Long-range scattering in the position representation

J. Dereziński *
C. Gérard
Centre de Mathématiques, URA 169 CNRS,
Ecole Polytechnique
91128 Palaiseau Cedex, France

Abstract

For a large class of long-range potentials we prove the asymptotic completeness of modified wave operators constructed using solutions of the eikonal equation $-\partial_t\Psi(t,x) = \frac{1}{2}(\nabla_x\Psi(t,x))^2 + V(x)$.

1 Introduction

In the scattering theory for Schrödinger operators one considers the free Hamiltonian

$$H_0 = \frac{1}{2}D^2 \text{ acting on } L^2(X),$$

where $X = \mathbf{R}^n$, and the full Hamiltonian

$$H = \frac{1}{2}D^2 + V(x),$$

where V is a real potential tending in some weak sense to zero when x tends to ∞. As is well known, potentials V fall naturally into two classes: the short-range potentials where roughly speaking

$$|V(x)| \leq C\langle x\rangle^{-\mu},\ \mu > 1, \tag{1.1}$$

and the long-range potentials, where:

$$|\partial_x^\alpha V(x)| \leq C\langle x\rangle^{-\mu-|\alpha|},\ 1 \geq \mu > 0,\ |\alpha| = 0, 1, 2. \tag{1.2}$$

*on leave from Department of Mathematical Methods in Physics, Warsaw University, Hoża 74, 00-682 Warszawa, Poland.

For short-range potentials the wave operators

$$\Omega_{\rm sr}^{\pm} = \text{s}-\lim_{t\to\pm\infty} \mathrm{e}^{itH}\mathrm{e}^{-itH_0} \tag{1.3}$$

exist and are complete

$$\mathrm{Ran}\Omega_{\rm sr}^{\pm} = \mathcal{H}_{\rm c}(H),$$

where $\mathcal{H}_c(H)$ denotes the continuous spectral subspace of the operator H. For long-range potentials, the limits (1.3) typically do not exist and the definition of wave operators has to be modified.

Several different constructions of wave operators in the long-range case can be found in the literature. Probably the most popular approach is the so-called momentum approach. It consists in replacing the free dynamics e^{-itH_0} by a modified free dynamics of the form $\mathrm{e}^{-iS(t,D)}$, where $\mathbf{R}\times X' \ni (t,\xi) \mapsto S(t,\xi)$ is an exact or approximate solution of the Hamilton-Jacobi equation (see [Hö], [DG]).

$$\partial_t S(t,\xi) = \frac{1}{2}\xi^2 + V(\nabla_\xi S(t,\xi)). \tag{1.4}$$

In [Hö], the existence of the modified wave operators

$$\Omega_S^{\pm} = \text{s}-\lim_{t\to\pm\infty} \mathrm{e}^{itH}\mathrm{e}^{-iS(t,D)} \tag{1.5}$$

and their completeness

$$\mathrm{Ran}\Omega_S^{\pm} = \mathcal{H}_{\rm c}(H)$$

is shown under condition (1.2) using the stationary approach. In [DG] the same result is shown under slightly weaker conditions by the time-dependent approach.

Another approach to constructing modified wave operatrors, is that of Isozaki-Kitada [IK], [DG].

This paper is devoted to yet another approach, which we call the “position approach” and is due to Yafaev [Yaf1].

In order to describe Yafaev’s construction let us start with the short-range case. Let us recall that

$$\mathrm{e}^{-itH_0}u(x) = \mathrm{e}^{i\pi n/4}(2\pi t)^{-n/2}\int \mathrm{e}^{i\frac{(x-y)^2}{2t}}u(y)dy. \tag{1.6}$$

Let $\mathcal{F}$ be the Fourier transformation:

$$\mathcal{F}u(x) = (2\pi)^{-n/2}\int \mathrm{e}^{-ix\cdot x'}u(x')\mathrm{d}x'.$$

It follows directly from (1.6) that if we set

$$U_{\Psi_0}(t)u(x) := \mathrm{e}^{i\pi n/4}t^{-n/2}e^{i\frac{x^2}{2t}}\mathcal{F}u\left(\frac{x}{t}\right),$$

then

$$\mathrm{e}^{-itH_0}u = U_{\Psi_0}(t)u + o(t^0), \tag{1.7}$$

in L^2 norm when t tends to ∞. Let now $V(x)$ be a short range potential satisfying (1.1). It follows then from (1.7) that the wave operators in the short range case have an alternative definition

$$\Omega^{\pm}_{\mathrm{sr}} = \mathrm{s-}\lim_{t\to\infty} \mathrm{e}^{itH} U_{\Psi_0}(t).$$

To handle long-range potentials, Yafaev proposed in [Yaf1] to replace the phase function $\Psi_0(t,x) = \frac{x^2}{2t}$ by a solution $\Psi(t,x)$ of the *eikonal equation*

$$-\partial_t \Psi(t,x) = \frac{1}{2}(\nabla_x \Psi(t,x))^2 + V(t,x). \tag{1.8}$$

This is analogous to the replacement of the function $\frac{1}{2}t\xi^2$ by a solution $S(t,\xi)$ of the Hamilton-Jacobi equation (1.4) in the momentum approach to the long-range scattering.

[Yaf1], [Yaf2] contain the proof of the existence of the limits

$$\Omega^{\pm}_{\Psi} := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{itH} U_{\Psi}(t), \tag{1.9}$$

where

$$U_{\Psi}(t)u(x) := \mathrm{e}^{i\pi n/4} t^{-n/2} \mathrm{e}^{i\Psi(t,x)} \mathcal{F}u(\frac{x}{t}).$$

under rather strong conditions on $V(x)$. In our paper we would like to give a direct proof of the completeness of the operators Ω^{+}_{Ψ}.

The proof of existence and completeness of wave operators in the time-dependent approach can be split in two independent steps.

In the first step one proves some rough propagation estimates which pinpoint the difference between bound states and scattering states. The Mourre estimate [Mo] or the RAGE theorem are two examples of tools used in this first step (see for example [E1], [E2], [SS], [Gr], [DG]). In this step there is no essential difference between the short-range and the long-range case and the choice of a wave operator does not play a role.

In the second step, one has to prove some sharper propagation estimates for scattering states which are of a semiclassical nature. The technical details of the second step depend crucially on which construction of wave operators we use. If we use the momentum approach (1.5), then the pseudodifferential calculus enters in an essential way, for example to estimate quantities like $[V(x), iS(t,D)]$. Under minimal regularity hypotheses on the potentials this involves some rather delicate symbol classes (see [Hö] for the stationary approach and [DG] for the time-dependent approach).

In the Isozaki-Kitada approach one has to estimate some Fourier integral operators.

