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GENERATING FUNCTIONAL IN CFT AND EFFECTIVE ACTION FOR TWO-DIMENSIONAL QUANTUM GRAVITY ON HIGHER GENUS RIEMANN SURFACES

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ABSTRACT. We formulate and solve the analog of the universal Conformal Ward Identity for the stress-energy tensor on a compact Riemann surface of genus $g > 1$, and present a rigorous invariant formulation of the chiral sector in the induced two-dimensional gravity on higher genus Riemann surfaces. Our construction of the action functional uses various double complexes naturally associated with a Riemann surface, with computations that are quite similar to descent calculations in BRST cohomology theory. We also provide an interpretation of the action functional in terms of the geometry of different fiber spaces over the Teichmüller space of compact Riemann surfaces of genus $g > 1$.

1. INTRODUCTION

Conformal symmetry in two dimensions, according to Belavin, Polyakov, and Zamolodchikov [8], is generated by the holomorphic and anti-holomorphic components $\mathbf{T}(z)$ and $\bar{\mathbf{T}}(\bar{z})$ of the stress-energy tensor of a Conformal Field Theory. These components satisfy the Operator Product Expansions [8, 15]

$$\begin{aligned}
&\mathbf{T}(z)\,\mathbf{T}(w) \sim \frac{c/2}{(z-w)^4} + \left(\frac{2}{(z-w)^2} + \frac{1}{z-w}\,\frac{\partial}{\partial w}\right)\mathbf{T}(w),\\
&\bar{\mathbf{T}}(\bar{z})\,\bar{\mathbf{T}}(\bar{w}) \sim \frac{c/2}{(\bar{z}-\bar{w})^4} + \left(\frac{2}{(\bar{z}-\bar{w})^2} + \frac{1}{\bar{z}-\bar{w}}\,\frac{\partial}{\partial \bar{w}}\right)\bar{\mathbf{T}}(\bar{w}),\\
&\mathbf{T}(z)\,\bar{\mathbf{T}}(\bar{w}) \sim 0,
\end{aligned}$$

where c is the central charge of the CFT and $\sim$ means "up to the terms that are regular as $z \to w$". These OPE, together with the regularity condition $\mathbf{T}(z) \sim 1/z^4$ as $|z| \to \infty$, are used to construct Verma modules for the Virasoro algebra that correspond to the holomorphic and anti-holomorphic sectors of a CFT. The operator content of the CFT is specified by the highest weight vectors of the Virasoro algebra that correspond to the primary fields $\mathbf{O}_l(z,\bar{z})$ with conformal weights $(h_l, \bar{h}_l)$, satisfying

$$\mathbf{T}(z)\,\mathbf{O}_l(w,\bar{w}) \sim \left(\frac{h_l}{(z-w)^2} + \frac{1}{z-w}\,\frac{\partial}{\partial w}\right)\mathbf{O}_l(w,\bar{w})\,,$$

and similar OPE with $\bar{\mathbf{T}}(\bar{z})$.

A CFT is determined by the complete set of correlation functions among the primary fields, which are built up of conformal blocks: the correlation functions for the holomorphic sector. The conformal blocks are defined by the Conformal Ward Identities of BPZ [8], which follow from the OPE for the primary fields. Introducing the generating functional for the n-point correlation functions

$$\exp\{-W[\mu](z_1,\dots,z_n)\} = \Big\langle \mathbf{O}_1(z_1)\dots\mathbf{O}_n(z_n)\exp\Big(-\frac{1}{\pi}\int_{\mathbb{C}}\mu(z,\bar z)\,\mathbf{T}(z)\,\mathrm{d}^2 z\Big)\Big\rangle \stackrel{\text{def}}{=} \langle \mathbf{O}_1(z_1)\cdots\mathbf{O}_n(z_n)\rangle_\mu ,$$

where the integration goes over the complex plane $\mathbb{C}$ and $\mathrm{d}^2 z = \frac{i}{2}\,\mathrm{d}\,z\wedge\mathrm{d}\,\bar z = \mathrm{d}\,x\wedge\mathrm{d}\,y$, $z = x+iy, \bar z = x-iy$, the CWI can be written in the following "universal form" (cf. [31, 30])

$$(\bar\partial - \mu\,\partial - 2\,\mu_z)\frac{\boldsymbol{\delta} W}{\boldsymbol{\delta}\mu(z)} = \frac{c}{12\pi}\mu_{zzz} + \sum_{l=1}^{n}\{h_l\,\delta_z(z-z_l) + \delta(z-z_l)\,\frac{\partial W}{\partial z_l}\},$$

where $\partial = \partial/\partial z$, $\bar\partial = \partial/\partial\bar z$. Describing the complete solution of this equation, as well as of its generalization for higher genus Riemann surfaces, is one of the major problems of CFT.

This problem remains non-trivial even in the simplest case of conformal blocks without primary fields, when the generating functional $W[\mu]$ takes the form

$$\exp\{-W[\mu]\} = \Big\langle\exp\Big\{-\frac{1}{\pi}\int_{\mathbb{C}}\mu(z,\bar z)\,\mathbf{T}(z)\;\mathrm{d}^2 z\Big\}\Big\rangle \stackrel{\text{def}}{=} \langle \mathrm{I}\rangle_\mu\,. \tag{1.1}$$

It gives the expectation value of the unit operator I in the presence of the Schwinger's source term μ, which is a characteristic feature of all CFT with the same central charge c. The corresponding universal CWI reduces to the equation

$$(\bar\partial - \mu\,\partial - 2\,\mu_z)\frac{\boldsymbol{\delta} W}{\boldsymbol{\delta}\mu(z)} = \frac{c}{12\pi}\mu_{zzz} \tag{1.2}$$

for the expectation value of the stress-energy tensor

$$\langle\mathbf{T}(z)\rangle_\mu \stackrel{\text{def}}{=} \frac{\boldsymbol{\delta} W}{\boldsymbol{\delta}\mu(z)}\,.$$

It is remarkable that the functional $W[\mu]$, for $|\mu| < 1$, can be determined in closed form and that it turns out to be the Euclidean version of Polyakov's action functional for two-dimensional induced quantum gravity [26].

To see this, let μ be a Beltrami coefficient on $\mathbb{C}$—a bounded function μ with the property $|\mu| < 1$—to which one can associate a self-mapping $f : \mathbb{C}\to\mathbb{C}$ as a unique normalized (fixing $0, 1$ and ∞) solution of the Beltrami equation

$$f_{\bar z} = \mu\, f_z\,.$$

Denote by

$$T(z) = \{f, z\} = \frac{f_{zzz}}{f_z} - \frac{3}{2}\Big(\frac{f_{zz}}{f_z}\Big)^2$$

the Schwarzian derivative of f—"the stress-energy tensor associated with f". Then (see, e.g. [22, 31]), equation (1.2) is equivalent to the following Cauchy-Riemann equation

$$(\bar\partial - \mu\partial)\Big(\big(\frac{\boldsymbol{\delta} W}{\boldsymbol{\delta}\mu(z)} - \frac{c}{12\pi}T(z)\big)/(f_z)^2\Big) = 0$$

with respect to the complex structure on $\mathbb{C}$ defined by the coordinates $\zeta = f(z,\bar z)$, $\bar\zeta = \overline{f(z,\bar z)}$. Using the regularity of the stress-energy tensor at ∞ one gets that

$$\frac{\boldsymbol{\delta} W}{\boldsymbol{\delta}\mu(z)} = \langle \mathbf{T}(z)\rangle_\mu = \frac{c}{12\pi} T(z)\,. \tag{1.3}$$

This variational equation for determining W was explicitly solved by Haba [18]. Specifically, let $f^{t\mu}$ be the family of self-mappings of $\mathbb{C}$ associated to the Beltrami coefficients $t\mu$, $0 \le t \le 1$. Then

$$W[\mu] = \frac{c}{12\pi}\int_0^1 \mathrm{d}\,t \int_{\mathbb{C}} T^{t\mu}\,\mu\;\mathrm{d}^2 z$$

solves (1.3). The functional W can be considered as a WZW type functional since its definition requires an additional integration over a path in the field space.

Next, consider Polyakov's action functional for two-dimensional induced quantum gravity in the light-cone gauge [26], applied to the quasi-conformal map f:

$$S[f] = -\int_{\mathbb{C}} \frac{f_{zz}}{f_z}\left(\frac{f_{\bar z}}{f_z}\right)_z \mathrm{d}^2 z\,. \tag{1.4}$$

It has the property

$$\frac{\boldsymbol{\delta} S}{\boldsymbol{\delta}\mu(z)} = 2\,T(z) = 2\{f,z\},$$

so that $c\,S[f]/24\pi$, considered as a functional of $\mu = f_{\bar z}/f_z$, also solves equation (1.2). Therefore, one has the fundamental relation

$$W[\mu] = \frac{c}{24\pi} S[f], \tag{1.5}$$

which expresses W as a local functional of f and which can be verified directly. This relation provides the interpretation (cf. [31, 7, 27]) of two-dimensional induced gravity in the conformal gauge in terms of a gravitational WZNW model (and hence in terms of a Chern-Simons functional as well).

In the present paper we formulate and solve the analog of the equation (1.2) for the stress-energy tensor on a compact Riemann surface of genus $g > 1$. As in the genus zero case, it provides an invariant formulation of the chiral sector in two-dimensional induced gravity on higher genus Riemann surfaces, a solution to the problem discussed in [30]. From a different point of view, this problem was also considered in [34, 35].

First, it should be noted that it is trivial to generalize the genus zero treatment to the case of elliptic curves—compact Riemann surfaces of genus 1. Namely, let X be an elliptic curve realized as the quotient $L\backslash\mathbb{C}$ of the complex plane $\mathbb{C}$ by the action of a rank 2 lattice L generated by 1 and τ, with $\operatorname{Im}\tau > 0$. The analog of the equation (1.2) has the same form, where μ is now a doubly-periodic function on $\mathbb{C}$, while the corresponding normalized solution f of the Beltrami equation has the property

$$f(z+1) = f(z) + 1\,,\quad f(z+\tau) = f(z) + \tilde\tau\,,$$

where $\tilde\tau = f(\tau)$, $\operatorname{Im}\tilde\tau \neq 0$. It follows that

$$f\circ\gamma = \tilde\gamma\circ f \quad \text{for all } \gamma\in L,$$

where $\tilde{\gamma} \in \tilde{L}$, the rank 2 lattice in $\mathbb{C}$ generated by 1 and $\tilde{\tau}$. As a result, the functional $S[f]$ has the same form as in (1.4), where now the integration goes over the fundamental parallelogram Π of the lattice L.

Having thus addressed the genus 1 case, we start by formulating equation (1.2)—the same applies to the universal CWI as well—on a compact Riemann surface X of genus $g > 1$. In order to do it one needs to use projective connections on X (see, e.g., [17] for details). Namely, recall [14] that the stress-energy tensor $\mathbf{T}$ of a CFT on a Riemann surface is $c/12$ times a projective connection. Therefore the expectation value

$$\langle \mathbf{T}(z) \rangle = \frac{c}{12} Q(z),$$

is a holomorphic projective connection on X which depends on the particular CFT. The difference between two projective connections on X is a quadratic differential, so that in order to define the generating functional for the stress-energy tensor on X, one can choose a "background" holomorphic projective connection R and set

$$\exp\{-W[\mu]\} = \Big\langle \exp\Big\{ -\frac{1}{\pi} \int_X \mu(z,\bar{z}) \, (\mathbf{T}(z) - \frac{c}{12} R(z)) \, \mathrm{d}^2 z \Big\} \Big\rangle ,$$

where μ is a Beltrami differential on X. The analog of equation (1.2) takes the form [6, 22]

$$(\bar{\partial} - \mu\partial - 2\mu_z) \frac{\boldsymbol{\delta} W}{\boldsymbol{\delta}\mu(z)} = \frac{c}{12\pi} (\mu_{zzz} + 2R\mu_z + R_z \mu),$$

where z is a local complex coordinate on X, and was used in [34, 35]. As it follows from the definition of W,

$$\left. \frac{\boldsymbol{\delta} W}{\boldsymbol{\delta}\mu(z)} \right|_{\mu=0} = \langle \mathbf{T}(z) - \frac{c}{12} R(z) \rangle = \frac{c}{12} (Q(z) - R(z))$$

and this expectation value can be set to zero if one chooses $Q = R$. However, when working with all conformal field theories on X having the same central charge c, it is preferrable to have a canonical choice of the holomorphic projective connection R. One possibility, which is the choice we will adopt in this paper, is to use a Fuchsian projective connection. It is defined by the Fuchsian uniformization of the Riemann surface X, i.e. by its realization as a quotient $\Gamma\backslash\mathbb{H}$ of the upper half-plane $\mathbb{H}$ by the action of a strictly hyperbolic Fuchsian group Γ with $2g$ generators. The upper half plane is isomorphic to the universal cover of X, while Γ, as an abstract group, is isomorphic to $\pi_1(X)$, the fundamental group of the surface X. Note that the Fuchsian uniformization of Riemann surfaces plays a fundamental role in the geometric approach to the two-dimensional quantum gravity through quantum Liouville theory (see [29] and references therein).

The covering $\mathbb{H} \to X$ allows to pull-back geometric objects from X to $\mathbb{H}$. Since the Fuchsian projective connection tautologically vanishes on $\mathbb{H}$, the stress-energy tensor $\mathbf{T}(z)$ becomes a quadratic differential for the Fuchsian group Γ

$$\mathbf{T} \circ \gamma \, (\gamma')^2 = \mathbf{T} \quad \text{for all } \gamma \in \Gamma,$$

whereas the source term μ becomes a Beltrami differential for Γ

$$\mu \circ \gamma \, \frac{\overline{\gamma'}}{\gamma'} = \mu \quad \text{for all } \gamma \in \Gamma.$$

The product $\mathbf{T}\mu$ is a $(1,1)$-tensor for Γ, so that the integral

$$\int_F \mathbf{T}\mu \,\mathrm{d}\,z\wedge \mathrm{d}\,\bar{z}$$

—the natural pairing between quadratic and Beltrami differentials—is well-defined, i.e. it does not depend on the choice of the fundamental domain $F\subset\mathbb{H}$ of the Fuchsian group Γ. As a result, the functional $W[\mu]$ retains the same form as in formula (1.1), where now the integration goes over the domain F, and satisfies the same equation (1.2), with $z\in\mathbb{H}$. It should be noted that the expectation value of $\langle\mathbf{T}(z)\rangle_\mu$ is no longer zero when $\mu=0$, but rather is $c/12$ times a holomorphic quadratic differential q, which is the pull-back to $\mathbb{H}$ of the quadratic differential $Q-R$ on X and characterizes a particular CFT. Thus, as it was observed in [34, 35], the generating functional for the stress-energy tensor on a higher genus Riemann surface is no longer a universal feature of all conformal field theories with the same value of c. However, as we shall show in the paper, one can still find the general solution of the equation (1.2).

Next, in order to solve the universal CWI and to define an action functional for the chiral sector in two-dimensional induced gravity on X, one could first try to extend Polyakov's functional (1.4) from $\mathbb{C}$ to X by considering the following integral

$$\frac{1}{2i}\int_F \omega[f]\,, \tag{1.6}$$

where

$$\omega[f]=\frac{f_{zz}}{f_z}\left(\frac{f_{\bar z}}{f_z}\right)_z \mathrm{d}\,z\wedge\mathrm{d}\,\bar z\,,$$

which was the correct choice for the genus 1 case. In this expression $\mu=f_{\bar z}/f_z$ should be a Beltrami differential for Γ, which is necessary for an invariant definition of the generating functional $W[\mu]$. This imposes strong conditions on the possible choices of the mapping f. It should be noted in the first place that, contrary to the genus zero case, the correspondence $f\mapsto\mu(f)=f_{\bar z}/f_z$ is no longer one-to-one. Indeed, the solution of the Beltrami equation

$$f_{\bar z}=\mu f_z$$

on $\mathbb{H}$ depends on the extension of the Beltrami coefficient μ to the lower half-plane $\overline{\mathbb{H}}$ of the complex plane $\mathbb{C}$. There are two canonical choices compatible with the action of Γ. In the first case

$$\mu(\bar z,z)\overset{\mathrm{def}}{=}\overline{\mu(z,\bar z)},\quad z\in\mathbb{H},$$

whereas in the second case

$$\mu(z,\bar z)\overset{\mathrm{def}}{=}0,\quad z\in\overline{\mathbb{H}}\,.$$

In both cases, the property of μ being a Beltrami differential for Γ is equivalent to the following equivariance property of f (the solution of the Beltrami equation in $\mathbb{C}$). There should exist an isomorphism $\Gamma\ni\gamma\mapsto\tilde\gamma\in\tilde\Gamma\subset\mathrm{PSL}(2,\mathbb{C})$, such that

$$f\circ\gamma=\tilde\gamma\circ f\quad\text{for all }\gamma\in\Gamma. \tag{1.7}$$

In the first case, the restriction of f to $\mathbb{H}$ yields a self-mapping of $\mathbb{H}$ with $\tilde\Gamma$ a Fuchsian group (thus defining a Fuchsian deformation of Γ), whereas in the second case f maps $\mathbb{H}$ onto the interior of a simple Jordan curve in $\mathbb{C}$ with $\tilde\Gamma$ a quasi-Fuchsian group (thus defining a quasi-Fuchsian deformation of Γ).

