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STATISTICS IN SPACE DIMENSION TWO

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ABSTRACT

We construct as a selfadjoint operator the Schroedinger hamiltonian for a system of N identical particles on a plane, obeying the statistics defined by a representation π_1 of the braid group. We use quadratic forms and potential theory, and give details only for the free case; standard arguments provide the extension of our approach to the case of potentials which are small in the sense of forms with respect to the laplacian.
We also comment on the relation between the analysis given here and other approaches to the problem, and also on the connection with the description of a quantum particle on a plane under the influence of a shielded magnetic field (Aharanov-Bohm effect).

0. INTRODUCTION

For quantum particles in space dimension two (as can be considered, at least to first approximation, particles confined in a very thin layer) statistics has a richer structure than in three dimensions. Loosely speaking, this is due to the fact that on a plane one can define a relative winding number for two impenetrable particles: given a configuration of the particles on the plane, the relative winding number counts the number of times one of the particles has gone around the other before reaching again the initial configuration. It is a topological invariant (given the impenetrability).
This richer topological structure has an influence on the quantum mechanical description of a collection of N identical particles in a two-dimensional world and in particular on the definition of the Schroedinger hamiltonian as a self-adjoint operator in the suitable Hilbert space (see e.g. [BCMS],[GS],[LM],[MS],[S.2],[W],[Wu]).
Knot theory is the natural tool to study the topology of the problem. In this paper we shall advocate the point of view that the natural functional-analytic tool are quadratic forms.

There is a connection between the analysis given here and a possible description of a quantum particle on a plane under the influence of a shielded magnetic field, as in the Aharanov-Bohm effect for a particle in R^2 in presence of a singular magnetic field produced by a finite number of solenoids ([AB],[R],[S.1]). We shall touch briefly upon this relation in Section 3.

Throughout this note, we shall consider only the free case, paying special attention to the description of statistics; standard arguments allow to extend the results to the case in which there is a perturbation which is small in the sense of forms. Singular perturbations, such as point interaction, can be introduced but a detailed treatment requires further work and will not be discussed here (see e.g. [MT]).

1. GENERAL FORMULATION

Consider N identical particles of mass m in R^2 and denote by

$$D_N \equiv \{(R^2)^N \ni x \equiv \{x^k,\ k=1,\dots N\} :\ \exists\, i<k \ \ for\ which\ \ x^i = x^k\}$$

the "coincidence set".
To simplify notations, we choose units in which $2m = \hbar = 1$.
Let S_N be the permutation group of N elements, which acts on $(R^2)^N$ in an obvious way, leaving D_N invariant.
The particles are identical and impenetrable; therefore, as a set, a natural candidate for configuration space is the quotient

$$Y_N \equiv \left((R^2)^N - D_N\right)/S_N \qquad 1.1$$

The space Y_N has a natural differential structure and also a natural measure μ induced by the Lebesgue measure on $(R^2)^N$.
The space Y_N is not simply connected; its first homology group is the "braid group" for N elements on R^2, denoted B_N (this is one way of stating in mathematical terms the presence of a nontrivial relative winding number).
We denote by $\tilde{Y}_N$ be the universal covering space of Y_N and denote by π_0 the representation of B_N as the fundamental group of $\tilde{Y}_N$; $\pi_0(g),\ g \in B_N$ is a map on $\tilde{Y}_N$ which projects to the identity in Y_N.

Let π_1 be a finite-dimensional unitary representation of B_N on a complex vector space V of dimension d with scalar product $<,>$.
Identical quantum particles with statistics given by the representation π_1 are described by (complex valued) functions f from $\tilde{Y}_N$ to V satisfying the equivariance relation

$$f(\pi_0(g)\tilde{y}) = \pi_1(g)f(\tilde{y}) \qquad g \in B_N \tag{1.3}$$

One refers to these particles as *plektons;* if V is one-dimensional they are called *anyons.*

We want to define a Schroedinger operator which describes the dynamics of plektons.
A possible line of approach is the following [MS].
A Hilbert space structure is introduced on functions which satisfy (1.3) through the scalar product

$$(f, g) \equiv \int_{Y_N} < f(\tilde{y}), g(\tilde{y}) > d\mu(y) \tag{1.4}$$

where $\tilde{y}$ is any point on the orbit of π_0 through y. Since $\pi_1(g)$ is unitary, the integrand is constants on the orbits of π_0 and therefore (1.4) is well defined.

We shall denote by $\mathcal{H}$ the closure with respect to (1.4) of continuous V-valued functions on $\tilde{Y}_N$ with compact support and satisfying (1.3).
The next step is to introduce dynamics; this is done by choosing a "hamiltonian", i.e. a self-adjoint operator H on $\mathcal{H}$. This can be done by defining on smooth functions on $\tilde{Y}_N$ an operator $\tilde{H}$ which commutes with the representation $\mathcal{R}$ of B_N which is defined by

$$(\mathcal{R}(g)f)(\pi_0(g)\tilde{y}) \equiv \pi_1(g)f(\tilde{y}) \tag{1.5}$$

and then restricting $\tilde{H}$ to $\mathcal{H}$.
Recall that locally one has

$$\tilde{Y}_N \simeq Y_N \otimes W \tag{1.6}$$

where W is a discrete set.
One can then consider the special case in which $\tilde{H}$ acts locally as $(-\Delta + U) \otimes I$ where Δ is the Laplacian on Y_N) and the operator $U(y)$ (the "potential energy) acts multiplicatively on functions on Y_N.
One can also envisage more general situations, for example Δ could be a covariant second order differential operator. We shall not consider these generalizations here.
If one identifies Y_N with a section of $\tilde{Y}_N$ and one sets $\tilde{U}(\pi_0(g)\tilde{y}) = U(y)$ for every $g \in B_N$ one can easily extend U to be a multiplication operator on smooth functions on $\tilde{Y}_N$; by construction it leaves invariant the linear space of functions satisfying (1.3) and therefore can be restricted to $\mathcal{H}$.
If U is sufficiently regular the resulting multiplication operator is self-adjoint.

It is not entirely obvious how to define the differential part of the operator in such a way that H is selfadjoint.
The reason is that $\tilde{H}$ was defined as acting as differential operator on smooth functions on $\tilde{Y}_N$ but, unless the action of π_0 on W can be reduced to an action on a finite set (as is the case for Bose and Fermi statistics) it is not clear how to find a countably additive measure which is invariant under π_0 (so that (1.5) defines a unitary representation) and with respect to which $\tilde{H}$ is self-adjoint.