The main advantage of the position approach (1.9) is that we can make use of the fact that the Schrödinger operator is a partial differential operator and we do not use neither Fourier integral nor pseudodifferential calculus. Our goal in this paper is to show that the position approach can be used to prove the asymptotic completeness for long-range potentials, under the same conditions as the one used in [DG, Chapter 4], using rather elementary differential calculus.

In the short-range case wave operators are uniquely defined by (1.3). In the long-range case modified wave operators are no longer unique. A quantity, which is uniquely defined, is the asymptotic velocity

$$\begin{aligned} P^{\pm} \ &:= \mathrm{s}{-}C_\infty - \lim_{t\to\pm\infty} \mathrm{e}^{itH}\tfrac{x}{t}\mathrm{e}^{-itH} \\ &= \mathrm{s}{-}C_\infty - \lim_{t\to\pm\infty} \mathrm{e}^{itH} D \mathrm{e}^{-itH}\mathbf{1}^{\mathrm{c}}(H), \end{aligned}$$

where $\mathrm{s}{-}C_\infty - \lim$ denotes the so called "strong C_∞ limit" (see [DG] and Theorem 5.1). One expects that modified wave operators $\Omega^{\pm}$ have the following properties:

$$\begin{aligned} &\Omega^{\pm}\Omega^{\pm *} = \mathbf{1}^{\mathrm{c}}(H), \qquad \Omega^{\pm}\Omega^{\pm *} = \mathbf{1}, \\ &\Omega^{\pm} D = P^{\pm}\Omega^{\pm}, \qquad \Omega^{\pm} H_0 = H\Omega^{\pm}. \end{aligned} \tag{1.10}$$

Note that if two operators $\Omega_1^{\pm}$, $\Omega_2^{\pm}$ satisfy (1.10), then

$$\Omega_1^{\pm} = \Omega_2^{\pm}\alpha(D)$$

for some $\alpha \in L^\infty(X)$ such that $|\alpha| = 1$. So in some sense, the non-uniqueness of $\Omega^{\pm}$ is quite weak (in particular, it does not influence the value of scattering cross-sections).

It is shown in [DG] that under the assumption (1.2) the operators $\Omega_S^{\pm}$ satisfy (1.10). In this paper we will give an independent proof of (1.10) for the operators $\Omega_\Psi^{\pm}$.

To a function $S(t,\xi)$ satisfying the Hamilton-Jacobi equation (1.4) we can naturally associate a solution $\Psi(t,x)$ of the eikonal equation (1.8) by setting

$$\Psi(t,x) = \mathrm{vc}_\xi(\langle x, \xi\rangle - S(t,\xi)),$$

where vc means the critical value. It is tempting to conjecture that for such a pair of functions $S(t,\xi)$ and $\Psi(t,x)$ we have

$$\Omega_S^{\pm} = \Omega_\Psi^{\pm}, \tag{1.11}$$

It is not difficult to show that (1.11) is true for potential satisfying

$$|\partial_x^\alpha V(x)| \le C_\alpha \langle x\rangle^{-\mu-|\alpha|},\ 1 \ge \mu > 0.\ |\alpha| \ge 0. \tag{1.12}$$

However, both $\Omega_S^{\pm}$ and $\Omega_\Psi^{\pm}$ can be constructed under the condition (1.2), which is much weaker than (1.12). We conjecture, that (1.11) is true under condition (1.2), although we have not been able to prove it.

In our proof we follow the general philosophy of [DG]. First we consider time-decaying potentials, which roughly satisfy

$$|\partial_x^\alpha V(t,x)| \le C\langle t\rangle^{-\mu-|\alpha|}, \quad |\alpha| = 0,1,2, \quad \mu > 0.$$

We prove that for such potentials position-type modified wave operators exist and are complete (unitary). This is the subject of Sections 2 and 3. In Sections 4 and 5 we apply those results to time-independent potentials.

2 Eikonal equation I

This section is devoted to the construction of solutions of the eikonal equation for long-range time-dependent potentials.

Proposition 2.1 *Let $V(t,x)$ be a time-dependent potential such that*

$$\int_0^{+\infty} \langle t\rangle^{|\alpha|-1} \|\partial_x^\alpha V(t,\cdot)\|_\infty \mathrm{d}t < \infty,\ |\alpha| = 1,2. \tag{2.1}$$

Then for sufficiently big T_1 there exists a real function $\Psi(t,x)$ such that:

$$-\partial_t \Psi(t,x) = \frac{1}{2}(\nabla_x \Psi(t,x))^2 + V(t,x), \quad x \in X, \quad t \geq T_1, \tag{2.2}$$

satisfying:

$$\partial_x^\alpha(\Psi(t,x) - \tfrac{x^2}{2t}) \in o(t^{1-|\alpha|}) \cap \langle t\rangle^{2-|\alpha|} L^1(\mathrm{d}t), \quad |\alpha| = 1,2. \tag{2.3}$$

Proof. In [DG, Sect. 1.7] we constructed a function $S(t,\xi)$ that solves of the following Hamilton-Jacobi equation:

$$\begin{cases} \partial_t S(t,\xi) = \frac{1}{2}\xi^2 + V(t, \nabla_\xi S(t,\xi)),\ x \in X, t \geq T, \\ S(T,\xi) = 0, \end{cases} \tag{2.4}$$

and satisfies the estimates

$$\partial_\xi^\alpha \left(S(t,\xi) - \frac{1}{2} t\xi^2 \right) \in o(t),\ |\alpha| = 1,2. \tag{2.5}$$

We will define the function $\Psi(t,x)$ by:

$$\Psi(t,x) = \mathrm{vc}_\xi(\langle x,\xi\rangle - S(t,\xi)), \tag{2.6}$$

where vc means the critical value. In fact if the critical point equation:

$$x = \nabla_\xi S(t,\xi), \tag{2.7}$$

has a unique solution $\xi = \xi(t,x)$ for t large enough, then $\Psi(t,x)$ solves the eikonal equation (2.2).