However, using the equivariance property of f it is easy to see that the "naive" expression (1.6) can not be considered as a correct choice for the action functional in higher genus. Indeed, it follows from (1.7) that:

1. The density $\omega[f]$ is not a $(1,1)$-tensor for Γ, so that the integral (1.6) depends on any particular choice of the fundamental domain F.
2. The formal variation of (1.6) depends on the values of $\boldsymbol{\delta} f$ on the boundary ∂F of F.

One may try to overcome these difficulties and resolve the second problem by adding suitable "correction terms" to the functional (1.6); these can be determined by performing the formal variation of (1.6). Specifically, all local computations will be the same as in the genus zero case (see Lemma 2.6), except that now (1.7) does not allow to get rid of the boundary terms in the Stokes formula by setting the variations $\boldsymbol{\delta}\mu$ or $\boldsymbol{\delta} f$ to zero on ∂F. Therefore, besides the local "bulk" term, the variation of (1.6) will contain "total derivative" terms localized at ∂F. This suggests the addition of "counterterms", which depend only on the edges of F, such that their variation cancels the boundary terms coming from the variation of (1.6). Such counterms can be determined; it should be noted that a similar, though much simpler procedure was used in [33], where the Liouville action functional on the fundamental domain of a Schottky group was defined. In our case, however, the actual construction goes one step further: the variation of the edge terms produces additional quantities localized at the vertices of ∂F. In turn, their cancellation requires counterterms that depend on the vertices of ∂F, which can be determined as well.

It turns out that this rather complicated procedure, which solves problem 2, can be carried out in a canonical way using standard tools from homological algebra, namely various double complexes naturally associated with the Riemann surface X. It is remarkable that at the same time it solves problem 1 as well!

By using the action of the group Γ on $\mathbb{H}$, we extend the singular chain boundary differential and the de Rham differential on $\mathbb{H}$ to act on chains and cochains for the group homology and cohomology of Γ. The corresponding group boundary and coboundary differentials give rise to two double complexes such that the fundamental domain F and the density $\omega[f]$ can be extended to representatives of suitable homology and cohomology classes $[\Sigma]$ and $[\Omega_f]$ and the pairing between them becomes Γ-invariant. Subsequently, we define the action functional $S[f]$ as the result of such pairing, i.e. as the evaluation of $[\Omega_f]$ on $[\Sigma]$. Quite naturally, the actual computation of these representatives goes exactly like descent calculations, familiar from BRST cohomology (see, e.g. [20]). This is more than a simple analogy in the following sense. The appropriate tool for linearizing the action of a discrete group is the group ring, which leads to the group (co)homology that we are using for the action of the Fuchsian group Γ on $\mathbb{H}$. The corresponding concept in the case of a continuous (Lie) group is the Lie algebra and its (co)homology, which is used in BRST theory.

The action functional $S[f]$ resulting from this construction looks as follows. Let F be a canonical fundamental domain for Γ in the form of a closed non-Euclidean polygon in $\mathbb{H}$ with $4g$ edges. For any $\gamma \in \Gamma$ and any pair $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma$, let $\theta_\gamma[f]$ and $\Theta_{\gamma_1,\gamma_2}[f]$ be a 1-form and a function on $\mathbb{H}$ given by the following explicit expressions:

$$\begin{aligned}\theta_{\gamma^{-1}}[f] &= \log(\tilde{\gamma}'\circ f)\,\mathrm{d}\log f_z - \log(f_z\circ\gamma)\,\mathrm{d}\log\gamma' - 2\frac{\gamma''}{\gamma'}\mu\,\mathrm{d}\,\bar{z}\\ \mathrm{d}\,\Theta_{\gamma_2^{-1},\gamma_1^{-1}}[f] &= f^*\big(\log\big(\tilde{\gamma}_1\circ\tilde{\gamma}_2\big)'\,\mathrm{d}\log\tilde{\gamma}_2'\big) + \log\gamma_2'\;\mathrm{d}\log\big(\gamma_1\circ\gamma_2\big)'\\ &\quad - \frac{1}{2}f^*\big(\mathrm{d}\big(\log\tilde{\gamma}_2'\big)^2\big) - \frac{1}{2}\,\mathrm{d}\big(\log\gamma_2'\big)^2\,,\end{aligned}$$

where f^* denotes the pull-back of differential forms on $\mathbb{H}$ by the mapping f. Then

$$\begin{aligned}2iS[f] = &\int_F \omega[f] - \sum_{i=1}^{g}\int_{b_i}\theta_{\beta_i}[f] + \sum_{i=1}^{g}\int_{a_i}\theta_{\alpha_i}[f]\\ &+\sum_{i=1}^{g}\Big(\Theta_{\alpha_i,\beta_i}[f](a_i(0)) - \Theta_{\beta_i,\alpha_i}[f](b_i(0)) + \Theta_{\gamma_i^{-1},\alpha_i\beta_i}[f](b_i(0))\Big)\\ &-\sum_{i=1}^{g-1}\Theta_{\gamma_g^{-1}\cdots\gamma_{i+1}^{-1},\gamma_i^{-1}}[f](b_g(0))\,.\end{aligned}\tag{1.8}$$

Here a_i and b_i are the standard cycles on X viewed as edges of F with initial points $a_i(0)$ and $b_i(0)$, α_i and β_i are the corresponding generators of the group Γ, and γ_i stands for the commutator $[\alpha_i,\beta_i]\overset{\text{def}}{=}\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$.

Observe that one can formally set $g=1$ in the representation (1.8), replacing the non-abelian groups Γ and $\tilde{\Gamma}$ by the lattices L and $\tilde{L}$, respectively. Since in this case $\gamma'=\tilde{\gamma}'=1$ identically, the differential forms θ and $\mathrm{d}\,\Theta$ vanish and the action functional $S[f]$ is given by the bulk term only.

It is also instructive to compare our construction with that presented in [34, 35]. Namely, in [34, 35] a solution of (1.2) was written directly on a higher genus Riemann surface equipped with additional algebro-geometric and/or dissection data. Formally, this solution also features a bulk term derived from the genus zero Polyakov action plus contributions of lower degree, but a rather complicated series of prescriptions is involved in its definition. In our construction, the functional $S[f]$ is written down on the universal cover $\mathbb{H}$ and it only depends on the choice of the normalized solution f of the Beltrami equation on $\mathbb{H}$. As a result, it enjoys the same nice variational properties as in the genus zero case. Specifically, we summarize our main results as follows.

Theorem A. *The functional $S[f]$ does not depend on either the choice of the fundamental domain F, or the choice of standard generators for the Fuchsian group Γ. It has a geometrical interpretation as a result of the evaluation map given by the canonical pairing*

$$H^2(X,\mathbb{C})\times H_2(X,\mathbb{Z})\longrightarrow\mathbb{C}$$

where $\omega[f]-\theta[f]-\Theta[f]$ represents an element in $H^2(X,\mathbb{C})$ depending on f and F is canonically extended to a representative of the fundamental class of X in $H_2(X,\mathbb{Z})$.

Since the action functional $S[f]$ is independent of all the choices made, the corresponding variational problem is well-defined. We shall consider two versions of it, depending on whether we choose either μ or f, related through the Beltrami equation, to be the independent functional variable. In the first case, the independent variable belongs to the linear space of Beltrami differentials for Γ and the "source" Fuchsian group Γ uniquely determines the "target" Fuchsian (or quasi-Fuchsian) group $\tilde{\Gamma} = f \circ \Gamma \circ f^{-1}$ through the solution of the Beltrami equation ("variation with free endpoint"). In the second case, the "target" group $\tilde{\Gamma}$ and the homomorphism $\Gamma \to \tilde{\Gamma}$ are fixed a priori ("variation with fixed endpoints") and the independent variable f is a self-mapping of $\mathbb{H}$ (or a mapping of $\mathbb{H}$ onto the interior of a simple Jordan curve) satisfying the equivariance property (1.7). In both cases it is guaranteed that the boundary terms arising from (1.6) are taken care of by the counterterms in (1.8), so that we have

Theorem B. *The variation of the action $S[f]$ with respect to μ or f is given by the formulas*

$$\boldsymbol{\delta} S[f] = 2 \int_F T(z)\, \boldsymbol{\delta}\mu(z)\, \mathrm{d}^2 z$$

and

$$\boldsymbol{\delta} S[f] = -2 \int_F \mu_{zzz}\, \frac{\boldsymbol{\delta} f}{f_z}\, \mathrm{d}^2 z\,,$$

respectively.

Needless to say, the variational derivatives of $S[f]$—the quantities $T(z)$ and μ_{zzz}—are, respectively, $(2,0)$ and $(2,1)$-tensors for Γ (see Lemma 4.2) and can be therefore pushed down to the Riemann surface $X \simeq \Gamma\backslash\mathbb{H}$.

Note that the critical points of the functional $S[f]$, considered for the mappings f that intertwine a given Fuchsian group Γ and a Fuchsian (or quasi-Fuchsian) group $\tilde{\Gamma}$, consist of those maps f such that the corresponding $\mu = f_{\bar{z}}/f_z$ satisfies the "equation of motion"

$$\mu_{zzz} = 0\,. \tag{1.9}$$

For a given pair $\Gamma, \tilde{\Gamma}$, determining the critical set of $S[f]$ seems to be a very difficult problem. However, it is rather easy to find the dimension of the solution space of the equation (1.9) without imposing any conditions on the target group $\tilde{\Gamma} = f \circ \Gamma \circ f^{-1}$. We shall show in section 4, using the Riemann-Roch theorem, that this dimension is actually $4g - 3$.

Critical points of the functional $S[f]$ with respect to the variation with free endpoint satisfy the equation of motion $T(z) = 0$. They are a subset of the previous "fixed-end" critical set (cf. Lemma 2.3 and Proposition 5.2). Again, determining this set seems to be a non simple task.

As in the genus zero case, it follows from Theorem B that $c\, S[f]/24\pi$, considered as a functional of $\mu = f_{\bar{z}}/f_z$, solves equation (1.2), and is a solution local in the map f. However, in the higher genus case the correspondence $\mu \mapsto f$ is no longer one-to-one and, at least, there are two canonical choices for f producing a Fuchsian or a quasi-Fuchsian deformation of the Fuchsian group Γ. Both the functionals $c\, S[f]/24\pi$ corresponding to these mappings solve equation (1.2). We shall show in section 4.2.2 that the difference of the corresponding stress-energy tensors is a quadratic differential for Γ, which is holomorphic with respect to the complex structure on X determined by the Fuchsian and the quasi-Fuchsian deformations of Γ.

As we already mentioned, in genus zero it is possible to express the solution of (1.3) by integrating along a linear path in the space of Beltrami coefficients. Actually, as we show in 2.2, any path $\mu(t)$ that connects μ to 0 leads to the same functional. In the higher genus case, we denote by $f^{\mu(t)}$ the corresponding solutions of the Beltrami equation on $\mathbb{H}$ producing either a Fuchsian or a quasi-Fuchsian deformation of Γ, depending on the given terminal mapping f, and set

$$T^t(z) = \{f^{\mu(t)}, z\} .$$

According to Lemma 4.2, the definition

$$W[\mu] \stackrel{\text{def}}{=} \frac{c}{12\pi} \int_0^1 \left(\int_X T^t \, \dot{\mu}(t) \, \mathrm{d}^2 z \right) \mathrm{d}\, t \,, \tag{1.10}$$

where $\dot{\mu}(t) = d\mu(t)/dt$, makes perfect sense since the integrand in (1.10), being a product of a Beltrami and a quadratic differential for Γ, is a $(1,1)$-tensor for Γ. We have

Theorem C. (i) *Let f be either a Fuchsian or a quasi-Fuchsian solution of the Beltrami equation on $\mathbb{H}$. Then*

$$W[\mu] = \frac{c}{24\pi} S[f] \,,$$

so that the functional $W[\mu]$ does not depend on the choice of the homotopy $\mu(t)$ and

$$\boldsymbol{\delta} W = \frac{c}{12\pi} \int_X T(z) \boldsymbol{\delta}\mu(z) \, \mathrm{d}^2 \, z \,.$$

(ii) *The functional $W[\mu]$ is a holomorphic functional of μ in the quasi-Fuchsian case, while in the Fuchsian case*

$$\left. \frac{\partial^2 W[\epsilon\mu]}{\partial\epsilon\partial\bar{\epsilon}} \right|_{\epsilon=0} = -\frac{c}{48\pi} \int_F |\mu|^2 y^{-2} \, \mathrm{d}^2 \, z \,,$$

for Bers harmonic Beltrami differentials μ.

It is worth stressing again that W, as defined in (1.10), is but one possible solution to the universal CWI on X: we have already noted that the solution corresponding to a given CFT with central charge c may differ from (1.10) by a term involving a Γ-quadratic differential, which is the expectation value of the stress-energy tensor of that CFT. (Similar observations about the lack of uniqueness in the solution to the CWI due to holomorphic quadratic differentials appear in [34, 35].) Moreover, the fact that in higher genus the correspondence $\mu \mapsto f$ ceases to be one-to-one clearly affects the value of (1.10), which will depend on the prescription used to solve the Beltrami equation. These observations lead to the question of what features of conformal field theories at central charge c are actually conveyed by (1.10). Since, according to Theorem C, the solution of (1.10) featuring a quasi-Fuchsian deformation depends holomorphically on μ, it is therefore natural to conjecture that the corresponding functional $W[\mu]$ (or $(c/24\pi)S[f]$, through Theorem C) represents a universal feature of all conformal field theories with central charge c.

We also observe that (1.10) can be considered as a WZW type functional, since it is obtained integrating over a path in the field space. Theorem C says that this term has also a local representation in two dimensions. This parallels the genus zero situation, where the Polyakov's action in the light cone gauge can be actually derived from a WZNW model [2]. (See also [31, 32] for the analogous situation in the conformal gauge.) In that case, one obtains a local functional in two dimensions as a consequence of the topological triviality of the WZW term for the group $\mathrm{SL}_2(\mathbb{R})$.

The organization of this paper is as follows. In section 2 we present a consistent formulation of the two-dimensional induced gravity in the conformal gauge using quasi-conformal (even smooth) mappings of $\mathbb{C}$ and without using any analytic continuation from the light-cone gauge or treating z and $\bar{z}$ as independent variables. There we gather all results, based on local computations, that will be used in the subsequent sections. Needless to say, essentially all these results are known (see [18, 26, 31, 32]) and we present them mainly for the convenience of the reader and in order to make the paper self-contained. We also discuss in detail the formulation based on the functional $W[\mu]$ from [18], prove that it coincides with the Polyakov's action functional (which was implicitly contained in [31]) and compute the Hessians of the action functionals $S[f]$ and $W[\mu]$.

We start section 3 by briefly discussing the genus 1 case. Next, we recall the standard concepts from homological algebra and differential topology that are needed to treat the case of higher genus Riemann surfaces, relegating the proofs of some rather technical results to the appendix. We then present the explicit construction of the representatives of the fundamental class $[\Sigma]$ and the cohomology class $[\Omega_f]$ corresponding to the fundamental domain F and the density $\omega[f]$, respectively.

In section 4 we finally define an analog of the Polyakov's action functional for the Riemann surface X of genus $g > 1$ and prove Theorems A, B and C. We also prove that the solution space of the equation $\mu_{zzz} = 0$ is $4g - 3$-dimensional and compute the Hessians of the action functionals $S[f]$ and $W[\mu]$.

The relation of the constructions presented in sections 3 and 4 with the geometry of various fiber spaces over the Teichmüller space is analyzed in section 5. There we describe $\exp(-W[\mu])$ as a section of a line bundle over Teichmüller space, making contact with previous work on the subject. In the last subsection we draw our conclusions and set some directions for future work.

2. Generating functional and Polyakov's action in genus zero

2.1. Let f be a normalized self-mapping of the complex plane $\mathbb{C}$, i.e. an orientation preserving diffeomorphism of the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ fixing $0, 1, \infty$. Define a map $f \mapsto \mu = \mu(f) = f_{\bar{z}}/f_z$, where μ is a smooth Beltrami coefficient on $\mathbb{C}$: a smooth bounded function such that $|\mu| < 1$. The following basic result of the theory of quasi-conformal mappings guarantees that the correspondence $f \mapsto \mu$ is one-to-one and onto.

