 0} \frac{d^q f(x_n)}{dx_n^q} = -\infty$$

then for given $M_2 > 0$ we can find positive integer N such that $d^q f(x_n)/dx_n^q \leq -M_2$ for all $n > N$. Therefore

$$\begin{aligned} f(x_n) &\leq \frac{-M_2}{\Gamma(q)} \int_0^{x_n} \frac{dy}{(x_n - y)^{-q+1}} \\ &= \frac{-M_2}{\Gamma(q+1)} x_n^q \end{aligned}$$

Again if we write $\delta = x_N$, there exists $x < \delta$ such that

$$f(x) \leq -k_2 \delta^q \tag{28}$$

Therefore by (27) and (28)there exists $x < \delta$ such that, for $q > \alpha$,

$$|f(x)| \geq K\delta^q.$$

Again for any $y \in (0,1)$ there exists x such that for $q > \alpha$ and $|x - y| < \delta$

$$|f(x) - f(y)| \geq k\delta^q.$$

Hence we have [10]

$$dim_B f(x) \geq 2 - \alpha.$$

Notice that part (a) of the theorem above is the generalization of the statement that C^1 functions are locally Lipschitz (hence their graphs have dimension 1) to the case when the function has Hölder type upper bound (hence their dimension is greater than one).

Here the function is required to have the same critical order throughout the interval. We can weaken this condition slightly. Since we are dealing with box dimension which is finitely stable [10], we can allow finite number of points having different critical order so that we can divide the set in finite parts having same critical order in each part.

The example of a polynomial of degree n having critical order one and dimension one is consistent with the above result, as we can divide the graph of the polynomial in finite number of parts such that at each point in every part the critical order is one. Using finite stability of box dimension, the dimension of the whole curve will be one.

We can also prove a partial converse of the above theorem.

Theorem 2 *Let $f : [0,1] \to I\!R$ be a continuous function.*

a) Suppose

$$|f(x) - f(y)| \leq c|x - y|^\alpha,$$

where $c > 0$, $0 < \alpha < 1$ *and* $|x - y| < \delta$ *for some* $\delta > 0$. *Then*

$$\lim_{x \to y} \frac{d^q(f(x) - f(y))}{[d(x - y)]^q} = 0 \quad \text{for} \ \ q < \alpha$$

for all $y \in (0, 1)$

b) Suppose that for each $y \in (0, 1)$ *and for each* $\delta > 0$ *there exists* x *such that* $|x - y| \leq \delta$ *and*

$$|f(x) - f(y)| \geq c\delta^{\alpha},$$

where $c > 0$, $\delta \leq \delta_0$ *for some* $\delta_0 > 0$ *and* $0 < \alpha < 1$. *Then there exists a sequence* $x_n \to y$ *as* $n \to \infty$ *such that*

$$\lim_{n \to \infty} \frac{d^q(f(x_n) - f(y))}{[d(x_n - y)]^q} = \pm\infty \quad \text{for} \ \ q > \alpha$$

for all y.

Proof

a)Assume that there exists a sequence $x_n \to y$ as $n \to \infty$ such that

$$\lim_{n \to \infty} \frac{d^q(f(x_n) - f(y))}{[d(x_n - y)]^q} = \pm\infty \quad \text{for} \ \ q < \alpha \ \ ,$$

for some y. Then by arguments between eqn(26) and eqn(27) of the second part of the previous theorem it is a contradiction. Therefore

$$\lim_{x \to y} \frac{d^q(f(x) - f(y))}{[d(x - y)]^q} = const \ \text{ or } \ 0 \quad \text{for} \ \ q < \alpha$$

Now if

$$\lim_{x \to y} \frac{d^q(f(x) - f(y))}{[d(x - y)]^q} = const \quad \text{for} \ \ q < \alpha,$$

then we can write

$$\frac{d^q(f(x) - f(y))}{[d(x - y)]^q} = K + \eta(x, y)$$

where $K = const$ and $\eta(x, y) \to 0$ sufficiently fast as $x \to y$. Now taking ϵ-derivative of both sides, for sufficiently small ϵ we get

$$\frac{d^{q+\epsilon}(f(x) - f(y))}{[d(x - y)]^{q+\epsilon}} = \frac{K(x - y)^{-\epsilon}}{\Gamma(1 - \epsilon)} + \frac{d^{\epsilon}\eta(x, y)}{[d(x - y)]^{\epsilon}} \quad \text{for} \ \ q + \epsilon < \alpha$$

As $x \to y$ the right hand side of the above equation goes to infinity (term involving η doesn't matter since η goes to 0 sufficiently fast) which again is a contradiction. Hence the proof.

b)Proof follows by the method used in the previous section to show that the fractional derivative of order greater than $2 - \alpha$ of the Weierstrass function does not exist.

