Semiclassical eigenvalue estimates for the Pauli operator with strong non-homogeneous magnetic fields

I. Non-asymptotic Lieb-Thirring type estimate

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Abstract

We give the first Lieb-Thirring type estimate on the sum of the negative eigenvalues of the Pauli operator that behaves as the corresponding semiclassical expression even in the case of strong non-homogeneous magnetic fields. This enables us, in the companion paper [ES-II], to obtain the leading order semiclassical eigenvalue asymptotic, which, in turn, leads to the proof of the validity of the magnetic Thomas-Fermi theory of [LSY-II]. Our work generalizes the results of [LSY-II] to non-homogeneous magnetic fields.

Contents

1 Introduction

In this paper and its companion [ES-II] we shall study the semiclassical limit of the Pauli operator with both electric and magnetic fields. Our main concern is to allow for non-homogeneous magnetic fields. As we hope to illustrate, the transition from homogeneous to non-homogeneous fields is highly non-trivial. This is so not only because of technical difficulties but also because the case of non-homogeneous field is qualitatively different.

We shall be concerned here mainly with dimension three. In the follow up paper we will consider also dimension two. The three dimensional Pauli operator is the following operator acting on the space $L^2(\mathbf{R}^3;\mathbf{C}^2)$ of spinor valued functions.

$$H(h,\mathbf{A},V) := [\boldsymbol{\sigma}\cdot(-ih\nabla+\mathbf{A}(x))]^2+V(x) = (-ih\nabla+\mathbf{A}(x))^2+V(x)+h\boldsymbol{\sigma}\cdot\mathbf{B}(x), \quad (1.1)$$

where $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ is the vector of Pauli spin matrices, i.e.,

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

The magnetic field $\mathbf{B}:\mathbf{R}^3\to\mathbf{R}^3$ is a divergence free field related to the vector potential $\mathbf{A}:\mathbf{R}^3\to\mathbf{R}^3$ by $\mathbf{B}=\nabla\times\mathbf{A}$. The potential $V(x)$ describes the electric field. As usual h is the semiclassical parameter. Throughout the paper we shall use the convention of writing $-i\nabla=\mathbf{p}$.

The two dimensional Pauli operator has essentially the same form as the three dimensional operator above. The modifications are rather obvious. The magnetic field is a function $B:\mathbf{R}^2\to\mathbf{R}$, the vector potential is a vector field $\mathbf{A}:\mathbf{R}^2\to\mathbf{R}^2$ and we shall write as before $B=\nabla\times\mathbf{A}$ with the obvious interpretation. We may then write

$$H^{(2)}(h,\mathbf{A},V) := [\boldsymbol{\sigma}\cdot(h\mathbf{p}+\mathbf{A}(x))]^2+V(x) = (h\mathbf{p}+\mathbf{A}(x))^2+V(x)+h\sigma_3B(x). \quad (1.2)$$

The Pauli operator describes the motion of a non-relativistic electron, where the electron spin is important because of its interaction with the magnetic field. For simplicity we have not included any physical parameters (i.e., the electron mass, the electron charge, the speed of light, or Planck's constant $\hbar$) in the expressions for the operators. In place of Planck's constant we have the semiclassical parameter h, which we let tend to zero.

The last identities in (1.1) and (1.2) can easily be checked. If we note that $\boldsymbol{\sigma}\cdot(h\mathbf{p}+\mathbf{A}(x))$ is in fact the three dimensional Dirac operator, we recognize the last identity in (1.1) as the Lichnerowicz formula. As a consequence of these identities one sees a significant difference between the Pauli operator and the 'magnetic' Schrödinger operator $(h\mathbf{p}+\mathbf{A}(x))^2+V(x)$. In particular, the location of the essential spectrum can be very different for the two operators (recall that the physically most interesting eigenvalues are those below the essential spectrum). Typically, the bottom of the essential spectrum for the Schrödinger operator depends on the magnetic field. But it turns out that, under very general circumstances, the essential spectrum for the Pauli operator in both two and three dimensions starts at zero. It was proven in [HNW] for $V=0$, but the proof works as well for potentials vanishing at infinity. Therefore we can simply study the negative eigenvalues. For the Schrödinger operator the right question would be to study the eigenvalues below the bottom of the essential spectrum, but since the essential spectrum is in general not known a-priori (for a constant magnetic field it is known) these eigenvalues are very difficult to locate. Note that it is a bonus from Nature that the Pauli operator, which describes the electron correctly, including its spin, behaves in a more stable way.

The physically as well as mathematically interesting quantities connected with the eigenvalues are the number and the sum of the negative eigenvalues. Recall that the sum of the negative eigenvalues represents the energy of the non-interacting fermi gas in the external potential V and magnetic field $\mathbf{B}$.

In the case of a constant magnetic field it is known (e.g. [Sol], [Sob-1986]) that even for a smooth compactly supported potential V, which is negative, there will be *infinitely* many negative eigenvalues. This holds in both two and three dimensions. In fact, the asymptotics of how the number of eigenvalues accumulate near the essential spectrum is studied in [I, Sob-1986]. It was, however, proved in [LSY-II] (three dimensions) and [LSY-III] (two dimensions) that the *sum* of the negative eigenvalues is finite. In fact, [LSY-II, LSY-III] establish Lieb-Thirring type estimates on the eigenvalue sum which only require the negative part of the potential to be in appropriate L^p spaces.

The goal in [LSY-II] was to analyze the eigenvalue sum in the semiclassical limit, i.e., as the semiclassical parameter h tends to zero. In the case where one fixes the magnetic field $\mathbf{B}$ and let $h \to 0$ one finds that the leading order contribution to the sum of the eigenvalues becomes independent of $\mathbf{B}$. It is therefore equal to the non-magnetic Weyl term, which in three dimensions is $-2(15\pi^2)^{-1}h^{-3}\int_{\mathbf{R}^3}[V]_-^{5/2}$ and in two dimensions is $-h^{-2}(8\pi)^{-1}\int_{\mathbf{R}^2}[V]_-^2$ ($[V]_-$ denotes the negative part of the function V). This type of semiclassical limit is therefore not very well suited to the study of the effect of magnetic fields. One could maybe hope that higher order terms in the expansion would reveal information about the magnetic field. In this context we should point out, however, that without some assumptions on the classical Hamiltonian flow one cannot establish non-vanishing higher order corrections to the above Weyl term.

The observation made in [LSY-II] for homogeneous magnetic fields is that one can establish a semiclassical expression for the sum of the negative eigenvalues which is asymptotically exact uniformly in the magnetic field strength. Note that it is a great simplification that we, as discussed above, can concentrate on the eigenvalues in a fixed interval (the negative eigenvalues) and not have to consider eigenvalues in a $\mathbf{B}$ dependent interval, as it would be the case for the 'magnetic' Schrödinger operator. In contrast to the above standard semiclassical Weyl term, the generalized semiclassical expression, indeed, depends on the magnetic field. In case of three dimensions this formula is given by

$$E_{scl}(h, \mathbf{B}, V) := -h^{-3}\int_{\mathbf{R}^3} P(h|\mathbf{B}(x)|, [V(x)]_-)\mathrm{d}x \tag{1.3}$$

with

$$P(B,W) := \frac{B}{3\pi^2}\left(W^{3/2} + 2\sum_{\nu=1}^{\infty}[2\nu B - W]_-^{3/2}\right) = \frac{2}{3\pi}\sum_{\nu=0}^{\infty} d_\nu B[2\nu B - W]_-^{3/2} \tag{1.4}$$

being the pressure of the three dimensional Landau gas ($B, W \geq 0$). Here $d_0 = (2\pi)^{-1}$ and $d_\nu = \pi^{-1}$ if $\nu \geq 1$. Observe that if $\|\mathbf{B}\| = o(h^{-1})$ then E_{scl} reduces to leading order to the standard Weyl term as $h \to 0$. If $B(x)h \to \infty$ for all x, then only the lowest Landau band gives the main contribution, i.e. E_{scl} reduces to leading order to a similar expression where only the first term ($\nu = 0$) is kept in (1.4). Here and throughout the paper $\|\cdot\|$ refers to the supremum norm.

In [LSY-III] the two dimensional problem was studied, but since this paper was aimed mainly at applications the semiclassical formula did not appear explicitly. It is

$$E_{scl}^{(2)}(h, B, V) := -h^{-2}\int_{\mathbf{R}^2} P^{(2)}(hB(x), [V(x)]_-)\mathrm{d}x \tag{1.5}$$

with

$$P^{(2)}(B,W) := \frac{B}{2\pi}\left(W + 2\sum_{\nu=1}^{\infty}[2\nu B - W]_-\right) = \sum_{\nu=0}^{\infty} d_\nu B[2\nu B - W]_- \tag{1.6}$$

being the pressure of the two dimensional Landau gas (d_ν is as above). Again if $\|\mathbf{B}\| = o(h^{-1})$ then $E_{scl}^{(2)}$ reduces to the standard Weyl term.

Our goal in this paper and its continuation is to show that these semiclassical formulas are exact also for non-homogeneous fields. Of course, for non-homogeneous fields it is more subtle exactly what one means by uniformity in the field, since the field is now no longer determined by just one parameter. We shall return to this question later.

First we would like to motivate the importance of studying these issues. Historically, magnetic fields have occured as small perturbations in the Hamiltonian, e.g., as in the standard treatment of the Zeeman effect. In later years, however, the investigation of strong magnetic fields has become relevant in important physical systems.

In the early 70's it became clear to astrophysicists that the strong magnetic fields expected to exist on a neutron star would have a significant influence on the structure of the atoms existing on the surface of the star. This was one of the main motivations for the study in [LSY-II]. (A fairly extensive reference to the physical literature can be found in [LSY-II]).

In the 80's strong magnetic fields became of importance even in laboratory experiments. This occured as a result of new experimental techniques which allowed one to study effectively two dimensional systems, e.g., the quantum Hall experiments and the heterostructures such as quantum dots. In these systems the electrons move in crystal structures where their effective mass and charge change to such an extent that laboratory magnetic fields become effectively very strong and can no longer be treated perturbatively.

In all of these systems it is a good approximation to consider the magnetic field as homogeneous. There are several reasons why one would still like to extend the analysis to non-homogeneous fields. First of all it is of course natural to ask whether the features found for homogeneous fields are really stable. Furthermore, a detailed mathematical study often requires one to be able to locally vary the field, even if one is mainly interested in the constant field case. Even though we will find that the semiclassical results known for constant fields really carry over to non-constant fields, we will also see that, in fact, not all features of the constant case are stable.

Of course the problem is also of independent interest and raises, as we shall see, many extremely interesting mathematical issues. The homogeneous field case is comparatively simple because the kinetic energy part is an exactly solvable quantum mechanical model.

The semiclassical analysis of eigenvalues is really twofold. One must first of all establish non-asympotic estimates allowing one to control errors and contributions coming from non-semiclassical regions. These estimates, which for the sum of the eigenvalues are often refered to as Lieb-Thirring estimates, show that the sum of the negative eigenvalues is finite and can be controlled from below by expressions, which behave like the semiclassical formula one would like to prove. They need, however, not be asymptotically exact since they are only used for controlling errors.

The second part of the semiclassical study is to show that, when all the errors have been controlled, one can indeed get the asymptotic formulas.

Our approach to the problem, which we treat in the continuation to this paper, is the coherent state method also used in [LSY-II, LSY-III].

The present paper treats the somewhat more fundamental problem of establishing a Lieb-Thirring estimate. This part requires many novel ideas and most of them will also be needed in the subsequent asymptotic study of the second paper.

Having proved Lieb-Thirring estimates, one can prove leading order semiclassics with very weak regularity assumptions on the potential and the magnetic field. In particular, we can treat the case of atomic potentials with the Coulomb singularity. The explicit treatment of the leading order energy for large atoms in strong non-homogeneous magnetic fields can be found in the second paper, where we prove, as an application of the semiclassical result, that the 'Magnetic Thomas Fermi (MTF) theory' introduced even for non-homogeneous fields in [LSY-II] gives the asymptotically correct ground state energy of atoms for large nuclear charge.