We shall overcome this difficulty by constructing a unitary map between $\mathcal{H}$ and a suitable Hilbert space $\mathcal{H}'$ and then defining the Schroedinger operator on $\mathcal{H}'$.
Let

$$\Omega_N \equiv \{x \in (R^2)^N \mid x_2^k - x_2^{k+1} < 0,\ k = 1, \dots N-1\} \qquad 1.7$$

where the index 2 denotes the second coordinate with respect to a fixed cartesian frame.
Define

$$\Sigma_N^{\pm} \equiv \cup_k \Sigma_N^{k,\pm}$$

$$\Sigma_N^{h,\pm} \equiv \{x \in (R^2)^N : \ x_2^1 < \dots < x_2^h = x_2^{h+1} < \dots < x_2^N,\ \ \pm(x_1^{h+1} - x_1^h) > 0\} \qquad 1.8$$

$$\Sigma_N^k \equiv \Sigma_N^{k,+} \cup \Sigma_N^{k,-} \qquad \Sigma_N \equiv \Sigma_N^+ \cup \Sigma_N^-$$

Then

$$\Sigma_N^+ \cap \Sigma_N^- = \emptyset, \qquad \bar{\Sigma}_N^+ \cup \bar{\Sigma}_N^- \equiv \partial\Omega_N$$

Moreover

$$\bar{\Sigma}_N^+ \cap \bar{\Sigma}_N^- = \bar{\Omega}_N \cap D_N$$

Let ϕ be the product of the maps ϕ_k which take each point on $\Sigma_N^{k,-}$ to that point on $\Sigma_N^{k,+}$ which is obtained by exchanging the numerical values of x_1^k and x_1^{k+1}.
Then as a set Y_N can be identified with $\Omega_N \cup \Sigma_N^+$ and as a topological space Y_N can be identified with $\Omega_N \cup (\Sigma_N)/\phi)$

REMARK

A simple example is the case $N = 2$. Using coordinates $x^1 + x^2$ and $x^1 - x^2$ one readly sees that Y_2 is diffeomorphic to the product of R^2 with a cone with vertex in the origin, which can be obtained from the closed upper half plane minus the origin identifying by reflection every point on R^- with the corresponding one on R^+.

Denote by g_k the element of the braid group which can be identified with $\phi_k \cdot R_{\pi,k}$ as a map of $\Sigma_N^{k,+}$ onto itself, where $R_{\pi,k}$ denotes rotation of ninety degrees in the anticlockwise direction (i.e. going through Ω_N); these elements generate the entire group, so that the representation π_1 is completely determined by its restriction to the g_k.
Then there is a one-to-one correspondence between continuous functions f on $\tilde{Y}_N$ which satisfy (1.3) and continuous functions F on $\Omega_N \cup \Sigma_N$ which satisfy the boundary condition

$$F(x) = \pi_1(g_k) F(\phi_k(x)), \quad x \in \cup_k \Sigma_N^{k,-} \qquad 1.9$$

It is obtained by identifying $\Omega_N \cup \Sigma_N^+$ with Y_N, as described above, and then Y_N with a subset of $\tilde{Y}_N$. Condition (1.9) guarantees that F can be extended uniquely to a function f on $\tilde{Y}_N$ which satisfies the equivariance requirement (1.3).
This correspondence is an isometry for the corresponding scalar products and extends therefore to a unitary equivalence between $\mathcal{H}$ and the closure in $L^2(\Omega_N, V)$ (Lebesgue measure is understood) of the continuous compactly supported functions satisfying (1.9). Notice that the latter concides with $L^2(\Omega_N, V)$ (in fact the subset of functions which have support in Ω_N, and therefore satisfy (1.9), is dense in $L^2(\Omega_N, V)$). We shall therefore define the Schroedinger operator as a self-adjoint operator on $L^2(\Omega_N, V)$.
Statistics, i.e. the representation π_1 of the braid group B_N, will then enter through the explicit structure of the operator (and in particular through its domain). More precisely, the

free Schroedinger operator will be identified with the self-adjoint operator whose quadratic form is the restriction of the energy form [S.2]

$$Q^{\pi_1}(f,f) \equiv \int_{(R^2)^N} < \nabla f(x), \nabla f(x) > dx \tag{1.10}$$

to the domain

$$D(Q^{\pi_1}) = \{ f \in H^1(\Omega_N, V) |\ f\ satisifies\ (1.9) \}$$

We have denoted by H^s the Sobolev space of order s (see e.g. [A])
This is the *definition* of statistics we shall advocate; it has the advantage of allowing for a simple treatment of the issue of essential self-adjiointness.
A similar approach has been suggested by P.Stovicek in [S.2].

REMARK 1

The distributions in the domain of the quadratic form (1.10) are in general not continuous, but they have a trace on Σ_N which belongs to $H^{1/2}(\Sigma_N, V)$; condition (1.9) will be understood to hold in this slightly more general sense.

¿From potential theory we know that for the laplacian in Ω_N with smooth boundary $\bar{\Sigma}_N$, boundary conditions which are local can be described by double-layer potentials, i.e. by distributions of "dipoles" on the boundary.
Also, as we will discuss in detail in the next Section, it is known that there is a one-to one correspondence (continuous in a suitable topology) between dipole charges and the restriction to Σ_N of the corresponding potential.

The explicit construction of the closed quadratic form which defines the free laplacian for the given choice of statistics will be given in Section 2. The form domain will be the disjoint union of $H_0^1(\Omega_N, V)$ and of a suitable Hilbert space of $V-$valued (dipole charge) distributions on Σ_N.
The dependence on the statistics (i.e. on the representation π_1 of B_N) will be through the expression of the quadratic form on dipole distributions.

REMARK 2

We have already remarked that Σ_N^k does not intersect the subspace $\{x \mid x^k = x^{k+1}\}$. Therefore a boundary condition is not placed directly at coincidence points; since functions in $H^1((R^2)^N, V)$ are not continuous in general, the introduction of such "boundary condition" must be understood in a more general sense [GFT] and would correspond to the introduction of a "point interaction" among plektons. While there seems to be no particular difficulty in applying the methods of [GFT] to achieve this construction, we shall not do it here.

In the remaining part of this Section we shall describe briefly the relation of our approach to one which can be found in the literature for the study of anyons (in this case the representation π_1 is abelian). In this approach one selects an open dense subset $Z_N \subset (R^2)^N$, chosen so that its Riemann covering has B_N as fundamental group. This allows for the introduction of an equivariance condition in the spirit of (1.3).
Define

$$\eta_N \equiv \cup_{k<h} \zeta_N^{k,h}, \qquad \zeta_N^{k,h} \equiv \{x | x_2^k - x_2^h = 0,\ x_1^k < x_1^h\} \tag{1.11}$$

and set

$$Z_N \equiv (R^2)^N - \bar{\eta}_N$$

Denote by $\tilde{Z}_N$ its Riemann covering. Then one can verify that $\tilde{Y}_N$ is diffeomorphic to the quotient of $\tilde{Z}_N$ by S_N and in this way the braid group B_N is identified with the fundamental group of $\tilde{Z}_N$.
We shall denote by $g \to \pi_2(g)$ this identification.
While this presentation of $\tilde{Y}_N$ is not 'canonical" (it depends on the arbitrary choice of η_N) it leads to a concrete construction of the Schroedinger operator.
One considers the Hilbert space completion of functions on $\tilde{Z}_N$ which satisfy the equivariance requirement

$$\phi(\pi_2(g)\tilde{z}) = \pi_1(g)\phi(\tilde{z}), \qquad \tilde{z} \in \tilde{Z}_N \tag{1.12}$$

