Spectral Theory of Thermal Relaxation

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Abstract. We review some results obtained in a recent series of papers on thermal relaxation in classical and quantum dissipative systems. We consider models where a *small system* $\mathscr{S}$, with a finite number of degrees of freedom, interacts with a *large environment* $\mathscr{R}$ in thermal equilibrium at positive temperature T. The zeroth law of thermodynamics postulates that, independently of its initial configuration, the system $\mathscr{S}$ approaches a unique stationary state as $t \to \infty$. By definition, this limiting state is the equilibrium state of $\mathscr{S}$ at temperature T. Statistical mechanics further identifies this state with the Gibbs canonical ensemble associated with $\mathscr{S}$. For *simple* models we prove that the above picture is correct, provided the equilibrium state of the environment $\mathscr{R}$ is itself given by its canonical ensemble. In the quantum case we also obtain an exact formula for the thermal relaxation time.

1. Introduction

This paper is an informal outline of the results proven by the authors in a recent series of papers [1 – 8]. It is based on a lecture given by the second author on the occasion of the "Institut Henri Poincaré/Gauthier-Villars prizes 1996."

We have attempted to explain our results on simple models and to emphasize the physical content of the ideas involved. Interested reader should consult the original papers for a mathematically precise account of the theory. We note that this paper is not a review of the subject. In particular, we will only discuss works which motivated, or are directly related to our own.

The project discussed here began while the authors were post-doctoral fellows at the University of Toronto. We are grateful to I.M. Sigal for suggesting the problem to us, and providing the necessary *impetus*. We were helped by many colleagues and friends, and we use this occasion to thank them all. To B. Simon and J.-P. Eckmann we owe much more. Without their constant support this work would not have been possible. Finally we thank the editorial board of the "Annales de l'Institut Henri Poincaré" for distinguishing our article [1] among many well deserving papers.

2. Motivation

We are interested in the long-time behavior of the dissipative dynamics which occurs when a system with finitely many degrees of freedom is brought into contact with an infinitely extended environment in thermal equilibrium. This is a standard topic in non-equilibrium statistical mechanics. Two well-known examples are the motion of a Brownian particle in a fluid and the dynamics of an atom coupled to the electro-magnetic field.

We denote the finite system by $\mathscr{S}$ and the environment (conventionally called the reservoir) by $\mathscr{R}$. Under *normal conditions* the system $\mathscr{S}$ slightly perturbs the reservoir, and the interacting system $\mathscr{S}+\mathscr{R}$ will return to equilibrium [9]. Furthermore, this equilibrium state is the Gibbs canonical ensemble of the combined system at the temperature of the reservoir. This fundamental fact is a major ingredient of the phenomenological theory of thermal relaxation and underlies the very definition of temperature for the finite system $\mathscr{S}$ [10].

We briefly review the phenomenological theory on two famous examples. The first is the dynamical theory of Brownian motion based on the Langevin equation. The second is Einstein's theory of spontaneous and induced radiative transitions. The common feature of these two examples is an approximate description of the dynamics by a Markoff process.

Dynamical theory of Brownian motion. A heavy particle $\mathscr{S}$ of mass m and position $q \in \mathbf{R}^3$ is moving in a fluid $\mathscr{R}$ of lighter particles. The phenomenological equation which describes the dynamics of the Brownian particle $\mathscr{S}$ is the Langevin equation [11]. In its simplest form, this equation is

$$m\ddot{q}_t = -W\dot{q}_t + \xi_t, \tag{2.1}$$

where W is a friction constant and $\xi_t = (\xi_t^{(1)}, \xi_t^{(2)}, \xi_t^{(3)})$ a Gaussian stochastic process with mean zero and covariance $\langle \xi_t^{(j)}\, \xi_s^{(k)} \rangle = \mu\delta_{jk}\delta(t-s)$. We will identify the coefficient μ later.

The Langevin equation departs from Newton's equation for a free particle by the addition of two forces. The first is a frictional force due to the reaction of the reservoir to the motion of the Brownian particle. The second is a random force due to the thermal motion of the particles of the fluid.

Equ. (2.1) is of course exactly solvable, and one gets

$$\lim_{t\to\infty}\frac{\langle (q_t-q_0)^2\rangle}{t}=\frac{3\mu}{W^2},\qquad \lim_{t\to\infty}\frac{m}{2}\langle |\dot q_t|^2\rangle=\frac{3\mu}{4W}.$$

It follows form the first equation that $\mu = 2DW^2$, where D is the diffusion constant of the Brownian particle. The mean value of the kinetic energy of a particle in thermal equilibrium at temperature T is $3kT/2$, and the second equation yields *the Einstein relation*

$$D=\frac{kT}{W}. \tag{2.2}$$

If the friction constant W is known (for example from Stokes' law), then a measurement of D yields the value of the Boltzmann constant k and thus of the Avogadro number. The historical importance of these arguments in the establishment of the atomistic nature of matter is well-known [11].

Matter-radiation equilibrium. We consider a simple model where a two-level atom $\mathscr{S}$, with energy levels $E_1 < E_2$, interacts with a radiation field $\mathscr{R}$. Let $u(\nu)$ be the equilibrium spectral energy density of this field at temperature T. When immersed in the radiation field, the atom continuously makes transitions between its energy levels by absorbing and emitting radiation. A deep insight into the statistics of these transitions has lead Einstein to a new derivation of Planck's law [12]. We briefly paraphrase this well-known argument.

According to Bohr's frequency rule, a transition from level E_i to level E_k occurs with the emission ($i > k$) or absorption ($i < k$) of a photon of frequency $h\nu = |E_i - E_k|$. Let Γ_{ik} be the probability per unit time that the atom will make such a transition, and denote by $p_i(t)$ the occupation probability of the level E_i. The corresponding phenomenological equation of motion

$$\begin{aligned}\dot p_1(t)&=\Gamma_{21}\,p_2(t)-\Gamma_{12}\,p_1(t),\\ \dot p_2(t)&=\Gamma_{12}\,p_1(t)-\Gamma_{21}\,p_2(t),\end{aligned} \tag{2.3}$$

is the simplest example of Pauli's master equation. Einstein made the following Ansatz for the general form of the transition rates Γ_{ik}:

$$\begin{aligned}\Gamma_{12}&=u(\nu)B_{12},\\ \Gamma_{21}&=u(\nu)B_{21}+A_{21}.\end{aligned} \tag{2.4}$$

The coefficient A_{21} corresponds to spontaneous emission, and is independent of the spectral energy density of the radiation. The B's correspond to induced emission and absorption [13]. It follows from Equ. (2.3) that $p_i(t) \to \overline{p}_i$ as $t \to \infty$, with a stationary probability distribution satisfying the detailed balance relation $\overline{p}_1\Gamma_{12} = \overline{p}_2\Gamma_{21}$. On the other hand, if the system $\mathscr{S} + \mathscr{R}$ is in thermal equilibrium at temperature T then

$$\overline{p}_i=\frac{\mathrm{e}^{-E_i/kT}}{\mathrm{e}^{-E_1/kT}+\mathrm{e}^{-E_2/kT}}.$$

Combining these facts with the Rayleigh-Jeans law (which was known to hold in the limit $T \to \infty$) one derives the relation

$$u(\nu) = \frac{8\pi}{c^3} \frac{\nu^2 (E_2 - E_1)}{\mathrm{e}^{(E_2 - E_1)/kT} - 1}.$$

The substitution $h\nu = E_2 - E_1$ yields Planck's law for spectral energy density of a black-body radiation in thermal equilibrium at temperature T

$$u(\nu) = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{\mathrm{e}^{h\nu/kT} - 1}.$$

Each photon contributes an energy $h\nu$ to this density, therefore the corresponding spectral photon density is

$$\varrho(\nu) = \frac{8\pi\nu^2}{c^3} \frac{1}{\mathrm{e}^{h\nu/kT} - 1}. \tag{2.5}$$

In general, the phenomenological equations only approximate the motion of the system $\mathscr{S}$ for suitable values of the physical parameters and on a suitable time scale. The surprising fact is that these equations are in some sense universal: They arise in a variety of physical contexts, with very different microscopic structure. For example, the Langevin equation is usually a good approximation of the motion of a fluctuating mode in a macroscopic environment [14]. Pauli's master equation is believed to describe the motion of general, weakly interacting quantum systems on the appropriate time scale [15].

From a modern point of view, a more satisfactory theory of thermal relaxation should be based on microscopic considerations. In this framework we expect the traditional phenomenological theory to appear as a first approximation, in suitable limiting situations. The related program can be roughly summarized as follows.

(i) Introduce a realistic microscopic model of the system $\mathscr{S} + \mathscr{R}$ and construct a dynamical system describing this model near thermal equilibrium.

(ii) Show that this system enjoys strong ergodic properties (mixing, K-property, *etc.*).

(iii) Derive the exact transport equation of thermal relaxation and obtain the phenomenological theory as a first approximation to this equation.

Realistic reservoirs arise as thermodynamic limits of interacting particle systems. Their intrinsic complexity appears to compromise the above program at an early stage. To proceed anyhow, one considers ideal reservoirs, *i.e.* non-interacting particle systems. Since it is usually easy to show that an ideal reservoir is a strongly ergodic system, there is at least a hope that the program could be carried out.

The reservoir we consider is an infinitely extended gas of free bosons in thermal equilibrium. The main physical example is the free electro-magnetic field. In this case, the system $\mathscr{S}$ is a finite assembly of confined charged particles and the coupling is introduced according to the rules of non-relativistic electrodynamics. Similar models arise in solid state physics (*e.g.* the spin-boson system [16]). The Ford-Kac-Mazur model of a particle coupled to a chain of harmonic oscillators [17] is another well known example.

In the sequel we outline how the above program is carried out for such models. We will use the terms *friction* and *dissipation* strictly in this limited context. For brevity and clarity, we will consider very simple (but still physically acceptable) models.

3. Friction in classical physics

The system $\mathscr{S}$ is a particle of mass m moving in $\mathbf{R}^3$ under the influence of a smooth confining potential $V(q)$ [18]. Its Hamiltonian function is

$$H_{\mathscr{S}}(q,p) \equiv \frac{|p|^2}{2m} + V(q),$$

and the corresponding equation of motion is

$$\begin{aligned} m\dot{q}_t &= p_t, \\ \dot{p}_t &= -\nabla V(q_t). \end{aligned}$$

The system $\mathscr{R}$—an infinitely extended gas of free classical phonons—is described by the classical field theory associated to the 3-dimensional wave equation. Its Hamiltonian is

$$H_{\mathscr{R}}(\varphi,\pi) \equiv \frac{1}{2}\int \left(|\nabla\varphi(x)|^2 + |\pi(x)|^2\right)\, d^3x,$$

where φ, π are canonically conjugate field variables. In the sequel we will use the shorthand

$$\phi \equiv \begin{pmatrix} \varphi \\ \pi \end{pmatrix}.$$

The equation of motion of the system $\mathscr{R}$ is

$$\dot{\phi}_t = L_{\mathscr{R}}\phi_t, \tag{3.6}$$

where

$$L_{\mathscr{R}} \equiv \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}.$$

Let $\Gamma_{\mathscr{R}}$ be the phase space of finite energy configurations of the reservoir. It is a Hilbert space with the inner product $(\phi,\phi) \equiv 2H_{\mathscr{R}}(\phi)$ [19]. In this space, the initial value problem (3.6) is solved by the unitary group $\mathrm{e}^{L_{\mathscr{R}}t}$.