The first generalizations of the Lieb-Thirring inequality of [LSY-II] and [LSY-III] to the case of non-homogeneous magnetic fields were first given in [E-1995] and later improved in [Sob-1996(1)] and [Sob-1996(2)]. These works were mainly concerned with allowing fields growing at infinity. However, for the purpose of applications in semiclassical proofs, it is more important that the size of the Lieb-Thirring estimate be comparable with the semiclassical expression. In the case of dimension two and in the case of constant *direction* field in dimension three, the estimates obtained in [E-1995], [Sob-1996(1)] and [Sob-1996(2)] are adequate for the purpose of studying the semiclassical limit. For three dimensions in general, none of the previously established Lieb-Thirring estimates (including the one in [LLS]) can be applied to the semiclassical problem, because none of them scale correctly for large magnetic fields. The purpose of the present paper is to establish an estimate which is adequate in three dimensions. The issue is somewhat complicated and we shall now try to illucidate it.

The three dimensional case is complicated by the fact that there exist fields $\mathbf{B}$ (nonconstant direction) such that the corresponding Pauli-operator with $V = 0$, although non-negative, can have eigenfunctions with eigenvalue zero, we shall refer to them as zero modes. Examples of such fields were given in [LY]. The same is true in two dimensions, but there these eigenfunctions are expected from the index theorem or more precisely the Aharonov-Casher Theorem (see [CFKS]). Even in two dimensions the zero modes make the problem non-trivial, but at least there one has a control on the density of zero modes (see [E-1993]). In three dimensions the issue is much more complicated.

From (1.3) it is natural to conjecture that the correct Lieb-Thirring estimate would bound the sum of the negative eigenvalues from below by

$$-Ch^{-3}\int_{\mathbf{R}^3}|h\mathbf{B}(x)|[V(x)]_-^{3/2}\mathrm{d}x - Ch^{-3}\int_{\mathbf{R}^3}[V(x)]_-^{5/2}\mathrm{d}x.$$

This is indeed the result for constant magnetic field proved in [LSY-II]. The existence of zero modes together with the variational principle, however, immediately implies that such an estimate is *wrong* in general. Thus there is a fundamental difference here between homogeneous and non-homogeneous fields. In fact, if $u_1, \dots$ denote the (orthonormalized) zero modes, i.e., if $\boldsymbol{\sigma}\cdot(h\mathbf{p}+\mathbf{A}(x))u_i = 0$, then the variational principle implies that the sum of the negative eigenvalues is bounded above by

$$\sum_i \langle u_i|H(h,\mathbf{A},V)|u_i\rangle = \int_{\mathbf{R}^3}\sum_i |u_i(x)|^2 V(x)\mathrm{d}x.$$

Note that for small V this expression can in general not be dominated by the two terms above. The Lieb-Thirring inequality must therefore contain a term of the form

$$-\int_{\mathbf{R}^3} n(x)[V(x)]_-\mathrm{d}x,$$

where $n(x)$ is an upper bound to, $\sum_i |u_i(x)|^2$, the density of states of zero modes.

We shall not go into details about the results proved in [E-1995, Sob-1996(1), Sob-1996(2), LLS], but just remark that the function $n(x)$ in all of these behaves qualitatively like $|\mathbf{B}(x)|^{3/2}$. Indeed, $|\mathbf{B}(x)|^{-1/2}$ has the dimension of a length and therefore $|\mathbf{B}(x)|^{3/2}$ behaves dimensionally as a density, i.e., like a negative third power of a length. Of course the function $n(x)$, which appears must have the the correct dimensionality. [When we say that $|\mathbf{B}(x)|^{-1/2}$ has the dimension of a length, we of course simply refer to how it behaves under scaling transformations.] Now we immediately see that if we replace $n(x)$ by $|\mathbf{B}(x)|^{3/2}$ then the above term has a completely different behavior than the semiclassical expression. In fact, if we insert the semiclassical parameter h (i.e., replace $\mathbf{B}$ by $h\mathbf{B}$ and multiply the eigenvalue sum with h^{-3}) we see that the extra term above with $n(x) \sim \|\mathbf{B}\|^{3/2}$ would behave like $h^{-3}h^{3/2}\|B\|^{3/2} = h^{-3/2}\|B\|^{3/2}$. It is, however, clear that (if, say, $|\mathbf{B}|$ is bounded above and below away from zero then) $E_{scl} \sim h^{-2}\|B\| + h^{-3}$. It is therefore obvious that unless $\|B\| \leq O(h^{-1})$ the above term will be bigger than the semiclassical expression. Thus, to be able to derive the semiclassical formula (1.3) for $\|\mathbf{B}\| \gg h^{-1}$, we must prove a new Lieb-Thirring estimate. Recall, that $\|B\| \sim h^{-1}$ is exactly the borderline for when the semiclassical expression (1.3) no longer behaves like the Weyl term.

We shall throughout the paper assume the following conditions on the magnetic field ($\|\cdot\|$ denotes the supremum norm).

$$\|B\| < \infty, \tag{1.7}$$

$$l(\mathbf{B})^{-1} := \|\nabla \mathbf{n}\| = \|\nabla \frac{\mathbf{B}}{B}\| < \infty, \tag{1.8}$$

$$L(\mathbf{B})^{-1} := \|\frac{|\nabla B|}{B}\| < \infty. \tag{1.9}$$

Here $l = l(\mathbf{B})$ describes the length scale on which the field line geometry changes, while $L = L(\mathbf{B})$ is the length scale on which the field strength varies. Note that the conditions imply that $B(x) := |\mathbf{B}(x)|$ never vanishes. We are especially interested in the case when $l \gg L$. In particular, for $l = \infty$, $L < \infty$, we obtain the constant direction (but non-homogeneous) case.

The potential V is assumed to be in $L^{5/2} \cap W^{1,1}$, and we restrict ourselves to the case when $V \geq 0$ (since the Lieb-Thirring inequality is a lower bound on the eigenvalue sum one can always replace V by its negative part $-[V]_-$). The following theorem, which we state for $h = 1$, is the main result of this paper.

Theorem 1.1 *With the notations above, there exists a universal constant C_1 such that the following estimate is valid for the sum of the negative eigenvalues $\{e^{(m)}_{\mathrm{Pauli}}\}_{m=1,2,\ldots}$ of the Pauli operator $[\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A})]^2 - V$*

$$\sum_m |e^{(m)}_{\mathrm{Pauli}}| \tag{1.10}$$

$$\leq C_1 \left(\int_{\mathbf{R}^3} V^{5/2} + \int_{\mathbf{R}^3} (B + d_0^{-2}) V^{3/2} + \int_{\mathbf{R}^3} (B + d_0^{-2}) d_0^{-1} V + \int_{\mathbf{R}^3} (B + d_0^{-2}) |\nabla V| \right),$$

where

$$d_0 := \min\{\|B\|^{-1/4} l^{1/2}, L, l\}. \tag{1.11}$$

Remark 1. (Comparison with semiclassics) Inserting the semiclassical parameter $h < 1$, assuming $\|B\| \geq \max\{l^{-2}, L^{-2}\}$, and bounding $B \leq \|B\|$, we obtain a bound of order

$$h^{-3} \int_{\mathbf{R}^3} V^{5/2} + \|B\| h^{-2} \int_{\mathbf{R}^3} V^{3/2} + \|B\|^{5/4} h^{-5/4} l^{-1/2} \int_{\mathbf{R}^3} V + \|B\| h^{-1} \int_{\mathbf{R}^3} (|\nabla V| + V). \tag{1.12}$$

The sum of the first two terms is exactly of the same order of magnitude as the semiclassical expression. The point is that the error terms (last two terms) are smaller than the semiclassical terms if

$$\|B\| l(\mathbf{B})^{-2} = o(h^{-3})$$

(assuming that V and its first derivatives are fixed). Thus this LT inequality will suffice to show the semiclassical formula when this condition is satisfied. This is the real application of our theorem and the methods are developed to satisfy the needs of a semiclassical statement. In particular, we obtain semiclassics uniformly in $\|B\|$ for constant direction field ($l = \infty$).

The reason for the restriction $\|B\| l(\mathbf{B})^{-2} = o(h^{-3})$ is the following. Part of the motivation for believing that the semiclassical formula for homogeneous or even constant direction fields should generalize to fully non-homogeneous fields is that these fields on the relevant quantum scales should behave approximately like constant direction fields. This is, however, not true if the field is too strong. A charged quantum particle moving in a magnetic field essentially occupies a region in space of the shape of a cylinder with axis parallel to the magnetic field. For particles of fixed energy e the radius of the cylinder is the Landau radius $r_L = B^{-1/2} h^{1/2}$ and the height is of order $s = he^{-1/2}$ (particles localized in regions of length $he^{-1/2}$ in one dimension have energies of order e). The condition that one can approximate the magnetic field within this region by a constant direction field is that the field lines remain within this cylinder, i.e., that $l^{-1} s^2 \ll r_L$. This condition is simply that $Bl^{-2} \ll h^{-3} e^2$.

Although the above restriction on the magnetic field might seem natural, we believe that it can be removed by an additional geometrical analysis which is beyond the scope of the present work. We intend to return to this issue in the future.

Remark 2. (The last term) The last term above which depends on the gradient of V is a conceptual error coming from our method and we have no reason to believe that it could not possibly be removed. It, however, scales in a way that it still allows for proving the semiclassical asymptotics.

Remark 3. (Zero modes) In this Lieb-Thirring inequality the function corresponding to the density $n(x)$ discussed above is the function $(B(x)+d_0^{-2})d_0^{-1}$. In our inequality, however, this is not the only term which is linear in V. The last gradient term also has this property. We can therefore not say that $(B(x)+d_0^{-2})d_0^{-1}$ bounds the density of zero modes.

Remark 4. (Constant field) The Lieb-Thirring inequality for a constant magnetic field ([LSY-II]) with our notations asserts the standard bound

$$C_1\left(\int V^{5/2}+\int B\,V^{3/2}\right). \tag{1.13}$$

To get a constant field, we have to let l and L go to infinity. This reduces our bound to (1.13) with the extra error term of size $\int B\,|\nabla V|$.

Remark 5. (Large field.) We know that if the field has a large gradient, the standard estimate can be violated (see [E-1995]). This is reflected by the fact that d_0^{-1} blows up in (1.10) as l^{-1} or L^{-1} increases.

Remark 6. (Constant direction field.) In case of the constant direction field with a uniform positive lower bound, we expect the standard (1.13) to hold. This case is equivalent to $l=\infty$ in our setup. This does not suppress the extra terms in our theorem. We do not focus on this point since the constant direction LT under the assumption (1.7) (no gradient bounds) has been proven by much simpler methods in [E-1995]. With more involved methods, the condition (1.7) was weakened first in [E-1995], then later [Sob-1996(2)] gave an essentially optimal result.

Before going into the proof of the Lieb-Thirring estimate (1.10) we now state the semiclassical results that are proved in the second paper [ES-II]. Proving these fundamental theorems is the real motivation for establishing the Lieb-Thirring estimate of the present paper.

In the two dimensional case we have a completely uniform statement.

Theorem 1.2 (2D Semiclassics) *Assume that the potential V satisfies $V\in L^1(\mathbf{R}^2)\cap L^2(\mathbf{R}^2)$ and the magnetic field B satisfies $\|B\|<\infty$, $\|\,|\nabla B|/|B|\,\|<\infty$. For $h,b>0$ let $e_1(h,b), e_2(h,b),\ldots$ denote the negative eigenvalues of the operator $H=H^{(2)}(h,b\mathbf{A},V)$ in (1.2). Then*

$$\lim_{h\to 0}\left|\frac{\sum_k e_k(h,b)}{E^{(2)}_{scl}(h,bB,V)}-1\right|=0$$

uniformly in b.

As discussed above the three dimensional case requires a condition on the magnetic field strength.