These two theorems give an equivalence between Hölder exponent and the critical order of fractional differentiability.

5 Local Fractional Derivative as a tool to study pointwise regularity of functions

Motivation for studying pointwise behavior of irregular functions and its relevance in physical processes was given in the Introduction. There are several approaches to studying pointwise behavior of functions. Recently wavelet transforms [29, 30] were used for this purpose and have met with some success. In this section we argue that LFDs is a tool that can be used to characterize irregular functions and has certain advantages over its counterpart using wavelet transforms in aspects explained below. Various authors [32, 40] have used the following general definition of Hölder exponent. The Hölder exponent $\alpha(y)$ of a function f at y is defined as the largest exponent such that there exists a polynomial $P_n(x)$ of order n that satisfies

$$|f(x) - P_n(x-y)| = O(|x-y|^{\alpha}), \tag{29}$$

for x in the neighborhood of y. This definition is equivalent to equation (25), for $0 < \alpha < 1$, the range of interest in this work.

It is clear from theorem I that LFDs provide an algorithm to calculate Hölder exponents and dimensions. It may be noted that since there is a clear change in behavior when order q of the derivative crosses critical order of the function it should be easy to determine the Hölder exponent numerically. Previous methods using autocorrelations for fractal signals [10] involve an additional step of finding an autocorrelation.

5.1 Isolated singularities and masked singularities

Let us first consider the case of isolated singularities. We choose the simplest example $f(x) = ax^{\alpha}, \quad 0 < \alpha < 1, \quad x > 0$. Critical order at $x = 0$ gives the order of singularity at that point whereas the value of the LFD $I\!\!D^{q=\alpha} f(0)$, viz $a\Gamma(\alpha+1)$, gives strength of the singularity.

Using LFD we can detect a weaker singularity masked by a stronger singularity. As demonstrated below, we can estimate and subtract the contribution due to stronger singularity from the function and find out the critical order of the remaining function. Consider, for example, the function

$$f(x) = ax^{\alpha} + bx^{\beta}, \qquad 0 < \alpha < \beta < 1, \quad x > 0. \tag{30}$$

LFD of this function at $x = 0$ of the order α is $I\!\!D^{\alpha} f(0) = a\Gamma(\alpha+1)$. Using this estimate of stronger singularity we now write

$$G(x;\alpha) = f(x) - f(0) - \frac{I\!\!D^{\alpha} f(0)}{\Gamma(\alpha+1)} x^{\alpha}$$

which for the function f in eqn(30) is

$$\frac{d^q G(x'\alpha)}{dx^q} = \frac{b\Gamma(\beta+1)}{\Gamma(\beta-q+1)} x^{\beta-q} \tag{31}$$

Therefore the critical order of the function G, at $x = 0$, is β. Notice that the estimation of the weaker singularity was possible in the above calculation just because the LFD gave the coefficient of $x^{\alpha}/\Gamma(\alpha+1)$. This suggests that using LFD, one should be able to extract secondary singularity spectrum masked by the primary singularity spectrum of strong singularities. Hence one can gain more insight into the processes giving rise to irregular behavior. Also, one may note that this procedure can be used to detect singularities masked by regular polynomial behavior. In this way one can extend the present analysis beyond the range $0 < \alpha < 1$, where α is a Hölder exponent.

Comparison of two methods of studying pointwise behavior of functions, one using wavelets and other using LFD, shows that characterization of Hölder classes of functions using LFD is direct and involves fewer assumptions. Characterization of Hölder class of functions with oscillating singularity, e.g. $f(x) = x^\alpha sin(1/x^\beta)$ ($x > 0$, $0 < \alpha < 1$ and $\beta > 0$), using wavelets needs two exponents [31]. Using LFD, owing to theorem I and II critical order directly gives the Hölder exponent for such a function.