To set a unitary correspondence with the Hilbert space $\mathcal{H}$ one must now find a bijection between functions on $\tilde{Y}_N$ and a linear subset of functions on $\tilde{Z}_N$.
Once this is done, the definition of a second order differential operator presents no problems. The proof that its restriction to functions satisfying (1.12) defines a self-adjoint operator is less simple. This is clear from the fact that Z_N is $(R^2)^N$ with a subset of codimension one removed. The definiton of Δ as a selfadjoint operator on $\tilde{Z}_N$ requires the specification of boundary conditions on η_N.
To construct a bijection, one can extend to Z_N and then to $\tilde{Z}_N$ the functions defined in Ω_N and for which (1.9) is satisfied.
Any choice of extension is in principle legitimate, under some unitarity conditions. One can for example extend every function from Ω_N to a dense subset of Z_N by setting $f(\sigma(x)) = f(x)$ for every element σ of the permutation group. In this way one obtains functions which have discontinuities on η_N and the definiton of the laplacian requires boundary conditions on subsets of η_N.
Notice that this is the way in which Bose statistics is usually introduced, with the representation π_1 chosen to be $\pi_1(g) = 1 \ \ \forall g \in B_N$.
To give an example, we shall discuss at the end of the next Section the presentation of Fermi statistics which one obtains if one chooses the extension given by

$$f(\sigma(x)) = f(x) \qquad \forall\, \sigma \in S_N \qquad x \in \Omega_N \tag{1.13}$$

The choice (1.13) provides a presentation in which the Hilbert space is made of symmetric functions, but there are functions in the form domain of the Schroedinger operator which are not in $H^1((R^2)^N)$.
In the abelian case (anyons) there is a natural construction which associates to continuous functions on $\tilde{Y}_N$ continuous functions on $\tilde{Z}_N$.
Let $w(x)$ be a differentiable function on Z_N which extends by equivariance (for the abelian representation π_1) to the covering space $\tilde{Z}_N$ as a differentiable function and satisfies

$$w(\pi_2(g)\tilde{x}) = (\omega(g))^{-1} w(\tilde{x}), \quad \tilde{x} \in \tilde{Z}_N \tag{1.14}$$

where we have set

$$S^1 \ni \omega(g) \equiv \pi_1(g), \quad g \in B_N$$

to stress the fact that π_1 is abelian, and therefore $\pi_1(g)$ is a phase factor, in view of unitarity (if the representation is not irreducible, one can consider separately the irreducible components).

A function w with the property (1.14) can be constructed by exploiting the identification of R^2 with the complex plane and choosing for w a suitable meromorphic function.
Then the map

$$W \,:\, f \to f\, w \tag{1.15}$$

takes continuous functions on $\tilde{Z}_N$ which satisfy (1.12) to continuous functions on $\tilde{Z}_N$ which transform under the identity representation of the fundamental group (and correspond therefore to continuous functions on Z_N).
Since $|w(x)| = 1$ the map (1.15) provides an isometry.
In deriving the transformation properties of $f \cdot w$ essential use is made of the fact that the representation π_1 of B_N is abelian. A direct extension of this construction to the nonabelian case does not seem to be possible.

One can now choose the natural bijection between continuous symmetric functions on $(R^2)^N$ and their restriction to the closure of Ω_N.
This leads to consider the linear subspace of functions f on Z_N which are of the form

$$f(x) = F(x)w(x), \qquad F(x) = F(\sigma(x)) \tag{1.16}$$

where σ is any element of the permutation group S_N.
Notice that for this class of functions, the restriction to $\bar{\Omega}_N$ satisfies

$$f(\pi_0(x)) = \omega(g_k)^{1/2} f(x) = \omega(g_k^{1/2}) f(x), \qquad x \in \Sigma_N^{k,+}$$

(also here we have used the fact that the representation is abelian) and therefore the statistics defined by this choice of extension corresponds, in the notations we have used previously, to the representation $g_k \to \pi_1(g_k)^{1/2}$.
One can now proceed to define the Schroedinger operator on the linear space of functions on Z_N which satisfy (1.16).

A possible approach is to define it through its quadratic form, and describe the domain as the direct sum of $H_0^1(Z_N)$ and of potentials of charges concentrated at η_N. This approach has the advantage that an explicit description can be given of domains of selfadjointness. Moreover this construction closely parallels the description of a quantum particle in the field generated by several solenoids perpendicular to a fixed plane to which the particle is restricted (Aharanov-Bohm effect); we shall briefly discuss this in Section 3.
As we shall see there, in the Aharanov-Bohm problem the role of the set Z_N is taken by

$$R^2 - \gamma \qquad \gamma \equiv \cup_{j=1,\dots N}\{x \in R^2 \mid x_2 - y_2^j \in R^+\}$$

where y^j are the positions of the N solenoids (more precisely, the intersections of the N solenoids with the plane).

For example, if only one solenoid were present at the point y, the self-adjoint Schroedinger operator corresponding to the presence of the solenoid is defined by requiring that the functions in its form domain satisfy at γ the boundary condition

$$f^- = e^{ie\alpha/\hbar c} f^+ \tag{1.17}$$

where e is the electric charge of the particle, α is the "strength" of the solenoid (the total flux through any domain which contains y in its interior), $f^{\pm}$ are the traces of f on the two sides of γ.

This is a "local condition", and is treated by the introduction of a dipole density on γ, as we will explain in Section 3.
In the case of a particle interacting with N solenoids this procedure allows to give a rather explicit characterization of the domain of the hamiltonian and of the resolvent.
We point out here that the use of boundary conditions to study the dynamics of a quantum particle in a plane interactiong with N point solenoids has been suggested by P.Stovichek in [S.1] for the case when the N points y_k are colinear.

An alternative way to define the Schroedinger operator starting from the condition (1.16) on functions in its domain proceeds as follows.
Let Δ_0 be the restriction of Δ on $C^2(\tilde{Z}_N)$ to functions which have support disjoint from the image of $\bar{\eta}_N$ in $\tilde{Z}_N$.
The map W takes these functions to functions which are twice differentiable on Z_N and have support disjoint from $\bar{\eta}_N$. We denote by $\mathbf{F}_N$ this class of functions.
Under W the image of the operator $-\Delta$ is

$$K \equiv -\Delta + 2\alpha \cdot \nabla + \beta, \quad \alpha(x) \equiv w^{-1}\nabla w, \quad \beta(x) = w^{-1}\Delta w \tag{1.19}$$

a symmetric operator defined on $\mathbf{F}_N$, a dense subset of $L^2((R^2)^N, dx)$
Notice that the coefficients $\alpha(x)$ and $\beta(x)$ are singular on η_N so that the hamiltonian is now a singular perturbation of the laplacian.
The proof that K defined in (1.19) is essentially selfadjoint on $\mathbf{F}_N$ presents now serious difficulties, and to our knowledge it has been carried out in detail only in the case when there is an additional hard core potential, and the particles are confined to a compact subset of R^2 so that the resulting operator has a compact resolvent and therefore pure point spectrum [BCMS].