Theorem 1.3 (3D Semiclassics) *Assume that the potential V satisfies $V \in L^{5/2}(\mathbf{R}^3) \cap L^{3/2}(\mathbf{R}^3)$, $[V]_- \in W^{1,1}(\mathbf{R}^3)$ and the magnetic field $\mathbf{B}$ satisfies (1.7)–(1.9). For $h, b > 0$ let $e_1(h,b), e_2(h,b), \dots$ denote the negative eigenvalues of the operator $H = H(h, b\mathbf{A}, V)$ in (1.1). Then*

$$\lim_{\substack{h\to 0\\ bh^3\to 0}} \left| \frac{\sum_k e_k(h,b)}{E_{scl}(h, b\mathbf{B}, V)} - 1 \right| = 0. \tag{1.14}$$

In fact, the results proved in [ES-II] are somewhat stronger, e.g., in the three dimensional case V is allowed to depend on the limiting parameter τ in a certain way. This generalization may seem to be an unnecessary complication. It turns out, though, that it is vital to the application in the theory of atoms in magnetic fields.

In order to prove our Lieb-Thirring inequality we localize in regions of space where we can approximate by a constant magnetic field. In these regions we then use standard Lieb-Thirring estimates. As already mentioned, the optimal regions are cylinders parallel to the magnetic field. Approximating the magnetic field also requires approximating the vector potential. This involves choosing optimal gauges locally. One of the main ingredients in the proof is a particularly good gauge choice in a cylindrical domain. The well known Poincaré formula for choosing a gauge, is well suited for spherical domains, but is not optimal in elongated domains. We discuss approximating magnetic fields in Section 2 and the cylindrical gauge choice is described in 2.2.

When localizing we have to control two type of errors: the localization errors (the energy it costs to localize) and the approximation errors. Of course controlling the localization error require choosing large regions, whereas the approximation errors are small when the regions are small. It turns out that we must do the localization in two steps. The first localization uses essentially 'isotropic' regions (cubes). On these cubes we approximate the magnetic field, not by a constant field, but only by a constant *direction* field. This can in general be done on a larger region than if we were approximating by a constant field immediately. Since we now have a distinguished direction on each of these cubes we can separate the subspaces corresponding to spin up and spin down relative to this direction. This is done in Section 4.1.

The spin up subspace is fairly easy to treat since the free Pauli operator on this subspace can be controlled by the Schrödinger operator, which is then in turn controlled by the diamagnetic inequality. This is Section 4.2.

The spin down subspace is studied in Section 4.3. Here we need to localize further into cylinders (which are typically elongated ones, i.e., if $l \gg L$) and we approximate the magnetic field by a *constant* field on each cylinder. This is done by introducing Neumann boundary conditions on the cylinders (due to the narrowness of the cylinders, Dirichlet localization would cost too much). In each cylinder we then use a standard supersymmetry argument to separate the lowest Landau level from the higher levels. The higher levels are then treated somewhat similarly to the spin up space. The lowest level is studied using explicit estimates on the Neumann ground state density. These estimates are derived in Section 3. The main difficulty here is that the Neumann density diverges near the boundary. We get around this difficulty by choosing overlapping cylinders allowing us in each cylinder to consider the potential to be supported in what we call the cores, which are away from the boundary.

The reader may wonder why we need two localizations. The main reason is that localizing in narrow cylinders with Dirichlet boundary conditions is very expensive. We can only afford that on the orthogonal complement to the lowest level. Fortunately, for the lowest level the supersymmetry gives us additional information, so the Neumann problem is actually tractable. This requires, however, that we know the lowest level, and this we do only if we have a constant direction field.

This delicate sequence of steps is characteristic for the problem, one cannot afford being sloppy in the estimates. The structure of the proof is very tight and in each step we have to use the full strength of the estimate involved.

Finally, we explain how the errors are controlled. If we consider a cube, where the potential is in some sense large (see the beginning of Section 4.1.2 for the precise notion) we absorb the errors into the potential. If the potential is small on a cube we do not need the full kinetic energy to control it and we may simply ignore a fraction of the kinetic energy thereby only keeping a fraction of the errors. This simple, but crucial idea has been used several times before in studying magnetic problems (see, e.g., [F, FLL, LLS]). Since the localization errors from one cube often has to be controlled in the neighboring cubes (allocation of errors), we must have that the sizes of the potential in neighboring cubes are comparable. In Section 4.1.1 we show how to replace the original potential with a potential with this property.

2 The geometry of the magnetic field

In this section we prove two propositions. Their proofs are a bit technical, but the statements carry the core of our main proof. We suggest skipping the proofs in a first reading.

2.1 Approximating the magnetic field

The following proposition will be used to approximate a general magnetic field by a *constant direction* field. This statement will be used twice; the crucial advantage of a constant direction field is that it allows the separation of the third direction and that it carries a natural supersymmetric structure, which is essential to establish a spectral gap. We recall the definitions

$l(\mathbf{B})^{-1} = \|\nabla(\mathbf{B}/B)\|$ and $L(\mathbf{B})^{-1} = \||\nabla B|/B\|$.

Proposition 2.1 *Consider an arbitrary cube Ω with center Q and edge length λ and a non-vanishing C^1 magnetic (divergence free) field $\mathbf{B} : \mathbf{R}^3 \to \mathbf{R}^3$. Assume that*

$$(6 + 3\sqrt{3})\lambda l(\mathbf{B})^{-1} \leq 1. \tag{2.1}$$

Then there exists a magnetic (divergence free) field $\tilde{\mathbf{B}}$, with constant direction parallel to the field at the center Q of Ω, such that for all $x \in \Omega$

$$|\mathbf{B}(x) - \tilde{\mathbf{B}}(x)| \leq \lambda l(\mathbf{B})^{-1} \left\{ \sup_{|x-Q|\leq 5\lambda} B(x) \right\} \left(\frac{\sqrt{3} + 4\sqrt{6}}{\sqrt{2}} + \frac{6 + 3\sqrt{3}}{2}\lambda \left(l(\mathbf{B})^{-1} + L(\mathbf{B})^{-1} \right) \right) \tag{2.2}$$

and

$$|\nabla\tilde{\mathbf{B}}(x)| \leq |\nabla\mathbf{B}(x)| \leq \left\{ \sup_{|x-Q|\leq 5\lambda} B(x) \right\} (L(\mathbf{B})^{-1} + l(\mathbf{B})^{-1}). \tag{2.3}$$

Remarks. (i) The assumption (2.1) is a geometric condition, which states that the field lines of the field $\mathbf{B}$ should not vary too fast over the scale of the cube.

(ii) In our application, where typically $l \gg \lambda$, the approximation in (2.2) will be better than the straightforward choice $\tilde{\mathbf{B}}(x) := \mathbf{B}(Q)$ (constant field), since that would yield only $|\mathbf{B}(x) - \mathbf{B}(Q)| \leq (\sqrt{3}/2)\lambda \sup|\nabla\mathbf{B}|$, which is of order $\sup|B|\lambda(l^{-1} + L^{-1})$. This is worse by a factor of $l\lambda^{-1} \gg 1$ than the similar term in (2.2).

Before giving the proof of this proposition we first discuss another approximation result, which will be a simple corollary of the proposition. We shall, indeed, also need approximations of the magnetic field by a constant field and not just a constant direction field. In order to keep the same accuracy in the approximation we must restrict to a smaller region. It turns out that we can cover the cube by parallel cylinders such that within each of these we, without losing in the approximation, can approximate the magnetic field by a constant field along the cylinder axis.

To formulate this more precisely we choose an orthonormal coordinate system $\{\xi_i\}_{i=1}^3$ in $\mathbf{R}^3$, such that the center Q of the cube Ω is the origin and that $\mathbf{B}(Q) = \mathbf{B}(0)$ points in the positive third direction. Note that the sides of Ω need not be parallel with the coordinate planes in this new coordinate system. We shall refer to the plane $\mathcal{P} := \{\xi : \xi_3 = 0\}$ as the base plane of the cube. We consider cylinders, C_P, given in this new coordinate system by

$$C_P = \{\xi : |\xi_\perp - P| \leq w \ \ |\xi_3| \leq \sqrt{3}\lambda/2\}, \tag{2.4}$$

where $P \in \mathcal{P}$ and $w > 0$ (here $\xi_\perp := (\xi_1, \xi_2, 0)$). The point P is called the center of the cylinder.

Note that the cylinders are aligned along $\mathbf{B}(Q)$, the magnetic field at the center of the cube Ω and that the union of all these cylinders covers Ω. Moreover, all the cylinders C_P such that $C_P \cap \Omega \neq \emptyset$ are subsets of the larger cube Ω', that, in the new coordinate system, is defined by $[-w - \sqrt{3}\lambda/2, w + \sqrt{3}\lambda/2]^3$.

Corollary 2.2 *Let Ω, Ω' and C_P, for $P \in \mathcal{P}$ be as defined above. Assume that the magnetic field $\mathbf{B}$ satisfies*

$$(6 + 3\sqrt{3})(2w + \sqrt{3}\lambda)l(\mathbf{B})^{-1} \leq 1. \tag{2.5}$$

Then within each C_P such that $C_P \cap \Omega \neq \emptyset$, one can approximate the magnetic field $\mathbf{B}$ by a constant field, $\tilde{\mathbf{B}}_P$ pointing along the axis of the cylinder, with the following precision

$$|\mathbf{B}(x) - \tilde{\mathbf{B}}_P| \leq \tag{2.6}$$

$$(2w+\sqrt{3}\lambda)l(\mathbf{B})^{-1}\left\{\sup_{|x-Q|<5(\sqrt{3}\lambda+2w)} B(x)\right\}\left(\frac{\sqrt{3}+4\sqrt{6}}{\sqrt{2}}+\frac{6+3\sqrt{3}}{2}(2w+\sqrt{3}\lambda)\left(l(\mathbf{B})^{-1}+L(\mathbf{B})^{-1}\right)\right)$$

$$+w\left\{\sup_{|x-Q|<5(\sqrt{3}\lambda+2w)} B(x)\right\}\left(l(\mathbf{B})^{-1}+L(\mathbf{B})^{-1}\right),$$

for $x \in C_P$.

Proof. We apply the proposition to the large cube Ω'. Let $\tilde{\mathbf{B}}$ denote the approximating constant direction field. Within Ω' (in particular within C_P) it satisfies (2.1) with λ replaced by $2w + \sqrt{3}\lambda$. Since $\tilde{\mathbf{B}}$ is divergence free, and points along the third direction in the new coordinate system, we conclude that $\tilde{\mathbf{B}}$ is independent of ξ_3. From (2.3) we see therefore that $|\tilde{\mathbf{B}}(x) - \tilde{\mathbf{B}}(P)| \leq \left\{\sup_{|x-Q|<5(\sqrt{3}\lambda+2w)} B(x)\right\} w$ for $x \in C_P$. Thus we can choose $\tilde{\mathbf{B}}_P = \tilde{\mathbf{B}}(P)$. $\square$

Remark. Note that it is only the radius of the cylinder that appears in the last term in (2.6). This is important since in our applications, typically $w \ll \lambda$, i.e., the cylinder is very thin compared to the cube. The size λ of the cube appears only together with $l(\mathbf{B})^{-1}$ which, in our setup, will typically be small. It is in this way that we will achieve that the constant field approximation within C_P is as good as the constant direction field approximation within Ω.

Proof of Proposition 2.1. We work in the coordinate system described before Corollary 2.2. Consider the larger cube $\tilde{\Omega} := [-\sqrt{3}\lambda/2, \sqrt{3}\lambda/2]^3$, obviously $\Omega \subset \tilde{\Omega}$, but note that their sides may not be parallel.

We define the constant direction (divergence free) field

$$\tilde{\mathbf{B}}(\xi) := (0, 0, B_3(\xi_1, \xi_2, 0)). \tag{2.7}$$

For the proof of the gradient bound (2.3), we note that $|\nabla\tilde{\mathbf{B}}| \leq |\nabla\mathbf{B}|$ by construction, and

$$|\nabla\mathbf{B}| \leq |\nabla B| + |B||\nabla(\mathbf{B}/B)| \tag{2.8}$$

pointwise.