Let us prove that (2.7) has a unique solution for $t \geq T_1$. If we set

$$r(t,\xi) := t^{-1}(\nabla S(t,\xi) - t\xi), \tag{2.8}$$

we can rewrite (2.7) as

$$\xi + r(t,\xi) = \frac{x}{t}, \tag{2.9}$$

where using (2.5), we have:

$$r(t,\xi)\in o(t^0),\qquad \nabla_\xi r(t,\xi)\in o(t^0).$$

Applying the fixed point theorem we obtain a unique solution $\xi(t,x)$ to (2.9) for $t\geq T_1$. Note that if $\Psi(t,x)$ is defined by (2.6), one has:

$$\nabla_x\Psi(t,x)=\xi(t,x), \tag{2.10}$$

which shows that $\nabla_x\Psi(t,x)-\frac{x}{t}\in o(t^0)$. Next we have

$$\nabla_x^2\Psi(t,x)=\nabla_x\xi(t,x),$$

and

$$\nabla_x\xi(t,x)+\nabla_\xi r(t,\xi)\nabla_x\xi(t,x)=\frac{\mathbf{1}}{t},$$

which shows that

$$\nabla_x^2\Psi(t,x)-\frac{\mathbf{1}}{t}\in o(t^{-1}).$$

To complete the proof of (2.3), we will use the notation of [DG, Sect. 1.4]. We denoted there by

$$[t_1,t_2]\ni s\mapsto \tilde{y}(s,t_1,t_2,x,\xi)$$

the trajectory for the force $-\nabla_x V(t,x)$ having position x at $s=t_1$ and momentum ξ at $s=t_2$. We also put:

$$\tilde{z}(s,t_1,t_2,x,\xi):=\tilde{y}(s,t_1,t_2,x,\xi)-x-(s-t_1)\xi.$$

It is easy to check that $\tilde{z}(s)$ satisfies the following integral equation:

$$\tilde{z}(s)=\int_{t_1}^{t_2}\zeta_{t_1,s}(u)\nabla V(u,x+(u-t_1)\xi+\tilde{z}(u))\mathrm{d}u, \tag{2.11}$$

for

$$\zeta_{t_1,s}(u):=\begin{cases}0 \text{ for } u\leq t_1,\\ u-t_1 \text{ for } t_1\leq u\leq s,\\ s-t_1 \text{ for } s\leq u.\end{cases}$$

It follows from standard Hamilton-Jacobi theory (see eg [DG, Prop. 1.8.1]) that:

$$\tilde{y}(t,T,t,0,\xi)=\nabla_\xi S(t,\xi).$$

Hence

$$\nabla_x\Psi(t,x)-\frac{x}{t}=t^{-1}(T\xi-\tilde{z}(t,T,t,0,\xi)).$$

It follows then from hypothesis (2.1) that:

$$|\tilde{z}(t,T,t,0,\xi)| \le \int_T^t sf(s)\mathrm{d}s, \text{ for } f \in L^1(\mathrm{d}s). \tag{2.12}$$

Hence

$$\tilde{z}(t) \in \langle t\rangle^2 L^1(\mathrm{d}t) \cap o(t).$$

which ends the poof of (2.3) for $|\alpha| = 1$.

To finish the proof of (2.3) for $|\alpha| = 2$, we compute:

$$\begin{aligned}\nabla_\xi x(t,\xi) &= \nabla_\xi \tilde{y}(t,T,t,0,\xi)\\ &= (t-T)\left(\mathbf{1} + \tfrac{1}{t-T}\nabla_\xi \tilde{z}(t,T,t,0,\xi)\right),\end{aligned}$$

so that

$$\begin{aligned}\nabla_x^2\Psi(t,x) &= \nabla_x \xi(t,x)\\ &= (t-T)^{-1}\left(\mathbf{1} + \tfrac{1}{t-T}\nabla_\xi \tilde{z}(t,T,t,0,\xi)\right)^{-1}.\end{aligned} \tag{2.13}$$

It follows again from the equation (2.11) that

$$|\nabla_\xi \tilde{z}(t)| \le \int_T^t sf(s)ds, \text{ for } f \in L^1(\mathrm{d}s). \tag{2.14}$$

Hence,

$$\nabla_\xi \tilde{z}(t) \in \langle t\rangle^2 L^1(\mathrm{d}t) \cap o(t).$$

Note that since $\nabla_\xi \tilde{z}(t) = o(t)$, $(\mathbf{1} + \frac{1}{t-T}\nabla_\xi \tilde{z})^{-1}$ exists for $t \ge T_1$ and

$$\begin{aligned}\nabla_x^2\Psi(t,x) - \tfrac{\mathbf{1}}{t} &= (t-T)^{-1}\left(\mathbf{1} + \tfrac{1}{t-T}\nabla_\xi \tilde{z}(t,T,t,0,\xi)\right)^{-1} - \tfrac{\mathbf{1}}{t}\\ &= \langle t\rangle^{-2} O(|\nabla_\xi \tilde{z}(t)|) + O(t^{-2}).\end{aligned}$$

This implies (2.3) for $|\alpha| = 2$. □

We will also need the following lemma.

Lemma 2.2 *Assume in addition to (2.1) that*

$$\int_0^\infty \langle t\rangle^{3/2} \|\partial_x^\alpha V(t,\cdot)\|_\infty \mathrm{d}t < \infty, \ |\alpha| = 3. \tag{2.15}$$

Then the function $\Psi(t,x)$ satisfies

$$\partial_x^\alpha \Psi(t,x) \in o(t^{-3/2}), \ |\alpha| = 3.$$

Proof. ¿From [DG, Prop. 3.4.3] and its proof, we obtain that under assumptions (2.15), one has:

$$\partial_\xi^\alpha \tilde{z}(t) \in o(t^{3/2}),\ |\alpha| = 2.$$

So we obtain

$$\nabla_\xi \left(\mathbf{1} + \frac{1}{t-T}\nabla_\xi \tilde{z}(t)\right)^{-1} \in o(t^{1/2}).$$

¿From (2.13), we deduce that we have:

$$\nabla_x^3 \Psi(t,x) = \frac{1}{t-T}\nabla_x \xi(t,x) \nabla_\xi \left(\mathbf{1} + \frac{1}{t-T}\nabla_\xi \tilde{z}(t)\right)^{-1} \in o(t^{-3/2}),$$

which proves the lemma. □

3 Position-type wave operators for time-decaying potentials

In this section we consider the case of time-dependent potentials. For a real function $(t,x) \in \mathbf{R}^+ \times X \mapsto \Psi(t,x)$ we define the unitary operator $U_\Psi(t)$ by

$$U_\Psi(t)u(x) := \mathrm{e}^{i\pi n/4} t^{-n/2} \mathrm{e}^{i\Psi(t,x)} \mathcal{F}u\left(\frac{x}{t}\right),$$

where the Fourier transformation $\mathcal{F}$ is defined by

$$\mathcal{F}u(x) := (2\pi)^{-n/2} \int \mathrm{e}^{ix\cdot x'} u(x')\mathrm{d}x'.$$

Let the time-dependent Hamiltonian be defined as

$$H(t) := -\frac{1}{2}D^2 + V(t,x).$$

Let $U(t,s)$ denote the unitary dynamics generated by $H(t)$ in the sense described in [DG, Sect. B.3].

We first recall the existence of the asymptotic momentum observable for time-dependent potentials (cf [DG, Sect. 3.2]).