It has been shown in the context of wavelet transforms that one can detect singularities masked by regular polynomial behavior [32] by choosing analyzing wavelet with its first n (for suitable n) moments vanishing. If one has to extend the wavelet method for the unmasking of weaker singularities, one would then require analyzing wavelets with fractional moments vanishing. Notice that one may require this condition along with the condition on first n moments. Further the class of functions to be analyzed is in general restricted in these analyses. These restriction essentially arise from the asymptotic properties of the wavelets used. On the other hand, with truly local nature of LFD one doesn't have to bother about behavior of functions outside our range of interest.

5.2 Treatment of multifractal function

Multifractal measures have been object of many investigations [33, 34, 35, 36, 37]. This formalism has met with many applications. Its importance also stems from the fact such measures are natural measures to be used in the analysis of many phenomenon [38, 39]. It may however happen that the object one wants to understand is a function (e.g. a fractal or multifractal signal) rather than a set or a measure. For instance one would like to characterize the velocity of fully developed turbulence. We now proceed with the analysis of such multifractal functions using LFD.

Now we consider the case of multifractal functions. Since LFD gives the local and pointwise behavior of the function, conclusions of the theorem I will carry over even in the case of multifractal functions where we have different Hölder exponents at different points. Multifractal functions have been defined by Jaffard [40] and Benzi et. al. [41]. However as noted by Benzi et. al. their functions are random in nature and the pointwise behavior can not be studied. Since we are dealing with nonrandom functions in this paper, we shall consider a specific (but nontrivial) example of a function constructed by Jaffard to illustrate the procedure. This function is a solution F of the functional equation

$$F(x) = \sum_{i=1}^{d} \lambda_i F(S_i^{-1}(x)) + g(x). \tag{32}$$

where S_i's are the affine transformations of the kind $S_i(x) = \mu_i x + b_i$ (with $|\mu_i| < 1$ and b_i's real) and λ_i's are some real numbers and g is any sufficiently smooth function (g and its derivatives should have fast decay). For the sake of illustration we choose $\mu_1 = \mu_2 = 1/3$, $b_1 = 0$, $b_2 = 2/3$, $\lambda_1 = 3^{-\alpha}$, $\lambda_2 = 3^{-\beta}$ $(0 < \alpha < \beta < 1)$ and

$$\begin{aligned} g(x) &= sin(2\pi x) \qquad \text{if} \quad x \in [0, 1] \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Such functions are studied in detail in [40] using wavelet transforms where it has been shown that the above functional equation (with the parameters we have chosen) has a unique solution F and at any point F either has Hölder exponents ranging from α to β or is smooth. A sequence of points $S_{i_1}(0),\quad S_{i_2}S_{i_1}(0),\quad \cdots,\quad S_{i_n}\cdots S_{i_1}(0),\quad \cdots$, where i_k takes values 1 or 2, tends to a point in $[0,1]$ (in fact to a point of a triadic cantor set) and for the values of μ_is we have chosen this correspondence between sequences and limits is one to one. The solution of the above functional equation is given by [40] as

$$F(x) = \sum_{n=0}^{\infty} \sum_{i_1,\cdots,i_n=1}^{2} \lambda_{i_1}\cdots\lambda_{i_n} g(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(x)). \tag{33}$$

Note that with the above choice of parameters the inner in (33) sum reduces to a single term. Jaffard [40] has shown

$$h(y) = \liminf_{n\to\infty} \frac{\log(\lambda_{i_1(y)}\cdots\lambda_{i_n(y)})}{\log(\mu_{i_1(y)}\cdots\mu_{i_n(y)})}, \tag{34}$$

where $\{i_1(y)\cdots i_n(y)\}$ is a sequence of integers appearing in the sum in equation(33) at a point y, is the local Hölder exponent at y. It is clear that $h_{min} = \alpha$ and $h_{max} = \beta$. The function F at the points of a triadic cantor set have $h(x) \in [\alpha, \beta]$ and at other points it is smooth (where F is as smooth as g). Now