Next, we show the estimate (2.2) for the coordinate plane $\mathcal{P} := \{\xi : \xi_3 = 0\}$. We have, for $\xi \in \mathcal{P}$,

$$|\mathbf{B}(\xi) - \tilde{\mathbf{B}}(\xi)| = \sup_{\|\mathbf{u}\|=1, u_3=0} (\mathbf{B}(\xi) - \tilde{\mathbf{B}}(\xi)) \cdot \mathbf{u} = \sup_{\|\mathbf{u}\|=1, u_3=0} \mathbf{B}(\xi) \cdot \mathbf{u}. \tag{2.9}$$

On the other hand, since $\mathbf{B}(0) \cdot \mathbf{u} = 0$ if $u_3 = 0$,

$$|\mathbf{B}(\xi) \cdot \mathbf{u}| \leq B(\xi) \left| \left(\frac{\mathbf{B}(\xi)}{B(\xi)} - \frac{\mathbf{B}(0)}{B(0)} \right) \cdot \mathbf{u} \right| \leq \left\{ \sup_{|\xi| \leq 5\lambda} B(\xi) \right\} l(\mathbf{B})^{-1} \lambda \sqrt{\frac{3}{2}}, \tag{2.10}$$

using $|\xi| \leq \lambda\sqrt{3/2}$ which proves (2.2) for points in $\xi \in \mathcal{P} \cap \tilde{\Omega}$.

Now we consider an arbitrary point $\xi \in \Omega$ with $\xi_3 > 0$ (the case $\xi_3 < 0$ is treated similarly), and the arclength parametrized field line curve $\gamma_\xi(t)$, which passes through ξ at time $t = 0$, and satisfies $\mathrm{d}\gamma_\xi/\mathrm{d}t = -(\mathbf{B}/B)(\gamma_\xi(t))$. Clearly, $|\gamma_\xi(t) - \gamma_\xi(0)| \leq t$. We now show that

$$|\gamma_\xi(t) - (\xi_1, \xi_2, \xi_3 - t)| \leq t(3\lambda/2 + t)\|\nabla(\mathbf{B}/B)\|. \tag{2.11}$$

In fact, $\eta(t) := \gamma_\xi(t) - (\xi_1, \xi_2, \xi_3 - t)$ satisfies $\eta(0) = 0$, and

$$\begin{aligned} |\mathrm{d}\eta/\mathrm{d}t| &\leq |(\mathbf{B}/B)(\gamma_\xi)(t) - (0,0,1)| \leq \|\nabla(\mathbf{B}/B)\||\gamma_\xi(t)| \\ &\leq \|\nabla(\mathbf{B}/B)\|(|\gamma_\xi(0)| + |\gamma_\xi(t) - \gamma_\xi(0)|) \leq (3\lambda/2 + t)l(\mathbf{B})^{-1}, \end{aligned} \tag{2.12}$$

where we used that $\mathbf{B}/B(0) = (0, 0, 1)$.

Since we assumed (2.1), there exists a time $t_\xi < \sqrt{3}\lambda$ such that $P_\xi := \gamma_\xi(t_\xi) \in \mathcal{P}$, since for short times $t \leq \sqrt{3}\lambda$, the points $\gamma_\xi(t)$ and $(\xi_1, \xi_2, \xi_3 - t)$ are still $(3 + 3\sqrt{3}/2)\lambda^2 l(\mathbf{B})^{-1} \leq \lambda/2$

close to each other, but at time $t = \sqrt{3}\lambda$, the latter point is already at least at a distance $\sqrt{3}\lambda/2 > \lambda/2$ on the opposite side of $\mathcal{P}$ than ξ. Note that for $0 \le t \le t_\xi$ the curve $|\gamma_\xi(t)| \le 5\lambda$.

Armed with these observations and recalling the notation $\xi_\perp = (\xi_1, \xi_2, 0)$, we have

$$|\mathbf{B}(\xi) - \tilde{\mathbf{B}}(\xi)| \le |\mathbf{B}(\xi) - \mathbf{B}(P_\xi)| + |\mathbf{B}(P_\xi) - \mathbf{B}(\xi_\perp)| + |\mathbf{B}(\xi_\perp) - \tilde{\mathbf{B}}(\xi_\perp)| \tag{2.13}$$

since $\tilde{\mathbf{B}}(\xi_\perp) = \tilde{\mathbf{B}}(\xi)$. The last term has been treated above, since $\xi_\perp \in \mathcal{P} \cap \tilde{\Omega}$. The second term is bounded by

$$\begin{aligned}\left\{\sup_{|\xi'|\le 5\lambda} |\nabla\mathbf{B}(\xi')|\right\} |P_\xi - \xi_\perp| &\le \left\{\sup_{|\xi'|\le 5\lambda} |\nabla\mathbf{B}(\xi')|\right\} t_\xi(3\lambda/2 + t_\xi)\|\nabla(\mathbf{B}/B)\| \\ &\le (3 + 3\sqrt{3}/2)\lambda^2 \left\{\sup_{|\xi'|\le 5\lambda} B(\xi')\right\} (L(\mathbf{B})^{-1} + l(\mathbf{B})^{-1}) l(\mathbf{B})^{-1}\end{aligned} \tag{2.14}$$

using that $|P_\xi - \xi_\perp|^2 = |\eta(t_\xi)|^2 - |\xi_3 - t_\xi|^2 \le |\eta(t_\xi)|^2$.

To estimate the first term in (2.13), we use the fact that $\mathbf{B}$ is divergence free, in particular

$$0 = \operatorname{div}\left(B\frac{\mathbf{B}}{B}\right) = B\operatorname{div}\frac{\mathbf{B}}{B} + \frac{\mathbf{B}}{B}\cdot\nabla B, \tag{2.15}$$

which implies the pointwise estimate

$$\left|\left(\frac{\mathbf{B}}{B}\cdot\nabla\right)\mathbf{B}\right| \le \left|\frac{\mathbf{B}}{B}\cdot\nabla B\right| + B\left|\nabla\left(\frac{\mathbf{B}}{B}\right)\right| \le 4B|\nabla(\mathbf{B}/B)|. \tag{2.16}$$

Therefore

$$|\mathbf{B}(\xi) - \mathbf{B}(P_\xi)| \le t_\xi \left\{\sup_{|\xi|<5\lambda}\left|\left(\frac{\mathbf{B}}{B}(\xi)\cdot\nabla\right)\mathbf{B}(\xi)\right|\right\} \le 4\sqrt{3}\lambda\left\{\sup_{|\xi|<5\lambda} B(\xi)\right\} l(\mathbf{B})^{-1}, \tag{2.17}$$

which completes the proof of Lemma 2.1. □

2.2 Choice of a good gauge

The corollary of the previous proposition will provide us with a good approximating constant field within a cylinder (see (2.4). Here we show that the difference field (which is supposed to be small) can be generated by a small vector potential within this cylinder. In general, if one is given a magnetic field within a domain, then there exists a vector potential bounded by the supremum of the field times the largest linear size of the domain (see (2.19) below).

For instance, one can choose the gauge given by the Poincaré formula; see (2.21) later. This gives a very crude bound for domains which are elongated cylinders.

The crucial fact is that, assuming some bound on the first derivative of the field in addition to its supremum bound, one can choose a gauge independent of the longest linear size of the domain. In particular, we can choose a gauge within our cylinder which is bounded by a constant independent of the length of the cylinder.

Proposition 2.3 *Given a C^1 magnetic field $\beta : \mathbf{R}^3 \to \mathbf{R}^3$ and consider a cylinder C with radius w. Then there exists a vector potential $\alpha : \mathbf{R}^3 \to \mathbf{R}^3$, such that $\nabla \times \alpha = \beta$ and*

$$\sup_C \|\alpha\| \le 4(w \sup_C \|\beta\| + w^2 \sup_C \|\nabla\beta\|). \tag{2.18}$$

The bound is uniform in the length of the cylinder.

Remark. For comparison, the Poincaré formula (2.21) obviously yields a gauge α, for any domain D, satisfying the bound

$$\sup_D \|\alpha\| \le \sqrt{3} \sup_D \|\beta\| \cdot \mathrm{diam}(D) \tag{2.19}$$

Proof. We may assume that $C = \{ x \in \mathbf{R}^3 \; : \; |x_\perp| \le w \}$. The following formula gives α explicitly:

$$\alpha_1(y) = -\int_0^1 t\beta_3(ty)y_2 \mathrm{d}t \tag{2.20}$$

$$\alpha_2(y) = \int_0^1 t\beta_3(ty)y_1 \mathrm{d}t + \int_0^{y_1} \beta_3(u, y_2, y_3)\mathrm{d}u - \int_0^1 \mathrm{d}t \int_0^{y_1} \mathrm{d}u[t^2 u\partial_1\beta_3 + t^2 y_2\partial_2\beta_3 + 2t\beta_3](tu, ty_2, ty_3)$$

$$\alpha_3(y) = -\int_0^{y_1} \beta_2(u, y_2, y_3)\mathrm{d}u + \int_0^{y_2} \beta_1(0, v, y_3)\mathrm{d}v$$

$$+ \int_0^1 \mathrm{d}t \Big[-t\beta_2(ty)y_1 + t\beta_1(ty)y_2 + \int_0^{y_1} \mathrm{d}u[t\beta_2 + t^2\partial_2\beta_2 y_2 + t^2\partial_1\beta_2 u](tu, ty_2, ty_3)$$

$$- \int_0^{y_2} \mathrm{d}v[t\beta_1 + t^2\partial_2\beta_1 v](0, tv, ty_3) \Big].$$

One can verify directly that this vector potential generates the β field (one has to use $\mathrm{div}\beta = 0$ several times). It is more instructive to show how one can find such a formula. We can start with the Poincaré gauge choice $\alpha^{(P)}$:

$$\alpha_1^{(P)}(y) = \int_0^1 t\beta_2(ty)y_3 \mathrm{d}t - \int_0^1 t\beta_3(ty)y_2 \mathrm{d}t \tag{2.21}$$

$$\alpha_2^{(P)}(y) = \int_0^1 t\beta_3(ty)y_1\mathrm{d}t - \int_0^1 t\beta_1(ty)y_3\mathrm{d}t$$

$$\alpha_3^{(P)}(y) = \int_0^1 t\beta_1(ty)y_2\mathrm{d}t - \int_0^1 t\beta_2(ty)y_1\mathrm{d}t$$

The bad terms are those which contain y_3, therefore we eliminate them. The first term in $\alpha_1^{(P)}$ is eliminated by adding $\nabla\varphi$ to $\alpha^{(P)}$, where

$$\varphi(y) := -\int_0^1 ty_3 \int_0^{y_1} \beta_2(tu, ty_2, ty_3)\mathrm{d}u\mathrm{d}t. \tag{2.22}$$

After some integration by parts (and $\mathrm{div}\beta = 0$), one obtains, for the new gauge $\tilde{\alpha} := \alpha^{(P)} + \nabla\varphi$,

$$\tilde{\alpha}_1(y) = -\int_0^t t\beta_3(ty)y_2\mathrm{d}t \tag{2.23}$$

$$\tilde{\alpha}_2(y) = \int_0^1 t\beta_3(ty)y_1\mathrm{d}t - \int_0^1 t\beta_1(0, ty_2, ty_3)y_3\mathrm{d}t + \int_0^{y_1} \beta_3(u, y_2, y_3)\mathrm{d}u$$
$$- \int_0^1 \mathrm{d}t \int_0^{y_1} \mathrm{d}u(t^2u\partial_1\beta_3 + t^2y_2\partial_2\beta_3 + 2t\beta_3)(tu, ty_2, ty_3)$$
$$\tilde{\alpha}_3(y) = \int_0^1 t[-\beta_2(ty)y_1 + \beta_1(ty)y_2]\mathrm{d}t - \int_0^{y_1} \beta_2(u, y_2, y_3)\mathrm{d}u$$
$$+ \int_0^1 \mathrm{d}t \int_0^{y_1} \mathrm{d}u[t\beta_2 + t^2u\partial_1\beta_2 + t^2y_2\partial_2\beta_2](tu, ty_2, ty_3).$$