2. QUADRATIC FORMS FOR PLEKTONS

In this section we shall give the construction of the quadratic form corresponding to the free Schroedinger operator for plektons. Addition of a potential which is relatively small in form requires no essential additional work.
Consider N plektons in R^2. As configuration space we choose Ω_N as described in (1.7), Section 1.
The form domain will be a subset of $H^1(\bar{\Omega}_N, V)$, the collection of distributions with support in the closure of Ω_N and with square-integrable gradient; we denote by $\|\cdot\|_1$ the corresponding norm.

From now on we shall omit the index N in Ω_N and Σ_N (for their definition see (1.8)).

Any function f in $H^1(\bar{\Omega}, V)$ has a trace F_f on Σ and if f_1 and f_2 have the same trace, then $f_1 - f_2 \in H^1_0(\bar{\Omega}, V)$
Moreover (see e.g.[A]) $F_f \in H^{1/2}(\Sigma, V)$; we denote by $\|\cdot\|_{1/2}$ the norm for this space.

Conversely, for each $k = 1, \dots N$ let μ_k be a $V-$ valued function on Σ^k of class $H^{1/2}$ and consider the $V-$valued function defined in Ω as the double-layer potential due to the distributions μ_k (of "dipole charges").

$$(\hat{G}_\lambda * \mu)(\xi) \equiv \sum_k \int \hat{G}_\lambda(\xi - y)\mu_k(y)\delta(y_2^k - y_2^{k+1})d^{2N}y \tag{2.1}$$

where $\hat{G}_\lambda$ is the double-layer Green function with parameter λ, defined by

$$\hat{G}_\lambda(x-y)|_{y\in\Sigma^k} \equiv \frac{\partial G_\lambda}{\partial n(y)}(x-y)$$

where $n(y)$ is the normal to Σ^k at y directed toward Ω and

$$G_\lambda(x-y) \equiv (-\Delta+\lambda I)^{-1}(x-y) \qquad 2.2$$

Due to the properties of the double-layer potential (see e.g. [K]) the function $\hat{G}_\lambda * \mu$ belongs to $H^1(\bar{\Omega}, V)$; we denote by $\Gamma_\lambda \mu$ its trace on Σ, a function in $H^{1/2}(\Sigma, V)$,

$$\Gamma_\lambda \mu \equiv \left(\hat{G}_\lambda * \mu\right)|_\Sigma$$

and by $\Gamma_\lambda^k \mu$ the restriction of $\Gamma_\lambda \mu$ to Σ^k.
By construction Γ_λ is an integral operator with kernel

$$\Gamma_\lambda(\xi,\eta) \equiv \hat{G}_\lambda(\xi-\eta), \qquad \xi,\ \eta\in\Sigma \qquad 2.3$$

We prove now

PROPOSITION 2.1

For λ sufficiently large (depending only on N) the operator Γ_λ is bounded symmetric and invertible on $H^{1/2}(\Sigma, V)$.

Proof

By the properties of the double-layer potential one has

$$\Gamma_\lambda \mu = \frac{I}{2}\mu + R_\lambda \mu \qquad 2.4$$

where R_λ is an integral operator with matrix kernel $R_\lambda^{k,h}$, $k,h=1,\dots N$ given by

$$R_\lambda^{k,k}(\xi,\eta)=0 \quad \xi,\ \eta\in\Sigma^k$$

$$R_\lambda^{k,h}(\xi,\eta)=\frac{\partial}{\partial n_k(\xi)}\frac{\partial}{\partial n_h(\eta)}G_\lambda(\xi-\eta) \quad \xi\in\Sigma^k,\ \ \eta\in\Sigma^h,\ \ h\neq k.$$

If

$$\mu,\ \mu'\in H^{1/2}(\Sigma,V) \quad \mu_{\Sigma^k}\equiv\mu_k$$

one has, denoting by $(\cdot,\cdot)_\Sigma$ the scalar product for $L^2(\Sigma, V)$ (with respect to Lebesgue measure)

$$(\mu', R_\lambda\mu)_\Sigma = \sum_{k\neq h}\int_{\Sigma^k} d\xi \int_{\Sigma_h} d\eta\ \bar{\mu}'_k(\xi)\mu_h(\eta)\frac{\partial}{\partial n_k(\xi)}\frac{\partial}{\partial n_h(\eta)}G_\lambda(\xi-\eta) \qquad 2.5$$

Using the definition of $G_\lambda(\xi-\eta)$ and Schwartz's inequality the k,h term in the sum (2.5) can be estimated to be smaller in absolute value than

$$c_\lambda \|\mu'_h\|_{1/2}\|\mu_k\|_{1/2}$$

where

$$lim_{\lambda\to\infty} c_\lambda = 0$$

so that

$$\|R_\lambda\| \leq (N-1)c_\lambda \qquad 2.6$$

Therefore for λ sufficiently large Γ_λ is (bounded and) invertible. $\diamondsuit$

It follows from Proposition 2.1 that one can find $\lambda_0(N)$ such that for $\lambda > \lambda_0$ every function $f \in H^1(\bar\Omega, V)$ can be uniquely written as

$$f = f_0^\lambda + \hat G_\lambda * \mu, \qquad f_0^\lambda \in H_0^1(\bar\Omega, V), \qquad \mu \in H^{1/2}(\Sigma, V) \qquad 2.7$$

(notice that μ in (2.7) does not depend on λ).
The advantage of this decomposition lies in the fact that the two terms in the decomposition (2.7) are orthogonal in the scalar product defined by the energy form. This is due to the fact that $(-\Delta+\lambda I)(\hat G_\lambda * \mu)$ is a distribution supported by Σ and is therefore orthogonal to smooth functions vanishing in a neighbourhood of Σ.
One has therefore for any such function g

$$\Big(\nabla g, \nabla(\hat G_\lambda * \mu)\Big) + \lambda(g, (\hat G_\lambda * \mu)) = (g, (-\Delta+\lambda I)(\hat G_\lambda * \mu)) = 0 \qquad 2.8$$

where we have indicated by $(\cdot,\cdot)$ the scalar product in $L^2(\Omega, V)$
The proof of (2.8) for any function in $H_0^1(\Omega, V)$ is then obtained by a standard approximation procedure.
As a consequence, the quadratic form which represents the laplacian with the given boundary conditions will be written as the sum of the energy form on $H^1(\Omega, V)$ and of a bilinear form for functions on Σ.
This will simplify the study of the resolvent and of the spectral properties.
Using (2.7) and Green's theorem one has

$$(\nabla f, \nabla f) + \lambda(f,f) = (\nabla f_0^\lambda, \nabla f_0^\lambda) + \lambda(f_0^\lambda, f_0^\lambda) - \frac{1}{2}\int_\Sigma \mu \frac{\partial}{\partial n} G_\lambda * \mu \qquad 2.9$$

It is known from potential theory that the last term is positive, and bounded if $\mu \in H^{1/2}(\Sigma, V)$. This is seen explicitely by taking Fourier transforms, upon which this term can be written as

$$\frac{1}{4\pi}\sum_k \int \sqrt{\sum_{h\neq k,\, k+1} q_h^2 + \sum_i p_i^2 + \zeta_k^2 + \lambda}\ \ |\mu_k(q^{(k)}, p, \zeta_k)|^2 dq^{(k)} dp\, d\zeta_k + N_\lambda \qquad 2.10$$

where N_λ is a "nondiagonal" term (containing the sum over integrations on terms bilinear in μ_k,μ_h for $h \neq k$) which can be made arbitrarily small by taking λ large enough.