The phase space of finite energy configurations of the combined system $\mathscr{S}+\mathscr{R}$ is $\mathbf{R}^6 \times \Gamma_{\mathscr{R}}$, and its Hamiltonian function is

$$H(q,p,\varphi,\pi) \equiv H_{\mathscr{S}}(q,p) + H_{\mathscr{R}}(\varphi,\pi) + \lambda\, q \cdot\!\int\!\rho(x)\nabla\varphi(x)\, d^3x + \frac{\lambda^2}{6}|q|^2. \tag{3.7}$$

Here λ is a real coupling constant and ρ a normalized "charge density." For simplicity, we assume that ρ is spherically symmetric. The quadratic term $\lambda^2|q|^2/6$ is an appropriate renormalization of the potential. For notational purposes let

$$\phi(f) \equiv (\phi, f),$$

for $\phi, f \in \Gamma_{\mathscr{R}}$, and $\alpha \equiv (\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)})$, with

$$\widehat{\alpha}^{(j)}(k) \equiv \begin{pmatrix} (i|k|)^{-1}\widehat{\rho}(k) \\ 0 \end{pmatrix} \frac{k^{(j)}}{|k|}.$$

Here, and subsequently in this paper, the "hat" denotes Fourier transform

$$\widehat{f}(k) \equiv \frac{1}{(2\pi)^{3/2}} \int f(x) \mathrm{e}^{-ik\cdot x}\, d^3x.$$

In this notation, The Hamiltonian (3.7) becomes

$$H(q,p,\phi) = H_{\mathscr{S}}(q,p) + \frac{1}{2}\|\phi\|^2 + \lambda\, q\cdot\phi(\alpha) + \frac{\lambda^2}{6}|q|^2.$$

With $V_{\mathit{eff}}(q) \equiv V(q) + \lambda^2|q|^2/6$, the evolution equation for finite energy configurations is

$$\begin{aligned} m\dot{q}_t &= p_t, \\ \dot{p}_t &= -\nabla V_{\mathit{eff}}(q_t) - \lambda\phi_t(\alpha), \\ \dot{\phi}_t &= L_{\mathscr{R}}(\phi_t + \lambda q_t\cdot\alpha). \end{aligned} \tag{3.8}$$

The last equation is easily integrated, and insertion into the first two equations leads to

$$\begin{aligned} m\dot{q}_t &= p_t, \\ \dot{p}_t &= -\nabla V_{\mathit{eff}}(q_t) + \lambda^2 \int_0^t D(t-s) q_s\, ds + \lambda\xi_t, \end{aligned} \tag{3.9}$$

where the entries of the dissipation matrix D are given by

$$D_{ij}(t) \equiv \left(\alpha^{(i)}, \mathrm{e}^{-L_{\mathscr{R}}t} L_{\mathscr{R}}\alpha^{(j)}\right) = \frac{\delta_{ij}}{3} \int |\widehat{\rho}(k)|^2 |k| \sin(|k|t)\, d^3k,$$

and

$$\xi_t \equiv -\phi_0(\mathrm{e}^{-L_{\mathscr{R}}t}\alpha).$$

So far, we have only discussed finite energy initial configurations of the reservoir. In this situation the relevant physical process is radiative damping: The system $\mathscr{S}$ continuously dissipates its energy into the reservoir. As this energy propagates towards infinity, $\mathscr{S}$ comes to rest at some critical point $\overline{q}$, where $\nabla V(\overline{q}) = 0$. This phenomenon is analyzed in detail in a recent work [20].

If the reservoir is initially in thermal equilibrium at temperature $T > 0$, one has to allow for infinite energy initial configurations ϕ_0. These initial configurations are distributed according to a statistical law, and ξ_t becomes a random noise. More precisely, the thermodynamic limit yields that ξ_t is a Gaussian random process with mean zero and covariance

$$\langle \xi_t^{(i)}\, \xi_s^{(j)} \rangle_\beta \equiv \beta^{-1} C_{ij}(t-s) = \frac{\delta_{ij}}{3\beta} \int |\widehat{\rho}(k)|^2 \cos(|k|(t-s))\, d^3k.$$

The random integro-differential equation (3.9) is obviously a generalization of the Langevin equation discussed in Section 2. We will call it *the non-Markovian Langevin equation* (NML-equation), and the resulting random process *the non-Markovian Ornstein-Uhlenbeck process* (NMOU-process).

The equation

$$D(t) = -\dot{C}(t), \tag{3.10}$$

which holds by inspection, relates the dissipation mechanism to the random fluctuations of the reservoir in thermal equilibrium. It is an example of a more general principle known as "the fluctuation-dissipation Theorem." We remark that Equ. (3.10) and Einstein's relation (2.2) are of a completely different nature. The phenomenological Einstein relation is dynamical: It holds under the assumption that Brownian motion in a medium in thermal equilibrium also reaches thermal equilibrium [15]. In other words, as $t \to \infty$, the state of the system $\mathscr{S}$ converges towards the Gibbs canonical ensemble

$$d\mu_{\mathscr{S}}^{\beta} \equiv \frac{1}{Z^\beta} \mathrm{e}^{-\beta H_{\mathscr{S}}(q,p)}\, d^3q\, d^3p.$$

Relation (3.10) is purely kinematical: It follows entirely from the structure of the model and the form of the initial state of the reservoir. From the microscopic point of view, the principal assumption of the phenomenological theory becomes an ergodic problem for the NMOU-process. The main goal of our program is to study this ergodic problem.

The process generated by Equ. (3.9) is non-Markovian and for this reason difficult to study. The usual way around this problem is based on the following limiting procedure. For fixed $\kappa > 0$, choose sequences of coupling constants λ_n and coupling functions ρ_n in such a way that, for the corresponding sequence of covariances, one has $\lambda_n^2 C_n(t) \to \kappa\delta(t)$ as $n \to \infty$. Using Relation (3.10), and integrating by parts the memory term in Equ. (3.9) we obtain, in the limit, the equation

$$m\ddot{q}_t = -\nabla V(q_t) - \kappa \dot{q}_t - 2\kappa\delta(t) q_0 + \xi_t, \tag{3.11}$$

where ξ_t is the Gaussian white noise process with covariance $\langle \xi_t^{(i)}\, \xi_s^{(j)} \rangle_\beta = 2\kappa\beta^{-1}\delta_{ij}\delta(t-s)$. Except for a singular term which depends on the initial position of the particle [21], Equ. (3.11) is the well-known Markovian Langevin equation. Its ergodic properties can be studied by standard techniques [22].

Our goal is to develop the ergodic theory of the NMOU-process generated by Equ. (3.9). If the condition

$$\left\| \left(-\Delta + |x|^2 \right)^s \rho \right\| < \infty, \tag{3.12}$$

holds for some $s > 3/2$ then, for any λ and almost all ξ, the initial value problem associated with Equ. (3.9) has a global solution which defines a flow

$$T_\xi^t(q_0, p_0) \equiv (q_t, p_t),$$

on the phase space $\mathbf{R}^6$. This ensures the existence of the NMOU-process. We can now formulate the relevant ergodic problem. We say that the NMOU-process *returns to equilibrium* [23] if, for any initial state μ (an absolutely continuous probability measure on $\mathbf{R}^6$) and for any observable $f \in L^\infty(\mathbf{R}^6)$, one has

$$\lim_{t\to\infty} \int \left\langle f \circ T_\xi^t \right\rangle_\beta d\mu = \int f \, d\mu_{\mathscr{S}}^\beta. \tag{3.13}$$

We remark that return to equilibrium is equivalent to the mixing property of the NMOU-process. Our main result is that if (3.12) and

$$|\widehat{\rho}(k)| \geqslant \frac{C}{(1+|k|)^\nu}, \tag{3.14}$$

hold for some $s > 3/2$, some positive constants C and ν and all $k \in \mathbf{R}^3$ then, for all non-vanishing λ, the NMOU-process returns to equilibrium.

The NML-equation (3.9) is a contracted description of the dynamics of the system $\mathscr{S}$. It incorporates the reservoir in a relatively simple way, through the frictional and fluctuating forces which are related by Equ. (3.10). However, the NML-equation *per se* does not completely reveal the physical mechanism of thermal relaxation. Let us elucidate this point. The initial states we consider are local perturbations of the joint thermal equilibrium state. The combined system returns to equilibrium if such perturbations disperse to infinity. To understand this process, one has to take into account the motion of the reservoir. The strong ergodic properties of the combined system arise precisely through this motion. Thus, it is both conceptually and technically important to take the systems $\mathscr{S}$ and $\mathscr{R}$ on equal footing. The principal idea of our approach is to study the ergodic properties of the differentiable dynamical system $(\mathscr{G}, \Xi^t, \mu^\beta)$ which describes the combined system $\mathscr{S} + \mathscr{R}$ near thermal equilibrium. Here, $\mathscr{G}$ is the phase space, Ξ^t the Hamiltonian flow, and μ^β the Gibbs canonical ensemble of the combined system in thermal equilibrium at inverse temperature β. If this dynamical system is mixing then return to equilibrium (3.13) follows.