Theorem 3.1 *Assume that*

$$V(t,x) = V_\mathrm{s}(t,x) + V_\mathrm{l}(t,x)$$

with

$$\int_0^\infty \|V_\mathrm{s}(t,\cdot)\|_\infty \mathrm{d}t < \infty,$$

$$\int_0^\infty \|\nabla_x V_1(t,\cdot)\|_\infty \mathrm{d}t < \infty.$$

Then there exist the limit

$$\mathrm{s-}\lim_{t\to+\infty} U(0,t)f(D)U(t,0),\ f\in C_\infty(X). \tag{3.1}$$

There exists a vector D^+ of commuting self-adjoint operators with a dense domain such that (3.1) equals $f(D^+)$.

Moreover one has:

$$\mathrm{s-}\lim_{t\to+\infty} U(0,t)f\left(\frac{x}{t}\right)U(t,0) = f(D^+),\ f\in C_\infty(X).$$

The observable D^+ is called the *asymptotic momentum.*

The main result of this section is the following theorem.

Theorem 3.2 *Assume that*

$$V(t,x) = V_\mathrm{s}(t,x) + V_\mathrm{l}(t,x)$$

with

$$\int_0^\infty \|V_\mathrm{s}(t,\cdot)\|_\infty \mathrm{d}t < \infty,$$

$$\int_0^\infty \langle t\rangle^{|\alpha|-1}\|\partial_x^\alpha V_\mathrm{l}(t,\cdot)\|_\infty \mathrm{d}t < \infty,\quad |\alpha| = 1,2. \tag{3.2}$$

Then there exists a function $\Psi(t,x)\in C^{1,1}(X)$ satisfying

$$\partial_x^\alpha\left(\Psi(t,x) - \frac{x^2}{2t}\right) \in o(t^{1-|\alpha|})\cap\langle t\rangle^{2-|\alpha|}L^1(\mathrm{d}t),\ |\alpha| = 1,2.$$

such that the limits

$$\Omega_\Psi^+ := \mathrm{s-}\lim_{t\to\infty} U(0,t)U_\Psi(t) \tag{3.3}$$

$$\Omega_\Psi^{+*} = \mathrm{s-}\lim_{t\to\infty} U_\Psi(t)^* U(t,0) \tag{3.4}$$

exist. Moreover, Ω_Ψ^+ is unitary and

$$D^+ = \Omega_\Psi^+ D\Omega_\Psi^{+*}. \tag{3.5}$$

We will start the proof of Theorem 3.2 with the proof of the existence of the limit (3.3), which is easy.

Proof of the existence of (3.3).

By the chain rule of the wave operators, it suffices to prove that the limit

$$\mathrm{s-}\lim_{t\to\infty} U_1(0,t)U_\Psi(t)$$

exists, where $U_1(t,s)$ is the dynamics generated by $\frac{1}{2}D^2 + V_1(t,x)$. We can rewrite $U_\Psi(t)$ as:

$$U_\Psi(t) = \mathrm{e}^{i\pi n/4}\mathrm{e}^{i\Psi}t^{-iA}\mathcal{F}, \tag{3.6}$$

where Ψ is the operator of multiplication by $\Psi(t,x)$, and $A = \frac{1}{2}(\langle x, D\rangle + \langle D, x\rangle)$ is the generator of dilations. This shows that:

$$i\partial_t U_\Psi(t)\phi = \mathrm{e}^{i\pi n/4}\mathrm{e}^{i\Psi}\left(-\partial_t\Psi + \frac{A}{t}\right)t^{-iA}\mathcal{F}\phi.$$

We recall from Proposition 2.1 that $\Psi(t,x)$ satisfies the following estimates:

$$\partial_x^\alpha\left(\Psi - \frac{x^2}{2t}\right) \in o(t^{1-|\alpha|}) \cap \langle t\rangle^{2-|\alpha|}L^1(\mathrm{d}t),\ |\alpha| = 1, 2. \tag{3.7}$$

Using the eikonal equation (1.8), we compute for $\phi \in H^2(X)$:

$$\begin{aligned}
&(i\partial_t - \tfrac{1}{2}D^2 - V_1(t,x))U_\Psi(t)\phi\\
&= \mathrm{e}^{i\pi n/4}\mathrm{e}^{i\Psi}\left(-\partial_t\Psi + \tfrac{A}{t} - \tfrac{1}{2}(D+\nabla\Psi)^2 - V_1\right)t^{-iA}\phi\\
&= \mathrm{e}^{i\pi n/4}\mathrm{e}^{i\Psi}\left(-\partial_t\Psi - \tfrac{1}{2}(\nabla\Psi)^2 - V_1 + \tfrac{A}{t} - \tfrac{1}{2}D^2 - \nabla\Psi D - \tfrac{i}{2}\Delta\Psi\right)t^{-iA}\phi\\
&= \mathrm{e}^{i\pi n/4}\mathrm{e}^{i\Psi}\left(-\tfrac{1}{2}D^2 + (\tfrac{x}{t} - \nabla\Psi)D + \tfrac{i}{2}(\tfrac{n}{t} - \Delta\Psi)\right)t^{-iA}\phi\\
&= \mathrm{e}^{i\pi n/4}\left(t^{-2}\mathrm{e}^{i\Psi}t^{-iA}\tfrac{1}{2}D^2 + t^{-1}\mathrm{e}^{i\Psi}(\tfrac{x}{t} - \nabla\Psi)t^{-iA}D + \mathrm{e}^{i\Psi}\tfrac{i}{2}(\tfrac{n}{t} - \Delta\Psi)t^{-iA}\right)\phi \in L^1(\mathrm{d}t),
\end{aligned}$$

using the estimates (3.7). This proves the existence of the limit (3.3). □

To prove the unitarity of the wave operator Ω_Ψ^+, we will need more elaborate arguments which are close to those of [DG, Sect.3.4]. In particular, we will split the potential into a long-range and short-range part. To this end we recall the following result from [DG].