Proposition 2.1. *Let $\mu \in L^\infty(\mathbb{C})$ (the Banach space of measurable functions with finite* sup *norm) such that $||\mu||_\infty < 1$. Then the Beltrami equation*

$$f_{\bar{z}} = \mu f_z \tag{2.1}$$

has a unique solution f fixing $0, 1, \infty$ which is an orientation preserving quasi-conformal homeomorphism of $\mathbb{C}$. The solution is smooth (real-analytic) whenever μ is smooth (real-analytic).

Proof. See [1]. □

Let $\omega[f]$ be the following $(1,1)$-form

$$\omega[f] = \frac{f_{zz}}{f_z}\mu_z \, \mathrm{d}\, z \wedge \mathrm{d}\, \bar{z}, \tag{2.2}$$

which (see the introduction) we identify as the density of the Polyakov's action functional. Here and elsewhere it is understood that $\mu = \mu(f)$. From now on we also assume that $f(z,\bar{z}) - z \to 0$ as $|z| \to \infty$ in such a way that the $(1,1)$-form $\omega[f]$ is integrable on $\mathbb{C}$. (One can simply consider μ with finite support; other less restrictive conditions for the difference $f(z,\bar{z}) - z$ can be formulated in terms of Sobolev spaces.) Define the functional

$$S[f] = \frac{1}{2i}\int_{\mathbb{C}} \omega[f] = -\int_{\mathbb{C}} \frac{f_{zz}}{f_z}\mu_z \, \mathrm{d}^2 z, \tag{2.3}$$

Remark 2.2. The functional $S[f]$ is the Euclidean version of the Polyakov's action functional for the two-dimensional quantum gravity in the light-cone gauge [26]. Let us recall that it can be also formally obtained (cf. [30]) as a "chiral" version of the Liouville action

$$A[\phi] = \frac{1}{2}\int_{\mathbb{C}} \sqrt{h}\,(h^{ab}\partial_a\phi\,\partial_b\phi + \phi\,R_h),$$

(where $x_1 = x,\ x_2 = y$ and R_h is the curvature of the background metric h), in the following way. Consider the "metric" $h = (\mathrm{d}\,z + \mu\,\mathrm{d}\,\bar{z}) \otimes \mathrm{d}\,\bar{z},\ \mu = \mu(f)$ and set $\phi = \log f_z$. Since $R_h = 2\mu_{zz}$, the integrand in $A[\phi]$ is equal to

$$\phi_z\phi_{\bar{z}} + 2\mu\bigl(-\frac{1}{2}\phi_z^2 + \phi_{zz}\bigr) = -\frac{f_{zz}}{f_z}\,\mu_z + 2\left(\mu\frac{f_{zz}}{f_z}\right)_z .$$

Let $T = \{f, z\}$ be the Schwarzian derivative of the mapping f. We have the following identity, which could also be looked at as an "equation for the trace anomaly" [26, 32].

Lemma 2.3.

$$(\bar{\partial} - \mu\partial - 2\mu_z)T = \mu_{zzz}\,.$$

Proof. A direct computation using the definitions of μ and of the Schwarzian derivative. □

Lemma 2.4. *The functional $S[f]$ is smooth in the sense that its variational derivative $\boldsymbol{\delta} S/\boldsymbol{\delta}\mu(z)$, defined as*

$$\left.\frac{d}{dt}\right|_{t=0} S(\mu + t\,\boldsymbol{\delta}\mu) = \int_{\mathbb{C}} \frac{\boldsymbol{\delta} S}{\boldsymbol{\delta}\mu}\,\boldsymbol{\delta}\mu\,\mathrm{d}^2 z$$

exists and is given by

$$\frac{\boldsymbol{\delta} S}{\boldsymbol{\delta}\mu(z)} = 2\,T(z).$$

Proof. Starting with the formula

$$\boldsymbol{\delta}\mu = \frac{\boldsymbol{\delta} f_{\bar{z}}}{f_z} - \mu\frac{\boldsymbol{\delta} f_z}{f_z}\,, \tag{2.4}$$

that relates the variations of μ and f, we get by a straightforward computation

$$\boldsymbol{\delta}\omega = \left\{\left(\frac{\boldsymbol{\delta} f_z}{f_z}\right)_z \mu_z + \frac{f_{zz}}{f_z}\boldsymbol{\delta}\mu_z\right\}\mathrm{d}\,z \wedge \mathrm{d}\,\bar{z} = -2\,T\boldsymbol{\delta}\mu\,\mathrm{d}\,z \wedge \mathrm{d}\,\bar{z} - \mathrm{d}\,\eta, \tag{2.5}$$

where

$$\eta[f;\boldsymbol{\delta} f] = \left(\frac{f_{zz}\boldsymbol{\delta} f_{\bar{z}}}{f_z^2} + \frac{\mu_z\boldsymbol{\delta} f_z}{f_z} - \left(\frac{f_{zz}}{f_z}\right)_z \mu\right)\mathrm{d}\,z + \frac{f_{zz}\,\boldsymbol{\delta} f_z}{f_z^2}\,\mathrm{d}\,\bar{z}\,.$$

□

Proposition 2.5. *The functional $c\,S[f]/24\pi$ is the unique solution of the universal CWI for the stress-energy tensor.*

Proof. It follows immediately from Lemmas 2.3 and 2.4 that $cS[f]/24\pi$, considered as a functional of μ, satisfies the equation (1.2)

$$(\bar\partial - \mu\partial - 2\mu_z)\frac{\boldsymbol{\delta} W}{\boldsymbol{\delta}\mu(z)} = \frac{c}{12\pi}\mu_{zzz}\,.$$

To prove uniqueness, consider the difference

$$Q[\mu](z) = \left(\frac{\boldsymbol{\delta} W}{\boldsymbol{\delta}\mu(z)} - \frac{c}{24\pi}\frac{\boldsymbol{\delta} S}{\boldsymbol{\delta}\mu(z)}\right)(f_z)^{-2}$$

and observe (cf. [22, 31]) that it satisfies the following equation

$$(\bar\partial - \mu\partial)Q[\mu](z) = 0,$$

which shows that $Q[\mu](z,\bar z)$ is holomorphic with respect to the new complex structure $\zeta = f(z,\bar z), \bar\zeta = \overline{f(z,\bar z)}$ on $\mathbb{C}$ defined by the Cauchy-Riemann operator $\bar\partial - \mu\,\partial$. Recalling that $\boldsymbol{\delta} W/\boldsymbol{\delta}\mu(z)$, as well as $T(z)$, vanish as $|z|\to\infty$ (regularity of the stress-energy tensor at ∞) we conclude that $Q[\mu]$ is an entire function of ζ vanishing at ∞, so that $Q[\mu]=0$. Therefore, the functional

$$\frac{c}{24\pi}S[f] = -\frac{c}{24\pi}\int_{\mathbb{C}}\frac{f_{zz}}{f_z}\mu_z\,\mathrm{d}^2 z$$

solves the universal CWI (1.2) on $\mathbb{P}^1$. ∎

Next, we determine the variation of S with respect to f and determine the classical equations of motion: the critical points $\boldsymbol{\delta} S[f]=0$ of the functional S.

Lemma 2.6.

$$\boldsymbol{\delta} S[f] = -2\int_{\mathbb{C}}(T_{\bar z} - \mu\,T_z - 2\,\mu_z\,T)\,\frac{\boldsymbol{\delta} f}{f_z}\,\mathrm{d}^2 z = -2\int_{\mathbb{C}}\mu_{zzz}\,\frac{\boldsymbol{\delta} f}{f_z}\,\mathrm{d}^2 z\,,$$

so that the classical equation of motion is

$$\mu_{zzz} = 0\,.$$

Proof. It follows from the identity

$$T\,\boldsymbol{\delta}\mu\,\mathrm{d}\,z\wedge\mathrm{d}\,\bar z = (-T_{\bar z} + \mu\,T_z + 2\mu_z T)\,\frac{\boldsymbol{\delta} f}{f_z} - \mathrm{d}\,\eta',$$

where

$$\eta' = T\frac{\boldsymbol{\delta} f}{f_z}\,\mathrm{d}\,z + \mu\,T\frac{\boldsymbol{\delta} f}{f_z}\,\mathrm{d}\,\bar z\,,$$

and from Lemma 2.3. ∎

2.2. Let $\mu(t)$, $0\le t\le 1$, be the path in the space of Beltrami coefficients connecting 0 with the given Beltrami coefficient μ. It gives rise to a homotopy $f^t = f^{\mu(t)}$, $f^0 = \mathrm{id}$, $f^1 = f$ that consists of normalized quasi-conformal mappings satisfying the Beltrami equation

$$f^t_{\bar z} = \mu(t) f^t_z\,.$$

Denoting the corresponding Schwarzians as $T^t(z) = \{f^t, z\}$, so that $T^0 = 0$ and $T^1 = T$, we have the following useful variational formulas.

Lemma 2.7.

$$\mu(t)_{zzz} = (\bar\partial - \mu(t)\,\partial - 2\,\mu(t)_z)(T^t), \tag{i}$$

$$\boldsymbol{\delta} T^t = (\partial^3 + 2\,T^t\,\partial + T^t_z)(u^t), \tag{ii}$$

$$\boldsymbol{\delta}\mu(t) = (\bar\partial - \mu(t)\,\partial + \mu(t)_z)(u^t), \tag{iii}$$

where $u^t = \boldsymbol{\delta} f^t / f^t_z$.

Proof. Equation (i) is just a restatement of Lemma 2.3, applied to the map f^t. The variational formula (ii) is verified by a straightforward (though lengthy) computation using $T = \{f, z\}$ and the definition of the Schwarzian derivative. Finally, equation (iii) follows from the variational formula (2.4), written as

$$\boldsymbol{\delta}\mu = (\bar\partial - \mu\,\partial + \mu_z)\left(\frac{\boldsymbol{\delta} f}{f_z}\right)$$

and specialized to the map f^t. □

As it follows from Lemma 2.7, the differential operators

$$\mathcal{T} = \partial^3 + 2\,T\,\partial + T_z$$

and

$$\mathcal{M} = \bar\partial - \mu\,\partial + \mu_z.$$

play a fundamental role in the variational theory. In particular, the third-order differential operator $\mathcal{T}$ appears in many other different areas as well. It serves as a Jacobi operator for the second Poisson structure for the KdV equation [24] that is given by the Virasoro algebra and it plays an important role in Eichler cohomology on Riemann surfaces [17]. The operator $\mathcal{T}$ is skew-symmetric, $\mathcal{T}^\tau = -\mathcal{T}$, with respect to the inner product given by

$$(u, v) = \int_{\mathbb{C}} u\, v\ \mathrm{d}^2\, z\,, \tag{2.6}$$

whereas $\mathcal{M}^\tau = -\mathcal{D}$, where $\mathcal{D} \overset{\text{def}}{=} \bar\partial - \mu\partial - 2\mu_z$. However, we have the following result.

Lemma 2.8. *The operator* $\mathcal{T}\mathcal{M}$ *is symmetric.*

Proof. It reduces to the verification of the identity $(\mathcal{T}\mathcal{M})^\tau = \mathcal{D}\mathcal{T}$, or

$$(\partial^3 + 2\,T\,\partial + T_z)(\bar\partial - \mu\,\partial + \mu_z) = (\bar\partial - \mu\,\partial + 2\mu_z)(\partial^3 + 2\,T\,\partial + T_z),$$

which immediately follows from Lemma 2.3 and $T = \{f, z\}$. □

Now, let us introduce the functional

$$W[\mu] = \frac{c}{12\pi} \int_0^1 \int_{\mathbb{C}} T^t\, \dot\mu(t)\ \mathrm{d}^2 z\ \mathrm{d}\, t, \tag{2.7}$$

where the dot stands for d/dt. A priori it may depend on the choice of the homotopy $\mu(t)$. The following result shows that the variational derivative of W with respect to $\mu = \mu(1)$ does not depend on $\mu(t)$.

Lemma 2.9.

$$\frac{\boldsymbol{\delta} W}{\boldsymbol{\delta}\mu(z)} = \frac{c}{12\pi} T(z)\,.$$

Proof. Writing $\boldsymbol{\delta}(T^t\dot{\mu}(t)) = \boldsymbol{\delta}T^t\dot{\mu}(t)+T^t\boldsymbol{\delta}\dot{\mu}(t)$ and using (ii) in Lemma 2.7, together with the relation

$$\dot{\mu}(t) = \mathcal{M}^t(v^t), \tag{2.8}$$

(where $v^t = \dot{f}^t/f^t_z$) which follows from formula (iii) of Lemma 2.7 applied to $\boldsymbol{\delta} = d/dt$, we get

$$\boldsymbol{\delta}T^t\dot{\mu(t)} = \mathcal{T}^t(u^t)\mathcal{M}^t(v^t)\,.$$

Using Lemma 2.8, equations (2.8), (iii) and the equation

$$\dot{T}^t = \mathcal{T}^t(v^t)\,,$$

which follows from formula (ii) of Lemma 2.7 applied to $\boldsymbol{\delta} = d/dt$, we obtain

$$\begin{aligned}\int_{\mathbb{C}} \boldsymbol{\delta}T^t\dot{\mu}(t)\ \mathrm{d}^2 z &= (\mathcal{T}^t(u^t),\mathcal{M}^t(v^t)) = -(u^t,\mathcal{T}^t\mathcal{M}^t(v^t)) \\ &= (u^t,(\mathcal{M}^t)^\tau\mathcal{T}^t(v^t)) = (\mathcal{M}^t(u^t),\mathcal{T}^t(v^t)) \\ &= \int_{\mathbb{C}} \boldsymbol{\delta}\mu(t)\dot{T}^t\ \mathrm{d}^2 z\,.\end{aligned}$$

Substituting this into the expression for $\boldsymbol{\delta}W$, we get

$$\int_0^1 (\dot{T}^t\boldsymbol{\delta}\mu(t) + T^t\boldsymbol{\delta}\dot{\mu}(t))\ \mathrm{d}\,t = T^t\boldsymbol{\delta}\mu(t)\big|_{t=0}^{t=1} = T\boldsymbol{\delta}\mu,$$

which completes the proof. □

Moreover, as the next result shows, the functional W is actually independent of the choice of the path $\mu(t)$ connecting the points 0 and μ in the space of Beltrami coefficients.

Proposition 2.10.

$$W[\mu] = \frac{c}{24\pi}S[f]\,,$$

where f and μ are related through $\mu = f_{\bar z}/f_z$.

Proof. It is essentially the computation in Lemma 2.4, done in the reverse order. Namely, considering the families $\mu(t)$ and $f^{\mu(t)}$ and using the formula (2.5) for the case $\boldsymbol{\delta} = d/dt$, we get

$$2\,T^t\dot{\mu}(t)\,\mathrm{d}\,z\wedge\mathrm{d}\,\bar z = \frac{d}{dt}\left(\frac{f^t_{zz}}{f^t_z}\mu(t)_z\,\mathrm{d}\,z\wedge\mathrm{d}\,\bar z\right) + \mathrm{d}\,\eta[f^t;\dot{f}^t]\,,$$

which after integrating over $\mathbb{C}\times[0,1]$ yields the result. □

2.3. Here we compute the Hessian of the functional $S[f]$, i.e. its second variation with respect to f, evaluated at the critical point. Let $\boldsymbol{\delta}_1 f$ and $\boldsymbol{\delta}_2 f$ be two variations of f, defined through the two-parameter family $f_{s,t}$ with $f_{0,0} = f$ as

$$\boldsymbol{\delta}_1 f = \left.\frac{\partial f_{s,t}}{\partial s}\right|_{s=t=0}, \quad \boldsymbol{\delta}_2 f = \left.\frac{\partial f_{s,t}}{\partial t}\right|_{s=t=0}.$$

The second variation of $S[f]$ is

$$\boldsymbol{\delta}^2 S[f] = \left.\frac{d^2}{ds\,dt}S[f_{s,t}]\right|_{s=t=0},$$

and it can be computed using the first variation of $S[f]$ from Lemma 2.6

$$\boldsymbol{\delta}_1 S[f] = -2\int_{\mathbb{C}} \mu_{zzz}\, \frac{\boldsymbol{\delta}_1 f}{f_z}\, \mathrm{d}^2 z$$

by evaluating $\boldsymbol{\delta}_2(\mu_{zzz}[f])$. As it follows from Lemma 2.7,

$$\boldsymbol{\delta}_2\big(\mu_{zzz}[f]\big) = \big(\partial^3 \circ \mathcal{M}\big)\left(\frac{\boldsymbol{\delta}_2 f}{f_z}\right), \tag{2.9}$$

so that

$$\boldsymbol{\delta}^2\, S[f](\boldsymbol{\delta}_1 f, \boldsymbol{\delta}_2 f) = -2\int_{\mathbb{C}} \frac{\boldsymbol{\delta}_1 f}{f_z}\big(\partial^3 \circ \mathcal{M}\big)\left(\frac{\boldsymbol{\delta}_2 f}{f_z}\right)\, \mathrm{d}^2 z\,. \tag{2.10}$$

The Hessian is symmetric, so that the right hand side of (2.10) should be a symmetric bilinear form in $\boldsymbol{\delta}_1 f, \boldsymbol{\delta}_2 f$ whenever $\mu_{zzz} = 0$. This can be verified directly, as we have

Lemma 2.11. *The operator $\partial^3 \circ \mathcal{M}$ for $\mu_{zzz} = 0$ is symmetric with respect to the bilinear form* (2.6)*.*

Proof. Using $(\partial^3)^\tau = -\partial^3$ we have

$$\big(\partial^3 \circ \mathcal{M}\big)^\tau = \mathcal{D} \circ \partial^3\,,$$

where $\mathcal{D} = \bar\partial - \mu\,\partial - 2\,\mu_z$, and it is straightforward to verify the following identity when $\mu_{zzz} = 0$:

$$\partial^3 \circ \mathcal{M} = \mathcal{D} \circ \partial^3\,.$$

□

Similarly, one can compute the Hessian of the functional $W[\mu]$. We have

Lemma 2.12.

$$\boldsymbol{\delta}^2\, W[\mu](\boldsymbol{\delta}_1\mu, \boldsymbol{\delta}_2\mu) = \frac{c}{12\pi}\int_{\mathbb{C}} \boldsymbol{\delta}_1\mu\big(\partial^3 \circ \mathcal{M}^{-1}\big)(\boldsymbol{\delta}_2\mu)\, \mathrm{d}^2 z.$$

Remark 2.13. Since

$$\mathcal{M}\left(\frac{u \circ f}{f_z}\right) = \frac{\overline{f_z}}{f_z}\,(1 - |\mu|^2)\,(\bar\partial u) \circ f\,, \tag{2.11}$$

the operator $\mathcal{M}$ is invertible on the subspace of smooth functions on $\mathbb{C}$ vanishing at ∞.