To describe the dynamical system $(\mathscr{G}, \Xi^t, \mu^\beta)$, let ϕ be the Gaussian random field indexed by the Hilbert space $\Gamma_{\mathscr{R}}$, with covariance

$$\langle \phi(f)\phi(g)\rangle_\beta = \beta^{-1}(f,g),$$

for $f, g \in \Gamma_{\mathscr{R}}$. We denote by $(\mathscr{G}_{\mathscr{R}}, \mathscr{F}_{\mathscr{R}}, \mu_{\mathscr{R}}^\beta)$ the associated probability space. The phase space of the reservoir is $\mathscr{G}_{\mathscr{R}}$, and $\mu_{\mathscr{R}}^\beta$ is its thermal equilibrium state. The construction of this space is discussed in detail in [7], see also [24]. The phase space of the system $\mathscr{S} + \mathscr{R}$ is $\mathscr{G} \equiv \mathbf{R}^6 \times \mathscr{G}_{\mathscr{R}}$, and its Gibbs canonical ensemble is the probability measure

$$d\mu^\beta \equiv \mathrm{e}^{-\beta(\lambda q\cdot\phi(\alpha)+\lambda^2|q|^2/6)} \, d\mu_{\mathscr{S}}^\beta \, d\mu_{\mathscr{R}}^\beta.$$

The existence of the dynamics is guaranteed by the following result: If Condition (3.12) holds for some $s > 3/2$ then, for all λ and for $\mu_{\mathscr{R}}^\beta$-almost all $\phi_0 \in \mathscr{G}_{\mathscr{R}}$, the initial value problem associated with Equ. (3.8) has a global solution which defines a flow $\Xi^t(q_0, p_0, \phi_0) \equiv (q_t, p_t, \phi_t)$ on the phase space $\mathscr{G}$. The measure μ^β is invariant under this Hamiltonian flow. Let S^β be the set of states which are local perturbations of the thermal equilibrium state. In technical terms, S^β

is the set of all probability measures on $\mathscr{G}$ which are absolutely continuous with respect to μ^β. We say that combined system $\mathscr{S}+\mathscr{R}$ returns to equilibrium if the dynamical system $(\mathscr{G}, \Xi^t, \mu^\beta)$ satisfies

$$\lim_{t\to\infty} \int F \circ \Xi^t \, d\mu = \int F \, d\mu^\beta, \tag{3.15}$$

for all $\mu \in S^\beta$ and $F \in L^\infty(\mathscr{G}, d\mu^\beta)$. Clearly, (3.13) is a consequence of (3.15). Our main results is that if Conditions (3.12) and (3.14) hold then, for all non-vanishing λ, the system $\mathscr{S}+\mathscr{R}$ returns to equilibrium.

The strategy of our approach (Koopmanism) is to reduce the ergodic properties of the system $\mathscr{S}+\mathscr{R}$ to a spectral problem for a distinguished self-adjoint operator. This operator, which we call the Liouvillean, is constructed as follows. The Koopman space of the combined system is $L^2(\mathscr{G}, d\mu^\beta)$, and the map

$$U^t \colon F \mapsto F \circ \Xi^t,$$

is a strongly continuous unitary group on this space. The Liouvillean $\mathscr{L}$ is the skew-adjoint generator of this group [25]. Note that constant functions are always invariant, therefore zero is always an eigenvalue of $\mathscr{L}$. Return to equilibrium (3.15) is equivalent to

$$\mathrm{w}-\lim_{t\to\infty} U^t = |1\rangle\langle 1|. \tag{3.16}$$

By the well-known property of absolutely continuous spectrum, Relation (3.16) holds if zero is a simple eigenvalue of $\mathscr{L}$ and if the rest of its spectrum is absolutely continuous. We have proven this spectral result (and all its consequences) in [7].

The results discussed here hold in a more general setting. All what we require is that $\mathscr{S}$ is a Hamiltonian system whose configuration space is a finite dimensional smooth manifold. In particular, the system $\mathscr{S}$ could be a confined macroscopic gas of interacting particles. The reservoir could be any linear dynamical system for which there is an outgoing subspace in the sense of Lax-Phillips theory (*e.g.* wave equation, Maxwell's equations, *etc.*). Naturally, we do need some restrictions on the form of the coupling between the two systems. For details and additional information we refer the reader to [6 – 8].

4. Friction in quantum physics

4.1. Quantizing dissipative systems

The theory of classical friction is based on the Hamiltonian (3.7). After canonical quantization we obtain the starting point of the theory of quantum friction: The Hamiltonian

$$\begin{aligned} &\Big(-\frac{1}{2m}\Delta + V(q)\Big) + \int \epsilon(k)\, a^*(k) a(k)\, d^3k \\ &+ \lambda\, q \cdot \int ik \left(\overline{\widehat{\rho}(k)} a(k) - \widehat{\rho}(k) a^*(k)\right) \epsilon(k)^{-1/2}\, d^3k + \frac{\lambda^2}{6}|q|^2, \end{aligned} \tag{4.17}$$

acting on $L^2(\mathbf{R}^3) \otimes \mathscr{F}$, where $\mathscr{F}$ is the bosonic Fock space over $L^2(\mathbf{R}^3)$. The Bose field satisfies the canonical commutation relation $[a(k), a^*(k')] = \delta(k-k')$ and the photon dispersion relation is $\epsilon(k) = |k|$. The function $\widehat{\rho}$ is the Fourier transform of the normalized "charge density" figuring in Equ. (3.7). The Hamiltonian (4.17) is a self-adjoint operator bounded from below provided $\epsilon^{1/2}\widehat{\rho} \in L^2(\mathbf{R}^3)$.

At the moment, we can not prove all the results discussed below for the model defined by the Hamiltonian (4.17). To avoid technicalities and simplify the exposition we restrict ourselves to the case where the system $\mathscr{S}$ is a spin $1/2$ (or equivalently the two-level atom of Section 2). The resulting system $\mathscr{S}+\mathscr{R}$ is known as *the spin-boson model.* Let σ_x, σ_y and σ_z be the Pauli matrices and

$$\varphi(f) \equiv \frac{1}{\sqrt{2}}\int \left(\widehat{f}(k)a^*(k) + \overline{\widehat{f}(k)}a(k)\right)\, d^3k,$$

the Segal field operator of the reservoir. The spin-boson Hamiltonian is

$$H \equiv H_{\mathscr{S}} + H_{\mathscr{R}} + \lambda H_{int} \equiv \sigma_z + \int \epsilon(k)\, a^*(k)a(k)\, d^3k + \lambda\, \sigma_x \otimes \varphi(g), \tag{4.18}$$

acting on $\mathbf{C}^2 \otimes \mathscr{F}$. Here λ is a coupling constant and $g \in L^2(\mathbf{R}^3)$ a normalized coupling function. We will only discuss the ohmic case ($\widehat{g}(k) \sim |k|^{1/2}$ for small k) which is characteristic of most physical applications.

Observables of $\mathscr{S}$ are self-adjoint elements of the algebra M_2 of complex 2×2-matrices. A natural choice of observables for $\mathscr{R}$ is the C^*-algebra $\mathscr{O}_{\mathscr{R}}$ generated by

$$\{\mathrm{e}^{i\varphi(f)} : f \in \mathscr{D}_{loc}\},$$

where $\mathscr{D}_{loc}$ is the space of compactly supported functions $f(x) \in L^2(\mathbf{R}^3)$. Consequently, observables of the spin-boson system are self-adjoint elements of the C^*-algebra

$$\mathscr{O} \equiv M_2 \otimes \mathscr{O}_{\mathscr{R}}.$$

We recall that a state ω of the spin-boson system is a normalized, positive linear functional on this algebra.

The spin-boson system is considered as the simplest physically acceptable model of quantum friction, and gave rise to an enormous literature [16]. Nevertheless, mathematically rigorous results are scarce (essentially limited to the weak coupling regime) and our understanding of the model is still incomplete.

Before we turn to the discussion of the spin-boson model, we would like to briefly comment on an alternative approach to the quantization of classical friction.

4.2. The quantum Langevin equation

In the same way the Langevin equation was deduced from the Hamilton equations (3.8), the Heisenberg equations associated to the Hamiltonian (4.17) can be used to derive the quantum Langevin equation (QLE) [26]

$$m\ddot{q}(t) = -\nabla V(q(t)) - \kappa\dot{q}(t) + \xi_Q(t).$$

Here $\xi_Q(t)$ is a Gaussian *operator-valued* random force characterized by the commutation relation

$$\frac{1}{i}\left[\xi_Q^{(i)}(t),\xi_Q^{(j)}(s)\right]=\frac{2\kappa}{3}\delta_{ij}\int|k|\sin(|k|(t-s))\,d^3k,$$

and the symmetric correlation function

$$\frac{1}{2}\left\langle\xi_Q^{(i)}(t)\xi_Q^{(j)}(s)+\xi_Q^{(j)}(s)\xi_Q^{(i)}(t)\right\rangle_\beta=\frac{2\kappa}{3}\delta_{ij}\int|k|\coth(\beta|k|/2)\cos(|k|(t-s))\,d^3k.$$

QLE has an intuitive appeal as the generator of a "quantum Ornstein-Uhlenbeck process." Since the seminal work of Ford, Kac and Mazur [17], a substantial body of work has been devoted to it. However, unlike its classical counterpart, QLE is a very singular object and the existing rigorous results are limited to small perturbations of exactly solvable harmonic models [27].

In our approach the model for quantum friction is the quantization of the dynamical system $(\mathscr{G},\Xi^t,\mu^\beta)$ introduced in Section 3. In this way we bypass QLE and related difficulties. Since QLE plays no role in our scheme, we will not discuss it any further.

4.3. Master equations

In this subsection we present an informal account of the theory of master equations [28], a systematic approach to the phenomenological description discussed in Section 2.

We denote by $\omega_{\mathscr{R}}^\beta$ the thermal equilibrium state of the reservoir at inverse temperature β (see subsections 4.4 – 4.5 for a precise definition). We assume that the initial state of the system has the product structure $\eta\otimes\omega_{\mathscr{R}}^\beta$, where η is a state (a density matrix) of the system $\mathscr{S}$. We denote by P the projection on such states, $P\zeta\equiv\mathrm{Tr}_{\mathscr{R}}(\zeta)\otimes\omega_{\mathscr{R}}^\beta$, and set $Q\equiv 1-P$. We are interested in the reduced dynamics obtained by tracing out the reservoir variables:

$$\eta_\lambda(t)=T_\lambda^t\eta\equiv\mathrm{Tr}_{\mathscr{R}}\left(\mathrm{e}^{-iHt}\,\eta\otimes\omega_{\mathscr{R}}^\beta\,\mathrm{e}^{iHt}\right).$$

This reduced dynamics is governed by the generalized master equation

$$\frac{d\eta_\lambda(t)}{dt}=-iL_{\mathscr{S}}\eta_\lambda(t)-\lambda^2\int_0^t\mathrm{Tr}_{\mathscr{R}}\left(L_{int}Q\mathrm{e}^{-iQLQ(t-s)}QL_{int}\,\eta_\lambda(s)\otimes\omega_{\mathscr{R}}^\beta\right)ds,$$

where

$$L\equiv[H,\cdot\,],\quad L_{\mathscr{S}}\equiv[H_{\mathscr{S}},\cdot\,],\quad L_{int}\equiv[H_{int},\cdot\,].$$

Clearly, the evolution T_λ^t is non-Markovian. However, in certain limiting cases, memory effects disappear and T_λ^t becomes a semi-group. Among the various Markovian approximations discussed in the physics literature, the van Hove limit $\lambda\to 0$, $t\to\infty$, with $\bar{t}=\lambda^2 t$ fixed, is best understood. In the sequel we discuss some rigorous results of Davies [29]. Let

$$h(t)\equiv\int\mathrm{e}^{i\epsilon(k)t}|\widehat{g}(k)|^2\mathrm{cth}(\beta|k|/2)\,d^3k.$$