Lemma 3.3 *i) Suppose that $V_1(t,x)$ satisfies*

$$\int_0^\infty \langle t\rangle^{|\alpha|-1}\|\partial_x^\alpha V_1(t,\cdot)\|_\infty \mathrm{d}t < \infty,\ |\alpha| = 1, 2.$$

Let $j \in C_0^\infty(X)$ be a cutoff function such that

$$\int j(y)\mathrm{d}y = 1,\ \int yj(y)\mathrm{d}y = 0,$$

and let

$$\tilde{V}_{\rm l}(t,x) := \int V_{\rm l}(x+t^{\frac{1}{2}}y)j(y){\rm d}y.$$

Then one has:

$$\int_0^\infty \|\tilde{V}(t,\cdot)-V_{\rm l}(t,\cdot)\|_\infty {\rm d}t<\infty,$$

$$\int_0^\infty \|\partial_x^\alpha \tilde{V}_{\rm l}(t,\cdot)\|_\infty \langle t\rangle^{|\alpha|-1}{\rm d}t<\infty,\quad |\alpha|=1,2,$$

$$\int_0^\infty \|\partial_x^\alpha \tilde{V}_{\rm l}(t,\cdot)\|_\infty \langle t\rangle^{\frac{1}{2}|\alpha|}{\rm d}t<\infty,\quad |\alpha|\geq 2.$$

So by replacing $V_{\rm s}$ by $V_{\rm s}+\tilde{V}_{\rm s}$ we may assume in the rest of the section that $V_{\rm l}$ satisfies:

$$\begin{array}{l}\int_0^\infty \|\partial_x^\alpha V_{\rm l}(t,\cdot)\|_\infty \langle t\rangle^{|\alpha|-1}{\rm d}t<\infty,\quad |\alpha|=1,2,\\ \int_0^\infty \|\partial_x^\alpha V_{\rm l}(t,\cdot)\|_\infty \langle t\rangle^{\frac{1}{2}|\alpha|}{\rm d}t<\infty,\quad |\alpha|\geq 2.\end{array}\tag{3.8}$$

We choose for $\Psi(t,x)$ the solution of the eikonal equation constructed in Proposition 2.1. ¿From Lemma 2.2, it follows that Ψ satisfies in addition to (3.7), the estimate:

$$\|\partial_x^\alpha \Psi(t,\cdot)\|_\infty\in o(t^{-3/2}),\ |\alpha|=3.\tag{3.9}$$

Using the estimates (3.7), we obtain the following identity:

$$\begin{array}{rl}\frac{1}{2}D^2+V_{\rm l}(t,x) & =\frac{1}{2}(\nabla_x\Psi)^2+V_{\rm l}(t,x)+\frac{i}{2}\Delta\Psi+\frac{1}{2}(D+\nabla_x\Psi|D-\nabla_x\Psi)\\ & =-\partial_t\Psi+\frac{1}{2}(D+\nabla_x\Psi|D-\nabla_x\Psi)+\frac{ni}{2t}+L^1({\rm d}t).\end{array}$$

Here $(.|.)$ denotes the Euclidean scalar product on X'. We define:

$$H_1(t):=-\partial_t\Psi+\frac{1}{2}(D+\nabla_x\Psi|D-\nabla_x\Psi)+\frac{ni}{2t},$$

so that

$$\|H(t)-H_1(t)\|\in L^1({\rm d}t).\tag{3.10}$$

and denote by $U_1(t,s)$ the (non-unitary) dynamics generated by $H_1(t)$. It is easy to see that this dynamics exists and is uniformly bounded using (3.10). We will first prove some propagation estimates for the dynamics $U_1(t,s)$. We denote by $\mathbf{D}_1=\partial_t+[H_1(t),i\cdot]$ the Heisenberg derivative associated with $U_1(t,s)$.

Proposition 3.4 *The following estimates hold:*

i) $\|(D-\nabla\Psi(t,x))U_1(t,0)\langle D\rangle^{-1}\langle x\rangle^{-1}\|\in O(t^{-1})$,

ii) $\|(D-\nabla\Psi(t,x))^2U_1(t,0)\langle D\rangle^{-2}\langle x\rangle^{-2}\|\in O(t^{-3/2})$.

Proof. Let us first prove *i)*. We compute

$$\begin{aligned} \mathbf{D}_1(D-\nabla\Psi) &= -\partial_t\nabla_x\Psi + [H_1(t), i(D-\nabla\Psi)] \\ &= \tfrac{1}{2}[D+\nabla\Psi, i(D-\nabla\Psi)](D-\nabla\Psi) \\ &= -\nabla_x^2\Psi(D-\nabla\Psi). \end{aligned}$$

Since $\nabla_x^2\Psi = \mathbf{1} + R(t)$, where $R(t) \in L^1(\mathrm{d}t)$, we have:

$$\mathbf{D}_1(D-\nabla\Psi) = -\frac{1}{t}(D-\nabla\Psi) + R(t)(D-\nabla\Psi).$$

We introduce the observable

$$C_1(t) := t(D-\nabla\Psi(t,x)),$$

and we have

$$\mathbf{D}_1 C_1(t) = R(t)C_1(t). \tag{3.11}$$

If we put

$$f_1(t) := \|U_1(0,t)C_1(t)U_1(t,0)\langle D\rangle^{-1}\langle x\rangle^{-1}\|,$$

we deduce from (3.11) that

$$\frac{\mathrm{d}}{\mathrm{d}t}f_1(t) \le g(t)f_1(t),$$

for

$$g(t) = \|U_1(0,t)R(t)U_1(t,0)\| \in L^1(\mathrm{d}t).$$

By the Gronwall inequality, we obtain

$$f(t) \le Cf(T),$$

which proves *i)*.

Let us now prove *ii)*. We compute for $1 \le i \le j \le n$

$$\begin{aligned} \mathbf{D}_1(D-\nabla\Psi)_i(D-\nabla\Psi)_j &= -\tfrac{2}{t}(D-\nabla\Psi)_i(D-\nabla\Psi)_j \\ &\quad + \sum_k (D-\nabla\Psi)_i r_{jk}(t)(D-\nabla\Psi)_k \\ &\quad + \sum_k r_{ik}(t)(D-\nabla\Psi)_k(D-\nabla\Psi)_j \\ &= -\tfrac{2}{t}(D-\nabla\Psi)_i(D-\nabla\Psi)_j \\ &\quad + \sum_k r_{ik}(t)(D-\nabla\Psi)_i(D-\nabla\Psi)_k \\ &\quad + \sum_k r_{ik}(t)(D-\nabla\Psi)_k(D-\nabla\Psi)_j \\ &\quad + \sum_k b_{ijk}(t)(D-\nabla\Psi)_k, \end{aligned}$$

where

$$r_{ij}(t) := \nabla^2_{ij}\Psi(t,x) - \delta_{ij}\tfrac{1}{t} \in L^1(\mathrm{d}t),$$

$$b_{ijk}(t) := \nabla_x r_{ij}(t) \in o(t^{-3/2}),$$

using (3.9). Introducing the matrix valued observable

$$C_2(t) = t^2\left((D-\nabla\Psi)_i(D-\nabla\Psi)_j\right),$$

we obtain:

$$\mathbf{D}_1 C_2(t) = R_2(t)C_2(t) + R_1(t)C_1(t), \tag{3.12}$$

where $R_1(t) \in o(t^{-1/2})$ and $R_2(t) \in L^1(\mathrm{d}t)$. We define now:

$$f_2(t) := \|U_1(0,t)C_2(t)U_1(t,0)\langle D\rangle^{-2}\langle x\rangle^{-2}\|,$$

and we obtain using (3.12):

$$\frac{\mathrm{d}}{\mathrm{d}t}f_2(t) \le o(t^{-1/2})f_1(t) + g(t)f_2(t), \tag{3.13}$$

where $g(t) = \|R_2(t)\| \in L^2(\mathrm{d}t)$. Therefore by the Gronwall inequality, we obtain:

$$f_2(t) \le Ct^{1/2},$$

which proves *ii)*. $\Box$

End of the proof of Theorem 3.2.