In (2.10) the $p_k's$ are the variables conjugated to the x_1^k, the q_h are the variables conjugated to the x_2^h and in the integral over Σ^k, ζ_k is conjugated to $1/2\left(x_2^k + x_2^{k+1}\right)$ and $q^{(k)}$ denotes the collection of the coordinates q_h with the exception of q_k and q_{k+1}.
¿From (2.9) the form associated to the laplacian on $H^1(\bar{\Omega}, V)$ without restrictions due to statistics is

$$Q(f,f) = (\nabla f_0^\lambda, \nabla f_0^\lambda) - \frac{1}{2}\int_\Sigma \mu \frac{\partial}{\partial n} G_\lambda * \mu - \lambda(f,f) + \lambda(f_0^\lambda, f_0^\lambda) \qquad 2.11$$

defined on

$$H_0^1(\Omega, V) \times H^{1/2}(\Sigma^+, V)$$

The quadratic form for particles obeying the statistics defined by the representation π_1 of the braid group B_N is obtained restricting (2.11) to functions which satisfy (1.9).
Since by construction f_0^λ vanishes on Σ the restriction is entirely on $\hat{G}_\lambda * \mu$; in view of Proposition 2.1 it can be expressed as a relation in (2.12) between μ^- and μ^+, the restrictions of μ to $\Sigma^\pm$.
In order to work out the details of this construction, notice that if one sets, for functions ν defined on $\Sigma^\pm$

$$\nu_k^\pm \equiv \theta(\pm\xi_1^k)\nu_k \qquad 2.12$$

(where $\theta(x)$ is 1 for $x > 0$ and 0 for $x \leq 0$) one obtains

$$\Gamma_\lambda \mu = \frac{I}{2}\mu + R_\lambda \mu^+ + R_\lambda \mu^- \qquad 2.13$$

The boundary conditions which define the statistics have the form

$$\left(\Gamma_\lambda^k \mu\right)^- = \pi_1(g_k)\left(\Gamma_\lambda^k \mu\right)^+ \qquad 2.14$$

where g_k is the element of B_N which has been described in Section 1 and the $\pm$ sign indicates restriction of $\Gamma_\lambda \mu$ to $\Sigma_k^\pm$.
We write (2.14) in the concise form

$$(\Gamma_\lambda \mu)^- = A_\lambda(\pi_1)\,(\Gamma_\lambda \mu)^+$$

This *defines* a linear map $A_\lambda(\pi_1)$ on V^N, which depends on the representation π_1. ¿From (2.13) one has

$$\frac{1}{2}\mu^- + R_\lambda^{-,+}\mu^+ + R_\lambda^{-,-}\mu^- = A_\lambda(\pi_1)\left[\frac{1}{2}\mu^+ + R_\lambda^{+,+}\mu^+ + R_\lambda^{+,-}\mu^-\right] \qquad 2.15$$

where $R_\lambda^{\pm,\pm}(\xi,\eta)$ are integral operators with kernels given by $R_\lambda(\xi,\eta)$ for $\xi \in \Sigma^\pm$, $\eta \in \Sigma^\pm$.
After rearrangment of terms (2.15) becomes

$$\frac{1}{2}\mu^- + R_\lambda^{-,-}\mu^- - A_\lambda(\pi_1)R_\lambda^{+,-}\mu^- = \frac{1}{2}A_\lambda(\pi_1)\mu^+ + A_\lambda(\pi_1)R_\lambda^{+,+}\mu^+ - R_\lambda^{-,+}\mu^+ \qquad 2.16$$

For λ sufficiently large the $H^{1/2}$ norms of $R_\lambda^{\tau,\nu}\mu^\nu$, $(\tau,\ \nu = \pm)$ become arbitrarily small with respect to the norm of μ (the proof is identical to the one given above for the operator $R_\lambda^{\pm}$).
Therefore if one takes λ sufficiently large (depending only on N) one can write (2.16) in the form

$$\mu^- = \left[I + 2R_\lambda^{-,-} - 2A_\lambda(\pi_1)R_\lambda^{+,-}\right]^{-1} A_\lambda(\pi_1)\left[I + 2R_\lambda^{+,+} - 2A_\lambda^{-1}(\pi_1)R_\lambda^{-,+}\right]\mu^+ \qquad 2.17$$

which, in components, can be written as

$$\mu_k^- = \sum_h T_{k,h}^\lambda(\pi_1)\mu_h^+ \qquad 2.18$$

where the matrix T^{π_1} defines a bounded invertible linear map from $H^{1/2}(\Sigma^+, V)$ to $H^{1/2}(\Sigma^-, V)$ which depends on the representation π_1 chosen for the braid group.
Equation (2.18), or rather its explicit form given through (2.17) will be used to give the Dirichlet form Q^{π_1} which describes the Laplacian for a system of N identical impenetrable particles in R^2 which satisfy the statistic defined by the representation π_1 of the braid group B_N.
Substituting (2.18) in (2.12) one has

PROPOSITION 2.2

The quadratic form associated to the free hamiltonian for a system of N identical impenetrable particles in R^2 described by the representation π_1 of the braid group is given by

$$Q^{\pi_1}(f,f) = (\nabla f_0, \nabla f_0) - \lambda(f,f) + \lambda(f_0^\lambda, f_0^\lambda) - \frac{1}{2}\int_\Sigma \mu \frac{\partial}{\partial n} G_\lambda * \mu \qquad 2.19$$

$$\mu_k^- \equiv \sum_h T_{k,h}^\lambda(\pi_1)\mu_h^+ \qquad 2.20$$

where $T_{k,h}^\lambda(\pi_1)$ is defined in (2.17),(2.18).

The analysis given above can be easily extended to the study of the Schroedinger equation with a potential U which is relatively small as form with respect to the laplacian.
More singular potentials, and in particular interactions supported by $\cup_k\left(\bar{\Sigma}_k^+ \cap \bar{\Sigma}_k^-\right)$ ("zero range interactions") require more care. We shall not discuss them here, but weremark that the characterization of the statistics in terms of a quadratic form for the laplacian allows to introduce point interactions in a natural manner [GFT].

We shall now briefly outline the domain of the hamiltonian and its action on the domain. Along similar lines one can give an expression for the resolvent. The simplicity of the action of the operator on elements in its domain suggests that one can obtain a rather explicit form for the resolvent. We shall not work it out here, but in the next Section we shall consider more in detail the analogous problem for a quantum particle in R^2 under the influence of the magnetic field of N solenoids.

Recall that, if $Q(v,u)$ is a strictly positive bilinear form on a Hilbert space $\mathcal{H}$, the domain of the corresponding operator is the subset of those $u's$ in the form domain for which $Q(v,u)$ is continuous in v in the topology of $\mathcal{H}$.