For example, $\tilde{\alpha}_2$ is obtained in the following way:

$$\tilde{\alpha}_2(y) = \alpha_2^{(P)} + \partial_2\varphi = \int_0^1 t\beta_3(ty)y_1\mathrm{d}t - \int_0^1 t\beta_1(ty)y_3\mathrm{d}t - \int_0^1 t^2y_3 \int_0^{y_1} \partial_2\beta_2(tu, ty_2, ty_3)\mathrm{d}u\mathrm{d}t$$
$$= \int_0^1 t\beta_3(ty)y_1\mathrm{d}t - \int_0^1 t\beta_1(ty)y_3\mathrm{d}t \tag{2.24}$$
$$+ \int_0^1 t^2y_3 \int_0^{y_1} \partial_1\beta_1(tu, ty_2, ty_3)\mathrm{d}u\mathrm{d}t + \int_0^1 t^2y_3 \int_0^{y_1} \partial_3\beta_3(tu, ty_2, ty_3)\mathrm{d}u\mathrm{d}t$$
$$= \int_0^1 t\beta_3(ty)y_1\mathrm{d}t - \int_0^1 t\beta_1(ty)y_3\mathrm{d}t$$
$$+ \int_0^1 ty_3 \int_0^{ty_1} \partial_1\beta_1(v, ty_2, ty_3)\mathrm{d}v\mathrm{d}t + \int_0^{y_1} \int_0^1 t^2y_3\partial_3\beta_3(tu, ty_2, ty_3)\mathrm{d}t\,\mathrm{d}u$$
$$= \int_0^1 t\beta_3(ty)y_1\mathrm{d}t - \int_0^1 t\beta_1(0, ty_2, ty_3)y_3\mathrm{d}t + \int_0^{y_1} \int_0^1 t^2\frac{\mathrm{d}}{\mathrm{d}t}[\beta_3(tu, ty_2, ty_3)]\mathrm{d}t\,\mathrm{d}u$$

$$
\begin{aligned}
&- \int_0^{y_1} \int_0^1 t^2 (u\partial_1\beta_3 + y_2\partial_2\beta_3)(tu, ty_2, ty_3) \mathrm{d}t \, \mathrm{d}u \\
&= \int_0^1 t\beta_3(ty) y_1 \mathrm{d}t - \int_0^1 t\beta_1(0, ty_2, ty_3) y_3 \mathrm{d}t + \int_0^{y_1} \beta_3(u, y_2, y_3) \mathrm{d}u \\
&- \int_0^1 \mathrm{d}t \int_0^{y_1} \mathrm{d}u (t^2 u\partial_1\beta_3 + t^2 y_2 \partial_2\beta_3 + 2t\beta_3)(tu, ty_2, ty_3).
\end{aligned}
$$

The calculation for $\tilde{\alpha}_3$ is similar.

The only remaining 'bad' term in $\tilde{\alpha}$ is the second one in $\tilde{\alpha}_2$. The important thing to notice about this term is that it is independent of y_1. Adding $\nabla\psi$ to $\tilde{\alpha}$ with

$$
\psi(y) := \int_0^1 \mathrm{d}t \, t \int_0^{y_2} \mathrm{d}v \beta_1(0, tv, ty_3) y_3, \tag{2.25}
$$

one can eliminate this term as well, i.e. the final gauge $\alpha := \tilde{\alpha} + \nabla\psi$, after some calculation, yields (2.20). Note that since ψ is independent of y_1, $\tilde{\alpha}_1 = \tilde{\alpha}_1 + \partial_1\psi$.

The estimate (2.18) is straighforward. $\square$.

3 The Neumann problem with a constant field on a cylinder

3.1 Supersymmetry

Consider an arbitrary cylinder $C := \{\xi : |\xi_\perp - P| \le w, |\xi_3| \le \lambda/2\}$ with center $P \in \mathcal{P} = \{\xi : \xi_3 = 0\}$ as in Section 2.1. Let $\hat{\mathbf{B}}$ be a constant magnetic field which is parallel with the ξ_3-axis, i.e. in this coordinate system $\hat{\mathbf{B}}$ can be written as $\hat{\mathbf{B}} = (0, 0, \hat{B})$, with $\hat{B} \in \mathbf{R}_+$. Choose a gauge $\hat{\mathbf{A}}$, $\nabla \times \mathbf{A} = \hat{\mathbf{B}}$, which is independent of ξ_3 and has a special form $\hat{\mathbf{A}} = (\hat{A}_{\xi_1}, \hat{A}_{\xi_2}, 0)$ (for example the standard gauge, $\hat{\mathbf{A}}(\xi) = \frac{1}{2}\hat{\mathbf{B}} \times (\xi - P)$, would do).

Let us define the operator $T_C := (p_{\xi_1} + \hat{A}_{\xi_1}) - i(p_{\xi_2} + \hat{A}_{\xi_2})$ on C with Neumann boundary condition (more precisely, $D(T_C) := H^1(C)$). An easy calculation shows that its adjoint is $T_C^* = (p_{\xi_1} + \hat{A}_{\xi_1}) + i(p_{\xi_2} + \hat{A}_{\xi_2})$ with Dirichlet boundary condition on the mantel of the cylinder $C = C_P$ (i.e. on the set $M_P := \{\xi : |\xi_3| < \lambda/2, |\xi_\perp - P| = w\}$).

Furthermore, the operator $T_C^* T_C$ is equal to the second order operator $\sum_{j=1}^2 (p_{\xi_j} + \hat{A}_{\xi_j})^2 - \hat{B}$ with operator domain $D(T_C^* T_C) = \{\phi \in D(T_C) : T_C\phi \in D(T_C^*)\}$. Similarly, $T_C T_C^* = \sum_{j=1}^2 (p_{\xi_j} + \hat{A}_{\xi_j})^2 + \hat{B}$ on $D(T_C T_C^*) = \{\phi \in D(T_C^*) : T_C^*\phi \in D(T_C)\}$ and both operators are nonnegative and self-adjoint. Clearly

$$
T_C T_C^* = T_C^* T_C + 2\hat{B} \tag{3.1}
$$

on $D(T_C T_C^*) \cap D(T_C^* T_C)$, in particular, $T_C T_C^* \geq 2\hat{B}$ on $D(T_C T_C^*)$ (use that this relation is true on $D(T_C T_C^*) \cap D(T_C^* T_C)$, but $H_0^2(C)$, which is a subset of this intersection, is a core for $T_C T_C^*$ by the Kato-Rellich theorem). By supersymmetry, the spectra of $T_C T_C^*$ and $T_C^* T_C$ coincide except at zero, therefore $T_C^* T_C$ has a gap in the spectrum, i.e.,

$$\operatorname{spec}(T_C^* T_C) \subset \{0\} \cup [2\hat{B}, \infty). \tag{3.2}$$

Let Π_C be the spectral projection onto the zero energy level in the spectrum of the operator $T_C^* T_C$. Note that all the operators T_C, T_C^*, and therefore Π_C commute with p_{ξ_3}. In particular, T_C, therefore Π_C act trivially on functions depending only on the third direction, i.e. $\Pi_C = \Pi_C^{(2)} \otimes \mathrm{Id}_{\mathbf{R}}$, where $\Pi_C^{(2)}$ is the zero energy projection of $T_C^* T_C$ viewed as a two-dimensional operator acting on $L^2(C \cap \mathcal{P})$ with Neumann boundary conditions.

3.2 Estimate of the Neumann ground state density

We use the notations from Section 3.1, and, for simplicity, let $P = 0$.

Lemma 3.1 *Consider a cylinder C (see (2.4)), with center $P = 0$, then $\Pi_C^{(2)}$ is an integral operator with a continuous kernel on the open disk $D := int(C \cap \mathcal{P})$ of radius w, and its diagonal element satisfies*

$$\Pi_C^{(2)}(\xi_\perp, \xi_\perp) \leq e^{\hat{B} w^2/4} \frac{w^2}{\pi (w^2 - |\xi_\perp|^2)^2}. \tag{3.3}$$

Remark. With a slight abuse of notation, we set $\xi_\perp := (\xi_1, \xi_2)$, while previously it was $(\xi_1, \xi_2, 0)$.

Proof. Obviously $\operatorname{Ran} \Pi_C^{(2)} = \operatorname{Ker} T_C = \operatorname{Ker} T_C^* T_C$, where T_C is viewed as a two-dimensional operator with Neumann boundary conditions on the disk D of radius w. It is well known (and easy to verify) that any function $g \in L^2(D)$ of two variables, satisfying $T_C g = 0$, can be presented in the form $g = h\lambda_{\hat{B}}$, where

$$\lambda_{\hat{B}}(\xi_\perp) := \exp(-\hat{B}\xi_\perp^2/4) \tag{3.4}$$

and h is an analytic function on D. Notice that $\hat{B}\xi_\perp^2 \leq \hat{B} w^2$ on D, therefore

$$\exp(-\hat{B} w^2/4) \leq \lambda_{\hat{B}} \leq 1 \tag{3.5}$$

within the disk. This implies that $h \in \mathcal{A}^2(D)$, i.e. it belongs to the set of square-integrable analytic functions on D. Conversely, if $h \in \mathcal{A}^2(D)$, then $g = h\lambda_{\hat{B}}$ is in $L^2(D)$ and pointwise satisfies the equation $T_C g = 0$, therefore it is in $\mathrm{Ker}\, T_C$ (recall that the differential equation $T_C g = 0$ is equivalent to the Cauchy-Riemann equations for h). This establishes a bounded bijection between $\mathrm{Ran}\Pi_C^{(2)}$ and $\mathcal{A}^2(D)$ with bounded inverse.

Let $\mathcal{B}$ be the projection operator from $L^2(D)$ to $\mathcal{A}^2(D)$. It is an integral operator and we have an explicit formula for its kernel (Bergman kernel):

$$\mathcal{B}(\xi_\perp, \eta_\perp) = \frac{w^2}{\pi(w^2 - \xi_\perp \overline{\eta_\perp})^2}, \tag{3.6}$$

which is valid on the open disk (the Bergman kernel, in connection with the Pauli operator, was also used in [F]). The overline here means complex conjugation as we canonically imbed D into $\mathbf{C}$.

Let $h_m(z) := \alpha_m z^m$ be the standard orthonormal basis for $\mathcal{A}^2(D)$ (where α_m is the normalization), then $\mathcal{B}(\xi_\perp, \eta_\perp) = \sum_{m=0}^\infty h_m(\xi_\perp)\overline{h_m(\eta_\perp)}$, and the sum is absolutely convergent for the interior of the disk.

Therefore, by the bijection above, $\{h_m \lambda_{\hat{B}}\}_{m=0}^\infty$ is an orthogonal basis for $\mathrm{Ran}\, \Pi_C^{(2)}$ (orthogonality follows from integration in the angular variable). Moreover,

$$\|h_m \lambda_{\hat{B}}\|_{L^2(D)} \geq e^{-\hat{B}w^2/4} \|h_m\|_{L^2(D)} = e^{-\hat{B}w^2/4}. \tag{3.7}$$

This uniform lower bound shows that $\Pi_C^{(2)}$ is an integral operator on the open disk with continuous kernel

$$\Pi_C^{(2)}(\xi_\perp, \eta_\perp) = \sum_{m=0}^\infty \frac{\lambda_{\hat{B}}(\xi_\perp) h_m(\xi_\perp) \overline{\lambda_{\hat{B}}(\eta_\perp) h_m(\eta_\perp)}}{\|h_m \lambda_{\hat{B}}\|^2}, \tag{3.8}$$

since the sum above is absolutely convergent by the boundedness of the Bergman kernel.