$$\begin{aligned}
\frac{d^q(F(x)-F(y))}{[d(x-y)]^q} &= \sum_{n=0}^{\infty} \sum_{i_1,\cdots,i_n=1}^{2} \lambda_{i_1}\cdots\lambda_{i_n} \\
&\qquad \frac{d^q[g(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(x)) - g(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(y))]}{[d(x-y)]^q} \\
&= \sum_{n=0}^{\infty} \sum_{i_1,\cdots,i_n=1}^{2} \lambda_{i_1}\cdots\lambda_{i_n}(\mu_{i_1}\cdots\mu_{i_n})^{-q} \\
&\qquad \frac{d^q[g(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(x)) - g(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(y))]}{[d(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(x-y))]^q}.
\end{aligned} \tag{35}$$

Provided the RHS is uniformly bounded. Following the procedure described in section III the fractional derivative in the RHS can easily be seen to be uniformly bounded and series is convergent if $q < \min\{h(x), h(y)\}$. Further it vanishes in the limit as $x \to y$. Therefore if $q < h(y)$ $\mathbb{D}^q F(y) = 0$, as Liouville in the case of Weierstrass function, showing that $h(y)$ is lower bound on critical order.

The procedure of finding an upper bound is technical and lengthy. It is carried out in the Appendix below.

In this way an intricate analysis of finding out the lower bound on Hölder exponent has been replaced by a calculation involving few steps. This calculation can easily be generalized for more general functions $g(x)$. Summarizing, the LFD enables one to calculate the local Hölder exponent even for the case of multifractal functions. This fact, proved in theorems I and II is demonstrated with concrete illustration.

6 Conclusion

In this paper we have introduced the notion of local fractional derivative using Riemann- formulation (or equivalents such as Grunwald's) of fractional calculus. This definition was found to appear naturally in the Taylor expansion (with remainder) of functions and thus is suitable for approximating scaling functions. In particular we have pointed out a possibility of replacing the notion of a tangent as an equivalence class of curves passing through the same point and having the same derivative with a more general one. This more general notion is in terms of an equivalence class of curves passing through the same point and having same critical order and same LFD. This generalization has the advantage of being applicable to non-differentiable functions also.

We have established that (for sufficiently large λ) the critical order of the Weierstrass function is related to the box dimension of its graph. If the dimension of the graph of such a function is $1+\gamma$, the critical order is $1-\gamma$. When γ approaches unity the function becomes more and more irregular and local fractional differentiability is lost accordingly. Thus there is a direct quantitative connection between the dimension of the graph and the fractional differentiability property of the function. This is one of the main conclusions of the present work. A consequence of our result is that a classification of continuous paths (e.g. fractional Brownian paths) or functions according to local fractional differentiability properties is also a classification according to dimensions (or Hölder exponents).

Also the Lévy index of a Lévy flight on one dimensional lattice is identified as the critical order of the characteristic function of the walk. More generally, the Lévy index of a Lévy distribution is identified as the critical order of its characteristic function at the origin.

We have argued and demonstrated that LFDs are useful for studying isolated singularities and singularities masked by the stronger singularity (not just by regular behavior). We have further shown that the pointwise behavior of irregular, fractal or multifractal, functions can be studied using the methods of this paper.

We hope that future study in this direction will make random irregular functions as well as multivariable irregular functions amenable to analytic treatment, which is badly needed at this juncture. Work is in progress in this direction.

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Appendix

Our aim in this is appendix is to find an upper bound on the critical order and hence on the local Hölder exponent for the function of equation (33). Our procedure will be similar to that of Jaffard [40] the only difference being that we take fractional derivative instead of wavelet transform. We proceed as follows. The defining equation for $F(x)$ when iterated N times gives

$$\begin{aligned} F(x) &= \sum_{n=0}^{N-1} \sum_{i_1,\cdots,i_n=1}^{2} \lambda_{i_1}\cdots\lambda_{i_n} g(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(x)) \\ &+ \sum_{i_1,\cdots,i_N=1}^{2} \lambda_{i_1}\cdots\lambda_{i_N} F(S_{i_N}^{-1}\cdots S_{i_1}^{-1}(x)) \qquad x\in[0,1]. \end{aligned} \tag{36}$$