The fundamental result of Davies states that

$$\lim_{\lambda\to 0}\mathrm{e}^{iL_{\mathscr{S}}\bar{t}/\lambda^2}T_\lambda^{\bar{t}/\lambda^2}\eta=\mathrm{e}^{-\mathscr{K}\bar{t}}\eta,$$

provided

$$\int_0^\infty (1+t)^\delta |h(t)|\, dt < \infty,$$

holds for some $\delta > 0$. The generator $\mathscr{K}$ can be explicitly computed. Let

$$\begin{aligned} \Gamma_\pm^\beta &\equiv 2\pi \frac{\mathrm{e}^{\pm\beta}}{\sinh(\beta)} \int_{S^2} |\widehat{g}(2\hat{k})|^2 \, d\hat{k}, \\ \Pi_\pm^\beta &\equiv \pm\frac{1}{2}\mathrm{PV}\!\!\int_{\mathbf{R}\times S^2} \frac{\mathrm{e}^{\pm\beta s/2}}{|\sinh(\beta s/2)|} \frac{|\widehat{g}(|s|\hat{k})|^2}{2-s}\, ds\, d\hat{k}, \end{aligned} \tag{4.19}$$

where $S^2 \subset \mathbf{R}^3$ is the unit sphere, $d\hat{k}$ its surface measure and PV denotes Cauchy's *principal value*. Further define

$$\begin{aligned} \Gamma^\beta &\equiv \Gamma_-^\beta + \Gamma_+^\beta, \\ \Pi^\beta &\equiv \Pi_-^\beta + \Pi_+^\beta. \end{aligned}$$

Then, the action of $\mathscr{K}$ takes the form

$$\mathscr{K}\begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = \begin{pmatrix} 2\Gamma_+^\beta\eta_{++} - 2\Gamma_-^\beta\eta_{--} & (\Gamma^\beta + i\Pi^\beta)\eta_{+-} \\ (\Gamma^\beta - i\Pi^\beta)\eta_{-+} & -2\Gamma_+^\beta\eta_{++} + 2\Gamma_-^\beta\eta_{--} \end{pmatrix}.$$

Thus, in the interaction representation, the linear transport equation of thermal relaxation is the well-known Bloch equation

$$\frac{d\eta(\bar{t})}{d\bar{t}} = -\mathscr{K}\eta(\bar{t}). \tag{4.20}$$

If the initial state of the spin system is diagonal ($\eta_{+-} = \eta_{-+} = 0$), then Bloch's equation reduces to Pauli's equation (2.3). We remark that, in second order perturbation theory, $2\Gamma_\pm^\beta\lambda^2$ is the probability per unit time for the transition $\pm \to \mp$ with emission or absorption of a radiation quantum. Note also that the entries of $\mathscr{K}$ differ from their usual textbook values by a factor λ^2, which has been absorbed in $\bar{t}$. Of course, one immediately gets from Equ. (4.19) the values of the Einstein coefficients in (2.4).

The matrix $\mathrm{e}^{-\mathscr{K}\bar{t}}$ can also be computed. With

$$\eta^\beta \equiv \frac{1}{Z^\beta}\mathrm{e}^{-\beta H_{\mathscr{S}}}, \tag{4.21}$$

and using the fact that $\mathrm{Tr}(\eta) = 1$, we obtain

$$\mathrm{e}^{-\mathscr{K}\bar{t}}\eta = \eta^\beta + \frac{1}{\Gamma^\beta}\begin{pmatrix} \mathrm{e}^{-2\Gamma^\beta\bar{t}}\left(\Gamma_+^\beta\eta_{++} - \Gamma_-^\beta\eta_{--}\right) & \mathrm{e}^{-(\Gamma^\beta+i\Pi^\beta)\bar{t}}\,\Gamma^\beta\eta_{+-} \\ \mathrm{e}^{-(\Gamma^\beta-i\Pi^\beta)\bar{t}}\,\Gamma^\beta\eta_{-+} & \mathrm{e}^{-2\Gamma^\beta\bar{t}}\left(\Gamma_-^\beta\eta_{--} - \Gamma_+^\beta\eta_{++}\right) \end{pmatrix}.$$

Since η^β is invariant under the free evolution $\mathrm{e}^{iL_{\mathscr{S}}t}$, the spin system approaches thermal equilibrium exponentially fast on the time-scale $\bar{t}$,

$$\lim_{\bar{t}\to\infty}\lim_{\lambda\to 0} T_\lambda^{\bar{t}/\lambda^2}\eta = \eta^\beta,$$

provided $\Gamma^\beta > 0$.

Davies' results were the starting point of our work. The necessity to understand the large time behavior of T_λ^t beyond the van Hove limit has been realized quite early [30]. In particular, the following two questions were open for some years:

(i) Is there return to equilibrium at finite coupling?

(ii) What are the higher order corrections to the linear transport equations? More precisely: Is it possible to derive an exact transport equation, with a convergent expansion in powers of λ, which reproduces Bloch's equation at the order λ^2?

These two problems are still open at $T = 0$. Their solution at $T > 0$ is our main contribution to the subject.

4.4. Thermal relaxation at $T = 0$

The relevant physical process at zero-temperature is radiative decay: The spin radiates its energy into the "frozen gas." As this energy propagates towards infinity, the interacting system dissipates to its lowest energy state, the ground state.

The zero-temperature equilibrium state of the reservoir is the pure state

$$\omega_{\mathscr{R}}^\infty(A) \equiv (\Omega_F, A\Omega_F),$$

associated with the Fock vacuum Ω_F. If $\widehat{g}/\epsilon \in L^2(\mathbf{R}^3)$ then $e_-(\lambda)$, the lowest point in the spectrum of H, is a simple eigenvalue [31]. We denote the corresponding eigenvector—the *ground state*—by Ψ_G. The zero-temperature equilibrium state ω^∞ of the spin-boson system is the pure state associated with this ground state. Accordingly, we say that this system returns to equilibrium at $T = 0$ if, for any $A \in \mathscr{O}$ and any density matrix ζ on $\mathbf{C}^2 \otimes \mathscr{F}$, we have

$$\lim_{t\to\infty} \mathrm{Tr}\left(\zeta\, \tau^t(A)\right) = \omega^\infty(A) \equiv (\Psi_G, A\Psi_G). \tag{4.22}$$

Here,

$$\tau^t(A) \equiv \mathrm{e}^{iHt} A \mathrm{e}^{-iHt},$$

is the time evolution of the system. It is an outstanding open problem to show that the spin-boson system has this property [32]. Recently, Hübner and Spohn have developed the scattering theory of the spin-boson model. They showed that asymptotic completeness implies return to equilibrium at zero-temperature [33]. For a modified spin-boson model, with a cutoff on the number of bosons, asymptotic completeness was proved by Gérard [34].

A first step towards (4.22) should be a better understanding of the spectral properties of the spin-boson Hamiltonian H. In particular, the analysis of its resonances should shed some light on the mechanism of thermal relaxation. This turns out to be a hard technical problem. In the sequel we briefly review some of the existing results.

The spectrum of the uncoupled Hamiltonian $H_0 \equiv H_{\mathscr{S}} + H_{\mathscr{R}}$ is

$$\begin{aligned}\sigma_{ac}(H_0) &= [-1,\infty),\\ \sigma_{sc}(H_0) &= \varnothing,\\ \sigma_{pp}(H_0) &= \{-1,1\}.\end{aligned}$$

The eigenfunctions corresponding to the eigenvalues ± 1 are $\chi_\pm \otimes \Omega_F$, where $\chi_\pm$ are the eigenfunctions of σ_z. As the coupling term is switched on, one expects the eigenvalue -1 to move along the real axis to a new location $e_-(\lambda)$. Being embedded in the continuous spectrum, the other eigenvalue has a different fate. We expect $+1$ to turn into a resonance in the following sense: There are $\varepsilon > 0$, $\eta > 0$, and a dense set of vectors $\mathscr{E}$ [35] such that, for $|\lambda| < \varepsilon$ and $\Phi \in \mathscr{E}$, the functions

$$R_\Phi(z) \equiv \left(\Phi, (H-z)^{-1}\Phi\right),$$

have a meromorphic continuation from the upper half-plane onto the region $O \equiv \{z : |z-1| < \eta\}$. The functions R_Φ should be regular in O except for a simple pole at $e_+(\lambda)$, which is independent of the choice of Φ. Furthermore, the function $e_+(\lambda)$ should be analytic for small λ,

$$e_+(\lambda) = 1 + \sum_{n=1}^{\infty} a^{(2n)}\lambda^{2n},$$

with $\mathrm{Im}(a^{(2)}) = -\Gamma_+^\infty$. The higher order terms in this expansion account for higher order (multi-phonon) transition processes.

If there is a resonance $e_+(\lambda)$ such that $\mathrm{Im}(e_+(\lambda)) < 0$, then the spectrum of H in the interval $]1-\eta, 1+\eta[$ is purely absolutely continuous. One expects more, namely

$$\begin{aligned}\sigma_{ac}(H) &= [e_-(\lambda),\infty),\\ \sigma_{sc}(H) &= \varnothing,\\ \sigma_{pp}(H) &= \{e_-(\lambda)\},\end{aligned} \tag{4.23}$$

the eigenvalue $e_-(\lambda)$ being simple. Let us discuss some dynamical consequences of (4.23). The first is that the spin-boson system has the mixing property at zero-temperature: For $A, B \in \mathscr{O}$,

$$\lim_{t\to\infty} \omega^\infty(A\tau^t(B)) = \omega^\infty(A)\omega^\infty(B). \tag{4.24}$$

We remark that (4.22) implies (4.24) but the opposite is not true. The second (a consequence of the RAGE theorem [36]) is that compact "observables" (which are *not* elements of $\mathscr{O}$) return to equilibrium in the following sense: For any density matrix ζ and any compact operator K on $\mathbf{C}^2 \otimes \mathscr{F}$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{Tr}\left(\zeta\,\tau^t(K)\right)\,dt = (\Psi_G, \zeta\Psi_G)\,(\Psi_G, K\Psi_G). \tag{4.25}$$

The third observation is the following. If the system is initially in a pure state Φ, the probability to find it in a pure state Ψ at time t is $|(\Psi, \mathrm{e}^{-iHt}\Phi)|^2$. It follows from (4.24) that

$$\lim_{t\to\infty} \left|\left(\Psi, \mathrm{e}^{-iHt}\Phi\right)\right|^2 = |(\Psi,\Psi_G)|^2\,|(\Phi,\Psi_G)|^2. \tag{4.26}$$

To study the resonances of H it is natural to use the field-theoretic analog of the Aguilar-Combes dilation analyticity technique [37]. For $\theta \in \mathbf{R}$, let $(u(\theta)f)\widehat{\ }(k) = \mathrm{e}^{-3\theta/2}\widehat{f}(\mathrm{e}^{-\theta}k)$ be the dilation unitary group on $L^2(\mathbf{R}^3)$. Let $U(\theta)$ be its second quantization and $H(\theta) \equiv U(\theta)HU(-\theta)$. Explicitly,

$$H(\theta) = H_{\mathscr{S}} + \mathrm{e}^{-\theta}H_{\mathscr{R}} + \lambda\sigma_x \otimes \varphi(u(\theta)g).$$

If the function $\widehat{g}$ is analytic in a suitable complex cone centered at origin, then the family of operators $H(\theta)$ is defined for complex θ. The spectrum of $H_0(\theta) = H_{\mathscr{S}} + \mathrm{e}^{-\theta}H_{\mathscr{R}}$ can be explicitly determined—it consists of the two half-lines $\{\pm 1 + t\mathrm{e}^{-\theta} : t \geqslant 0\}$. The central difficulty is now transparent: The complex deformation does not uncover the eigenvalues of H_0. Since the perturbation term is not relatively compact, it is very hard to analyze the spectrum around the unperturbed eigenvalues ± 1. This difficulty is just a reflection of the well-known infrared problem which plagued quantum electrodynamics since its conception.