Let us first prove the existence of the limit (3.4). By the chain rule of the wave operators, it suffices to prove the existence of

$$\mathrm{s-}\lim_{t\to\infty} U^*_\Psi(t)U_1(t,0). \tag{3.14}$$

We compute

$$\begin{aligned} i^{-1}\partial_t U^*_\Psi(t) &= \mathrm{e}^{-i\pi n/4}\mathcal{F}^* t^{iA}\tfrac{A}{t}\mathrm{e}^{-i\Psi} - \mathrm{e}^{-i\pi n/4}\mathcal{F}^* t^{iA}\mathrm{e}^{-i\Psi}\partial_t\Psi \\ &= U^*_\Psi(t)\left(x(D-\nabla\Psi) - \tfrac{ni}{2t} - \partial_t\Psi\right). \end{aligned}$$

Let us now pick $\phi \in \mathcal{D}(D^2)\cap\mathcal{D}(x^2)$ and compute:

$$\begin{aligned} &i^{-1}\partial_t U^*_\Psi(t)U_1(t,0)\phi \\ &= U^*_\Psi(t)\left(\tfrac{x}{t} - \tfrac{1}{2}(D+\nabla\Psi)\right)(D-\nabla\Psi)\,U_1(t,0)\phi. \end{aligned} \tag{3.15}$$

Using (3.7), we have:

$$\left(\frac{x}{t} - \frac{1}{2}(D+\nabla\Psi)\right) = -\frac{1}{2}(D-\nabla\Psi) + R(t),$$

where $R(t) \in \langle t \rangle L^1(\mathrm{d}t)$. So we obtain:

$$\begin{aligned} & \|\partial_t U_\Psi^*(t) U_1(t,0)\phi\| \\ & \leq \|R(t)\| \|(D-\nabla\Psi) U_1(t,0)\phi\| + \|(D-\nabla\Psi)^2 U_1(t,0)\phi\| \in L^1(\mathrm{d}t), \end{aligned}$$

using Proposition 3.4. This proves the existence of the limit (3.15).

The identity (3.5) follows from

$$U_\Psi(t) f(D) U_\Psi^*(t) = f\left(\frac{x}{t}\right).$$

This completes the proof of the theorem. □

4 Eikonal equation II

In this section we prove some additional results on solutions of the eikonal equation. Although, strictly speaking, these results involve time-dependent potentials, they will be used in our construction of position-type wave operators for time-independent potentials.

Let us start with the following extension of Proposition 2.1.

Proposition 4.1 *Let $V(t,x)$ be a time-dependent potential such that for any $\epsilon > 0$*

$$\int_0^{+\infty} t^{|\alpha|-1} \sup_{|x|\geq \epsilon t} |\partial_x^\alpha V(t,x)| \mathrm{d}t < \infty, \ |\alpha| = 1,2. \tag{4.16}$$

Then there exists a real function $\Psi(t,x)$ such that for any $\epsilon > 0$:

$$-\partial_t \Psi(t,x) = \frac{1}{2}(\nabla_x \Psi(t,x))^2 + V(t,x), \ \textit{in } |x| \geq \epsilon t, t \geq T_\epsilon, \tag{4.17}$$

satisfying in $|x| \geq \epsilon t$:

$$\partial_x^\alpha (\Psi(t,x) - \tfrac{x^2}{2t}) = o(t^{1-|\alpha|}) \cap \langle t \rangle^{2-|\alpha|} L^1(\mathrm{d}t), \ |\alpha| = 1,2. \tag{4.18}$$

Proof. In [DG, Prop. 4.7.3], we proved under the hypotheses (4.16) the existence of a function $S(t,\xi)$ satisfying for any $\epsilon > 0$ the Hamilton-Jacobi equation

$$\partial_t S(t,\xi) = \frac{1}{2}\xi^2 + V(t, \nabla_\xi S(t,\xi)), \ \text{in } |\xi| \geq \epsilon, t \geq T_\epsilon, \tag{4.19}$$

and the estimates

$$\partial_\xi^\alpha \left(S(t,\xi) - \frac{1}{2} t \xi^2 \right) = o(t), \ |\alpha| = 1,2 \text{ in } |\xi| \geq \epsilon. \tag{4.20}$$

As above we will define $\Psi(t,x)$ as:

$$\Psi(t,x) = \mathrm{vc}_\xi(\langle x,\xi\rangle - S(t,\xi)).$$

Let us check that for $|x| \geq \epsilon t, t \geq T_\epsilon$, there exists a unique solution ξ of (2.9) with $|\xi| \geq \epsilon$. In fact if $|x| \geq \epsilon t$, the map $\xi \mapsto x - r(t,\xi)$ sends the set $\{\xi | |\xi| \geq \epsilon/2\}$ into itself for $t \geq T_\epsilon$ and is a contraction there. The estimates (4.18) can then be proved as in Proposition 2.1. □

The following proposition will be needed to compare two solutions of the same eikonal equation.

Proposition 4.2 *Let $V(t,x)$ be a time-dependent potential such that for any $\epsilon > 0$*

$$\int_0^{+\infty} t^{|\alpha|-1} \sup_{|x|\geq \epsilon t} |\partial_x^\alpha V(t,x)| \mathrm{d}t < \infty, \ |\alpha| = 1,2. \tag{4.21}$$

Let $\Theta \subset X\backslash\{0\}$ be a compact set. Suppose that for ξ in a certain neighborhood of Θ and $t \geq T_0$ the functions $S_i(t,\xi), i = 1,2$ are two solutions of the Hamilton-Jacobi equation

$$\partial_t S_i(t,\xi) = \frac{1}{2}\xi^2 + V(t, \nabla_x S_i(t,\xi))$$

such that

$$\partial_\xi^\alpha \left(S_i(t,\xi) - \frac{1}{2}t\xi^2 \right) \in o(t), \qquad |\alpha| = 1,2.$$

Let $\Psi_i(t,x), i = 1,2$ the two solutions of the eikonal equation (4.17) given for $\frac{x}{t}$ in a certain (maybe smaller) neighborhood of Θ and $t \geq T_1$ by

$$\Psi_i(t,x) = \mathrm{vc}_\xi(\langle x,\xi\rangle - S_i(t,\xi)).$$

Then the limit

$$\lim_{t\to+\infty} (\Psi_1(t,ty) - \Psi_2(t,ty))$$

exist uniformly for $y \in \Theta$.