3. Algebraic and topological constructions

3.1. Here we consider the genus 1 case. Let X be an elliptic curve, i.e. a compact Riemann surface of genus 1, realized as the quotient $X \cong L\backslash\mathbb{C}$, where L is a rank 2 lattice in $\mathbb{C}$, generated by the translations $\alpha(z) = z + 1$ and $\beta(z) = z + \tau$, where $\operatorname{Im}\tau > 0$. Let μ be a Beltrami coefficient for L, i.e. a $||\mu||_\infty < 1$ function on $\mathbb{C}$ satisfying

$$\mu \circ \gamma = \mu \quad \text{for all } \gamma \in L,$$

and let $f = f^\mu$ be the normalized (fixing $0, 1, \infty$) solution of the Beltrami equation on $\mathbb{C}$

$$f_{\bar z} = \mu f_z\,.$$

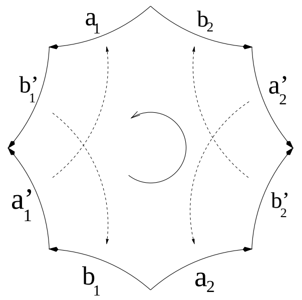

FIGURE 1. Conventions for the fundamental domain F

It is easy to see that $f \circ L = \tilde{L} \circ f$, where $\tilde{L}$ is the rank 2 lattice in $\mathbb{C}$ generated by 1 and $\tilde{\tau} = f(\tau)$. Indeed, $\tilde{\gamma} = f \circ \gamma \circ f^{-1}$ is a parabolic element in $\mathrm{PSL}(2,\mathbb{C})$ fixing ∞, i.e. a translation $z \mapsto z + h$, and it follows from the normalization that $f(z+1) = f(z) + 1$. Therefore the $(1,1)$-form $\omega[f]$ on $\mathbb{C}$ is well-defined on X so that the action functional takes the form

$$S[f] = \frac{1}{2i} \int_{\Pi} \omega[f] \,,$$

where Π is the fundamental parallelogram for the lattice L.

3.2. Here we consider the higher genus case and construct double complexes that extend the singular chain and the de Rham complexes on $\mathbb{H}$. We extend the fundamental domain F for Γ and the $(1,1)$-form $\omega[f]$ on $\mathbb{H}$ to representatives of the homology and cohomology classes $[\Sigma]$ and $[\Omega_f]$ for these double complexes.

3.2.1. Let $X \cong \Gamma\backslash\mathbb{H}$ be a compact Riemann surface of genus $g > 1$, realized as the quotient of the upper half-plane $\mathbb{H}$ by the action of a strictly hyperbolic Fuchsian group Γ. Recall that the group Γ is called marked if there is a chosen system, up to inner automorphism, of $2g$ free generators $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ satisfying the single relation

$$[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1 \,, \tag{3.1}$$

where $[\alpha_i, \beta_i] \stackrel{\mathrm{def}}{=} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$ and 1 is the unit element in Γ. For every choice of the marking there is a standard choice of a fundamental domain $F \subset \mathbb{H}$ for Γ as a closed non-Euclidean polygon with $4g$ edges, pairwise identified by suitable group elements. We will use the following normalization (see, e.g., [19] and Figure 1). The edges of F are labelled by a_i, a_i', b_i, b_i' and $\alpha_i(a_i') = a_i$, $\beta_i(b_i') = b_i$ for all $i = 1, \dots, g$; the orientation of the edges is chosen so that $\partial F = \sum_{i=1}^{g} (a_i + b_i' - a_i' - b_i)$. Also we set $\partial a_i = a_i(1) - a_i(0)$ and $\partial b_i = b_i(1) - b_i(0)$, where the label "1" represents the end point and the label "0" the initial point with respect to the edge's orientation. One has the following relations between the vertices of F and the generators: $a_i(0) = b_{i+1}(0)$, $\alpha_i^{-1}(a_i(0)) = b_i(1)$, $\beta_i^{-1}(b_i(0)) = a_i(1)$ and $[\alpha_i, \beta_i](b_i(0)) = b_{i-1}(0)$, where, in accordance with (3.1), $b_0(0) = b_g(0)$.

3.2.2. Let μ be a Beltrami differential for the Fuchsian group Γ, i.e. a bounded $(L^\infty(\mathbb{H}))$ function on $\mathbb{H}$ satisfying

$$\mu \circ \gamma \, \frac{\overline{\gamma'}}{\gamma'} = \mu \quad \text{for all } \gamma \in \Gamma.$$

In addition, it is called a Beltrami coefficient for Γ when $||\mu||_\infty < 1$. Denote by $f = f^\mu$ the normalized (fixing $0, 1$ and ∞) solution of the Beltrami equation on $\mathbb{H}$

$$f_{\bar{z}} = \mu f_z \,.$$

As it was already explained in the introduction, we consider f to be either a self-mapping of $\mathbb{H}$, or a mapping of $\mathbb{H}$ onto the interior of a simple Jordan curve in $\mathbb{C}$, uniquely determined by μ. These two choices can be realized by considering the Beltrami equation on the whole complex plane $\mathbb{C}$: in the former case the Beltrami coefficient μ is extended to the lower half-plane $\overline{\mathbb{H}}$ by reflecting it through the real line $\mathbb{R}$, while in the latter μ is extended by zero in $\overline{\mathbb{H}}$. In both cases there exists $\tilde{\Gamma} \subset \mathrm{PSL}(2, \mathbb{C})$, isomorphic to Γ as an abstract group and such that f intertwines between Γ and $\tilde{\Gamma}$

$$f \circ \gamma = \tilde{\gamma} \circ f \quad \text{for all } \gamma \in \Gamma,$$

which actually defines the isomorphism $\gamma \mapsto \tilde{\gamma}$. In the first case we have that $\tilde{\Gamma} \subset \mathrm{PSL}(2, \mathbb{R})$ and it is in fact a Fuchsian group, a Fuchsian deformation of Γ. In the second case $\tilde{\Gamma}$ is a so-called quasi-Fuchsian group, a special case of a Kleinian group. Its domain of discontinuity has two invariant components, the interior and the exterior of a simple Jordan curve in $\mathbb{C}$, which is the image of the real line $\mathbb{R}$ under the mapping f and is a limit set for $\tilde{\Gamma}$. These mappings, introduced and studied by Ahlfors and Bers, play a fundamental role in Teichmüller theory (see, e.g. [16]).

3.2.3. Let $\mathsf{S}_\bullet \equiv S_\bullet(X_0)$ be the standard singular chain complex of $\mathbb{H}$ with the differential ∂'. (From now on, we will denote the singular chain differential by ∂', as the symbol ∂ will be reserved for the total differential in a double complex, to be introduced below.) The group Γ acts on $\mathbb{H}$ and induces a left action on $\mathsf{S}_\bullet$ by translating the chains, hence $\mathsf{S}_\bullet$ becomes a complex of Γ-modules. Since the action of Γ on $\mathbb{H}$ is proper, $\mathsf{S}_\bullet$ is a complex of left *free* $\mathbb{Z}\Gamma$-modules [23], where $\mathbb{Z}\Gamma$ is the integral group ring of Γ: the set of finite combinations $\sum_{\gamma\in\Gamma} n_\gamma \gamma$ with coefficients $n_\gamma \in \mathbb{Z}$.

Let $\mathsf{B}_\bullet \equiv B_\bullet(\mathbb{Z}\Gamma)$ be the canonical "bar" resolution complex for Γ, with differential ∂''. Each $B_n(\mathbb{Z}\Gamma)$ is a free left Γ-module on generators $[\gamma_1|\dots|\gamma_n]$, with the differential $\partial'' : \mathsf{B}_n \to \mathsf{B}_{n-1}$ given by

$$\partial''[\gamma_1|\dots|\gamma_n] = \gamma_1 [\gamma_2|\dots|\gamma_n] + \sum_{i=1}^{n-1} (-1)^i [\gamma_1|\dots|\gamma_i \, \gamma_{i+1}|\dots|\gamma_n] \\ + (-1)^n [\gamma_1|\dots|\gamma_{n-1}]$$

for $n > 1$ and by

$$\partial''[\gamma] = \gamma[\,] - [\,]$$

for $n = 1$. Here $[\gamma_1|\dots|\gamma_n]$ is defined to be zero if any of the group elements inside $[\dots]$ equals the unit element 1 in Γ. $B_0(\mathbb{Z}\Gamma)$ is a $\mathbb{Z}\Gamma$-module on one generator $[\,]$, and can be identified with $\mathbb{Z}\Gamma$ under the isomorphism that sends $[\,]$ to 1.

Next, consider the double complex $\mathsf{K}_{\bullet,\bullet} = \mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet$. The associated total simple complex $\mathrm{Tot}\,\mathsf{K}$ is equipped with the total differential $\partial = \partial' + (-1)^p \partial''$ on $\mathsf{K}_{p,q}$. For the sake of future reference, we observe that $\mathsf{S}_\bullet$ is identified with $\mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_0$ under the correspondence $c \mapsto c \otimes [\,]$.

Remark 3.1. Since $\mathsf{S}_\bullet$ and $\mathsf{B}_\bullet$ are both complexes of left Γ-modules, in order to define their tensor product over $\mathbb{Z}\Gamma$ we need to endow each S_n with a right Γ-module structure. This is done in the standard fashion by setting $c \cdot \gamma \overset{\text{def}}{=} \gamma^{-1}(c)$. As a result $\mathsf{S} \otimes_{\mathbb{Z}\Gamma} \mathsf{B} = (\mathsf{S} \otimes_{\mathbb{Z}} \mathsf{B})_\Gamma$, so that the tensor product over integral group ring of Γ can be obtained as the set of Γ-invariants in the usual tensor product (over $\mathbb{Z}$) as abelian groups [9].

The application of standard spectral sequence machinery, together with the trivial fact that $\mathbb{H}$ is acyclic, leads to the following lemma, whose formal proof immediately follows, for example, from [23], Theorem XI.7.1 and Corollary XI.7.2.

Lemma 3.2. *There are isomorphisms*

$$H_\bullet(X, \mathbb{Z}) \cong H_\bullet(\Gamma, \mathbb{Z}) \cong H_\bullet(\operatorname{Tot} \mathsf{K}_{\bullet,\bullet}),$$

where the three homologies are the singular homology of X, the group homology of Γ and the homology of the complex $\operatorname{Tot} \mathsf{K}_{\bullet,\bullet}$ with respect to the total differential ∂.

We will use this lemma in the construction of the explicit cycle Σ in $\operatorname{Tot} \mathsf{K}$ that extends the fundamental domain F. For the convenience of the reader we present a simple minded proof of Lemma 3.2 in Appendix A.

3.2.4. We now turn to constructions dual to those in 3.2.3. Denote by $\mathsf{A}^\bullet \equiv A^\bullet_{\mathbb{C}}(X_0)$ the complexified de Rham complex on $\mathbb{H}$. Each A^n is a left Γ-module with the pull-back action of Γ, i.e. $\gamma \cdot \phi \overset{\text{def}}{=} (\gamma^{-1})^* \phi$ for $\phi \in \mathsf{A}^\bullet$ and for all $\gamma \in \Gamma$. Consider the double complex $\mathsf{C}^{p,q} = \operatorname{Hom}(\mathsf{B}_q, \mathsf{A}^p)$ with differentials d, the usual de Rham differential, and $\delta = (\partial'')^*$, the group coboundary. Specifically, for $\phi \in \mathsf{C}^{p,q}$

$$\begin{aligned}(\delta\phi)_{\gamma_1,\dots,\gamma_{q+1}} = \gamma_1 \cdot \phi_{\gamma_2,\dots,\gamma_{q+1}} &+ \sum_{i=1}^{q} (-1)^i \phi_{\gamma_1,\dots,\gamma_i\gamma_{i+1}\dots,\gamma_{q+1}} \\ &+ (-1)^{q+1} \phi_{\gamma_1,\dots,\gamma_q}.\end{aligned}$$

As usual, the total differential on $\mathsf{C}^{p,q}$ is $D = \mathrm{d} + (-1)^p \delta$. Either by dualizing Lemma 3.2 or working out the spectral sequences resulting from C, we obtain the

Lemma 3.3. *There are isomorphisms*

$$H^\bullet(X, \mathbb{C}) \cong H^\bullet(\Gamma, \mathbb{C}) \cong H^\bullet(\operatorname{Tot} \mathsf{C}^{\bullet,\bullet}),$$

where the the three cohomologies are the de Rham cohomology of X, the group cohomology of Γ and the cohomology of the complex $\operatorname{Tot} \mathsf{C}^{\bullet,\bullet}$ with respect to the total differential D.

As for Lemma 3.2, a simpler proof can also be found in Appendix A.

Finally, there exists a natural pairing between $\mathsf{C}^{p,q}$ and $\mathsf{K}_{p,q}$ which assigns to the pair $(\phi,\, c \otimes [\gamma_1|\dots|\gamma_q])$ the evaluation of the form $\phi_{\gamma_1,\dots,\gamma_q}$ over a cycle c

$$\langle \phi, c \otimes [\gamma_1|\dots|\gamma_q] \rangle = \int_c \phi_{\gamma_1,\dots,\gamma_q}. \tag{3.2}$$

By the very construction of the double complexes $\mathsf{C}^{\bullet,\bullet}$ and $\mathsf{K}_{\bullet,\bullet}$, the total differentials D and ∂ are transpose to each other

$$\langle D\Phi, C \rangle = \langle \Phi, \partial C \rangle \tag{3.3}$$

for all $\Phi \in \mathsf{C}^{\bullet,\bullet}$, $C \in \mathsf{K}_{\bullet,\bullet}$. Therefore the pairing (3.2) descends to the corresponding homology and cohomology groups and is non degenerate. It defines a pairing between $H^\bullet(\operatorname{Tot} \mathsf{C}^{\bullet,\bullet})$ and $H_\bullet(\operatorname{Tot} \mathsf{K}_{\bullet,\bullet})$ which we continue to denote by $\langle\ ,\ \rangle$.

3.3. Here we compute explicit representatives Σ and Ω_f, for the fundamental class of the surface X and a degree two cohomology class on X that extend the fundamental domain F and the 2-form $\omega[f]$, respectively.