If one considers massive bosons with a dispersion relation of the form $\epsilon(k) = |k| + \mu$, where $\mu > 0$, then the infrared problem disappears. The spectrum of $H_0(\theta)$ consists of two simple eigenvalues ± 1 and a sequence of half-lines $\{\pm 1 + n\mu + t\mathrm{e}^{-\theta} : n \geqslant 1, t \geqslant 0\}$. If $\mu/2$ is not an integer, the complex deformation unveils the eigenvalues, and the resonances can be studied by standard techniques (see [1] and [37] for details). The spectrum of $H(\theta)$ has the structure depicted in Fig. 1. For non-vanishing λ, singular spectrum can occur near the thresholds $\{\pm 1 + n\mu : n \geqslant 1\}$. However, as $\lambda \to 0$, its relative size shrinks down. In this limit we proved that Relations (4.24) – (4.26) hold [1].

The analyticity assumption on $\widehat{g}(k)$, although necessary for the study of resonances, is very restrictive, and one expects that (4.23) holds under weaker conditions. The natural way to approach this problem is to develop a field-theoretic version of Mourre theory [36]. If $\mu > 0$, this has been done in [38], and is still an open problem in the mass-less case.

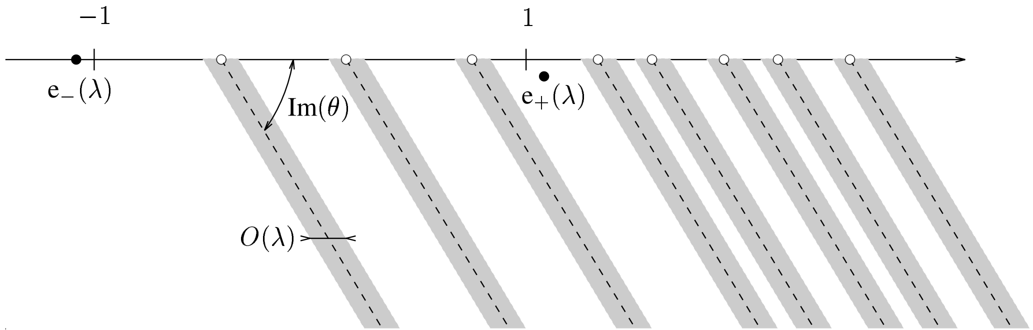

Figure 1: The spectrum of $H(\theta)$ for the massive spin-boson model. The essential spectrum is contained in the shaded areas

Recently, Bach, Fröhlich and Sigal [39] announced a number of results concerning the mathematical theory of resonances in mass-less, non-relativistic QED. Specialized to the spin-boson model, their results yield (4.23) under suitable assumptions on $\widehat{g}$.

We would like to emphasize that, in spite of the substantial recent progress on the spectral theory of the spin-boson system, the problem of return to equilibrium at zero-temperature is still open. For additional information we refer the reader to the excellent review [33].

4.5. Thermal relaxation at $T > 0$

At zero-temperature, classical friction is a deterministic phenomenon. There is no Koopman lemma and no spectral characterization of the global dynamics. We have a similar situation in quantum mechanics. Although one can learn a great deal from the spectral analysis of H, return to equilibrium at zero-temperature *cannot* be reduced to a spectral problem.

At positive temperature the situation is different. We have outlined in Section 3 how to construct the dynamical system $(\mathscr{G}, \Xi^t, \mu^\beta)$ which describes, in the framework of classical mechanics, the combined system $\mathscr{S} + \mathscr{R}$ near thermal equilibrium. Return to equilibrium is then equivalent to the mixing property of this dynamical system, and can be characterized in spectral terms. We would like to proceed in a similar fashion in quantum mechanics. To understand how, we begin with a simple case: The free reservoir.

In a finite volume Λ one easily computes the thermal equilibrium state of the reservoir in term of the finite volume Hamiltonian. In the thermodynamic limit $\Lambda \uparrow \mathbf{R}^3$, and in the absence of condensate, this state converges towards a state $\omega_{\mathscr{R}}^\beta$ of the infinite reservoir which is completely specified by the generating function [40]

$$\omega_{\mathscr{R}}^\beta(\mathrm{e}^{i\varphi(f)}) = \exp\left(-\frac{1}{2}\int (1 + 2\varrho_\beta(k))|\hat{f}(k)|^2\, d^3k\right). \tag{4.27}$$

Here ϱ_β—the equilibrium momentum distribution of the Bose gas—is given by Planck's law (2.5)

$$\varrho_\beta(k) = \frac{1}{\mathrm{e}^{\beta\epsilon(k)} - 1}. \tag{4.28}$$

Having a positive density $\int \varrho_\beta(k)\, d^3k$, the state $\omega_{\mathscr{R}}^\beta$ does not fit into the original Hilbert space: It is not represented by a density matrix on $\mathscr{F}$. To restore the familiar Hilbert space framework, one constructs the cyclic representation $(\mathscr{H}_{\mathscr{R}}, \pi_{\mathscr{R}}, \Omega_{\mathscr{R}})$ of the algebra $\mathscr{O}_{\mathscr{R}}$ associated with $\omega_{\mathscr{R}}^\beta$. This representation is characterized, up to unitary equivalence, by the formula

$$\omega_{\mathscr{R}}^\beta(A) = (\Omega_{\mathscr{R}}, \pi_{\mathscr{R}}(A)\Omega_{\mathscr{R}}), \tag{4.29}$$

for any $A \in \mathscr{O}_{\mathscr{R}}$, and the fact that the set $\pi_{\mathscr{R}}(\mathscr{O}_{\mathscr{R}})\Omega_{\mathscr{R}}$ is dense in $\mathscr{H}_{\mathscr{R}}$ (*i.e.* $\Omega_{\mathscr{R}}$ is cyclic).

The construction can be done on $\mathscr{H}_{\mathscr{R}} = \mathscr{F} \otimes \mathscr{F}$ with the help of the two-component Bose field

$$a_1(k) \equiv a(k) \otimes I, \qquad a_2(k) \equiv I \otimes a(k).$$

Define

$$\pi_{\mathscr{R}}(a(k)) = a_\beta(k) \equiv \sqrt{1 + \varrho_\beta(k)}\, a_1(k) + \sqrt{\varrho_\beta(k)}\, a_2^*(k), \tag{4.30}$$

and

$$\Omega_{\mathscr{R}} \equiv \Omega_F \otimes \Omega_F.$$

Then, for $f \in \mathscr{D}_{loc}$, one has $\pi_{\mathscr{R}}(\mathrm{e}^{i\varphi(f)}) = \mathrm{e}^{i\varphi_\beta(f)}$ with

$$\varphi_\beta(f) \equiv \frac{1}{\sqrt{2}}\int \left(f(k)a_\beta^*(k) + \overline{f(k)}a_\beta(k)\right)\, d^3k,$$

and one easily checks Relations (4.27) – (4.29). Cyclicity however requires some more work (see [41]).

In the cyclic representation, any automorphism for which the equilibrium state is invariant can be implemented by a unitary operator leaving the cyclic vector invariant. Thus such automorphisms have a natural extension to the von Neumann algebra $\mathfrak{M}_{\mathscr{R}}$ generated by $\pi_{\mathscr{R}}(\mathscr{O}_{\mathscr{R}})$ [42]. In particular, for any $A \in \mathscr{O}_{\mathscr{R}}$, one has

$$\pi_{\mathscr{R}}(\mathrm{e}^{iH_{\mathscr{R}}t}A\mathrm{e}^{-iH_{\mathscr{R}}t}) = \mathrm{e}^{i\mathscr{L}_{\mathscr{R}}t}\pi_{\mathscr{R}}(A)\mathrm{e}^{-i\mathscr{L}_{\mathscr{R}}t}, \tag{4.31}$$

where $\mathscr{L}_{\mathscr{R}}$ is a self-adjoint operator on $\mathscr{H}_{\mathscr{R}}$ satisfying

$$\mathscr{L}_{\mathscr{R}}\Omega_{\mathscr{R}} = 0. \tag{4.32}$$

Let us identify the generator $\mathscr{L}_{\mathscr{R}}$. A naive guess is

$$\pi_{\mathscr{R}}(H_{\mathscr{R}}) = \int \epsilon(k)a_\beta^*(k)a_\beta(k)\,d^3k.$$

However, even after the usual normal ordering renormalization, this operator is ill defined. More seriously it does not satisfy the invariance requirement (4.32). A closer look at the time evolution

$$\pi_{\mathscr{R}}(\mathrm{e}^{iH_{\mathscr{R}}t}a(k)\mathrm{e}^{-iH_{\mathscr{R}}t}) = \sqrt{1+\varrho_\beta(k)}\,a_1(k)\mathrm{e}^{-i\epsilon(k)t} + \sqrt{\varrho_\beta(k)}\,a_2^*(k)\mathrm{e}^{-i\epsilon(k)t},$$

leads to the correct formula

$$\mathscr{L}_{\mathscr{R}} = H_{\mathscr{R}}\otimes I - I\otimes H_{\mathscr{R}} = \int \epsilon(k)\ \left(a_1^*(k)a_1(k) - a_2^*(k)a_2(k)\right)\,d^3k.$$

To understand the structure of the cyclic representation it is convenient to think of the two copies of the Fock representation it contains as describing two kinds of pseudo-particles: Phonons in excess with respect to the equilibrium state (pseudo-phonons), and phonons missing from this state (pseudo-holes). Phonons being indistinguishable [43], it is not possible to differentiate the "members" of the equilibrium state from the others. Hence pseudo-phonons/holes have no physical existence. Nevertheless they are convenient for bookkeeping purposes. From Definition (4.30) we see that the creation of a "real" phonon is a coherent superpositions of two processes: The creation of a pseudo-phonon and the destruction of a pseudo-hole. The relative weight in this superposition is the germ of Einstein's A-B law (2.4). In this picture $\mathscr{L}_{\mathscr{R}}$ is the difference between the energy of the excess phonons and that of the missing ones. Therefore it describes the energy of the gas relative to the (infinite) energy of its equilibrium state, a quantity which can not be measured. Since the equilibrium state is invariant under space translations, a completely analogous construction leads to the formula

$$\mathscr{P}_{\mathscr{R}} = \int k\ \left(a_1^*(k)a_1(k) - a_2^*(k)a_2(k)\right)\,d^3k,$$

for the momentum operator which generates the unitary representation of the translation group in $\mathscr{H}_{\mathscr{R}}$.