Consequently we have for the diagonal element

$$\Pi_C^{(2)}(\xi_\perp, \xi_\perp) \leq e^{\hat{B}w^2/4} \mathcal{B}(\xi_\perp, \xi_\perp) \leq e^{\hat{B}w^2/4} \frac{w^2}{\pi(w^2 - |\xi_\perp|^2)^2}. \ \Box \tag{3.9}$$

3.3 Energy contribution from the Neumann ground states

If $\mathcal{U}$ is a non-negative function on $\mathbf{R}^3$ we shall study the negative spectrum of the operator

$$H_C^{low} = \Pi_C (p_{\xi_3}^2 - \chi_C \mathcal{U}) \Pi_C$$

on $L^2(C)$. We consider this operator defined as a quadratic form with Dirichlet boundary conditions on the top and bottom of the cylinder, i.e., defined on functions in $H^1(C)$, that vanish for $\xi_3 = \pm\lambda/2$.

Lemma 3.2 *If $0 \leq \mathcal{U} \in L^{3/2}(\mathbf{R}^3)$ then the sum of the negative eigenvalues of H_C^{low} is bounded below by*

$$-L_{1,1} \int_C \Pi_C^{(2)}(\xi_\perp, \xi_\perp) \mathcal{U}(\xi)^{3/2} \mathrm{d}\xi,$$

where $L_{1,1}$ is the constant in the one dimensional Lieb-Thirring inequality ((3.11) below).

Proof: For the sake of simplifying notations we shall assume that C is an infinite cylinder ($\lambda = \infty$). (This will only lower the sum of the negative eigenvalues.) Let $u_1, u_2, \ldots$ denote the (orthonormalized) eigenfunctions of H_C^{low} corresponding to negative eigenvalues. Then since $\Pi_C u_i = u_i$ we have that the sum of the negative eigenvalues of H_C^{low} is

$$\sum_i \int_C \left(|\partial_{\xi_3} u_i|^2 - \mathcal{U}(\xi)|u_i(\xi)|^2 \right) \mathrm{d}\xi.$$

We shall show below that for almost all $\xi_\perp$ in the disc D and all $f \in L^2(\mathbf{R})$ we have

$$0 \leq \sum_i \left| \int_{\mathbf{R}} f(\xi_3) u_i(\xi_\perp, \xi_3) \mathrm{d}\xi_3 \right|^2 \leq \Pi_C^{(2)}(\xi_\perp, \xi_\perp) \int_{\mathbf{R}} |f(\xi_3)|^2 \mathrm{d}\xi_3. \tag{3.10}$$

Let us first show how this estimate implies the lemma. For fixed $\xi_\perp \in D$ we consider the one dimensional operator $p_{\xi_3}^2 - \mathcal{U}(\xi_\perp, \cdot)$ acting on $L^2(\mathbf{R})$. It follows from the one dimensional Lieb-Thirring inequality that the sum of the negative eigenvalues of this operator is bounded below by

$$-L_{1,1} \int_{\mathbf{R}} \mathcal{U}(\xi)^{3/2} \mathrm{d}\xi_3, \tag{3.11}$$

which by assumption is finite for almost all $\xi_\perp \in D$. For almost all $\xi_\perp$ in D we therefore have the following inequality for $v \in L^2(\mathbf{R})$

$$\int_{\mathbf{R}} \left(|p_{\xi_3} v(\xi_3)|^2 - \mathcal{U}(\xi)|v(\xi_3)|^2 \right) \mathrm{d}\xi_3 \geq \sum_m \mu_{\xi_\perp}^{(m)} \left| \int_{\mathbf{R}} f_{\xi_\perp}^{(m)}(\xi_3) v(\xi_3) \mathrm{d}\xi_3 \right|^2,$$

where $f_{\xi_\perp}^{(m)}$ (the eigenfunctions of the one dimensional operator) are orthonormal in $L^2(\mathbf{R})$ and $\mu_{\xi_\perp}^{(m)}$ (the eigenvalues of the one dimensional operator) satisfy $\sum_m \mu_{\xi_\perp}^{(m)} \geq -L_{1,1} \int_{\mathbf{R}} \mathcal{U}(\xi)^{3/2} \mathrm{d}\xi_3$. The lemma follows easily from this.

We return to proving (3.10). We first use that since $u_i = \Pi_C u_i$ we have for fixed ξ_3

$$u_i(\xi_\perp, \xi_3) = \int_D \Pi_C^{(2)}(\xi_\perp, \eta_\perp) u_i(\eta_\perp, \xi_3) \mathrm{d}\eta_\perp.$$

Thus

$$\int_{\mathbf{R}} f(\eta_3) u_i(\xi_\perp, \eta_3) \mathrm{d}\eta_3 = \int_C f(\eta_3) \Pi_C^{(2)}(\xi_\perp, \eta_\perp) u_i(\eta) \mathrm{d}\eta.$$

It is easy to see from (3.8) that $f(\eta_3)\Pi_C^{(2)}(\xi_\perp, \eta_\perp)$ is in $L^2(C)$ as a function of η for each fixed $\xi_\perp \in D$. Since $u_1, u_2, \ldots$ are orthonormal it follows from Plancherel's formula that

$$\begin{aligned} \sum_i \left| \int_{\mathbf{R}} f(\eta_3) u_i(\xi_\perp, \eta_3) \mathrm{d}\eta_3 \right|^2 &= \sum_i \left| \int_C f(\eta_3) \Pi_C^{(2)}(\xi_\perp, \eta_\perp) u_i(\xi_\perp, \eta_3) \mathrm{d}\eta \right|^2 \\ &\le \int_C |f(\eta_3)|^2 |\Pi_C^{(2)}(\xi_\perp, \eta_\perp)|^2 \mathrm{d}\eta \\ &= \Pi_C^{(2)}(\xi_\perp, \xi_\perp) \int_{\mathbf{R}} |f(\eta_3)|^2 \mathrm{d}\eta_3, \end{aligned}$$

where in the last line we used that since $\Pi_C^{(2)}$ is a projection we have

$$\int_D |\Pi_C^{(2)}(\xi_\perp, \eta_\perp)|^2 \mathrm{d}\eta_\perp = \Pi_C^{(2)}(\xi_\perp, \xi_\perp). \ \Box$$

4 Proof of the Lieb-Thirring inequality

4.1 Localization into cubes

Consider a regular cubic tiling of the space by closed cubes $\{\Omega_i^0 : i \in \mathbf{Z}^3\}$ with edge-length

$$d := k_1 \min\{\|B\|^{-1/4} l^{1/2}, l, L\}, \tag{4.1}$$

and centers at the points of the lattice $(d\mathbf{Z})^3$. Here k_1 is an adjustable constant which we shall choose later, such that it satisfies $k_1 \le 1$. Let Ω_i denote the ten times bigger cubes, with the same center, we shall refer to Ω_i as cubes and to Ω_i^0 as small cubes. The cubes Ω_i cover the whole space with an overlap bounded by a universal constant.

4.1.1 Allocating the potential

We shall allocate the given potential V using the following lemma.

Lemma 4.1 *Given any non-negative function $V \in L^{5/2} \cap W^{1,1}$, there exists $\tilde{V} \in L^{5/2} \cap W^{1,1}$ with the following properties:*

$$\tilde{V} \ge V \tag{4.2}$$

$$\int_{\Omega_i} |\tilde V|^p \sim \int_{\Omega_j} |\tilde V|^p \quad \textit{if} \quad \Omega_i \cap \Omega_j \neq \emptyset \tag{4.3}$$

$$\int_{\mathbf{R}^3} |\tilde V|^p \le c \int_{\mathbf{R}^3} |V|^p \qquad \textit{and} \qquad \int_{\mathbf{R}^3} |\nabla \tilde V| \le c \int_{\mathbf{R}^3} |\nabla V|, \tag{4.4}$$

where $p = 1,\ 3/2$ *or* $5/2$, *and the notation* $X \sim Y$ *means that there is a universal constant* $c > 0$ *such that* $c^{-1} \le X/Y \le c$.

Proof: We define $\tilde V := V * \eta$, where η is the following distribution

$$\eta := \sum_{\mathbf{n} \in \mathbf{Z}^3} 2^{-|\mathbf{n}|} \cdot \delta_{d\mathbf{n}} \tag{4.5}$$

Property (4.2) is immediate since $\eta \ge \delta_0$; property (4.4) follows from $\|\eta\|_1 \le c$ and Young's inequality.

To check the property (4.3), first observe that it is enough to show

$$\int_{\Omega_i^0} |\tilde V|^p \sim \int_{\Omega_j^0} |\tilde V|^p \tag{4.6}$$

for the smaller (tiling) cubes with $\Omega_i^0 \cap \Omega_j^0 \neq \emptyset$, i.e. for neighboring small cubes.

Denoting the vector, pointing from the center of Ω_i^0 to that of Ω_j^0, by $d\mathbf{v}$, we have

$$\int_{\Omega_i^0} |\tilde V|^p = \int_{\Omega_i^0} \left(\sum_{\mathbf{n} \in \mathbf{Z}^3} 2^{-|\mathbf{n}|} V(x - d\mathbf{n}) \right)^p \mathrm{d}x = \int_{\Omega_j^0} \left(\sum_{\mathbf{n} \in \mathbf{Z}^3} 2^{-|\mathbf{n}|} V(x - d(\mathbf{n} + \mathbf{v})) \right)^p \mathrm{d}x$$

$$= \int_{\Omega_j^0} \left(\sum_{\mathbf{n} \in \mathbf{Z}^3} 2^{-|\mathbf{n}-\mathbf{v}|} V(x - d\mathbf{n}) \right)^p \mathrm{d}x \le 2^{p\sqrt{3}} \int_{\Omega_j^0} \left(\sum_{\mathbf{n} \in \mathbf{Z}^3} 2^{-|\mathbf{n}|} V(x - d\mathbf{n})) \right)^p \mathrm{d}x$$

$$= 2^{p\sqrt{3}} \int_{\Omega_j^0} |\tilde V|^p, \tag{4.7}$$

using that $|\mathbf{n}| \le \sqrt{3} + |\mathbf{n} - \mathbf{v}|$. □

We may therefore assume that our potential V satisfies the property (4.3), since, otherwise, in our theorem we can replace V by $\tilde V$ constructed above.

4.1.2 Splitting into overall spinup and spindown within a fixed cube

We begin with a definition.

Definition 4.2 *We call a cube Ω_j weak if*

$$k_2 \left(\int_{\Omega_j} V^{3/2} \right)^{2/3} < 1, \tag{4.8}$$

and we define

$$\delta_j := k_2 \left(\int_{\Omega_j} V^{3/2} \right)^{2/3}. \tag{4.9}$$

Otherwise, we call the cube Ω_j strong, and we let $\delta_j := 1$. Here k_2 denotes a constant to be chosen later.

Notice that if V satisfies property (4.3) then $\delta_i \sim \delta_j$ for overlapping Ω_i and Ω_j we call this the *regular variation property.*

Choose a partition of unity, consisting of C^∞-cutoff functions θ_i, such that $\operatorname{supp} \theta_j \subset \Omega_j$, $\theta_j \geq c$ on Ω_j^0, $\sum_j \theta_j^2 \equiv 1$, and $|\nabla \theta_j| \leq c d^{-1}$.

Lemma 4.3 *Let $P_-^{(j)}$ and $P_+^{(j)}$ denote the spin down and spin up projections relative to the magnetic field at the center Q_j of Ω_j (i.e. on the image of $P_\pm^{(j)}$ we have $\boldsymbol{\sigma} \cdot \mathbf{B}(Q_j) = \pm B(Q_j)$). Assume that the potential V satisfies (4.3). For any $f \in H^1(\mathbf{R}^3; \mathbf{C}^2)$ we have*

$$\int_{\mathbf{R}^3} \left(|\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}) f|^2 - V|f|^2 \right) \geq T_+(f) + T_-(f) \tag{4.10}$$

with

$$T_+(f) := \sum_j \delta_j T_+^{\Omega_j}(P_+^{(j)} \theta_j f), \quad T_-(f) := \sum_j \delta_j T_-^{\Omega_j}(P_-^{(j)} \theta_j f), \tag{4.11}$$

where

$$T_+^{\Omega_j}(g) := \int_{\Omega_j} \left(|\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}) g|^2 - (V \delta_j^{-1} + c d^{-2} + c k_1^4 B_j^{\#}) |g|^2 \right) \tag{4.12}$$

$$T_-^{\Omega_j}(g) := \int_{\Omega_j} \left(|\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}) g|^2 - (V \delta_j^{-1} + c d^{-2}) |g|^2 \right) \tag{4.13}$$

are quadratic forms defined on spinors $g \in H_0^1(\Omega_j, \mathbf{C}^2)$ and $P_+^{(j)} g = g$, $P_-^{(j)} g = g$, respectively. Here

$$B_j^{\#} = \sup_{|x - Q_j| < 200L} B(x). \tag{4.14}$$

The upper indices in the spin projections indicate that they depend on Ω_j, as they are determined by the direction of the magnetic field in the center of the cube. Sometimes, for shortness, we shall drop the index j and use the notation $T_\pm^\Omega(g)$.