3.3.1. *Homology computations.* Fix the marking of Γ and choose a fundamental domain F as in 3.2. We start by the observation that $F \cong F \otimes [\,] \in \mathsf{K}_{2,0}$. Furthermore, obviously $\partial'' F = 0$, and

$$\begin{aligned}\partial' F &= \sum_{i=1}^{g} (b'_i - b_i - a'_i + a_i) \\ &= \sum_{i=1}^{g} (\beta_i^{-1}(b_i) - b_i - \alpha_i^{-1}(a_i) + a_i)\,,\end{aligned}$$

which we can rewrite as $\partial' F = \partial'' L$ where $L \in \mathsf{K}_{1,1}$ is given by

$$L = \sum_{i=1}^{g} (b_i \otimes [\beta_i] - a_i \otimes [\alpha_i])\,. \tag{3.4}$$

This follows from $\gamma^{-1}(c) - c = c \cdot \gamma - c = c \otimes \gamma[\,] - c \otimes [\,] = c \otimes \partial''[\gamma]$ for any singular chain c and any $\gamma \in \Gamma$.

Let us now compute $\partial' L$. There exists $V \in \mathsf{K}_{0,2}$ such that $\partial' L = \partial'' V$; its explicit expression is given by

$$\begin{aligned}V = &\sum_{i=1}^{g} \left(a_i(0) \otimes [\alpha_i|\beta_i] - b_i(0) \otimes [\beta_i|\alpha_i] + b_i(0) \otimes [\gamma_i^{-1}|\alpha_i\beta_i]\right) \\ &- \sum_{i=1}^{g-1} b_g(0) \otimes [\gamma_g^{-1} \dots \gamma_{i+1}^{-1}|\gamma_i^{-1}]\,,\end{aligned} \tag{3.5}$$

where $[\alpha_i, \beta_i] = \gamma_i$. Indeed, a straightforward computation, using the relations between generators and vertices, yields

$$\partial' L = \partial'' V - b_g(0) \otimes [\gamma_g^{-1} \dots \gamma_1^{-1}]\,,$$

and the second term in the RHS vanishes by virtue of (3.1), since $[1] = 0$.

From the relations $\partial' F = \partial'' L$ and $\partial' L = \partial'' V$ it follows immediately that the element $\Sigma = F + L - V$ of total degree two is a cycle in Tot K, that is

$$\partial(F + L - V) = 0\,.$$

Thus we have the

Proposition 3.4. *The cycle $\Sigma \in (\mathrm{Tot}\ \mathsf{K})_2$ represents the fundamental class of the surface in $H_2(X, \mathbb{Z})$.*

Proof. This follows immediately from Lemma 3.2, provided the class $[\Sigma]$ is not zero, but this is not the case, since the cycle Σ is a "ladder" starting from the fundamental domain F. It follows from the arguments in Appendix A that the latter in fact maps under $\mathsf{S}_2 \ni F \mapsto F \otimes 1 \in \mathsf{S}_2 \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} \cong S_2(X)$ to a representative of the fundamental class. □

Remark 3.5. The existence of the elements L and V can be guaranteed a priori by the methods of Appendix A, using the fact that Γ has no cohomology except in degree zero.

As it follows from Proposition 3.4, the homology class $[\Sigma]$ is independent of the marking of the Fuchsian group Γ and of the choice of the fundamental domain F, whereas its representative Σ is not. Since this independence is a key issue in defining the action functional for the higher genus case, we will show explicitly that different choices lead to homologous Σ. Essentially, these choices are the following.

- Within the same marking choose another set of canonical generators α'_i, β'_i by conjugating α_i, β_i with $\gamma \in \Gamma$ so that $F' = \gamma F$ for the corresponding fundamental domains.
- Within the same marking make a different choice of the fundamental domain F' (which is always assumed to be closed in $\mathbb{H}$), not necessarily equal to the canonical $4g$ polygon F.
- Consider a different marking α'_i, β'_i and a fundamental domain F' for it.

Clearly, all the previous cases amount to an arbitary choice of the fundamental domain for Γ. However, if F and F' are two such choices, then there exist a suitable set of indices $\{\nu\}$, elements $\gamma_\nu \in \Gamma$ and singular two-chains c_ν such that

$$F' - F = \sum_\nu (\gamma_\nu^{-1}(c_\nu) - c_\nu)\,. \tag{3.6}$$

It follows, for instance, from the fact that the chain complex for $\mathbb{H}$ is a free Γ-module [23]. Then we have the following

Lemma 3.6. *If F and F' are two choices of the fundamental domain for Γ in $\mathbb{H}$, then $[\Sigma] = [\Sigma']$ for the corresponding classes in $H_\bullet(\operatorname{Tot}\mathsf{K}_{\bullet,\bullet})$.*

Proof. Let $\Sigma = F + L - V$ and $\Sigma' = F' + L' - V'$ be the cycles in $\operatorname{Tot}\mathsf{K}$ constructed according to the method of 3.3.1. It follows from (3.6) that

$$F' - F = \partial''\big(\sum_\nu c_\nu \otimes [\gamma_\nu]\big)\,,$$

and therefore

$$\begin{aligned} F' + L' - F - L &= \partial\big(\sum_\nu c_\nu \otimes [\gamma_\nu]\big) \\ &\quad + \big(L' - L - \sum_\nu \partial'(c_\nu) \otimes [\gamma_\nu]\big)\,. \end{aligned}$$

The second term in these expression is an element of $\mathsf{K}_{1,1}$ and its second differential is

$$\begin{aligned} \partial''\big(L' - L - \sum_\nu \partial'(c_\nu) \otimes [\gamma_\nu]\big) &= \partial'(F' - F) - \sum_\nu (\gamma_\nu^{-1}(\partial'(c_\nu)) - \partial'(c_\nu)) \\ &= 0\,. \end{aligned}$$

Since the higher homology of Γ with values in $\mathsf{S}_\bullet$ is zero (cf. Appendix A), there exists an element $C \in \mathsf{K}_{1,2}$ such that

$$L' - L - \sum_\nu \partial'(c_\nu) \otimes [\gamma_\nu] = \partial'' C\,,$$

so that

$$\Sigma' - \Sigma = \partial\big(\sum_\nu c_\nu \otimes [\gamma_\nu] - C\big) - V' + V + \partial' C\,.$$

Similarily, $\partial''(V' - V - \partial' C) = 0$, and therefore there exists $K \in \mathsf{K}_{0,3}$ such that $V' - V + \partial' C = \partial'' K$. Finally,

$$\Sigma' - \Sigma = \partial\big(\sum_{\nu} c_\nu \otimes [\gamma_\nu] - C - K\big)\,,$$

since, obviously, $\partial' K = 0$. □

3.3.2. *Cohomology computations.* Here we pass to the dual computations in cohomology. Let

$$\omega[f] = \frac{f_{zz}}{f_z}\mu_z \, \mathrm{d}\, z \wedge \mathrm{d}\, \bar{z}\,,$$

be the density of Polyakov's action functional in the genus zero case, where $\mu = f_{\bar{z}}/f_z$. Obviously, $\omega[f]$ can be considered as an element in $\mathsf{C}^{2,0}$, that is a two-form valued zero cochain on Γ. Then there exist elements $\theta[f] \in \mathsf{C}^{1,1}$ and $\Theta[f] \in \mathsf{C}^{0,2}$ such that

$$\delta\omega[f] = \mathrm{d}\,\theta[f] \quad \text{and} \quad \delta\theta[f] = \mathrm{d}\,\Theta[f]\,,$$

so that the f-dependent cochain $\Omega_f \overset{\text{def}}{=} \omega[f] - \theta[f] - \Theta[f]$ of total degree two is a cocycle in $\operatorname{Tot}\mathsf{C}$, that is

$$D(\omega[f] - \theta[f] - \Theta[f]) = 0\,.$$

Indeed, $\mathrm{d}\,\delta\omega[f] = \delta\,\mathrm{d}\,\omega[f] = 0$ because $\omega[f]$ is a top form on $\mathbb{H}$, and since $\mathbb{H}$ is contractible, it follows that there exists $\theta[f]$ such that $\delta\omega[f] = \mathrm{d}\,\theta[f]$. Similarly, $\mathrm{d}\,\delta\theta[f] = \delta\,\mathrm{d}\,\theta[f] = \delta\delta\omega[f] = 0$ and again, since $\mathbb{H}$ is acyclic, there exists $\Theta[f]$ such that $\delta\theta[f] = \mathrm{d}\,\Theta[f]$. Continuing along this way, we get $\mathrm{d}\,\delta\Theta[f] = 0$, so that $\delta\Theta[f]$ is a 3-cocycle on Γ with constant values. As it follows form Lemma 3.3, $H^3(\Gamma, \mathbb{C}) = \{0\}$, so that, shifting $\Theta[f]$ by a $\mathbb{C}$-valued group cochain, if necessary, one can choose the "integration constants" in the equation $\mathrm{d}\,\Theta[f] = \delta\theta[f]$ in such a way that $\delta\Theta[f] = 0$.

It is quite remarkable that explicit expressions for $\theta[f]$ and $\Theta[f]$ can be obtained by performing a straightforward calculation. Indeed, using

$$f \circ \gamma = \tilde{\gamma} \circ f \quad \text{and} \quad \mu \circ \gamma\, \frac{\overline{\gamma'}}{\gamma'} = \mu,$$

we get

$$\delta\omega_\gamma[f] = \omega[f] \circ \gamma^{-1} |(\gamma^{-1})'|^2 - \omega[f] = \mathrm{d}\,\theta_\gamma[f]. \tag{3.7}$$

A direct computation, using the property that $\{\gamma, z\} = 0$ for all fractional linear transformations, verifies that

$$\theta_{\gamma^{-1}}[f] = \log(\tilde{\gamma}' \circ f)\,\mathrm{d}\log f_z - \log(f_z \circ \gamma)\,\mathrm{d}\log\gamma' - 2\frac{\gamma''}{\gamma'}\mu\,\mathrm{d}\,\bar{z}. \tag{3.8}$$

Proceeding along the same lines one can work out an expression for $\Theta[f]$; in order to get a manageable formula, it is more convenient to write down its differential

$$\begin{aligned} \mathrm{d}\,\Theta_{\gamma_2^{-1},\gamma_1^{-1}}[f] = f^*\big(\log\big(\tilde{\gamma}_1 \circ \tilde{\gamma}_2\big)'\,\mathrm{d}\log\tilde{\gamma}_2'\big) + \log\gamma_2'\;\mathrm{d}\log\big(\gamma_1 \circ \gamma_2\big)' \\ - \frac{1}{2} f^*\big(\mathrm{d}\big(\log\tilde{\gamma}_2'\big)^2\big) - \frac{1}{2}\,\mathrm{d}\big(\log\gamma_2'\big)^2\,. \end{aligned} \tag{3.9}$$

It is easy to verify that the right hand side of this expression is indeed a closed one-form on $\mathbb{H}$ and, therefore, is exact.

Remark 3.7. One can obtain a formula for $\Theta[f]$ by integrating (3.9). The resulting expression will involve combinations of logarithms and dilogarithms, resulting from the typical integral

$$\int \log\gamma' \,\mathrm{d}\log\sigma',$$

where γ and σ are fractional linear transformations. The customary choice in defining this integral is to put branch-cuts from $-\infty$ to $\gamma^{-1}(\infty)$ and from $\sigma^{-1}(\infty)$ to ∞. When these elements belong to the Fuchsian group Γ, the branch-cuts should go along the real axis $\mathbb{R}$ which is the limit set of Γ. The same applies to the target group $\tilde{\Gamma}$ when the mapping f defines a Fuchsian deformation. If the target group $\tilde{\Gamma}$ is quasi-Fuchsian, the branch-cuts should go along the limit set of $\tilde{\Gamma}$, the simple Jordan curve that is the image of $\mathbb{R}$ under the mapping f. With this normalization, $\Theta_{\gamma_2^{-1},\gamma_1^{-1}}(f)$ is defined up to the "integration constants" $c_{\gamma_2^{-1},\gamma_1^{-1}}$ which are determined from the condition that $\delta\Theta[f]=0$.

Therefore we proved, in complete analogy with the homological computation, that the cochain $\Omega_f=\omega[f]-\theta[f]-\Theta[f]\in(\operatorname{Tot}\mathsf{C})^2$ is in fact a cocycle,

$$D\Omega_f=0\,.$$

Hence, from Lemma 3.3, we have

Proposition 3.8. *The cocycle $\Omega_f\in\ (\operatorname{Tot}\mathsf{C})^2$ represents a cohomology class in $H^2(X,\mathbb{C})\cong\mathbb{C}$, which depends on the mapping f.*

Remark 3.9. It might happen that the cohomology class $[\Omega_f]=0$ for some specific mapping(s) f.

4. Polyakov's action in higher genus

4.1. After the algebraic and topological preparations of section 3, here we finally define the Polyakov action functional and prove Theorems A, B, C. Let $X\simeq\Gamma\backslash\mathbb{H}$ be a Riemann surface of genus $g>1$ and f be a quasi-conformal mapping such that $\tilde{\Gamma}=f\circ\Gamma\circ f^{-1}$ is a Fuchsian or quasi-Fuchsian group isomorphic to Γ (see introduction and 3.2.2 for details). Using the pairing between $\mathsf{C}^{\bullet,\bullet}$ and $\mathsf{K}_{\bullet,\bullet}$, we set

$$\begin{aligned}
2iS[f] &= \langle\Omega_f,\Sigma\rangle\\
&= \langle\omega[f],F\rangle-\langle\theta[f],L\rangle+\langle\Theta[f],V\rangle\\
&= \int_F\omega[f]-\sum_{i=1}^{g}\int_{b_i}\theta_{\beta_i}[f]+\sum_{i=1}^{g}\int_{a_i}\theta_{\alpha_i}[f]\\
&\quad+\sum_{i=1}^{g}\Big(\Theta_{\alpha_i,\beta_i}[f](a_i(0))-\Theta_{\beta_i,\alpha_i}[f](b_i(0))+\Theta_{\gamma_i^{-1},\alpha_i\beta_i}[f](b_i(0))\Big)\\
&\quad-\sum_{i=1}^{g}\Theta_{\gamma_g^{-1}\cdots\gamma_{i+1}^{-1},\gamma_i^{-1}}[f](b_g(0))\,.
\end{aligned}\tag{4.1}$$

Proof of Theorem A. It follows at once from the constructions in section 3. First, the value of $S[f]$, for any given f, depends only on the classes defined by Ω_f and Σ and not on the explicit cocycles representing them. Indeed, because of the property (3.3) of the pairing $\langle\ ,\ \rangle$, shifting either Ω_f or Σ by (co)boundaries does not alter the value given in (4.1). Furthermore, by virtue of Lemma 3.6 and the above invariance, the action $S[f]$ does not depend on either the choice of the marking of Γ, or on the choice of the fundamental domain F. Finally, it follows from Propositions 3.4 and 3.8, which identify the (total) homology of the complexes $\mathsf{K}_{\bullet,\bullet}$ and $\mathsf{C}^{\bullet,\bullet}$ with that of the surface X, that the action $S[f]$ comes from the pairing

$$H^2(X,\mathbb{C}) \times H_2(X,\mathbb{Z}) \longrightarrow \mathbb{C}.$$

□

Remark 4.1. Since the action results from a pairing in homology, we write it as

$$S[f] = \frac{1}{2i}\langle [\Omega_f], [\Sigma]\rangle, \tag{4.2}$$

stressing its dependence on the (co)homology classes only.

4.2. Here we discuss the variational properties of the action functional (4.1) and prove Theorem B. As it was mentioned in the introduction, there are two versions of the variational problem for $S[f]$. In the first one, the free-end variation, we consider μ to be the independent variable, so that the target Fuchsian (or quasi-Fuchsian) group $\tilde{\Gamma}$ is determined by μ through the solution of the Beltrami equation. In the second case, the fixed-end variation, we fix the target Fuchsian (or quasi-Fuchsian) group $\tilde{\Gamma}$, together with the isomorphism $\Gamma \longrightarrow \tilde{\Gamma}$ and consider the set $\mathcal{QC}(\Gamma,\tilde{\Gamma})$ of all smooth quasi-conformal mappings f that intertwine between Γ and $\tilde{\Gamma}$.

In the first case, since the set of Beltrami coefficients for Γ is the interior of a ball of radius 1 (with respect to the $\|\ \|_\infty$ norm) in the linear space $\mathcal{B}(\Gamma)$ of all Beltrami differentials for Γ, the variation $\boldsymbol{\delta}\mu$ belongs to $\mathcal{B}(\Gamma)$.