Proof. As in (2.8) set

$$r_i(t,\xi) := \frac{\nabla_\xi S_i(t,\xi) - t\xi}{t}.$$

Recall that $r_i(t,\xi), \nabla r_i(t,\xi) \in o(t^0)$. Let $\xi_i(t,x)$ be the solution of

$$\nabla_\xi S_i(t,\xi_i) = ty,$$

or equivalently:

$$\xi_i + r_i(t,\xi_i) = y.$$

Such a solution exists for a sufficiently small neighborhood of Θ for t sufficiently big. Recall that if we set

$$\Psi_i(t,x) := x\xi_i(t,\tfrac{x}{t}) - S_i(t,\xi_i(t,\tfrac{x}{t})),$$

then

$$\nabla_x\Psi_i(t,x) = \xi_i(t,\tfrac{x}{t}).$$

It follows from [DG, Thm. 1.9.6] that $\nabla_\xi S_1(t,\xi) - \nabla_\xi S_2(t,\xi) \in O(1)$, which implies that $r_1(t,\xi) - r_2(t,\xi) \in O(t^{-1})$. We deduce then from

$$|\xi_1 - \xi_2| \le |r_1(t,\xi_1) - r_1(t,\xi_2)| + |r_1(t,\xi_2) - r_2(t,\xi_2)|. \tag{4.22}$$

that

$$\xi_1(t,y) - \xi_2(t,y) \in O(t^{-1}),\ y \in \Theta. \tag{4.23}$$

Now we compute for $y \in \Theta$ and t big enough:

$$\begin{aligned}
&\partial_t\left(\Psi_1(t,ty) - \Psi_2(t,ty)\right)\\
&= \partial_t\Psi_1(t,ty) - \partial_t\Psi_2(t,ty) + y\nabla_x\Psi_1(t,ty) - y\nabla_x\Psi_2(t,ty)\\
&= \left(\nabla_x\Psi_1(t,ty) - \nabla_x\Psi_2(t,ty)\right)\left(y - \tfrac{1}{2}\left(\nabla_x\Psi_1(t,ty) + \nabla_x\Psi_2(t,ty)\right)\right).
\end{aligned} \tag{4.24}$$

By the estimates (4.18) we have:

$$y - \partial_x\Psi_i(t,ty) \in \langle t\rangle L^1(\mathrm{d}t),$$

which using (4.23) implies that the rhs of (4.24) is in $L^1(\mathrm{d}t)$. This completes the proof of the proposition. □

5 Position-type wave operators for time-independent potentials

In this section we prove the existence and completeness of position-type wave operators for long-range time-independent potentials.

Theorem 5.1 *Assume that*

$$V(x)(1-\Delta)^{-1} \text{ is compact,} \tag{5.1}$$

and

$$\int_1^\infty \left\|(1-\Delta)^{-1}\nabla_x V(x)\mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R})(1-\Delta)^{-1}\right\|\mathrm{d}R < \infty. \tag{5.2}$$

and that $V(x)$ can be written as

$$V(x) = V_\mathrm{s}(x) + V_\mathrm{l}(x), \tag{5.3}$$

such that

$$\int_0^\infty \left\| (1-\Delta)^{-1} V_{\rm s}(x) \mathbf{1}_{[1,\infty[}\left(\frac{|x|}{R}\right)(1-\Delta)^{-\frac{1}{2}} \right\| {\rm d}R < \infty,$$

$$\int_0^\infty \sup_{|x|>R} |\partial_x^\alpha V_{\rm l}(x)| \langle R\rangle^{|\alpha|-1} {\rm d}R < \infty, \ |\alpha| = 1, 2. \tag{5.4}$$

Then there exists a real function $\Psi(t,x)$ such that the limits

$$\mathrm{s-}\lim_{t\to\infty} {\rm e}^{itH} U_\Psi(t) \tag{5.5}$$

and

$$\mathrm{s-}\lim_{t\to\infty} U_\Psi(t)^* {\rm e}^{-itH} \mathbf{1}^{\rm c}(H) \tag{5.6}$$

exist. If we denote (5.5) by Ω_Ψ^+ then (5.6) equals Ω_Ψ^{+*}. Moreover one has

$$\begin{aligned} &\Omega_\Psi^+ \Omega_\Psi^{+*} = \mathbf{1}^{\rm c}(H), \qquad \Omega_\Psi^{+*}\Omega_\Psi^+ = \mathbf{1}, \\ &P^+ = \Omega_\Psi^+ D \Omega_\Psi^{+*}, \qquad \mathbf{1}^{\rm c}(H) H = \Omega_\Psi^+ H_0 \Omega_\Psi^{+*}. \end{aligned} \tag{5.7}$$

Before starting the proof of Theorem 5.1, let us first recall some results from [DG]. In [DG, Chap. 4] we proved under hypotheses (5.1) and (5.2) the existence of the *asymptotic velocity* observable. Its definition and properties are recalled in the following theorem.

Theorem 5.2 *Assume (5.1) and (5.2). Then for all $f \in C_\infty(X)$ there exist the limit*

$$\mathrm{s-}\lim_{t\to+\infty} {\rm e}^{itH} f\left(\frac{x}{t}\right) {\rm e}^{-itH}. \tag{5.8}$$

Moreover there exist a vector of commuting self-adjoint operators P^+ with a dense domain called the asymptotic velocity *such that (5.8) equals $f(P^+)$.*

One has:

$$H \mathbf{1}_{X\setminus\{0\}}(P^+) = \frac{1}{2}(P^+)^2, \tag{5.9}$$

$$\mathbf{1}_{\{0\}}(P^+) = \mathbf{1}^{\rm pp}(H). \tag{5.10}$$

The proof of Theorem 5.1 consists in introducing an effective time-dependent potential and applying then the results of Section 3.