In our case, considering a sequence of functions v_n of the form

$$v_n = \hat{G}_\lambda * \mu_n, \qquad \mu_n \in H^{1/2}(\Sigma, V) \tag{2.21}$$

which converge in $L^2(\Omega, V)$ and requiring convergence of $Q^{\pi_1}(u, v_n)$ one finds the condition

$$u = u_0 + \hat{G}_\lambda * \mu, \qquad \mu \in H^{3/2}(\Sigma, V) \tag{2.22}$$

If the same procedure is applied to a sequence of functions of the form

$$v_n = \hat{G}_\lambda * \mu + h_n, \qquad \mu \in H^{1/2}(\Sigma, V), \qquad h_n \in H^1_0(\Omega, V)$$

which converge in $L^2(\Omega, V)$, and convergence of $Q^{\pi_1}(u, v_n)$ is required, one finds that u_0 must belong to $H^2(\bar{\Omega}, V)$ and that the normal derivative of u at Σ (which exists since $\mu \in H^{3/2}(\Sigma, V)$) must satisfy (1.9) (this condition cancels boundary terms which would appear in the integration by parts and for which the limit does not exist in general).
If these conditions on u are satisfied, by Riesz's representation theorem the quadratic form can be written, for every v in the form domain

$$Q^{\pi_1}(v, u) = (v, \xi_u) \tag{2.23}$$

for some $\xi_u \in L^2(\Omega, V)$, and then $Hu \equiv \xi_u$.
Setting $v = v_0 + \hat{G}_\lambda * \nu$ and performing the integrations by parts one concludes

PROPOSITION 2.3

The hamiltonian H corresponding to the quadratic form $Q^{\pi_1}(v, u)$ is characterized, for λ sufficiently large, by

$$D(H) \equiv$$

$$\{u = u_0 + \hat{G}_\lambda * \mu, \quad u_0 \in H^2(\bar{\Omega}, V) \cap H^1_0(\Omega, V), \quad \mu \in H^{3/2}(\Sigma, V), \quad \mu \simeq (2.20) \;\; \partial_n u \simeq (1.9)\}$$

(we have used the symbol $\simeq$ to indicate that the corresponding identity is satisfied).
If $u \in D(H)$ one has

$$(H + \lambda)u = (-\Delta + \lambda)u_0 \tag{2.24}$$

We conclude this Section giving some details of the presentation of the Schroedinger operator for N identical quantum particles satisfying Fermi statistics as an operator on a space of symmetric functions. It could be called pictorially "bosonization of free nonrelativistic fermions"
We shall treat for simplicity only the case $N = 2$.
One starts with the operator

$$\Delta = \Delta_1 + \Delta_2, \qquad \Delta_k \equiv \sum_{i=1,2} \frac{\partial^2}{\partial (x^k_i)^2}$$

defined on smooth functions on the open set

$$\Omega \equiv \{x : \; x^2_2 > x^1_2\}$$

Introducing the new coordinates $y,\ z$ defined by

$$y \equiv x^2 - x^1, \qquad z \equiv \frac{1}{2}(x^1 + x^2)$$

one has

$$\Delta = \Delta_z + \Delta_y \quad on \quad \{y, z :\ y_2 > 0\}$$

The operator Δ_z is essentially selfadjoint in the given domain and its closure is represented by the energy form on R^2. We shall then restrict attention to Δ_y, which is the Laplacian defined on smooth functions on the upper half plane $\{y :\ y_2 > 0\}$.
Here $\bar{\Sigma} \equiv \{y\ :\ y_2 = 0\}$ and we are looking for an extension with the property that every function in its form domain belongs to $H^1(R^2 - \bar{\Sigma})$ and is such that it has opposite traces on opposite sides of Σ. We describe this by requiring

$$f(-y_1, 0_-) = -f(y_1, 0_+), \quad y_1 \in R^+ \tag{2.25}$$

The isometry corresponding to bosonization extends to a map from smooth functions f in the open upper half plane, which have a limit (in $H^{1/2}(R)$) when $y_2 \to 0_+$ to smooth functions $\tilde{f}$ on $R^2 - \Sigma$ which have a limit from both sides and the two limits sum to zero:

$$lim_{y_2 \to 0_+} \tilde{f} + lim_{y_2 \to 0_-} \tilde{f} = 0 \tag{2.26}$$

The problem of finding the selfadjoint extension which corresponds to fermions is now posed as the problem of finding the self-adjoint extension of Δ^0 which has in its domain functions which satisfy (2.25).
Here we have denoted by Δ^0 the Laplacian as symmetric operator defined on smooth functions in R^2 with support which is compact and does not intersect the "horizontal" axis $\{y :\ y_2 = 0\}$.

According to (2.7) a function in the form domain of any of these self-adjoint extensions can be written (for λ sufficiently large) as

$$u(x) = u^0_\lambda(x) + \hat{G}_\lambda * \mu(x), \quad x \in R^2 - \bar{\Sigma} \tag{2.27}$$

The symmetric function u^0 belongs to $H^1(R^2 - \bar{\Sigma})$ (and therefore vanishes on $\bar{\Sigma}$), while the function $\mu \in H^{1/2}(R)$ is symmetric.
In fact, every symmetric function in $H^1(R^2)$ can be represented (and then uniquely) as in (2.27).
As a special case of (2.19) the energy form $Q(u)$ reads on functions represented as in (2.27), with the required symmetry

$$Q(u) = \left[(\nabla u^0_\lambda, \nabla u^0_\lambda) - \lambda(u, u) + \lambda(u^0_\lambda, u^0_\lambda)\right] + \Phi_\lambda(\mu) \tag{2.28}$$

$$\Phi_\lambda(\mu) \equiv \frac{1}{4\pi} \int_R \sqrt{p^2 + \lambda}\ |\hat{\mu}(p)|^2 dp \tag{2.29}$$

The selfadjoint operator defined by the quadratic form Q is unitarily equivalent to the laplacian defined on antisymmetric functions on R^2.

The unitary transformation is defined by

$$f'(x) = f(x), \;\; x_2 > 0, \qquad f'(x) = -f(x), \;\; x_2 < 0$$

Notice that, due to the symmetry requirement we have made on the function μ, the image of $\hat{G} * \mu$ is continuous at Σ and in fact a function in $H^1(R^2)$.