In the second case, since the target Fuchsian (or quasi-Fuchsian) group Γ is fixed, it follows from the equivariance property (1.7) that $\boldsymbol{\delta}f/f_z$ is $(-1,0)$-tensor for Γ, that is

$$\frac{\boldsymbol{\delta}f}{f_z}\circ\gamma = \frac{\boldsymbol{\delta}f}{f_z}\gamma' \quad \text{for all } \gamma\in\Gamma.$$

One can express $\boldsymbol{\delta}f/f_z$ in terms of a vector field on X as follows. Let $\mathcal{G}_0$ be the group of all orientation preserving diffeomorphisms of $\mathbb{H}$ fixing Γ and homotopic to the identity. Any path g^t in $\mathcal{G}_0$ connected to the identity defines a path $f^t = f\circ g^t$ in $\mathcal{QC}(\Gamma,\tilde{\Gamma})$ connected to $f\in\mathcal{QC}(\Gamma,\tilde{\Gamma})$, a deformation of the mapping f. Setting

$$\boldsymbol{\delta}f = \left.\frac{d}{dt}\right|_{t=0} f^t$$

and defining $v = v^z\,\partial_z + v^{\bar z}\,\partial_{\bar z}$ as the vector field generating the flow $t\mapsto g^t$, we get

$$\frac{\boldsymbol{\delta}f}{f_z} = v^z + \mu\, v^{\bar z}\,,$$

where $\mu = f_{\bar z}/f_z$ is the Beltrami coefficient for Γ corresponding to f.

Note that in the first case the corresponding variation $\boldsymbol{\delta}f/f_z$ is not necessarily a $(-1,0)$-tensor for Γ, since the target group $\tilde{\Gamma}$ "floats" under a generic variation of μ (variation with free end). Specifically,

$$\frac{\boldsymbol{\delta}f}{f_z}\circ\gamma\,\frac{1}{\gamma'} = \frac{\boldsymbol{\delta}f}{f_z} + \frac{1}{f_z}\left(\frac{\boldsymbol{\delta}\tilde{\gamma}}{\tilde{\gamma}'}\right)\circ f\,, \tag{4.3}$$

for all $\gamma\in\Gamma$. Objects on $\mathbb{H}$ with such tranformation property are pull-backs under the map f of non-holomorphic Eichler integrals of order -1 for the group $\tilde{\Gamma}$. By definition [21], the space $\mathsf{E}^{-1}_{\tilde{\Gamma}}$ of these Eichler integrals consists of smooth functions $\mathcal{E}$ on $\mathbb{H}$ such that

$$\mathcal{E} \circ \tilde{\gamma}\, \frac{1}{\tilde{\gamma}'} = \mathcal{E} + p_{\tilde{\gamma}}\,, \tag{4.4}$$

for all $\tilde{\gamma} \in \tilde{\Gamma}$, where $p_{\tilde{\gamma}}$ is a 1-cocycle of $\tilde{\Gamma}$ with coefficients in the linear space of polynomials P of order ≤ 2 with the action

$$P \mapsto ((\tilde{\gamma}^{-1})')^2 P \circ \tilde{\gamma}^{-1}.$$

Clearly the pull-back $(\mathcal{E} \circ f)/f_z$ of the Eichler integral $\mathcal{E}$ has the trasformation property (4.3).

In both cases the variations of f and μ are related by the same equation

$$\mathcal{M}\left(\frac{\boldsymbol{\delta} f}{f_z}\right) = \boldsymbol{\delta}\mu,$$

where $\mathcal{M} = \bar{\partial} - \mu\,\partial + \mu_z$ is the differential operator introduced in section 2. It has the remarkable property of mapping $(-1,0)$-tensors for Γ, and even objects of more complicated type such as pull-backs of Eichler integrals, into $(-1,1)$-tensors for Γ. There are other differentials operators with similar properties, collected in the following

Lemma 4.2. (i) *The operators $\mathcal{T} = \partial^3 + 2T\partial + T_z$ and $\mathcal{M} = \bar{\partial} - \mu\partial + \mu_z$, where T is a quadratic differential for Γ and μ is Beltrami differential for Γ, map $(-1,0)$-tensors for Γ into quadratic and Beltrami differentials for Γ, respectively.*

(ii) *The operators $\mathcal{T}$ and $\mathcal{M}$ from part (i) map pull-backs by the mapping f of Eichler integrals of order -1 for $\tilde{\Gamma}$ into quadratic and Beltrami differentials for Γ, respectively.*

(iii) *If f is mapping of $\mathbb{H}$ intertwining Γ and $\tilde{\Gamma}$, then $T = \{f, z\}$ is a quadratic differential for Γ.*

Proof. Part (i) is well-known (see, e.g. [17]) and the statements can be easily verified. In particular, setting $T = 0$ we get that μ_{zzz} is a $(2,1)$-tensor for Γ, which is also a known result (see, e.g. [21]).

In order to prove part (ii), note that for a holomorphic function p on $\mathbb{H}$ we have

$$\mathcal{T}\left(\frac{p \circ f}{f_z}\right) = f_z^2\,(\partial^3 p) \circ f\,,$$

which shows that the additional terms in the transformation law (4.3) belong to the kernel of $\mathcal{T}$. Similarly, (2.11) shows that these terms belong to the kernel $\mathcal{M}$ as well.

Part (iii) is another classical result, which can be easily verified as well. □

4.2.1. *Proof of Theorem B.* For concreteness, we first consider variations with respect to μ, though, as we shall see, the actual argument works for both kinds of variations.

The proof requires climbing the "ladder" in the double complex $\mathsf{C}^{\bullet,\bullet}$, together with the computation of the variation of $\omega[f]$. Since $\omega[f]$ is a local functional of f, we can just use the computation already done in genus zero so that, according to formula (2.5),

$$\boldsymbol{\delta}\omega = a - \mathrm{d}\,\eta\,, \tag{4.5}$$

where $a = -2\,T\,\boldsymbol{\delta}\mu\,\mathrm{d}\,z\wedge\mathrm{d}\,\bar{z}$ and the explicit expression for the 1-form η is not needed. (In order to simplify notations, we temporarily drop the dependence on f from the notation.) As it follows from Lemma 4.2, the 2-form a on $\mathbb{H}$ is a $(1,1)$-tensor for Γ, therefore it is closed with respect to the total differential, i.e. $Da = 0$.

Next observe that $D\boldsymbol{\delta}\Omega = \boldsymbol{\delta}D\Omega = 0$, therefore $D(\boldsymbol{\delta}\Omega - a) = 0$. We want to show that $\boldsymbol{\delta}\Omega - a$ is in fact D-exact up to a term whose contribution vanishes after pairing with Σ.

To this end, let us write

$$\boldsymbol{\delta}\Theta = \delta\chi\,,$$

where χ has degree $(0,1)$ in the total complex. This is possible, since, as it is shown in the appendix, the higher cohomology of Γ with coefficients in the de Rham complex vanishes. The equation $D\boldsymbol{\delta}\Omega = 0$ gives us the two relations

$$\mathrm{d}\,\boldsymbol{\delta}\Theta = \delta\boldsymbol{\delta}\theta\,, \quad \mathrm{d}\,\boldsymbol{\delta}\theta = \delta\boldsymbol{\delta}\omega\,, \tag{4.6}$$

of which the first one implies that

$$\boldsymbol{\delta}\theta = \mathrm{d}\,\chi + \delta\lambda\,,$$

where, again, the vanishing of $H^q(\Gamma, \mathsf{A}^p)$ for $q > 0$ has been used. Plugging this relation into the second one in (4.6), yields

$$\delta\boldsymbol{\delta}\omega = \delta\,\mathrm{d}\,\lambda\,.$$

Notice that this time we can at most conclude that $\boldsymbol{\delta}\omega - \mathrm{d}\,\lambda$ is a Γ-invariant form, since $H^0(\Gamma, \mathsf{A}^p)$ precisely gives the invariant p-forms (cf. the appendix). We write this invariant form as $a + b$, for some $(2,0)$ invariant element b, so that

$$\boldsymbol{\delta}\omega = \mathrm{d}\,\lambda + a + b$$

and, using (4.5),

$$b = -\,\mathrm{d}(\eta + \lambda)\,,$$

i.e. b is Γ-invariant and exact. Putting all together, we obtain

$$\begin{aligned}
\boldsymbol{\delta}\Omega &= \boldsymbol{\delta}\omega - \boldsymbol{\delta}\theta - \boldsymbol{\delta}\Theta \\
&= a - \mathrm{d}\,\eta - \mathrm{d}\,\chi - \delta\lambda - \delta\chi \\
&= a + b + D(\lambda - \chi)\,,
\end{aligned}$$

which, after evaluation against Σ, reduces to

$$\langle\boldsymbol{\delta}\Omega\,,\,\Sigma\rangle = \int_F a\,,$$

as wanted (the integral of b over F is obviously zero).

In order to complete the proof, notice that the variation of $\omega[f]$ always has the form (4.5), independently of whether either variable μ or f is varied. In the latter case, the variation $\boldsymbol{\delta}f/f_z$ is a $(-1,0)$-tensor for Γ, so that we can use (4.5) and the relation $\boldsymbol{\delta}\mu = \mathcal{M}(\boldsymbol{\delta}f/f_z)$ together with Lemma 2.3. $\square$

Remark 4.3. Note that the argument presented in the proof of Theorem B is quite general. It applies to any functional defined by an evaluation of a cocycle in $\mathrm{Tot}\,\mathsf{C}^2$ over a cycle Σ, provided that the cocycle is the extension of a 2-form on $\mathbb{H}$ with the property that its variation is a sum of D and d-exact terms.

4.2.2. As it was mentioned in the introduction, it follows from Theorem B that $c\,S[f]/24\pi$, considered as a functional of $\mu = f_{\bar z}/f_z$, solves equation (1.2), no matter what kind of deformation we are considering, be it Fuchsian or quasi-Fuchsian. Thus there are at least two possible solutions of (1.2) on a Riemann surface of genus higher than one. In order to clearly distinguish the two cases, let us adopt for a moment the customary notation in the theory of quasi-conformal mappings [1], so that f^μ and Γ^μ (respectively f_μ and Γ_μ) stand for the Fuchsian (respectively, quasi-Fuchsian) deformation of Γ.

There is a simple relationship between the variations of $S[f_\mu]$ and $S[f^\mu]$. First of all, observe that the mapping $g := f_\mu \circ (f^\mu)^{-1} : \mathbb{H} \to f_\mu(\mathbb{H})$ is conformal (note that $f^\mu(\mathbb{H}) = \mathbb{H}$). Indeed, it follows from the Beltrami equation that

$$\frac{\partial g}{\partial \bar\zeta} = \frac{\partial f^\mu}{\partial z}\left(\frac{\partial (f^\mu)^{-1}}{\partial \bar\zeta} + \mu\,\frac{\partial \overline{(f^\mu)^{-1}}}{\partial \bar\zeta}\right) = 0\,,$$

where $\zeta = f^\mu(z,\bar z)$ is the new complex coordinate on $\mathbb{H}$. Moreover, the map g intertwines Γ^μ and Γ_μ, thus it descends to a biholomorphic map

$$g : X^\mu = \Gamma^\mu\backslash\mathbb{H} \longrightarrow \Gamma_\mu\backslash f_\mu(\mathbb{H}) = X_\mu$$

showing that the Riemann surfaces X^μ and X_μ are conformally equivalent. Furthermore, we have

$$T_\mu(z) = \{f_\mu, z\} = \{g,\zeta\} \circ f^\mu\,(f^\mu_z)^2 + T^\mu(z)\,,$$

where $T^\mu(z) = \{f^\mu, z\}$. Thus the difference

$$Q = \frac{\boldsymbol{\delta} S[f_\mu]}{\boldsymbol{\delta}\mu} - \frac{\boldsymbol{\delta} S[f^\mu]}{\boldsymbol{\delta}\mu}$$

is just the pull-back under f^μ of the holomorphic quadratic differential obtained by taking the Schwarzian derivative of g with respect to the new complex coordinate ζ. Of course, the situation is completely symmetric under the exchange of f_μ and f^μ.

One can reach the same conclusion proceeding along a different line (cf. [32]). Namely, since both $S[f^\mu]$ and $S[f_\mu]$ satisfy (1.2), Q satisfies the equation

$$(\bar\partial - \mu\,\partial - 2\,\mu_z)Q = 0$$

which, using the Cauchy-Riemann operator

$$\frac{\partial}{\partial\bar\zeta} = \frac{\partial\bar z}{\partial\bar\zeta}\left(\frac{\partial}{\partial\bar z} - \mu\,\frac{\partial}{\partial z}\right)$$

can be written as

$$\partial_{\bar\zeta}\left(\frac{Q}{f_z^2}\right) = 0\,,$$

showing that Q is indeed the pull-back of a holomorphic quadratic differential with respect to the complex coordinate ζ.

Remark 4.4. The above argument actually shows that homogeneous solutions to the equation (1.2) on X are pull-backs under the mapping f^μ (or f_μ) of the holomorphic quadratic differentials on the "target" Riemann surface X^μ. According to the Riemann-Roch theorem, this space is $3g-3$-dimensional; therefore, the universal CWI (1.2) does not completely determine the generating functional for the stress-energy tensor in the higher genus case. As we mentioned in the introduction, additional information should be provided by the particular CFT.

4.2.3. According to Theorem B, the variation of the action with respect to the map f yields the classical equation of motion

$$\mu_{zzz} = 0\,. \tag{4.7}$$

Here we compute the dimension of the space of solutions of (4.7). It was observed in the introduction that determining the critical set of $S[f]$ in $\mathcal{QC}(\Gamma, \tilde{\Gamma})$ out of (4.7) seems to be a very difficult problem. However, the space of solutions to (4.7) is quite interesting since, as we show below, it contains the subspace of harmonic Beltrami differentials.

First, recall the definition of the so-called Maass operators (see, e.g. [13]). For $k, l \in \mathbb{Z}$, denote by $\mathsf{A}_\Gamma^{k,l} \equiv A_{\mathbb{C}}^{k,l}(\mathbb{H})^\Gamma \cong A_{\mathbb{C}}^{k,l}(X)$ the space of Γ-invariant (k,l)-forms on $\mathbb{H}$; by convention, $(\mathrm{d}z)^k$, for k negative, means $(\partial/\partial z)^{-k}$. Define

$$D_{k,l} : \mathsf{A}^{k,l} \longrightarrow \mathsf{A}^{k+1,l}$$

by

$$D_{k,l} = y^{-2k} \circ \partial \circ y^{2k}\,,$$

where $\partial = \partial/\partial z$. It is easy to verify that

$$\partial_z^3 = D_{1,1} \circ D_{0,1} \circ D_{-1,1}\,, \tag{4.8}$$

which once again shows that the operator ∂_z^3 maps Beltrami differentials into the $(2,1)$-tensors for Γ. Furthermore, a Beltrami differential $\nu \in \mathsf{A}_\Gamma^{-1,1}$ is called Bers harmonic if it is harmonic with respect to the ∂-Laplacian of the Poincaré metric on $\Gamma\backslash\mathbb{H}$, acting on $(-1,1)$-forms. It can be shown that

$$\nu = y^2\,\bar{q}\,,$$

where $q \in \mathsf{A}_\Gamma^{2,0}$ is a holomorphic quadratic differential. It follows from the Riemann-Roch theorem that Bers harmonic Beltrami differentials form a $(3g-3)$-dimensional complex vector space and play an important role in the Teichmüller theory [1, 16].

Proposition 4.5. *The space of solutions of equation* (4.7) *has complex dimension* $4g-3$*:*

$$\dim_{\mathbb{C}} \operatorname{Ker}_{\mathsf{A}_\Gamma^{-1,1}}(\partial_z^3) = 4g-3,$$

and contains the $3g-3$ *dimensional vector space of Bers harmonic Beltrami differentials.*

Proof. Using (4.8), we start by observing that the kernel of $D_{-1,1}$ coincides with the space of harmonic Beltrami differentials. Indeed, $\nu \in \operatorname{Ker}(D_{-1,1})$ if and only if $\partial(y^{-2}\nu) = 0$, which implies $\nu = y^2\,\bar{q}$, for q a holomorphic quadratic differential, since $y^{-2}\nu$ is a $(0,2)$-form.

Furthermore, $\operatorname{Ker}(D_{1,1}) \cap \operatorname{Im}(D_{0,1}) = \{0\}$. Indeed, an element in $\operatorname{Ker}(D_{1,1})$ is necessarily a multiple of the $(1,1)$-form y^{-2}. If it is non zero, then it cannot belong to $\operatorname{Im}(D_{0,1}) = \operatorname{Im}\partial$, since y^{-2} represents a cohomology class in $\Gamma\backslash\mathbb{H}$.