Let $\Theta \subset X$ be compact such that $0 \notin \Theta$. Fix $J \in C_0^\infty(X)$ such that $0 \notin {\rm supp} J$ and $J = 1$ on a neighborhood of Θ. Fix also $x_0 \in X$ such that $|x_0| \neq 0$. We introduce now the following effective time-dependent potential:

$$V_J(t,x) := (V_l(x) - V_l(tx_0)) J\left(\frac{x}{t}\right) + V_l(tx_0). \tag{5.11}$$

Obviously, for y in a neighborhood of Θ

$$V_l(ty) - V_J(t, ty) = 0. \tag{5.12}$$

¿From [DG, Prop. 4.7.5], we obtain that

$$\int_0^\infty \langle t\rangle^{|\alpha|-1} \|\partial_x^\alpha V_J(t,\cdot)\|_\infty \mathrm{d}t < \infty,\ \ |\alpha| = 1, 2. \tag{5.13}$$

We denote by $U_J(t,s)$ the unitary dynamics generated by $\frac{1}{2}D^2 + V_J(t,x)$ and by D_J^+ the asymptotic velocity associated with $U_J(t,s)$ (see [DG, Thm. 3.2.2]):

$$f(D_J^+) = \mathrm{s-}\lim_{t\to+\infty} U_J(0,t) f\left(\frac{x}{t}\right) U_J(t,0),\ \ f \in C_\infty(X).$$

The following result has been proved in [DG, Sect. 4.7]. It shows that on $\mathrm{Ran}\mathbf{1}_\theta(D^+)$ one can replace asymptotically the dynamics e^{-itH} by the effective dynamics $U_J(t,0)$.

Lemma 5.3 *There exist the limits*

$$\mathrm{s-}\lim_{t\to\infty} \mathrm{e}^{itH} U_J(t,0) \mathbf{1}_\Theta(D_J^+). \tag{5.14}$$

and

$$\mathrm{s-}\lim_{t\to\infty} U_J(0,t)\mathrm{e}^{-itH}\mathbf{1}_\Theta(P^+) \tag{5.15}$$

If we denote (5.14) by $\omega_{J,\Theta}^+$, then (5.15) equals $\omega_{J,\Theta}^{+*}$. Moreover,

$$\begin{aligned} &\omega_{J,\Theta}^+ \omega_{J,\Theta}^{+*} = \mathbf{1}_\Theta(P^+), \qquad \omega_{J,\Theta}^{+*}\omega_{J,\Theta}^+ = \mathbf{1}_\Theta(D_J^+), \\ &\omega_{J,\Theta}^+ D_J^+ = P^+ \omega_{J,\Theta}^+. \end{aligned} \tag{5.16}$$

Proof of Theorem 5.1. Let $\Theta_n \subset X$ be a sequence of compact sets such that $0 \notin \Theta_n$ and $\Theta_n \nearrow X\backslash\{0\}$. Since $\mathbf{1}_{\{0\}}(P^+) = \mathbf{1}^{\mathrm{pp}}(H)$, we have:

$$\mathbf{1}^{\mathrm{c}}(H) = \mathrm{s-}\lim_{n\to\infty} \mathbf{1}_{\Theta_n}(P^+).$$

Consequently to prove the existence of the limit (5.6), it suffices to prove the existence of

$$\mathrm{s-}\lim_{t\to+\infty} U_\Psi(t)^* \mathrm{e}^{-itH}\mathbf{1}_{\Theta_n}(P^+),$$

for all n.

Let us fix one such compact set Θ. We define $V_J(t,x)$ as in (5.11). It follows from (5.13) that $V_J(t,x)$ satisfies the hypotheses of Theorem 3.2. Consequently for the function $\Psi_J(t,x)$ described in Theorem 3.2 the limits:

$$\begin{aligned} &\mathrm{s-}\lim_{t\to\infty} U_J(0,t) U_{\Psi_J}(t), \\ &\mathrm{s-}\lim_{t\to\infty} U_{\Psi_J}(t)^* U_J(t,0). \end{aligned}$$

exists.

We define then as in Lemma 3.3:

$$\tilde{V}(t,x) := \int V_l(x + t^{\frac{1}{2}}y)j(y)\mathrm{d}y. \tag{5.17}$$

It is easy to see that $\tilde{V}(t,x)$ satisfies for all $\epsilon > 0$:

$$\textstyle\int_0^\infty \sup_{|x|\geq \epsilon t} |\partial_x^\alpha \tilde{V}(t,x)|\langle t\rangle^{|\alpha|-1}\mathrm{d}t < \infty, \ |\alpha| = 1,2,$$

Let $\Psi(t,x)$ be a solution of the eikonal equation described in Proposition 4.1. It remains to show that the limits

$$\begin{aligned} &\mathrm{s-}\lim_{t\to+\infty} U_\Psi(t)^* U_{\Psi_J}(t)\mathbf{1}_\Theta(D_J^+), \\ &\mathrm{s-}\lim_{t\to+\infty} U_{\Psi_J}(t)^* U_\Psi(t)\mathbf{1}_\Theta(P^+) \end{aligned} \tag{5.18}$$

exist.

To do this we recall from Section 3 that $\Psi_J(t,x)$ is the solution of the eikonal equation for the potential

$$\tilde{V}_J(t,x) = \int V_J(t, x + t^{\frac{1}{2}}y)j(y)dy,$$

constructed in Proposition 2.1. It is easy to show that and that $\tilde{V}(t,ty)$ and $\tilde{V}_J(t,ty)$ coincide for y in a neighborhood of Θ for t big enough. Using then Proposition 4.2 we obtain the existence of

$$\lim_{t\to+\infty}\left(\Psi(t,ty) - \Psi_J(t,ty)\right), \quad \text{for } y \in \Theta.$$

Using then the chain rule, we obtain the existence of the limits (5.5) and (5.6). The identities (5.7) follow then from (5.16) and (3.5).□

References

[DG] Dereziński J., Gérard, C.: *Asymptotic Completeness of $N-$Particle Systems*, Texts and Monographs in Physics, Springer (to appear).

[E1] Enss, V.: Asymptotic completeness for quantum-mechanical potential scattering, I Short range potentials, Comm. Math. Phys. **61** (1978) 285-291

[E2] Enss, V.: Asymptotic completeness for quantum-mechanical potential scattering, II Singular and long-range potentials, Ann. Phys. **119** (1979) 117-132

[Gr] Graf, G.M.: Asymptotic completeness for $N-$body short range quantum systems: A new proof, Comm. Math. Phys. **132** (1990) 73-101

[Hö] Hörmander, L.: *The analysis of linear partial differential operators*, vol IV Springer Verlag (1985)

[IK] Isozaki, H., Kitada, H.: Modified wave operators with time-independent modifiers, J. Fac. Sci. Univ. Tokyo, Sec1A **32** (1985) 77-104

[Mo] Mourre, E.: Absence of singular continuous spectrum for certain selfadjoint operators, Comm. Math. Phys. **78** (1981) 519-567

[SS] Sigal, I.M., Soffer, A.: The $N-$particle scattering problem: asymptotic completeness for short-range quantum systems, Ann. of Math. **125** (1987) 35-108

[Yaf1] Yafaev, D.R.: Wave operators for the Schrödinger equation, Theor. Math. Phys. 45 (1980) 992-998

[Yaf2] Yafaev, D.R.: New channels in two-body scattering, to appear in Mathematical Journal of Saint-Petersbourg (1996)