The system $\mathscr{R}$ has two important discrete symmetries: Parity and time reversal. The first one is implemented in $\mathscr{H}_{\mathscr{R}}$ by the unitary map P characterized by

$$\begin{array}{rcl} Pa_\ell(k) & = & a_\ell(-k)P, \quad (\ell = 1, 2) \\ P\Omega_{\mathscr{R}} & = & \Omega_{\mathscr{R}}. \end{array}$$

The time reversal operator T satisfies the same equations, but is anti-unitary. P and T map the algebra $\mathfrak{M}_{\mathscr{R}}$ into itself. They leave the operator $\mathscr{L}_{\mathscr{R}}$ invariant and flip the sign of the momentum operator $\mathscr{P}_{\mathscr{R}}$, therefore one has

$$\begin{array}{ll} P\mathrm{e}^{i\mathscr{L}_{\mathscr{R}}t} = \mathrm{e}^{i\mathscr{L}_{\mathscr{R}}t}P, & T\mathrm{e}^{i\mathscr{L}_{\mathscr{R}}t} = \mathrm{e}^{-i\mathscr{L}_{\mathscr{R}}t}T, \\ T\mathrm{e}^{i\mathscr{P}_{\mathscr{R}}\cdot x} = \mathrm{e}^{i\mathscr{P}_{\mathscr{R}}\cdot x}T, & P\mathrm{e}^{i\mathscr{P}_{\mathscr{R}}\cdot x} = \mathrm{e}^{-i\mathscr{P}_{\mathscr{R}}\cdot x}P. \end{array}$$

There is another natural symmetry in $\mathscr{H}_{\mathscr{R}}$ which, however, is non-physical (it does not preserve the algebra $\mathfrak{M}_{\mathscr{R}}$): The exchange of pseudo-phonons and pseudo-holes, $E\colon \Phi \otimes \Psi \mapsto \Psi \otimes \Phi$. It clearly flips the sign of both $\mathscr{L}_{\mathscr{R}}$ and $\mathscr{P}_{\mathscr{R}}$, thus the anti-unitary map $J_{\mathscr{R}} \equiv EPT$ has the remarkable property

$$J_{\mathscr{R}}\mathrm{e}^{i\mathscr{L}_{\mathscr{R}}t} = \mathrm{e}^{i\mathscr{L}_{\mathscr{R}}t}J_{\mathscr{R}}, \quad J_{\mathscr{R}}\mathrm{e}^{i\mathscr{P}_{\mathscr{R}}\cdot x} = \mathrm{e}^{i\mathscr{P}_{\mathscr{R}}\cdot x}J_{\mathscr{R}}.$$

It turns out that $J_{\mathscr{R}}$ is the modular conjugation associated to $(\mathfrak{M}_{\mathscr{R}}, \omega^\beta_{\mathscr{R}})$ by Tomita-Takesaki's theory of modular algebras [44] (this theory applies since, as a thermal equilibrium state, $\omega^\beta_{\mathscr{R}}$ is faithful [45]). One of the fundamental property of $J_{\mathscr{R}}$ is to map the algebra of observables into its commutant: $J_{\mathscr{R}}\mathfrak{M}_{\mathscr{R}}J_{\mathscr{R}} = \mathfrak{M}'_{\mathscr{R}}$. A formal computation shows that

$$\mathscr{L}_{\mathscr{R}} = \pi_{\mathscr{R}}(H_{\mathscr{R}}) - J_{\mathscr{R}}\pi_{\mathscr{R}}(H_{\mathscr{R}})J_{\mathscr{R}}.$$

Therefore, as already remarked, the operator $\mathscr{L}_{\mathscr{R}}$ is *not* observable: It differs from $\pi_{\mathscr{R}}(H_{\mathscr{R}})$ by an operator which commutes with all observables.

Unlike in the zero-temperature Fock representation, the generator of the dynamics is unbounded below. Apart from the zero eigenvalue reflecting the invariance of the equilibrium state, its spectrum is absolutely continuous and fills the entire real axis. Let us show how this spectral information, combined with the fundamental properties (4.31) – (4.32), leads to the mixing property of the reservoir. A simple calculation shows that

$$\omega^\beta_{\mathscr{R}}(A\mathrm{e}^{iH_{\mathscr{R}}t}B\mathrm{e}^{-iH_{\mathscr{R}}t}) - \omega^\beta_{\mathscr{R}}(A)\,\omega^\beta_{\mathscr{R}}(B) = (\pi_{\mathscr{R}}(A^*)\Omega_{\mathscr{R}}, (\mathrm{e}^{i\mathscr{L}_{\mathscr{R}}t} - |\Omega_{\mathscr{R}}\rangle\langle\Omega_{\mathscr{R}}|)\pi_{\mathscr{R}}(B)\Omega_{\mathscr{R}}).$$

Thus, by the cyclicity of $\Omega_{\mathscr{R}}$, the mixing property is equivalent to

$$\mathrm{w}-\lim_{t\to\infty} \mathrm{e}^{i\mathscr{L}_{\mathscr{R}}t} = |\Omega_{\mathscr{R}}\rangle\langle\Omega_{\mathscr{R}}|. \tag{4.33}$$

Invoking the same argument as in the classical case (see Equ. (3.16) and the remarks following it) we conclude that the free reservoir at positive temperature is mixing. Now we show more: Due to the above mentioned faithfulness of $\omega^\beta_{\mathscr{R}}$, the mixing property is equivalent to return to equilibrium [46]. Let ω_0 be a state of the form

$$\omega_0(A) = (B\Omega_{\mathscr{R}}, \pi_{\mathscr{R}}(A)B\Omega_{\mathscr{R}}), \tag{4.34}$$

for some $B \in \mathfrak{M}'$. Since B commutes with $\pi(\mathscr{O})$, we have

$$\begin{aligned}\omega_0(\mathrm{e}^{iH_{\mathscr{R}}t}A\mathrm{e}^{-iH_{\mathscr{R}}t}) &= (B\Omega_{\mathscr{R}}, \pi_{\mathscr{R}}(\mathrm{e}^{iH_{\mathscr{R}}t}A\mathrm{e}^{-iH_{\mathscr{R}}t})B\Omega_{\mathscr{R}}) \\ &= (B\Omega_{\mathscr{R}}, B\pi_{\mathscr{R}}(\mathrm{e}^{iH_{\mathscr{R}}t}A\mathrm{e}^{-iH_{\mathscr{R}}t})\Omega_{\mathscr{R}}) \\ &= (B^*B\Omega_{\mathscr{R}}, \mathrm{e}^{i\mathscr{L}_{\mathscr{R}}t}\pi_{\mathscr{R}}(A)\Omega_{\mathscr{R}}).\end{aligned}$$

From $\|B\Omega_{\mathscr{R}}\|^2 = \omega_0(I) = 1$, we see that

$$\omega_0(\mathrm{e}^{iH_{\mathscr{R}}t}A\mathrm{e}^{-iH_{\mathscr{R}}t}) - \omega_{\mathscr{R}}^{\beta}(A) = (B^*B\Omega_{\mathscr{R}}, (\mathrm{e}^{i\mathscr{L}_{\mathscr{R}}t} - |\Omega_{\mathscr{R}}\rangle\langle\Omega_{\mathscr{R}}|)\pi_{\mathscr{R}}(A)\Omega_{\mathscr{R}}).$$

Since $\Omega_{\mathscr{R}}$ is cyclic for $\mathfrak{M}'_{\mathscr{R}}$ [45], return to equilibrium for all states of the form (4.34) is equivalent to mixing. Moreover, a simple approximation argument shows that this characterization extends to all the states which are represented by a density matrix in $\mathscr{H}_{\mathscr{R}}$.

It is now clear how to obtain a spectral characterization of return to equilibrium for the spin-boson model (and for more general quantum systems, see Fig. 2):

Equilibrium state. In infinite dimensional quantum systems, thermal equilibrium states are characterized by the KMS condition [47]. Roughly, ω^{β} is a KMS-state at inverse temperature β for the dynamics τ^t if, for any observables $A, B \in \mathscr{O}$, the correlation function $\omega^{\beta}(\tau_t(A)B)$ is analytic for $0 < \mathrm{Im}(t) < \beta$ and, for $t \in \mathbf{R}$, satisfies the KMS boundary condition

$$\omega^{\beta}(\tau_{t+i\beta}(A)B) = \omega^{\beta}(A\tau_t(B)).$$

In the spin boson system, such a state exists and is unique [48].

Cyclic representation. The GNS construction [47] leads to a cyclic representation $(\mathscr{H}, \pi, \Omega)$ of the algebra of observables such that

$$\omega^{\beta}(A) = (\Omega, \pi(A)\Omega).$$

The dynamics is unitarily implemented in this representation by

$$\pi(\tau^t(A)) = \mathrm{e}^{i\mathscr{L}t}\pi(A)\mathrm{e}^{-i\mathscr{L}t}, \tag{4.35}$$

with a self-adjoint generator satisfying

$$\mathscr{L}\Omega = 0. \tag{4.36}$$

Let S^{β} be the set of states which have a density matrix in the cyclic representation [49]. We say that the system returns to equilibrium at inverse temperature β if, for any initial state $\omega_0 \in S^{\beta}$ and any observable A, one has

$$\lim_{t\to\infty} \omega_0(\tau^t(A)) = \omega^{\beta}(A).$$

The Liouvillean. Return to equilibrium, like other ergodic properties, are characterized by the spectrum of $\mathscr{L}$, in complete analogy with Koopman's lemma of classical mechanics [3]. It is therefore natural to call this operator the *Liouvillean* of the system. *A priori,* the abstract definition (4.35) – (4.36) does not provide an explicit formula for $\mathscr{L}$. The next task is therefore to compute the Liouvillean. For this purpose, the connection with Tomita-Takesaki's theory is essential [50].