Next, it is clear that $\operatorname{Ker}(D_{0,1})$ is complex anti-isomorphic to the linear space of Abelian differentials for X. Finally, the map $D_{-1,1}$ is onto: its image is the entire space of $(0,1)$-differentials. Namely, the operator adjoint to $D_{-1,1}$ with respect to the Hermitian scalar product on $\mathsf{A}_\Gamma^{k,l}$ induced by the Poincaré metric y^{-2} is $D_{-1,1}^* = -\bar{\partial} \circ y^2$, which has zero kernel since $g > 1$. Thus any element in $\operatorname{Ker}(D_{0,1})$ is the $D_{-1,1}$-image of an element in $\mathsf{A}_\Gamma^{-1,1}$, orthogonal to the subspace of harmonic Beltrami differentials, and it also belongs to the kernel of ∂_z^3. Counting $4g-3 = 3g-3+g$ proves the claim. □

Remark 4.6. As in the genus zero case, the equation of motion (4.7) is equivalent to the holomorphicity property of $T = \{f, z\}$ with respect to the new complex structure induced by f. Namely, when μ satisfies (4.7), the corresponding (1.2) becomes homogeneous so that, according to 4.2.2, we have

$$\partial_{\bar\zeta}\left(\frac{T}{(\partial_z\zeta)^2}\right) = 0 \tag{4.9}$$

for the stress-energy tensor in the new coordinates $\zeta, \bar\zeta$. This condition is well defined on the surface X as well as on the deformed Riemann surface $\tilde\Gamma\backslash f(\mathbb{H})$.

4.2.4. Here we briefly comment on the computation of the second variation. It follows from Lemma 4.2 that the differential operators used in the genus zero computation are tensorial; therefore, using Theorem B and the fact that the problem is local, we can just repeat the computations in 2.3 in order to get the

Proposition 4.7. *The Hessian of the Polyakov action* (4.1) *is given by the genus zero formula*

$$\boldsymbol{\delta}^2\, S[f](\boldsymbol{\delta}_1 f, \boldsymbol{\delta}_2 f) = -2\int_F \frac{\boldsymbol{\delta}_1 f}{f_z}(\partial^3\circ\mathcal{M})\left(\frac{\boldsymbol{\delta}_2 f}{f_z}\right)\,\mathrm{d}^2 z\,.$$

4.3. We now analyze how $S[f]$ relates to the functional $W[\mu]$ defined by (1.10), and prove Theorem C.

For $t \in [0,1]$, let μ^t be a homotopy in the space of Beltrami differentials connecting 0 to μ, and let f^t the solution of the Beltrami equation corresponding to μ^t. For the sake of convenience, let us rewrite (1.10) here:

$$W[\mu] = \frac{c}{12\pi}\int_0^1\left(\int_F T^t\,\dot\mu(t)\,\mathrm{d}^2 z\right)\mathrm{d}\,t\,. \tag{4.10}$$

The integration in (4.10) is extended to F, but, according to Lemma 4.2, the integrand is a $(1,1)$-tensor for Γ, hence the integral descends to X.

Proof of Theorem C. We want to proceed in a fashion similar to the proof of Theorem B.

Our construction of $S[f]$ applied to f^t produces ω^t, Ω^t and $S[f^t]$ for any $t \in [0,1]$. We can make use of formula (2.5) applied to $\boldsymbol{\delta} = d/dt$:

$$\dot\omega^t = -2\,T^t\dot\mu^t\;\mathrm{d}\,z\wedge\mathrm{d}\,\bar z - \mathrm{d}\,\eta(f^t;\dot f^t) \equiv a^t - \mathrm{d}\,\eta^t\,,$$

where, as before, $Da^t = 0$. On the other hand, $D\dot\Omega^t = 0$, since $D\Omega^t = 0$ for any t, and therefore the same arguments as in the proof of Theorem B lead us to conclude that

$$\langle\dot\Omega^t, \Sigma\rangle = \int_F a^t\,.$$

Integrating in t from 0 to 1 we get that $W[\mu] = (c/24\pi)S[f]$, which together with Theorem C proves part (i)..

First statement of the part (ii) follows from the is well-known [1] that the quasi-Fuchsian deformation $f = f_\mu$ depends holomorphically on μ. Finally, if $f = f^\mu$

is a Fuchsian deformation with harmonic Beltrami differential $\mu = y^2\bar{q}$, then the Ahlfors lemma (see, e.g., [33]) states

$$\left.\frac{\partial f^{\epsilon\mu}}{\partial\bar{\epsilon}}\right|_{\epsilon=0} = -\frac{1}{2}q.$$

Therefore, choosing a linear homotopy $\mu(t) = t\mu$, we have the following simple computation

$$\begin{aligned}\left.\frac{\partial^2 W[\epsilon\mu]}{\partial\epsilon\partial\bar{\epsilon}}\right|_{\epsilon=0} &= \frac{c}{12\pi}\int_0^1\int_F \left.\frac{\partial f^{\epsilon t\mu}}{\partial\bar{\epsilon}}\right|_{\epsilon=0}\mu\,\mathrm{d}^2 z\,\mathrm{d}\,t \\ &- \frac{c}{24\pi}\int_0^1 t\,\mathrm{d}\,t\int_F q\mu\,\mathrm{d}^2 z \\ &= -\frac{c}{48\pi}\int_F |\mu|^2 y^{-2}\,\mathrm{d}^2 z.\end{aligned}$$

□

Remark 4.8. Theorem C specifies the μ-dependence for two natural solutions for $W[\mu]$, defined by quasi-Fuchsian and Fuchsian deformations. In the former case the corresponding functional is holomorphic in μ, as a generating functional should be, while in the latter case it is not. Introducing the Weil-Petersson inner product in the space of Bers harmonic Beltrami differentials by

$$(\mu_1, \mu_2)_{\mathrm{WP}} = \int_F \mu_1\bar{\mu}_2\, y^{-2}\,\mathrm{d}^2 z,$$

the latter statement takes a quantative form

$$\left.\frac{\partial^2 W[\epsilon\mu]}{\partial\epsilon\partial\bar{\epsilon}}\right|_{\epsilon=0} = -\frac{c}{48\pi}||\mu||^2_{\mathrm{WP}},$$

that once again characterizes the Weil-Petersson metric as a "holomorphic anomaly". Finally, for arbitary Beltrami differential one should replace μ by $P\mu$ in the above formula, where P stands for the orthogonal projection (with respect to the Weil-Petersson metric) onto the space of harmonic Beltrami differentials.

4.4. Here we compute the Hessian of the action functional W as a functional of μ. For this end we need to extend the linear mapping $\mathcal{M} : \mathsf{A}_\Gamma^{-1,0} \to \mathsf{A}_\Gamma^{-1,1}$ to the space of pull-backs by the mapping f of Eichler integrals of order -1 for $\tilde{\Gamma}$. This mapping has no kernel on the subspace of normalized Eichler integrals (i.e. vanishing at $0, 1, \infty$) and, according to Bers, it is onto (see [21]). We denote, slightly abusing the notations, the inverse of thus extended mapping $\mathcal{M}$ by $\mathcal{M}^{-1}$.

Proposition 4.9. *The second variation of the functional $W[\mu]$ is given by*

$$\boldsymbol{\delta}^2\, W[\mu](\boldsymbol{\delta}_1\mu, \boldsymbol{\delta}_2\mu) = \frac{c}{12\pi}\int_F \boldsymbol{\delta}_1\mu\left(\mathcal{T}\circ\mathcal{M}^{-1}\right)(\boldsymbol{\delta}_2\mu)\,\mathrm{d}^2 z\,,$$

where, according to Lemma 4.2, the operator $\mathcal{T}\circ\mathcal{M}^{-1}$ maps Beltrami differentials for Γ into quadratic differentials. The Hessian of $W[\mu]$ at the point μ is given by the operator $\partial^3\circ\mathcal{M}^{-1}$.

Proof. It is the same as the genus zero computations using Lemma 4.2. Note that at the critical point $T(z) = 0$, so that $\mathcal{T} = \partial^3$. □

5. Fiber spaces over Teichmüller space. Discussion and conclusions

In the preceding sections we have defined the Polyakov's action for the chiral sector in the induced gravity on a Riemann surface X of genus $g>1$ and explored some of its properties. We have also pointed out the possible interpretation of $W[\mu]=(c/24\pi)\,S[f]$ as the universal part of the generating functional for the correlation functions of the stress-energy tensor for a CFT on X.

However, the most compelling interest in $W[\mu]$ (or $S[f]$) stems in its relation with the geometry of the various fiber spaces over Teichmüller space. We want to elaborate more on this point.

5.1. Recall that the Teichmüller space $\mathcal{T}(X)$ of the Riemann surface X of genus $g>1$ is naturally realized as the quotient of the open unit ball $\mathcal{B}(X)$ (with respect to the L^∞ norm) in the Banach space of Beltrami differentials on $X=\Gamma\backslash\mathbb{H}$ by the group of quasi-conformal self-mappings of $\mathbb{H}$ pointwise fixing the group Γ. If one replaces $\mathcal{B}(X)$ by its subset $\mathcal{P}(X)$ consisting of smooth Beltrami differentials and considers the identity component $\mathcal{G}_0(X)$ of the group $\mathcal{G}(X)$ of orientation preserving diffeomorphisms of X (elements in $\mathcal{G}_0(X)$ point-wise fix Γ while acting on $\mathbb{H}$), then one gets Earle and Eells [11] fiber space $\pi:\mathcal{P}(X)\to\mathcal{T}(X)$ over the Teichmüller space. It is a smooth (in the Frechét topology) principal $\mathcal{G}_0(X)$-bundle over $\mathcal{T}(X)$. The group action on $\mathcal{P}(X)$ can be written as $\mu=\mu(f)\mapsto\mu^g=\mu(f\circ g)$, for $g\in\mathcal{G}_0(X)$ [11], where $f=f^\mu$ is a Fuchsian deformation associated with μ. Explicitly, the above action is [1]:

$$\mu^g=\frac{\overline{g_z}}{g_z}\left(\frac{\mu-\mu(g^{-1})}{1-\mu\,\overline{\mu(g^{-1})}}\right)\circ g\,.$$

Consider now the tangent bundle exact sequence

$$0\longrightarrow T_V\mathcal{P}(X)\xrightarrow{\ i\ }T\mathcal{P}(X)\xrightarrow{\ \mathrm{d}\,\pi\ }\pi^*(T\mathcal{T}(X))\longrightarrow 0$$

determined by the Earle-Eells fibration. (Observe that since $\mathcal{P}(X)$ is a ball in the vector space $\mathsf{A}_\Gamma^{-1,1}$ of all smooth Beltrami differentials, the tangent space to it at any given point μ is canonically identified with $\mathsf{A}_\Gamma^{-1,1}$.) According to the description of the fixed-end variation given in 4.2, the deformation $f^t=f\circ g^t$, for $t\mapsto g^t\in\mathcal{G}_0(X)$, results in a vertical curve $t\mapsto\mu^t$ above the point $\pi(\mu)\in\mathcal{T}(X)$. Thus the corresponding variation $\boldsymbol{\delta}\mu=\dot\mu$ lies in the vertical tangent space $T_V\mathcal{P}(X)$ at point μ, which is isomorphic to $\mathrm{Im}(\mathcal{M})$, where $\mathcal{M}=\bar\partial-\mu\partial+\mu_z:\mathsf{A}_\Gamma^{-1,0}\to\mathsf{A}_\Gamma^{-1,1}$. Next, the tangent space $T_\mu\mathcal{P}(X)$ can also be identified with the space of smooth $\tilde\Gamma$-Beltrami differentials; an easy computation proves the following (well-known) lemma.

Lemma 5.1. *For any $\nu\in\mathsf{A}_\Gamma^{-1,1}$ the correspondence*

$$\nu\mapsto\left(\frac{f_z}{\overline{f_z}}\frac{\nu}{1-|\mu|^2}\right)\circ f^{-1}$$

maps $\mathsf{A}_\Gamma^{-1,1}$ isomorphically onto $\mathsf{A}_{\tilde\Gamma}^{-1,1}$. Under this map $\mathcal{M}$ becomes $\bar\partial_{\pi(\mu)}$, the $\bar\partial$-operator relative to the new complex structure on the Riemann surface X defined by μ.

This implies at once that the kernel of $\mathcal{M}$ is trivial, and therefore the correspondence

$$v = v^z \partial_z + v^{\bar{z}} \partial_{\bar{z}} \mapsto \mathcal{M}(v^z + \mu v^{\bar{z}})$$

explicitly gives the injection in the tangent bundle sequence above. Furthermore, it realizes $T_V \mathcal{P}(X)$ (and its quotient by $\mathcal{G}_0(X)$) as a bundle of Lie algebras, as usual in a principal fibration [4]. Here the Lie algebra in question is the Lie algebra $\mathrm{Vect}(X)$ of smooth vector fields on X, which can be identified—as a real vector space—with $\mathsf{A}_\Gamma^{-1,0}$.

With these definitions at hand, the following reinterpretation of the formulas in the statement of Theorem B becomes obvious.

Proposition 5.2. *For any smooth functional $\mathcal{F} : \mathcal{P}(X) \to \mathbb{C}$,*

1. *the open-end variation $\boldsymbol{\delta}\mathcal{F}$ computes its total differential on $\mathcal{P}(X)$;*
2. *the fixed-end variation computes its vertical differential.*

In particular, for the action functional W,

$$\mathrm{d}\, W|_\mu = \frac{c}{12\pi} T \in T_\mu^* \mathcal{P}(X).$$

Remark 5.3. The second point in the proposition can be verified by the following explicit computation, that uses Theorems B, C and Lemma 2.3.

$$\begin{aligned} \frac{\boldsymbol{\delta} W}{\boldsymbol{\delta} f(z)} &= -\frac{c}{12\pi} \int_F \mu_{zzz} \frac{\boldsymbol{\delta} f}{f_z} \,\mathrm{d}^2 z = -\frac{c}{12\pi} \int_F \mathcal{D}T(z) \frac{\boldsymbol{\delta} f}{f_z} \,\mathrm{d}^2 z \\ &= \frac{c}{12\pi} \int_F T(z) \mathcal{M}\left(\frac{\boldsymbol{\delta} f}{f_z}\right) \mathrm{d}^2 z. \end{aligned}$$

Remark 5.4. The description of the vertical bundle as the image of $\mathcal{M}$ immediately implies that

$$T_{\pi(\mu)} \mathcal{T}(X) \cong \mathsf{A}_\Gamma^{-1,1} / \operatorname{Im}(\mathcal{M}) \,,$$

so that we get the well-know result [11]

$$T_{\pi(\mu)} \mathcal{T}(X) \cong H_{\bar{\partial}}^{0,1}(X^\mu, T_{X^\mu}) \cong H^1(X^\mu, \Theta_{X^\mu}) \,,$$

where the last group gives the Kodaira-Spencer infinitesimal deformations. (Θ_{X^μ} is the holomorphic tangent sheaf to the Riemann surface X^μ.)

5.2. It is fundamental to investigate how the function $W : \mathcal{P}(X) \to \mathbb{C}$ relates to the geometry of the bundle $\pi : \mathcal{P}(X) \to \mathcal{T}(X)$. A long but straightforward computation using the definition (1.10) of W proves

Lemma 5.5. *There exists $A : \mathcal{P}(X) \times \mathcal{G}_0(X) \to \mathbb{C}$ such that*

$$W[\mu^g] = W[\mu] + A[\mu, g] \,. \tag{5.1}$$

The functional A depends only on the point (μ, g) and is local in μ and μ^g; in particular, it is independent of any possible choice of the solution of the Beltrami equation involved in the definition of W.

It trivially follows from (5.1) that the functional A satisfies the cocycle identity:

$$A[\mu, gh] = A[\mu^g, h] + A[\mu, g] \,.$$

Next, according to [30], the functional $\Psi[\mu] = \exp(-W[\mu])$ is to be interpreted as a conformal block for a CFT defined on X. Thus it is more convenient to work with the exponential version of (5.1). Namely, defining

$$C[\mu, g] = \exp(-A[\mu, g])\,,$$

we get

$$\Psi[\mu^g] = C[\mu, g]\,\Psi[\mu]\,. \tag{5.2}$$

The cocycle condition takes the form

$$C[\mu, gh] = C[\mu^g, h]\,C[\mu, g]\,,$$

which defines a 1-cocycle on $\mathcal{G}_0(X)$ with values in the group of non vanishing complex valued functions on $\mathcal{P}(X)$. We denote by $[C]$ the class of C in the cohomology group $H^1(\mathcal{G}_0(X), \mathbb{C}^*(\mathcal{P}(X)))$.

Proposition 5.6. *There is an injective map of the group $H^1(\mathcal{G}_0(X), \mathbb{C}^*(\mathcal{P}(X)))$ into the group of isomorphism classes of line bundles over $\mathcal{T}(X)$. The line bundle $L_{[C]}$ over $\mathcal{T}(X)$, defined by $[C]$ is, in particular, holomorphic.*

Proof. The existence of a map

$$0 \to H^1(\mathcal{G}_0(X), \mathbb{C}^*(\mathcal{P}(X))) \to H^2(\mathcal{T}(X), \mathbb{Z})$$

is an application of the well-known concept of G-vector bundle as presented in [5, 28]. We define an action by $\mathcal{G}_0(X)$ on the trivial line bundle $\tilde{L} = \mathcal{P}(X) \times \mathbb{C}$ by

$$(\mu, z) \mapsto (\mu^g, C[\mu, g]z)\,. \tag{5.3}$$

The action is free since it is so on the first factor, hence $L = \tilde{L}/\mathcal{G}_0(X)$ is a line bundle over $\mathcal{T}(X)$. As it is easily checked, cohomologous cocycles yield isomorphic bundles, and so $L_{[C]}$ is trivial if and only if $[C]$ is trivial.