To get the statement of the main Theorem 1.1, we have to sum up (4.10) for an arbitrary finite set of orthonormal spinors $\{f_m\}_{m=1}^N$, and give a uniform lower bound for the right hand side of

$$\sum_{m=1}^N \int_{\mathbf{R}^3} \Big(|\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})f_m|^2 - V|f_m|^2\Big) \geq \sum_{m=1}^N \left(T_+(f_m) + T_-(f_m)\right). \tag{4.15}$$

The rest of this section contains the proof of Lemma 4.3. In Section 4.2 we estimate T_+ in Section 4.3 we study T_- and finally in Section 4.4 we complete the proof of the main Theorem 1.1.

Proof of Lemma 4.3: Let f be an arbitrary spinor, then, by the localization formula and $\delta_j \leq 1$,

$$\int_{\mathbf{R}^3} \Big(|\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})f|^2 - V|f|^2\Big) \geq \sum_j \int_{\mathbf{R}^3} \Big(\delta_j\theta_j^2 |\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})f|^2 - V|\theta_j f|^2\Big)$$

$$= \sum_j \delta_j \int_{\mathbf{R}^3} \Big(|\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})(\theta_j f)|^2 - |\nabla\theta_j|^2|f|^2 - V\delta_j^{-1}|\theta_j f|^2\Big)$$

$$\geq \sum_j \delta_j \int_{\mathbf{R}^3} \Big(|\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})(\theta_j f)|^2 - (V\delta_j^{-1} + cd^{-2})|\theta_j f|^2\Big), \tag{4.16}$$

where, in the last step, we used the finite overlapping property of the cubes and the regular variation of the δ's to reallocate the localization errors. Recall that the assumption (4.3) on the potential implies the regular variation of the δ's.

Now we estimate the contribution of each cube individually. We consider a fixed cube $\Omega := \Omega_j$, and let ψ (in applications, $\psi = \theta_j f$) be a spinor with Dirichlet boundary conditions on Ω. We omit all references to the index j.

We can split the kinetic energy as follows (all integrals are over Ω, but can be considered over $\mathbf{R}^3$, since we may extend ψ to be zero outside Ω):

$$\int |\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})\psi|^2 = \int |(\mathbf{p}+\mathbf{A})\psi|^2 - \int (\psi, \boldsymbol{\sigma}\cdot\mathbf{B}\psi)$$

$$= \int |(\mathbf{p}+\mathbf{A})P_-\psi|^2 - \int (P_-\psi, \boldsymbol{\sigma}\cdot\mathbf{B}P_-\psi) + \int |(\mathbf{p}+\mathbf{A})P_+\psi|^2 - \int (P_+\psi, \boldsymbol{\sigma}\cdot\mathbf{B}P_+\psi)$$

$$- \int (P_-\psi, \boldsymbol{\sigma}\cdot\mathbf{B}P_+\psi) - \int (P_+\psi, \boldsymbol{\sigma}\cdot\mathbf{B}P_-\psi)$$

$$= \int |\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})P_-\psi|^2 + \int |\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})P_+\psi|^2 - \int (P_-\psi, \boldsymbol{\sigma}\cdot\mathbf{B}P_+\psi) - \int (P_+\psi, \boldsymbol{\sigma}\cdot\mathbf{B}P_-\psi). \quad (4.17)$$

We here used that $P_\pm$ commute with $\mathbf{p} + \mathbf{A}$. The cross terms are estimated by using $|\mathbf{B}(x)/B(x) - \mathbf{B}(Q)/B(Q)| \leq 20l^{-1}d$ which follows from (1.8) for any $x \in \Omega$ (as $|x - Q| \leq 10\sqrt{3}d < 20d$) and that $(P_+\psi, \boldsymbol{\sigma}\cdot\mathbf{B}(Q)P_-\psi) = 0$:

$$|(P_+\psi(x), \mathbf{B}(x)\cdot\boldsymbol{\sigma}P_-\psi(x))| = B(x)\left|\left(P_+\psi(x), \boldsymbol{\sigma}\cdot\left(\frac{\mathbf{B}(x)}{B(x)} - \frac{\mathbf{B}(Q)}{B(Q)}\right)P_-\psi(x)\right)\right|$$

$$\leq 20B^{\#}l^{-1}d|P_-\psi(x)||P_+\psi(x)| \leq 10B^{\#}l^{-1}d(\alpha|P_-\psi(x)|^2 + \alpha^{-1}|P_+\psi(x)|^2) \quad (4.18)$$

by the arithmetic-geometric mean inequality. Choosing $\alpha := d^{-3}l(B^{\#})^{-1}$ gives, for any $x \in \Omega$,

$$|(P_+\psi(x), \boldsymbol{\sigma}\cdot\mathbf{B}(x)P_-\psi(x))| \leq 10d^{-2}|P_-\psi(x)|^2 + 10k_1^4B^{\#}|P_+\psi(x)|^2,$$

where we have used that $d^4l^{-2}(B^{\#})^2 \leq k_1^4\|B\|^{-1}(B^{\#})^2 \leq k_1^4B^{\#}$.

We also split the potential term (using the pointwise identity $|\psi(x)|^2 = |P_+\psi(x)|^2 + |P_-\psi(x)|^2$):

$$\int (V\delta^{-1} + cd^{-2})|\psi|^2 = \int (V\delta^{-1} + cd^{-2})|P_-\psi|^2 + \int (V\delta^{-1} + cd^{-2})|P_+\psi|^2. \quad (4.19)$$

Altogether, we have

$$\int \Big(|\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})\psi|^2 - (V\delta^{-1} + cd^{-2})|\psi|^2\Big)$$

$$\geq \int \Big(|\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})P_+\psi|^2 - (V\delta^{-1} + cd^{-2} + ck_1^4B^{\#})|P_+\psi|^2\Big)$$

$$+ \int \Big(|\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})P_-\psi|^2 - (V\delta^{-1} + cd^{-2})|P_-\psi|^2\Big). \quad (4.20)$$

The lemma follows if we insert the formula above for each cube Ω_j and for $\psi = \theta_j f$ into (4.16). □

4.2 Treating the spin-up part in a cube

First, we focus on the contribution $T_+^{\Omega_j}(P_+^{(j)}\theta_j f)$, of a single cube Ω_j, in $T_+(f)$, so for a while we drop the indices and Ω will denote any of the Ω_j's. The notations P_+ and θ are used accordingly.

Theorem 4.4 *Let us consider a cube Ω with center Q and edge length $10d$, where $d = k_1 \min\{\|B\|^{-1/4} l^{1/2}, L, l\}$, and let P_+ be the spin-up projection according to the magnetic field in the center of Ω. Let $\delta := \min\{1, k_2 \left(\int_\Omega V^{3/2}\right)^{2/3}\}$ and $B^{\#} = \left\{\sup_{|x-Q|\leq 200L} B(x)\right\}$. Define a quadratic form on spinors $g \in H_0^1(\Omega, \mathbf{C}^2)$, $P_+ g = g$ by*

$$T_+^\Omega(g) := \int_\Omega \left(|\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})g|^2 - (V\delta^{-1} + cd^{-2} + ck_1^4 B^{\#})|g|^2 \right). \tag{4.21}$$

Then there exists a universal constant c' such that if $k_1 < c'$ and if $2^{3/2} L_{3,1} k_2^{-3/2} < 1$ [where $L_{3,1}$ is the Lieb-Thirring constant (see [LT])] then for any orthonormal family of spinors $\{f_m\}_{m=1}^N$ and for any $\theta \in C_0^\infty(\Omega)$, $0 \leq \theta \leq 1$, we have

$$\delta \sum_{m=1}^N T_+^\Omega(P_+\theta f_m) \geq -cd^{-2}(k_2^{-1/2} + k_2)\left(d^{-1}\int_\Omega V + \int_\Omega |\nabla V|\right) \tag{4.22}$$

if $\delta < 1$, and

$$\delta \sum_{m=1}^N T_+^\Omega(P_+\theta f_m) = \sum_{m=1}^N T_+^\Omega(P_+\theta f_m) \geq -c\int_\Omega V^{5/2} - ck_2 d^{-2}\left(d^{-1}\int_\Omega V + \int_\Omega |\nabla V|\right) \tag{4.23}$$

if $\delta = 1$, uniformly in N.

Before the proof we state a corollary, which shows how we are going to use this lemma to estimate T_+ (see (4.11)).

Corollary 4.5 *There exists $c'c > 0$ such that if $k_1 < c'$ and $k_2 > c'^{-1}$ (recall that T_+ depends on k_1 and k_2 through d and δ) then we have for any orthonormal family of spinors $\{f_m\}_{m=1}^N$*

$$\sum_{m=1}^N T_+(f_m) \geq -c\int_{\mathbf{R}^3} V^{5/2} - c(k_2^{-1/2} + k_2)d^{-2}\left(d^{-1}\int_{\mathbf{R}^3} V + \int_{\mathbf{R}^3} |\nabla V|\right). \tag{4.24}$$

Proof of Corollary 4.5. We can apply Theorem 4.4 for each cube $\Omega = \Omega_j$ and $\theta = \theta_j$. Putting together (4.22) and (4.23), using the universally bounded overlap of the cubes $\{\Omega_j\}_j$, we get

$$\sum_{m=1}^N T_+(f_m) = \sum_{m=1}^N \sum_j \delta_j T_+^{\Omega_j}(P_+^{(j)}\theta_j f_m) \geq -c\int_{\mathbf{R}^3} V^{5/2} - c(k_2^{-1/2} + k_2)d^{-2}\left(d^{-1}\int_{\mathbf{R}^3} V + \int_{\mathbf{R}^3} |\nabla V|\right), \tag{4.25}$$

which completes the proof. □.

Proof of Theorem 4.4. If $k_1 < (10(6+3\sqrt{3}))^{-1}$ then

$$10(6+3\sqrt{3})dl^{-1} \leq 10(6+3\sqrt{3})k_1 \leq 1, \tag{4.26}$$

This allows us to use Proposition 2.1 for the cube Ω, with edge length $10d$.

Using Lemma 2.1 for Ω (i.e. $\lambda = 10d$), by the Poincaré formula (2.21) and the estimate (2.19), there exists a vectorpotential $\tilde{\mathbf{A}}$ (depending on the fixed cube Ω), generating a constant direction field $\tilde{\mathbf{B}}$ (see (2.7)), parallel with $\mathbf{B}(Q)$, such that

$$\begin{aligned} |\mathbf{A}(x) - \tilde{\mathbf{A}}(x)| &\leq 10\sqrt{3}d \sup_{\Omega} |\mathbf{B}(x) - \tilde{\mathbf{B}}(x)| \leq cd^2 B^{\#} l^{-1}(1 + d(l^{-1} + L^{-1})) \\ &\leq ck_1^2 (B^{\#}/\|B\|)^{1/2} (B^{\#})^{1/2} \leq ck_1^2 (B^{\#})^{1/2} \end{aligned}$$

for all $x \in \Omega$. In the last step we used (2.2), and the definition of d. For brevity, we did not keep track of the explicit constant, but we applied the convention on the constants c mentioned at the beginning.