Spectral analysis. Once the Liouvillean is known, it remains to explore its spectral properties. Here, as in the zero-temperature case, spectral deformation techniques enter the game. The remarkable fact is that complex resonances of the Liouvillean determine the exact transport equation of thermal relaxation. In particular, Davies' generator $\mathscr{K}$ arises as Fermi's Golden Rule (second order perturbation theory) for these resonances.

Implementing the above program we have shown in [3] that, in the weak-coupling/high temperature regime, the spin-boson system returns to equilibrium. Moreover this return is exponentially fast in time. In the remaining part of this section we sketch some of the ideas involved in the proof of these results.

Since it is difficult to obtain the KMS state ω^β from the thermodynamic limit, we start with the uncoupled system ($\lambda = 0$). We construct its KMS state, the induced cyclic representation and the associated modular structure. Then we invoke perturbation theory to compute the cyclic representation and modular structure of the coupled system. We have already discussed the cyclic representation of the reservoir, thus we turn directly to the system $\mathscr{S}$. Its KMS state is the Gibbs state η^β of Equ. (4.21). The cyclic representation $(\mathscr{H}_\mathscr{S}, \pi_\mathscr{S}, \Omega_\mathscr{S})$ is given by

$$\begin{aligned}
\mathscr{H}_\mathscr{S} &\equiv \mathbf{C}^2 \otimes \mathbf{C}^2,\\
\Omega_\mathscr{S} &\equiv (2\cosh\beta)^{-1/2}\left(\mathrm{e}^{-\beta/2}\chi_+ \otimes \overline{\chi}_+ + \mathrm{e}^{\beta/2}\chi_- \otimes \overline{\chi}_-\right),\\
\pi_\mathscr{S}(X) &\equiv X \otimes I.
\end{aligned}$$

One easily checks that $\mathrm{Tr}(\eta^\beta X) = (\Omega_\mathscr{S}, \pi_\mathscr{S}(X)\Omega_\mathscr{S})$, and that the Liouvillean is

$$\mathscr{L}_\mathscr{S} = \sigma_z \otimes I - I \otimes \sigma_z.$$

Note that the exchange map $\phi \otimes \psi \mapsto \psi \otimes \phi$ flips the sign of the Liouvillean. Thus the anti-linear operator $J_\mathscr{S}\colon \phi \otimes \psi \mapsto \overline{\psi} \otimes \overline{\phi}$ satisfies

$$J_\mathscr{S}\mathrm{e}^{i\mathscr{L}_\mathscr{S}t} = \mathrm{e}^{i\mathscr{L}_\mathscr{S}t}J_\mathscr{S}.$$

Here again, $J_\mathscr{S}$ is the modular conjugation, and one easily checks that $\mathscr{L}_\mathscr{S} = \pi_\mathscr{S}(H_\mathscr{S}) - J_\mathscr{S}\pi_\mathscr{S}(H_\mathscr{S})J_\mathscr{S}$.

Clearly, the KMS state of the uncoupled spin-boson system is $\eta^\beta \otimes \omega^\beta_\mathscr{R}$, and the induced cyclic representation of $\mathscr{O}$ is $(\mathscr{H}_0, \pi_0, \Omega_0) = (\mathscr{H}_\mathscr{S} \otimes \mathscr{H}_\mathscr{R}, \pi_\mathscr{S} \otimes \pi_\mathscr{R}, \Omega_\mathscr{S} \otimes \Omega_\mathscr{R})$. It turns out that the cyclic representation $(\mathscr{H}, \pi, \Omega)$ associated with the state ω^β is given by $\mathscr{H} = \mathscr{H}_0$ and $\pi = \pi_0$. The construction of the vector Ω rests on deep results of Tomita-Takesaki's theory of modular algebras and Araki's perturbation theory of W^*-dynamical systems [51]. Here we just mention that this vector is defined by a suitable perturbation expansion and that $\Omega = \Omega_0 + O(\lambda)$. Let $\mathfrak{M} \equiv M_2 \otimes M_2 \otimes \mathfrak{M}_\mathscr{R}$, then the anti-linear map $J \equiv J_\mathscr{S} \otimes J_\mathscr{R}$ is the modular conjugation of $(\mathfrak{M}, \omega^\beta)$ and

$$\mathscr{L} = \mathscr{L}_\mathscr{S} + \mathscr{L}_\mathscr{R} + \lambda\pi(H_{int}) - \lambda J\pi(H_{int})J = \pi(H) - J\pi(H)J,$$

is the Liouvillean of the combined system. Note that, here again, the modular conjugation satisfies

$$J\mathrm{e}^{i\mathscr{L}t} = \mathrm{e}^{i\mathscr{L}t}J.$$

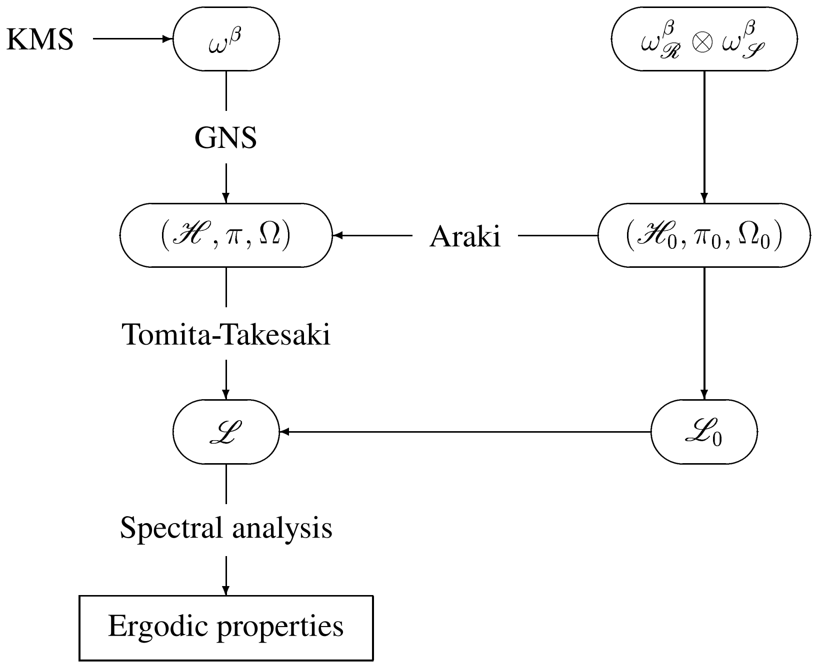

Figure 2: The logic of quantum Koopmanism

We proceed to analyze the spectrum of $\mathscr{L}$. Without interaction, the situation is the following. The spectrum of $\mathscr{L}_0 \equiv \mathscr{L}_{\mathscr{S}} + \mathscr{L}_{\mathscr{R}}$ is

$$\begin{aligned}
\sigma_{ac}(\mathscr{L}_0) &= \mathbf{R}, \\
\sigma_{sc}(\mathscr{L}_0) &= \varnothing, \\
\sigma_{pp}(\mathscr{L}_0) &= \{-2, 0, 2\},
\end{aligned}$$

where ± 2 are simple eigenvalues while 0 is two-fold degenerate. After the interaction term is switched on, a simple eigenvalue remains at 0 while the others turn into complex resonances. Thus, for small non-vanishsing λ, the Liouvillean $\mathscr{L}$ has absolutely continuous spectrum except for the simple eigenvalue 0, and return to equilibrium follows. The thermal relaxation time is determined by the width of the complex resonances of $\mathscr{L}$.

We now sketch some of the ideas involved in the analysis of these resonances. Roughly speaking, the pseudo-phonons/holes emission and absorption processes conspire together to smooth out the infrared singularity. To realize that we introduce the operator

$$b(s, \hat{k}) \equiv \begin{cases} s\, a_2(|s|\hat{k}) & \text{for} \quad s < 0, \\ s\, a_1(|s|\hat{k}) & \text{for} \quad s > 0, \end{cases}$$

where $s \in \mathbf{R}$ and $\hat{k} \in S^2$, the unit sphere in $\mathbf{R}^3$. A simple calculation shows that the commutation relation $[b(s,\hat{k}), b^*(s',\hat{k}')] = \delta(s-s')\delta(\hat{k},\hat{k}')$ holds. The Liouvillean can now be rewritten as

$$\begin{aligned}\mathscr{L} &= \mathscr{L}_{\mathscr{S}} + \int s\, b^*(s,\hat{k})b(s,\hat{k})\ ds\, d\hat{k}\\ &+ \lambda(\sigma_x\otimes I)\otimes\int\Big(\alpha_+(s,\hat{k})b^*(s,\hat{k}) + \overline{\alpha_+(s,\hat{k})}b(s,\hat{k})\Big)\ ds\, d\hat{k}\\ &- \lambda(I\otimes\sigma_x)\otimes\int\Big(\alpha_-(s,\hat{k})b^*(s,\hat{k}) + \overline{\alpha_-(s,\hat{k})}b(s,\hat{k})\Big)\ ds\, d\hat{k},\end{aligned}$$

where the functions $\alpha_\pm$ are given by

$$\alpha_\pm(s,\hat{k}) \equiv \frac{1}{2}\,\mathrm{e}^{\pm\beta s/4}\left(\frac{s}{\sinh(\beta s/2)}\right)^{1/2}\mathrm{sign}(s)\,|s|^{1/2}\,\widehat{g}(s\hat{k}).$$

Translations of the s variable are implemented in $\mathscr{H}$ by the unitary group $\mathrm{e}^{i\mathscr{A}\theta}$, where

$$\mathscr{A} \equiv \int b^*(s,\hat{k})\,\frac{1}{i}\partial_s\, b(s,\hat{k})\, ds\, d\hat{k}.$$

One easily shows that

$$\begin{aligned}\mathscr{L}(\theta) \equiv \mathrm{e}^{i\mathscr{A}\theta}\mathscr{L}\mathrm{e}^{-i\mathscr{A}\theta} &= \mathscr{L}_{\mathscr{S}} + \int (s+\theta)\, b^*(s,\hat{k})b(s,\hat{k})\, ds\, d\hat{k}\\ &+ \lambda(\sigma_x\otimes I)\otimes\int\Big(\alpha_+(s+\theta,\hat{k})b^*(s,\hat{k}) + \overline{\alpha_+(s+\theta,\hat{k})}b(s,\hat{k})\Big)\ ds\, d\hat{k}\\ &- \lambda(I\otimes\sigma_x)\otimes\int\Big(\alpha_-(s+\theta,\hat{k})b^*(s,\hat{k}) + \overline{\alpha_-(s+\theta,\hat{k})}b(s,\hat{k})\Big)\ ds\, d\hat{k}.\end{aligned}$$

A simple (and critical) observation is that the function

$$\left(\frac{s}{\sinh(\beta s/2)}\right)^{1/2}, \tag{4.37}$$

is analytic in the strip $|\mathrm{Im}(s)| < 2\pi/\beta$. Thus, if $\mathrm{sign}(s)\,|s|^{1/2}\widehat{g}(s\hat{k})$ has an analytic continuation there (*e.g.* $\widehat{g}(k) = |k|^{1/2}\mathrm{e}^{-|k|^2}$), then $\mathscr{L}(\theta)$ is a well-defined family of operators for complex θ, as long as $|\mathrm{Im}(\theta)| < 2\pi/\beta$.