Next, observe that $C[\mu, g]$ can be defined using the quasi-Fuchsian prescription, which, according to Theorem C, yields a holomorphic W. Moreover, μ^g is holomorphic in μ, as it follows from the explicit expression. Thus, $C[\,\cdot\,, g]$ is holomorphic and so is the action (5.3). ☐

Remark 5.7. The construction of the line bundle L is well known from works on anomalies [3, 10, 12]. An explicit construction of the map $H^1(\mathcal{G}_0(X), \mathbb{C}^*(\mathcal{P}(X))) \to H^2(\mathcal{T}(X), \mathbb{Z})$ using Čech cohomology appears in [12].

It follows from general arguments (cf. [28]) that sections of $L_{[C]}$ can be identified with the $\mathcal{G}_0(X)$-invariant sections of $\tilde{L}$, namely with those functions $\Phi : \mathcal{P}(X) \to \mathbb{C}$ satisfying

$$\Phi[\mu^g] = C[\mu, g]\,\Phi[\mu]\,.$$

Since the conformal block $\Psi = \exp(-W)$ does not vanish, the foregoing proves the following

Proposition 5.8. *The conformal block Ψ descends to a non-vanishing section of $L_{[C]}$, thereby providing a trivializing isomorphism $L_{[C]} \to \mathcal{T}(X) \times \mathbb{C}$.*

Observe (cf. [35]) that the line bundle $L_{[C]}$ is holomorphically trivial due to a general property of the Teichmüller space being a contractible domain of holomorphy [25]. Our construction provides an instance of this general fact, as well as an explicit trivializing map. Also note that, due to the universal nature of the cocycle

C, the ratio of two different conformal blocks, in accordance with [30], is $\mathcal{G}_0(X)$-invariant and, therefore, descends to a non-vanishing function on the Teichmüller space $\mathcal{T}(X)$.

5.3. The preceding observations bring in several additional questions concerning the geometrical significance of $\exp(-W[\mu])$. For instance, we can define the trivial connection on the trivial line bundle $\tilde{L}$ on $\mathcal{P}(X)$:

$$\begin{aligned}\nabla\Phi &= \Psi\,\mathrm{d}(\Psi^{-1}\Phi)\\ &= \mathrm{d}\,\Phi - (\Psi^{-1}\,\mathrm{d}\,\Psi)\Phi\,.\end{aligned}$$

This connection is easily verified to be $\mathcal{G}_0(X)$-invariant, hence it descends onto $L_{[C]}$. It follows from Proposition 5.2 and Theorem B that the connection form coincides with $\mathrm{d}\,W = c\,T/12\pi$.

This is very reminiscent of Friedan and Shenker's modular geometry program for CFT [14], where the vacuum expectation value of the stress-energy tensor is interpreted as a connection on a line bundle over the moduli space. As a further development, this suggests studying the action of the full group $\mathcal{G}(X)$ on the presented construction. As it is well known [11], the quotient of $\mathcal{P}(X)/\mathcal{G}(X)$ (the action being the same as in the previous case) is precisely the moduli space of compact Riemann surfaces of genus $g > 1$. All the local formulas will stay the same, while the action of the modular group $\mathcal{G}(X)/\mathcal{G}_0(X)$ on $\mathcal{T}(X)$ will introduce the topological "twisting". All of this should be fundamental for the differential-geometrical realization of Friedan and Shenker's program. In this respect it is important, as we proved in the paper, that the functional $W[\mu]$ is independent of the marking of a Riemann surface X.

Another direction, more directly related to the Earle-Eells fibration consists in finding the geometric interpretation of the critical points $T = 0$ and "vertical critical" points $\mu_{zzz} = 0$ of the functional $W[\mu]$.

Finally, the question of the relation of $W[\mu]$ with the full induced gravity action on X is also very important. Recall the genus zero factorization [30]

$$\int R\Delta^{-1}R = W[\mu] + \overline{W}[\bar{\mu}] + K[\phi,\mu,\bar{\mu}]\,,$$

where the term $K[\phi,\mu,\bar{\mu}]$ is further decomposed as a sum

$$K[\phi,\mu,\bar{\mu}] = S_L[\phi,\mu,\bar{\mu}] + K_{\mathrm{BK}}[\mu,\bar{\mu}]$$

of the Belavin-Knizhnik-like anomaly term plus the Liouville action in the background $|\,\mathrm{d}\,z + \mu\,\mathrm{d}\,\bar{z}|^2$. After having properly defined $W[\mu]$ on X, it is natural to ask whether such a decomposition holds in higher genus as well. We observe that the general (co)homological techniques applied in this paper can also be used to give a mathematically rigorous construction of the Liouville action (in various backgrounds) in the form of a "bulk" term plus boundary and vertex corrections, as in the spirit of [29, 33]. A construction of this kind should provide a meaning also to the full action $\int R\Delta^{-1}R$ in terms of a Liouville action in the "target" complex structure, provided one can actually define K_{BK} in higher genus as well. A full understanding of the geometrical properties of $W[\mu]$ and K_{BK} and their exponentials would be relevant in order to put the Geometric Quantization approach of ref. [30] and, more generally, the three-dimensional approach to two-dimensional gravity on a more conventional mathematical basis. Finally, similar construction can be carried out for defining the WZW functional on the higher genus Riemann surfaces. We are planning to address these questions in the next publications.

Appendix A. Some facts from Homological Algebra

We give a brief account on the use of double complexes as applied to our situation. We shall mainly focus on homology and just indicate the required modifications to discuss the cohomological counterpart of the various statements. For a full account cf. any book on homological algebra, like, for instance, [23].

A.1. The framework we put ourselves in is sufficiently simple that one can in fact avoid the use of spectral sequences altogether in the proof of Lemmas 3.2 and 3.3, provided one takes into account a few simple facts from homological algebra. The key point is that the various double complexes we are interested in have trivial (co)homology in higher degrees with respect to either the first or second differentials, so the arguments can be given in general, without referring to specific examples.

Let $\mathsf{K}_{\bullet,\bullet}$ a double complex with differentials $\partial' : \mathsf{K}_{p,q} \to \mathsf{K}_{p-1,q}$ and $\partial'' : \mathsf{K}_{p,q} \to \mathsf{K}_{p,q-1}$, and total differential $\partial\big|_{\mathsf{K}_{p,q}} = \partial' + (-1)^p \partial''$. According to our discussion, let us make the assumption that

$$H_q^{\partial''}(\mathsf{K}_{p,\bullet}) = \begin{cases} C_p & q = 0 \\ 0 & q > 0\,. \end{cases}$$

Then $\mathsf{C}_\bullet \overset{\text{def}}{=} \oplus C_p$ inherits a differential[1] $\partial : C_p \to C_{p-1}$ from the first differential ∂' in the double complex, and since

$$\cdots \xleftarrow{\partial''} \mathsf{K}_{p,q-1} \xleftarrow{\partial''} \mathsf{K}_{p,q} \xleftarrow{\partial''} \mathsf{K}_{p,q+1} \xleftarrow{\partial''} \cdots$$

is exact except in degree zero, we can "augment" $\mathsf{K}_{\bullet,\bullet}$ inserting the projection $\varepsilon : \mathsf{K}_{p,0} \to C_p$ to obtain the exact sequence

$$0 \longleftarrow \mathsf{C}_\bullet \longleftarrow \mathsf{K}_{\bullet,\bullet}\,.$$

Proposition A.1.

$$H_\bullet(\operatorname{Tot}\mathsf{K}) \cong H_\bullet(\mathsf{C})\,.$$

Proof. This is a routine check of the definitions. Suppose $c \in C_p$ is closed, i.e. $\partial c = 0$. This means that it exists a chain $c_0 \in \mathsf{K}_{p,0}$ such that $\varepsilon(\partial' c_0) = 0$, but $\varepsilon(\partial' c_0)$ is the class represented by $\partial' c_0$, since we clearly have $\partial''\partial' c_0 = 0$. So, this class is zero, and therefore we have

$$\partial' c_0 = \partial'' c_1 \quad \text{for } c_1 \in \mathsf{K}_{p-1,1}\,.$$

Now, $\partial''(\partial' c_1) = \partial'(\partial'' c_1) = \partial'\partial' c_0 = 0$, and since the ∂''-homology of $\mathsf{K}_{\bullet,\bullet}$ is concentrated only in dimension zero, it must exist a $c_2 \in \mathsf{K}_{p-2,2}$ such that

$$\partial' c_1 = \partial'' c_2\,,$$

and so on. The procedure stops at the p-th step. Thus the chain

$$C = c_0 + \sum_{i=1}^{p} (-1)^{\sum_{k=0}^{i-1}(p-k)} c_i$$

is a cycle in $\operatorname{Tot}\mathsf{K}$, that is, $\partial C = 0$.

[1] The use of the same symbol to denote the differentials in C and $\operatorname{Tot}\mathsf{K}$ should not generate any confusion.

Conversely, suppose $C = c_0 + \sum_{i=1}^{p} (-1)^{\sum_{k=0}^{i-1}(p-k)} c_i \in \operatorname{Tot} \mathsf{K}$ is ∂-closed. Then $c \equiv \varepsilon(c_0)$ is a degree p cycle in C_p. Indeed, in degree $(p-1, 0)$ we have $\partial' c_0 = \partial'' c_1$ and

$$\varepsilon(\partial' c_0) = \varepsilon(\partial'' c_1) = 0\,,$$

since the augmentation is exact.

That the cycle $c \in C_p$ is a boundary if and only if $C \in \operatorname{Tot} \mathsf{K}$ is a boundary can be proven along the same lines. This completes the argument. $\square$

A.2. Recall from section 3 the various double complexes we used. In particular, $\mathsf{K}_{\bullet,\bullet} = \mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet$ is the double complex obtained tensoring the singular chain complex on $X_0 \cong \mathbb{H}$ with the "bar" complex

$$0 \longleftarrow \mathsf{B}_0 \xleftarrow{\partial''} \mathsf{B}_1 \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \mathsf{B}_n \xleftarrow{\partial''} \cdots \tag{A.1}$$

which is exact except in degree zero. Its definition has been given in the main text. Being B_0 a Γ-module on the generator $[\,]$, introducing the augmentation $\varepsilon : \mathsf{B}_0 \to \mathbb{Z}$, $\varepsilon([\,]) = 1$, we can rewrite it as the exact sequence

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathsf{B}_0 \xleftarrow{\partial''} \mathsf{B}_1 \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \mathsf{B}_n \xleftarrow{\partial''} \cdots\,. \tag{A.2}$$

The above exact sequence is usually referred to as a "resolution" of the integers. Since every B_q is a *free* Γ-module, the sequence is a free resolution.

The singular chain complex $\mathsf{S}_\bullet \equiv S_\bullet(X_0)$ needs little description. Since Γ acts on the space, $\mathsf{S}_\bullet$ acquires a Γ-module structure simply by translating around the chains. That this actually is a complex of *free* Γ-modules is proven in [23] or [9]. A choice of free generators is to take those chains whose first vertex lies in a suitably chosen fundamental domain in X_0. The differential, which we called ∂' in the main text, is just the usual boundary homomorphism.

The homology of Γ with coefficients in any Γ-module M is by definition the homology of the complex $M \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet$. (Any other resolution of $\mathbb{Z}$ would be adequate.) In fact, tensor product does not preserve exactness in general. As a matter of terminology, a module M such that any exact sequence remains exact after tensoring with it, is called *flat*. Therefore, all the higher homology groups of Γ with coefficient in a flat module will be zero. A free Γ-module is in particular flat, as it is very easy to see. So, in our case, we have

$$H_q(\Gamma, \mathsf{S}_p) = \begin{cases} \mathsf{S}_p \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} & q = 0 \\ 0 & q > 0 \end{cases}$$

where $\mathbb{Z}$ is considered as a trivial Γ-module. Moreover, note that $\mathsf{S}_p \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} \equiv S_p(X_0) \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} \cong S_p(X)$ the space of singular chains on the surface. Indeed, if c is any chain on X_0 and γ is any group element, we have $c \cdot \gamma \otimes 1 = c \otimes \gamma \cdot 1 = c \otimes 1$, and therefore $c \otimes 1$ can be identified with a singular chain on the surface, as claimed.

After this preparations, we can exploit the exact complex (A.2) to build the augmented double complex

$$0 \longleftarrow \mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} \xleftarrow{\mathrm{id} \otimes \varepsilon} \mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet \tag{A.3}$$

with exact rows. According to the foregoing, the leftmost column in (A.3) is to be identified with the singular chain complex on the surface. (Or, more generally, of the quotient space.)

The complex (A.3) satisfies the hypotheses of Proposition A.1, and since the group homology is the ∂''-homology of the double complex, we conclude that $H_\bullet(\operatorname{Tot}\mathsf{K}) \cong H_\bullet(X,\mathbb{Z})$ thereby proving one half of Lemma 3.2.

In order to prove the other half, let us observe that actually all the columns in A.3, except the first one, are exact, $X_0 \cong \mathbb{H}$ being a contractible space. Indeed, the complex $\mathsf{S}_\bullet$ carries no homology except in degree zero, and we can "augment" it as well to obtain another resolution of the integers:

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathsf{S}_0 \xleftarrow{\partial'} \mathsf{S}_1 \xleftarrow{\partial'} \cdots \xleftarrow{\partial'} \mathsf{S}_n \xleftarrow{\partial'} \cdots .$$

Now the situation is completely symmetric and we can just "transpose" the above constructions to build the augmented complex

$$\begin{array}{c} \mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet \\ \big\downarrow{\scriptstyle \varepsilon\otimes\mathrm{id}} \\ \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet \\ \big\downarrow \\ 0 \end{array}$$

and apply Proposition A.1 to it to show that $H_\bullet(\operatorname{Tot}\mathsf{K}) \cong H_\bullet(\Gamma,\mathbb{Z})$.

A.3. The cohomological picture has a very similar structure. The cohomology of Γ with coefficients in M is by definition the homology of the complex $\operatorname{Hom}_{\mathbb{Z}\Gamma}(\mathsf{B}_\bullet, M)$. (Notice that Hom is contravariant in the first variable, thus it reverses the arrows.) We will be in position to apply the analogous of Proposition A.1 with the arrows reversed to the complex $\mathsf{C}^{\bullet,\bullet} = \operatorname{Hom}(\mathsf{B}_\bullet, \mathsf{A}^\bullet)$ provided we show that $H^q(\Gamma, \mathsf{A}^p) = 0$ for $q > 0$, that is, $\operatorname{Hom}(\,\cdot\,, \mathsf{A}^p)$ must preserve exactness, so that the higher cohomology groups are zero. An *injective* module M is by definition a Γ-module such that $\operatorname{Hom}(\,\cdot\,, M)$ preserves exactness, hence the higher cohomology groups of Γ with coefficients into an injective are zero. Thus we have to show that A^p is injective as a Γ-module. In fact, more can be done, namely it can be shown that $\mathsf{A}^p \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, A^p_{\mathbb{C}}(X))$, where $A^p_{\mathbb{C}}(X)$ is the vector space of (complex valued) differential forms on the Riemann surface X. The (easy) proof of this assertion requires the construction of an equivariant partition of unity on $\mathbb{H}$, see [21]. Then A^p has no higher cohomology since

$$\begin{aligned} \operatorname{Hom}_{\mathbb{Z}\Gamma}(\mathsf{B}_\bullet, \mathsf{A}^p) &\cong \operatorname{Hom}_{\mathbb{Z}\Gamma}(\mathsf{B}_\bullet, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, A^p_{\mathbb{C}}(X))) \\ &\cong \operatorname{Hom}_{\mathbb{Z}}(\mathsf{B}_\bullet, A^p_{\mathbb{C}}(X)), \end{aligned}$$

and the last complex has no cohomology, except in degree zero. Thus we have

$$H^q(\Gamma, \mathsf{A}^p) = \begin{cases} A^p_{\mathbb{C}}(X) & q = 0 \\ 0 & q > 0, \end{cases}$$

and applying Proposition A.1 to the double complex $\mathsf{C}^{\bullet,\bullet}$ we can prove that

$$H^\bullet(\operatorname{Tot}\mathsf{C}) \cong H^\bullet(X,\mathbb{C}).$$

To prove the rest of Lemma 3.3 we need only use the contractibility of $X_0 \cong \mathbb{H}$, so that $\mathsf{A}^\bullet$ has no cohomology, and apply Proposition A.1 to the transposed double complex.

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