3. THE AHARANOV – BOHM HAMILTONIAN

We exploit here the method of layer potentials to analyse the Aharanov-Bohm hamiltonian describing the interaction of a charged quantum particle on a plane Π, interacting with M thin solenoids which are orthogonal to the plane. We identify the plane with R^2 and denote by $y \equiv \{y_1, \; y_2\}$ the coordinates relative to a fixed reference frame.
Let $y^1, \ldots y^M$ the coordinates of the intersections of the solenoids with Π and let α_k be the magnetic flux associated to the k^{th} solenoid.
To simplify notation, we choose unit for which $\hbar = c = e = 2m = 1$ where e is the charge of the particle and m is its mass.
We agree to choose an ordering (and a reference frame) such that

$$y_2^k < y_2^{k+1}, \qquad 1 \leq k \leq M-1$$

and for each point consider the half-line

$$\gamma_k \equiv \{x \in R^2 | \; x_1 > y_1^k, \; x_2 = y_2^k\} \tag{3.1}$$

A singular gauge transformation maps the formal hamiltonian for a particle interacting with M solenoids into the free laplacian in

$$\Xi \equiv R^2 - \cup_k \gamma_k$$

with the following boundary conditions on each γ_k :

$$u_k^- = e^{i\alpha_k} u_k^+, \qquad \left(\frac{\partial u}{\partial x_2}\right)_k^- = e^{i\alpha_k} \left(\frac{\partial u}{\partial x_2}\right)_k^+ \tag{3.2}$$

where α_k are the preassigned magnetic fluxes and for a function $g(x)$ we have denoted by g_k^+ the trace on γ_k from above and by g_k^- the trace on γ_k from below (see e.g. [S.1],[R]).

In order to describe a self-adjoint realization in $L^2(R^2)$ of this formal operator, we introduce the quadratic form F defined on complex-valued functions as follows

$$D(F) \equiv \{u \in L^2(R^2) | \; u \in H^1(\Xi), \;\; u_k^- = e^{i\alpha_k} u_k^+\} \tag{3.3}$$

$$F(u) \equiv \int_\Xi |\nabla u|^2 dx \tag{3.4}$$

This form is obviously closed and positive, and defines a positive self-adjoint operator H which is by definition the hamiltonian of a charged particle in R^2 interacting with M solenoids orthogonal to the plane.

In the remaining part of this section we shall give a rather detailed study of this form and of the associated operator, as well as a fairly explicit form of its resolvent.
Our results generalize those of Stovicek [S.1] where the same hamiltonian is analysed in the case when the points y^k are colinear.
The analysis parallels to a large extent the one given in the previous section for the case of N plektons, with some simplifying features which allow for a more explicit description.
One has then

PROPOSITION 3.1

For λ sufficiently large one has

$$D(F) \equiv \{u \in L^2(R^2) |\ u = v + \sum_k \hat{G}_\lambda * \mu_k, \quad v \in H^1(R^2), \ \ \mu_k \in H^{1/2}(\gamma_k),$$

$$\frac{i\mu_k}{2} cotg \frac{\alpha_k}{2} - \sum_{j \neq k} \left(\hat{G}_\lambda * \mu_j \right)_k = v_k \} \qquad 3.5$$

and, on its domain, the quadratic form F is given by

$$F(u) \equiv \int_{R^2} \left(|\nabla v|^2 + \lambda |v|^2 - \lambda |u|^2 \right) dx$$

$$- \sum_k \int_{\gamma_k} \bar{\mu}_k(z) \frac{\partial}{\partial n} \left(\hat{G}_\lambda * \mu_k \right)_k (z) dz - \sum_{j \neq k} \int_{\gamma_k} \bar{\mu}_k(z) \left(\frac{\partial}{\partial n} \hat{G}_\lambda * \mu_j \right)_k (z) dz \qquad 3.6$$

where n is the normal at γ_k oriented to increasing values of y_2.

Proof

Let u belong to the domain of F. Then $u \in H^1(\Xi)$ and, making use of the discontinuity properties of $\hat{G}_\lambda * \mu_k$ at γ_k one computes

$$u_k^- - e^{i\alpha_k} u_k^+ = v_k + \sum_{j \neq k} \left(\hat{G}_\lambda * \mu_j \right)_k - \frac{\mu_k}{2} - e^{i\alpha_k} \left[v_k + \sum_{j \neq k} \left(\hat{G}_\lambda * \mu_j \right)_k + \frac{\mu_k}{2} \right]$$

$$= (1 - e^{i\alpha_k}) \left[v_k - \frac{i}{2} \mu_k cotg(\frac{\alpha_k}{2}) + \sum_{j \neq k} \left(\hat{G}_\lambda * \mu_j \right)_k \right] = 0$$

Conversely, given $u \in D(F)$ define

$$\mu_k \equiv u_k^+ - u_k^-, \qquad v \equiv u - \sum_k \hat{G}_\lambda * \mu_k \qquad 3.7$$

Then $\mu_k \in H^{1/2}(\gamma_k)$ and moreover, by direct computation, $v_k^+ = v_k^-$ which implies $v \in H^1(R^2)$.
Still by direct inspection one verifies that the relation between μ_k and v_k given in (3.5) is satisfied.

¿From (3.4) and (3.7), integrating by parts and using the discontinuity properties of double layer potentials one derives the expression for $F(u)$ given in Proposition 3.1. $\diamondsuit$

REMARK 3.2

In the case of only one solenoid the expression for the form F and for its domain $D(F)$ can be simplified.

¿From (3.5) one has (setting $\mu_1 \equiv \mu, \ldots$)

$$\mu = -2iv \; tg(\frac{\alpha}{2})$$

and, substituting in (3.6)

$$D(F) = \{u \in L^2(R^2) \mid u = v - 2i \; tg(\frac{\alpha}{2}) \; \hat{G}_\lambda * v, \quad v \in H^1(R^2)\}$$

$$F(u) = \int_{R^2} \left(|\nabla v|^2 + \lambda |v|^2 - \lambda |u|^2\right) dx - 4 \; tg^2\left(\frac{\alpha}{2}\right) \int_\gamma \bar{v}(z) \left(\frac{\partial}{\partial n} \hat{G}_\lambda * v\right)(z) dz$$

$$= \int_{R^2} \left(|\nabla v|^2 + \lambda |v|^2 - \lambda |u|^2\right) dx - 4 \; tg^2\left(\frac{\alpha}{2}\right) \int_R \sqrt{k^2 + \lambda} |\tilde{v}(k)|^2 dk$$

where $\tilde{v}$ is the Fourier transform of v.

In the present context, in order to describe the operator and its domain it turns out to be more convenient to introduce, in addition to the double-layer potentials, also the single-layer potentials associated to charge distributions σ_k on γ_k, defined by

$$\left(\tilde{G}_\lambda * \sigma_k\right)(x) \equiv \int_{\gamma_k} G_\lambda(x - z) \sigma_k(z) dz$$

In this notation one has

PROPOSITION 3.3

For λ sufficiently large the domain of the operator H associated to the form $F(u)$ is given by

$$D(H) = \{u \in L^2(R^2) \mid u = w + \sum_k \tilde{G}_\lambda * \sigma_k + \sum_k \hat{G}_\lambda * \mu_k\} \qquad 3.8$$

where

$$w \in H^2(R^2) \cap H^1(\Xi), \qquad \sigma_k \in H^{1/2}(\gamma_k), \qquad \mu_k \in H^{3/2}(\gamma_k)$$

$$\frac{i}{2} \mu_k cotg\left(\frac{\alpha_k}{2}\right) - \sum_{j \neq k} \left(\hat{G}_\lambda * \mu_j\right)_k - \sum_j \left(\tilde{G}_\lambda * \sigma_j\right)_k = 0$$

$$\frac{i}{2} \sigma_k cotg \frac{\alpha_k}{2} - \sum_{j \neq k} \left(\frac{\partial}{\partial n} \tilde{G}_\lambda * \sigma_j\right)_k - \sum_j \left(\frac{\partial}{\partial n} \hat{G}_\lambda \mu_j\right)_k = \left(\frac{\partial w}{\partial n}\right)_k$$

(recall that elements in $H^1(\Xi)$ have zero trace on all γ_k).