To describe the spectrum of $\mathscr{L}_0(\theta) \equiv \mathrm{e}^{i\mathscr{A}\theta}(\mathscr{L}_{\mathscr{S}} + \mathscr{L}_{\mathscr{R}})\mathrm{e}^{-i\mathscr{A}\theta}$, we introduce the operator

$$\mathscr{N}_{tot} \equiv \int \left(a_1^*(k)a_1(k) + a_2^*(k)a_2(k)\right)\, d^3k. \tag{4.38}$$

This is the ordinary number operator in $\mathscr{F}\otimes\mathscr{F}$, which counts the total number of pseudo-phonons and pseudo-holes [52]. A simple calculation shows that

$$\mathscr{N}_{tot} = \int b^*(s,\hat{k})b(s,\hat{k})\, ds\, d\hat{k},$$

so that $\mathscr{L}_0(\theta) = \mathscr{L}_{\mathscr{S}} + \mathscr{L}_{\mathscr{R}} + \theta\mathscr{N}_{tot}$. We immediately conclude that, for complex θ, the spectrum of $\mathscr{L}_0(\theta)$ consists of discrete eigenvalues $\{\pm 2, 0\}$, and a set of lines $\{t + i\,n\,\mathrm{Im}(\theta) : t\in\mathbf{R}, n > 0\}$ filled with absolutely continuous spectrum. Thus, the complex deformation unveils the eigenvalues of $\mathscr{L}_0$. One can now use perturbative arguments to show that these eigenvalues turn into the resonances after addition of the perturbation term, see Fig. 3. These resonances are the eigenvalues of the matrix

$$\Sigma_\lambda \equiv W_\lambda(\theta)P_\lambda(\theta)\mathscr{L}(\theta)P_\lambda(\theta)W_\lambda(\theta)^{-1}, \tag{4.39}$$

where $P_\lambda(\theta)$ is the spectral projection of $\mathscr{L}(\theta)$ on the resonance eigenvalues, $P_0 \equiv I \otimes |\Omega\rangle\langle\Omega|$ and

$$W_\lambda(\theta) \equiv [P_0P_\lambda(\theta)P_0]^{-1/2}P_0P_\lambda(\theta).$$

Of course, one can show that Σ_λ is independent of θ. To complete the argument, it remains to

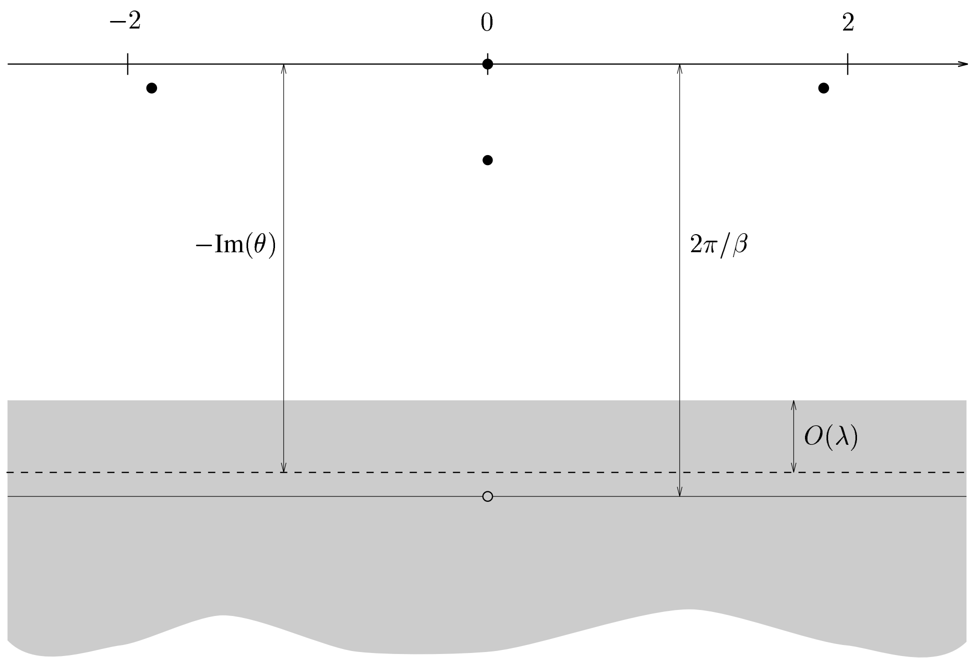

Figure 3: The spectrum of $\mathscr{L}(\theta)$ for $-2\pi/\beta < \mathrm{Im}(\theta) < 0$. The essential spectrum is contained in the shaded area

relate the family of operators $\mathscr{L}(\theta)$, defined for $-2\pi/\beta < \mathrm{Im}(\theta) < 0$, to the physical Liouvillean $\mathscr{L}$. It is here that we encounter the *infrared problem.* Let us briefly comment on this since it is one of the central technical point in our analysis. It is well known that the infrared catastrophe occurs when infinitely many "soft bosons" (in our case pseudo-phonons/holes) are created. This is dynamically possible since the total number operator $\mathscr{N}_{tot}$ can not be bounded by the relative energy $\mathscr{L}$, *i.e.* there are vectors $\Psi \in \mathscr{H}$ such that $|(\Psi, \mathscr{L}\Psi)| < \infty$ but $(\Psi, \mathscr{N}_{tot}\Psi) = \infty$. As long as the imaginary part of θ does not vanish, it acts as a mass term [53] which prevent the catastrophic creation of soft bosons. However, the problem shows up in the limit $\mathrm{Im}(\theta) \to 0$: The domain of the operator $\mathscr{L}(\theta)$ suddenly changes at $\mathrm{Im}(\theta) = 0$. To solve this difficulty we prove that, for any $\Psi \in \mathscr{H}$ and for $\mathrm{Im}(z)$ large enough,

$$\lim_{\mathrm{Im}(\theta)\to 0}(\mathscr{L}(\theta) - z)^{-1}\Psi = (\mathscr{L}(\mathrm{Re}(\theta)) - z)^{-1}\Psi.$$

It follows that the resonances of $\mathscr{L}$ are the eigenvalues of Σ_λ given by Equ. (4.39). This matrix is analytic, with a power expansion of the form

$$\Sigma_\lambda = \mathscr{L}_{\mathscr{S}} + \sum_{n=1}^{\infty} \Sigma^{(2n)} \lambda^{2n}. \tag{4.40}$$

Any term in this expansion can be computed. The matrix Σ_λ is directly related to the generator $\mathscr{K}_\lambda$ of the exact transport equation of thermal relaxation [5]

$$\frac{d\eta(t)}{dt} = -\mathscr{K}_\lambda \eta(t). \tag{4.41}$$

This generator also has a power expansion

$$\mathscr{K}_\lambda = iL_{\mathscr{S}} + \sum_{n=1}^{\infty} \mathscr{K}^{(2n)} \lambda^{2n},$$

each term being related to corresponding term in the expansion (4.40). In particular, one can show that $\mathscr{K}^{(2)}$ is identical to Davies' generator $\mathscr{K}$.

We finish with three remarks. First, as $\beta \to \infty$, $\mathscr{L}$ tends towards the zero-temperature Liouvillean, namely $H \otimes I - I \otimes H$. In the same limit $\Omega \to \Psi_G \otimes \Psi_G$, thus we recover the zero-temperature model. However, from the point of view of resonances, this limit is highly non-trivial. To understand why, remark that the function (4.37) has poles at the points $s = -i\beta^{-1}2\pi n$, for $n = 1, 2, \cdots$. These singularities obstruct the analytic continuation of the resolvent $(\mathscr{L} - z)^{-1}$, and hence the access to the resonances (see Fig. 3). Consequently, our method does not yield any information on the zero-temperature model. Second, although some analyticity assumption on the coupling function $\widehat{g}(k)$ is necessary to study the resonances of $\mathscr{L}$ and to derive the exact transport equation (4.41), it is not needed for return to equilibrium. We are presently working on a version of Mourre's theory for the Liouvillean which should yield return to equilibrium under much weaker assumptions. Of course, in this case one cannot hope to obtain any information concerning the rate of return to equilibrium. Finally, the results described here are valid in more general situations. In particular, we can treat realistic models in QED which are based on the dipole approximation. For additional information we refer the reader to [2–5].

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[42] Recall that the set of operators commuting with all elements of a given set of operators $\mathscr{B}$ is called the commutant of $\mathscr{B}$ and denoted by $\mathscr{B}'$. Von Neumann's theorem asserts that $\mathfrak{M}_{\mathscr{R}}$ is identical to the double commutant $\pi_{\mathscr{R}}(\mathscr{O}_{\mathscr{R}})''$, see [47]. From the mathematical point of view, it is natural (and convenient) to consider $\mathfrak{M}_{\mathscr{R}}$ as the algebra of observables of the reservoir at inverse temperature β.

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[45] A state ω on a C^*-algebra $\mathscr{O}$ is faithful if, for any $A \in \mathscr{O}$, $\omega(A^*A) = 0$ implies $A = 0$. A simple calculation shows that finite volume Gibbs states share this property. It survives the thermodynamic limit, and is a general feature of thermal equilibrium states at positive temperature [47]. Faithfulness of a state ω has far reaching consequences for the associated cyclic representation $(\mathscr{H}, \pi, \Omega)$ since, in this case, the vector Ω is not only cyclic for $\pi(\mathscr{O})$, but also for its commutant $\pi(\mathscr{O})'$, *i.e.* $\pi(\mathscr{O})'\Omega$ is dense in $\mathscr{H}$.

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[50] For the reader familiar with Tomita-Takesaki's theory, we mention that $\Delta \equiv \mathrm{e}^{-\beta\mathscr{L}}$ is the modular operator of $(\mathfrak{M}, \omega^\beta)$.

[51] In a more mathematical perspective, this construction shows that the KMS state (4.25) exists.

[52] Notice the '$+$' sign in Definition ((4.38)). $\mathscr{N}_{tot}$ should not be confused with the relative number operator $\int (a_1^*(k)a_1(k) - a_2^*(k)a_2(k))\ d^3k$ which counts the number of *physical* phonons relative to the equilibrium state.

[53] In this case one easily shows that $(\Psi, \mathscr{N}_{tot}\Psi) \leq |\mathrm{Im}(\theta)|^{-1}|(\Psi, \mathscr{L}(\theta)\Psi)|$.