We now consider

$$\begin{aligned} \frac{d^q(F(x)-F(y))}{[d(x-y)]^q} &= \sum_{n=0}^{N-1} \sum_{i_1,\cdots,i_n=1}^{2} \lambda_{i_1}\cdots\lambda_{i_n}(\mu_{i_1}\cdots\mu_{i_n})^{-q} \\ &\qquad \frac{d^q[g(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(x)) - g(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(y))]}{[d(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(x-y))]^q} \\ &+ \sum_{i_1,\cdots,i_N=1}^{2} \lambda_{i_1}\cdots\lambda_{i_N}(\mu_{i_1}\cdots\mu_{i_N})^{-q} \\ &\qquad \frac{d^q[F(S_{i_N}^{-1}\cdots S_{i_1}^{-1}(x)) - F(S_{i_N}^{-1}\cdots S_{i_1}^{-1}(y))]}{[d(S_{i_N}^{-1}\cdots S_{i_1}^{-1}(x-y))]^q}. \end{aligned} \tag{37}$$

Let us denote the first term on RHS by A and the second term by B. In the following $y\in(0,1)$. Choose N such that $3^{-(N+1)} < |x-y| < 3^{-N}$. Denote $\lambda_{n(y)} = \lambda_{i_1(y)}\cdots\lambda_{i_n(y)}$. For the values of μ_is we have chosen $\mu_{i_1}\cdots\mu_{i_n} = 3^{-n}$. Now since g is smooth $|g(x)-g(y)| \le C|x-y|$.

$$\frac{d^q[g(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(x)) - g(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(y))]}{[d(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(x-y))]^q} \le C\left|\frac{x-y}{\mu_{i_1}\cdots\mu_{i_n}}\right|^{1-q} \tag{38}$$

From the way we chosen μ_is and $|x-y|$ this term is bounded by $C3^{n(1-q)}3^{-N(1-q)}$ Therefore the first term in equation (37) above is bounded by

$$\begin{aligned} A &\le C\sum_{n=0}^{N-1} \lambda_{n(y)} 3^{nq} 3^{n(1-q)} 3^{-N(1-q)} \\ &= C3^{-N(1-q)} \sum_{n=0}^{N-1} \lambda_{n(y)} 3^n \\ &\le C3^{-N(1-q)} \lambda_{(N-1)(y)} 3^{N-1}\left(1 + \frac{\lambda_{(N-2)(y)}}{3\lambda_{(N-1)(y)}} + \frac{\lambda_{(N-3)(y)}}{3^2\lambda_{(N-1)(y)}} + \cdots\right) \end{aligned} \tag{39}$$

The quantity in the bracket is bounded. Therefore

$$A \le C3^{Nq}\lambda_{N(y)} \tag{40}$$

Now we consider the second term in the RHS of equation (37) and find a lower bound on that term. We assume that $F(x)$ is not Lipschitz at y (for otherwise LHS of equation (37) will be zero in the limit $x\to y$ and the case is uninteresting). Therefore there exists a sequence of points x_n approaching y such that

$$|F(x_n) - F(y)| \ge c|x_n - y| \tag{41}$$

The function is not Lipschitz at y implies that it not Lipschitz at $S_{i_N}^{-1}\cdots S_{i_1}^{-1}(y)$. Therefore we get (one may recall that only one of the 2^n terms in the summation $\sum_{i_1,\cdots,i_N=1}^{2}$ is nonzero)

$$\begin{aligned} B &\geq c\lambda_{N(y)}3^{Nq}3^{N(1-q)}3^{-N(1-q)} \\ &\geq c\lambda_{N(y)}3^{Nq} \end{aligned} \tag{42}$$

Therefore there exists a sequence x_n such that

$$|\frac{d^q(F(x_n)-F(y))}{[d(x_n-y)]^q} - c\lambda_{N(y)}3^{Nq}| \leq C\lambda_{N(y)}3^{Nq} \tag{43}$$

Since (41) is valid for every c for large enough n

$$\frac{d^q(F(x_n)-F(y))}{[d(x_n-y)]^q} \geq C'\lambda_{N(y)}3^{Nq}$$

where $C' = c - C$ Now from the expression of $h(y)$ it is clear that when $q > h(y)$ LFD does not exist and the critical order is bounded from above by $h(y)$.

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