On $D(H)$ the action of H is given by

$$(H + \lambda) u = (-\Delta + \lambda) w \qquad 3.9$$

The proof follows the same scheme outlined in the previous section and will be omitted.

REMARK 3.4

One can rephrase Proposition 3.1, making use of both single and double layer potentials, so that the condition on the domain coincides with the first condition in Proposition 3.3.

We conclude this section giving a rather explicit expression for the resolvent. We point out that a good representation for the resolvent is the starting point for a detailed analysis of the spectrum and for scattering theory. We plan to come back to these problems making use of the formalism developped here.

For $f \in L^2(R^2)$ we want to construct $(H + \lambda\, I)^{-1}\, f$ for λ sufficiently large.
This is equivalent to solving the following boundary value problem

$$(-\Delta + \lambda\, I)u = f \qquad on\;\; R^2 - \cup_k \gamma_k \tag{3.10}$$

$$u_k^- = e^{i\alpha_k} u_k^+, \qquad \left(\frac{\partial u}{\partial n}\right)_k^- = e^{i\alpha_k}\left(\frac{\partial u}{\partial n}\right)_k^+ \tag{3.11}$$

Taking into account Proposition 3.2 it is natural to represent the solution of (3.10), (3.11) as

$$u = G_\lambda * f + \sum_k \tilde{G}_\lambda * \sigma_k + \sum_k \hat{G}_\lambda * \mu_k \tag{3.12}$$

REMARK 3.5

Strictly speaking, this prescription is different from the one used in Proposition 3.1, since the first term does not vanish on $\cup_k \gamma_k$. The representation we use now is more convenient in this context than the previous one, and they only differ for a different definition of the charge distributions.

If one imposes the boundary conditions (3.11) in (3.12) one obtains a system of equations for the charge distributions

$$\frac{i}{2}\mu_k cotg\left(\frac{\alpha_k}{2}\right) - \sum_{j \neq k}\left(\hat{G}_\lambda * \mu_j\right)_k - \sum_j \left(\tilde{G}_\lambda * \sigma_j\right)_k = (G_\lambda * f)_k \tag{3.13}$$

$$-\frac{i}{2}\sigma_k cotg\left(\frac{\alpha_k}{2}\right) - \sum_{j \neq k}\left(\frac{\partial}{\partial n}\tilde{G}_\lambda * \sigma_j\right)_k - \sum_j \left(\frac{\partial}{\partial n}\hat{G}_\lambda * \mu_j\right)_k = \left(\frac{\partial G_\lambda * f}{\partial n}\right)_k \tag{3.14}$$

We now prove

PROPOSITION 3.6

For λ sufficiently large the system (3.13), (3.14) has a unique solution

$$\mu_k \in H^{3/2}(\gamma_k), \qquad \sigma_k \in H^{1/2}(\gamma_k) \tag{3.15}$$

Proof

Observe that, since $G_\lambda * f \in H^2(R^2)$, the r.h.sides of (3.13), (3.14) have the regularity described in (3.8).

We rewrite (3.13),(3.14) as

$$\sum_j \Lambda^D_{k,j}(\mu_j,\sigma_j) + \sum_j \Lambda^{ND}_{k,j}(\mu_j,\sigma_j) = \Big((G_\lambda * f)_k, (\frac{\partial}{\partial n} G_\lambda * f)_k \Big) \qquad 3.16$$

where

$$\sum_j \Lambda^D_{k,j}(\mu_j,\sigma_j) = \Big(\frac{i}{2}\mu_k cotg\left(\frac{\alpha_k}{2}\right) - (\tilde G_\lambda * \sigma_k)_k, -\frac{i}{2}\sigma_k cotg\left(\frac{\alpha_k}{2}\right) - (\hat G_\lambda * \mu_k)_k \Big) \qquad 3.17$$

$$\sum_j \lambda^{ND}_{k,j}(\mu_j,\sigma_j) =$$

$$\Bigg(-\sum_{j\neq k}\Big[(\hat G_\lambda * \mu_j)_k + (\tilde G_\lambda * \sigma_j)_k \Big] \ , \ -\sum_{j\neq k}\Big[(\frac{\partial}{\partial n}\tilde G_\lambda * \sigma_j)_k + (\frac{\partial}{\partial n}\hat G_\lambda * \mu_j)_k \Big] \Bigg) \qquad 3.18$$

We prove first that the operator Λ^D is a bijection, i.e. the following system has a unique solution

$$\frac{i}{2}\mu_k cotg\left(\frac{\alpha_k}{2}\right) - \left(\tilde G_\lambda * \sigma_k\right)_k = \xi_k, \quad -\frac{i}{2}\sigma_k cotg\frac{\alpha_k}{2} - \left(\frac{\partial}{\partial n}\hat G_\lambda * \mu_k\right)_k = \zeta_k \qquad 3.19$$

where

$$\xi_k \in H^{3/2}(\gamma_k), \qquad \zeta_k \in H^{1/2}(\gamma_k)$$

Operating with G_λ on the second equation and substituting in the first, and making use of the identity

$$-\Big[\tilde G_\lambda * (\frac{\partial}{\partial n}\hat G_\lambda * \mu_k) \Big]_k = \frac{\mu_k}{4} \qquad 3.20$$

(see e.g. [K]) one finds as unique solution

$$\mu_k = \frac{-\frac{i}{2}\xi_k cotg\left(\frac{\alpha_k}{2}\right) + (\tilde G_\lambda * \zeta_k)_k}{\frac14 (1 + cotg^2\left(\frac{\alpha_k}{2}\right))} \in H^{3/2}(\gamma_k) \qquad 3.21$$

$$\sigma_k = \frac{\frac{i}{2}\zeta_k cotg\left(\frac{\alpha_k}{2}\right) + (\frac{\partial}{\partial n}\hat G_\lambda * \xi_k)_k}{\frac14 (1 + cotg^2\left(\frac{\alpha_k}{2}\right))} \in H^{1/2}(\gamma_k) \qquad 3.22$$

The off-diagonal term Λ^{ND} is easily seen to be a bounded operator in the spaces indicated, with norm decreasing to zero when $\lambda \to \infty$.
This concludes the proof of Proposition 3.6. $\diamondsuit$

REMARK 3.7

In case $M = 1$ the off diagonal operator is not present and the above procedure leads to the explicit solutions of the equations for μ and σ and thus to a completely explicit formula for the resolvent:

$$(H + \lambda\, I)^{-1} f = G_\lambda * f + \tilde G_\lambda * \sigma + \hat G_\lambda * \mu \qquad 3.23$$

where μ and σ are given in (3.21) and (3.22).

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