Therefore using a Cauchy-Schwarz inequality

$$\int_{\Omega} |\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A})g|^2 \geq \frac{1}{2} \int_{\Omega} \left(|\boldsymbol{\sigma} \cdot (\mathbf{p} + \tilde{\mathbf{A}})g|^2 - ck_1^4 B^{\#} |g|^2 \right). \tag{4.27}$$

We shall now prove that if k_1 is sufficiently small then there exists a universal constant $\tilde{c}$ such that $|\tilde{\mathbf{B}}(x)| \geq \tilde{c} B^{\#}$ for any $x \in \Omega$. To see this use that since $d \leq k_1 l$ and $d \leq k_1 L$, we have

$$||\mathbf{B}(x)| - |\tilde{\mathbf{B}}(x)|| \leq |\mathbf{B}(x) - \tilde{\mathbf{B}}(x)| \leq cdl^{-1} B^{\#}(1 + d(L^{-1} + l^{-1})) \leq ck_1 B^{\#}. \tag{4.28}$$

On the other hand for $x \in \Omega$ we have

$$\log(B^{\#}/B(x)) \leq 210L \|\nabla \log B\| = cLL^{-1} \leq c,$$

therefore

$$B(x) \geq \exp(-c) B^{\#}. \tag{4.29}$$

Thus we may find $0 < \tilde{c} < 1$ and then choose k_1 small enough to ensure that $\tilde{\mathbf{B}}(x)$, which points in the same direction as $\mathbf{B}(Q)$ for all $x \in \Omega$, satisfies

$$\tilde{c} B^{\#} \leq \tilde{B}(x) \leq (\tilde{c})^{-1} B^{\#} \tag{4.30}$$

Using the lower bound in (4.30), we shall now prove the inequality

$$(g, [\boldsymbol{\sigma} \cdot (\mathbf{p} + \tilde{\mathbf{A}})]^2 g) \geq (g, (\mathbf{p} + \tilde{\mathbf{A}})^2 g) + \tilde{c} B^{\#}(g, g) \tag{4.31}$$

on spinors g in the range of P_+ and with Dirichlet boundary condition on the cube Ω. The inequality (4.31) follows from

$$P_+[\boldsymbol{\sigma} \cdot (\mathbf{p} + \tilde{\mathbf{A}})]^2 P_+ = P_+[(\mathbf{p} + \tilde{\mathbf{A}})^2 + \boldsymbol{\sigma} \cdot \tilde{\mathbf{B}}]P_+ = P_+[(\mathbf{p} + \tilde{\mathbf{A}})^2 + |\tilde{\mathbf{B}}|]P_+, \tag{4.32}$$

where, in the last step, we used that $\tilde{\mathbf{B}}$ points in the same direction as $\mathbf{B}(Q)$.

This implies, along with (4.27) that

$$\begin{aligned} T_+^{\Omega}(g) &= \int_{\Omega} \Big(|\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}) g|^2 - (V\delta^{-1} + cd^{-2} + ck_1^4 B^{\#})|g|^2 \Big) \\ &\geq \frac{1}{2}(g, ((\mathbf{p} + \tilde{\mathbf{A}})^2 - W)g), \end{aligned} \tag{4.33}$$

with

$$W := 2V\delta^{-1} + cd^{-2} + (ck_1^4 - \tilde{c})B^{\#}. \tag{4.34}$$

Thus if $ck_1^4 \leq \tilde{c}$ we obtain

$$H(W) \geq (\mathbf{p} + \tilde{\mathbf{A}})^2 - 2V\delta^{-1} - cd^{-2} \tag{4.35}$$

where

$$H(W) := (\mathbf{p} + \tilde{\mathbf{A}})^2 - W \tag{4.36}$$

with Dirichlet boundary conditions on Ω.

Weak cube ($\delta < 1$):

Using the assumption $2^{3/2} L_{3,1} k_2^{-3/2} < 1$, the operator $(\mathbf{p} + \tilde{\mathbf{A}})^2 - 2V\delta^{-1}$, with Dirichlet boundary conditions on Ω, is seen to be positive by the CLR-bound (recall that the CLR bound [L] is true for Schrödinger operators with the magnetic Laplacian, $(\mathbf{p} + \tilde{\mathbf{A}})^2$, as well, as it was pointed out in [AHS], see also [MR]) since $L_{3,1} \int_{\Omega} (2V\delta^{-1})^{3/2} \leq 2^{3/2} L_{3,1} k_2^{-3/2} < 1$. Note that we could use the CLR-bound on $\mathbf{R}^3$, since the Dirichlet boundary condition for $P_+\theta f$ allows us to extend the functions onto the whole space.

Therefore, we have a lower bound $H(W) = (\mathbf{p} + \tilde{\mathbf{A}})^2 - W \geq -cd^{-2}$; in particular this is a lower bound on all eigenvalues. On the other hand, again by the CLR bound, the dimension of the spectral projection $\Pi_{\{H(W)<0\}}$ onto the negative spectrum of $H(W)$ (in this case this is just the number of negative eigenvalues of $H(W)$) is bounded by

$$\dim \Pi_{\{H(W)<0\}} = \operatorname{Tr} \Pi_{\{H(W)<0\}} \le L_{3,1} \int_\Omega (2V\delta^{-1} + cd^{-2})^{3/2} \le ck_2^{-3/2} + c. \tag{4.37}$$

Let $\{f_m\}_{m=1}^N$ be an arbitrary orthonormal set of spinors and $\theta \in C_0^\infty(\Omega)$, $0 \le \theta \le 1$. Then, from (4.33), we have for weak cubes Ω (i.e. $\delta \le 1$)

$$\delta \sum_{m=1}^N T_+^\Omega(P_+\theta f_m) = \sum_{m=1}^N \delta \int_\Omega \Big(|\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})(P_+\theta f_m)|^2 - (V\delta^{-1} + cd^{-2} + ck_1^4 B^\#)|P_+\theta f_m|^2 \Big)$$

$$\ge c\delta \sum_{m=1}^N (P_+\theta f_m, ((\mathbf{p}+\tilde{\mathbf{A}})^2 - W)P_+\theta f_m) = c\delta \operatorname{Tr}(H(W)P_+\theta\gamma_N\theta P_+) \tag{4.38}$$

where γ_N is the projection (written in Dirac notation)

$$\gamma_N := \sum_{m=1}^N |f_m\rangle\langle f_m|. \tag{4.39}$$

Since we have the operator inequality $0 \le P_+\theta\gamma_N\theta P_+ \le 1$ and $H(W) \ge -cd^{-2}$ we have

$$\operatorname{Tr}(H(W)P_+\theta\gamma_N\theta P_+) \ge -cd^{-2}\operatorname{Tr}\Pi_{\{H(W)<0\}} \tag{4.40}$$

and using (4.37) and the definition of δ for weak cubes, we obtain

$$\delta \sum_{m=1}^N T_+^\Omega(f_m) \ge -c\delta d^{-2}(k_2^{-3/2} + c) \ge -cd^{-2}(k_2^{-1/2} + k_2) \Big(\int_\Omega V^{3/2} \Big)^{2/3} \tag{4.41}$$

$$\ge -c(k_2^{-1/2} + k_2)d^{-2} \Big(d^{-1}\int_\Omega V + \int_\Omega |\nabla V| \Big),$$

which completes the proof of (4.22). Here, in the last step, we used Sobolev's inequality, which asserts that for any nonnegative function V on an arbitrary cube Ω, with edge length λ,

$$\Big(\int_\Omega V^{3/2} \Big)^{2/3} \le |\Omega|^{-1/3} \int_\Omega V + \int_\Omega |\nabla V| = \lambda^{-1} \int_\Omega V + \int_\Omega |\nabla V|. \tag{4.42}$$

The reason for using Sobolev's inequality is that finally we will have to add up the contributions from each cube, and $(\int_\Omega V^{3/2})^{2/3}$ is not additive. Note that this step introduces the extra terms, $\int V$ and $\int |\nabla V|$, which do not appear in Lieb-Thirring type estimates for constant direction field [E-1995], [Sob-1996(2)].

Strong cube ($\delta = 1$):

From (4.33) and (4.35) we have, for strong cubes Ω,

$$\sum_{m=1}^{N} T_+^{\Omega}(P_+\theta f_m) = \sum_{m=1}^{N} \int_\Omega \left(|\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})(P_+\theta f_m)|^2 - (V\delta^{-1} + cd^{-2} + ck_1^4 B^{\#})|P_+\theta f_m|^2 \right)$$

$$\geq c\sum_{m=1}^{N} (P_+\theta f_m, ((\mathbf{p}+\tilde{\mathbf{A}})^2 - 2V - cd^{-2})P_+\theta f_m) = \mathrm{Tr}(P_+\theta\gamma_N\theta P_+ H(2V + cd^{-2})), \qquad (4.43)$$

where $H(2V + cd^{-2}) = (\mathbf{p}+\tilde{\mathbf{A}})^2 - 2V - cd^{-2}$ with Dirichlet boundary conditions on Ω and γ_N was given (4.39).

Again using the operator inequality $0 \leq P_+\theta\gamma_N\theta P_+ \leq 1$ we can estimate $\mathrm{Tr}(P_+\theta\gamma_N\theta P_+ H(2V + cd^{-2}))$ below, using the variational principle, by the sum of the negativ eigenvalues of $H(2V + cd^{-2})$. By the standard Lieb-Thirring inequality (which is also valid for the magnetic Laplacian by the diamagnetic inequality) this eigenvalue sum is bounded below by $-2^{5/2}L_{3,1}\int_\Omega (V + cd^{-2})^{5/2}$. Therefore

$$\sum_{m=1}^{N} T_+^{\Omega}(P_+\theta f_m) \geq -c\int_\Omega (V + cd^{-2})^{5/2} \geq -c\int_\Omega V^{5/2} - cd^{-2} \qquad (4.44)$$

$$\geq -c\int_\Omega V^{5/2} - ck_2 d^2 \left(\int_\Omega V^{3/2}\right)^{2/3} \geq -c\int_\Omega V^{5/2} - ck_2 d^{-2}\left(d^{-1}\int_\Omega V + \int_\Omega |\nabla V|\right),$$

again using Sobolev's inequality (4.42) in the last step. $\square$

4.3 Treating the spin-down part in a fixed cube

Similarly to Section 4.2, first we focus on the contribution $T_-^{\Omega_j}(f)$, of a single cube, in $T_-(f)$ (see (4.11)), so we drop the indices (similarly for P_- and θ).

Theorem 4.6 *We again consider a cube Ω with center Q and edge length $10d$, where $d = k_1 \min\{\|B\|^{-1/4}l^{1/2}, L, l\}$, and let P_- be the spin-down projection according to the magnetic field in the center of Ω. Again, let $\delta := \min\{1, k_2\left(\int_\Omega V^{3/2}\right)^{2/3}\}$. Define a quadratic form on spinors $g \in H_0^1(\Omega, \mathbf{C}^2)$, with $P_- g = g$,*

$$T_-^{\Omega}(g) := \int_\Omega \left(|\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})g|^2 - (V\delta^{-1} + cd^{-2})|g|^2 \right). \qquad (4.45)$$

Define

$$w := \min\{k_3 (B^{\#})^{-1/2}, 10d\}, \tag{4.46}$$

where $B^{\#} = \sup_{|x-Q|\leq 200L} B(x)$. There exists a universal constant c' such that if $k_1 < c'$ and $k_2, k_3 > c'^{-1}$ then for any orthonormal family of spinors $\{f_m\}_{m=1}^N$ and for any $\theta \in C_0^\infty(\Omega)$, $0 \leq \theta \leq 1$, we have

$$\delta \sum_{m=1}^N T_-^\Omega(P_- \theta f_m) \geq -C_1(k_1, k_2, k_3) w^{-2} \left(d^{-1} \int_\Omega V + \int_\Omega |\nabla V| \right) \tag{4.47}$$

if $\delta < 1$ (weak cube), and

$$\delta \sum_{m=1}^N T_-^\Omega(P_- \theta f_m) = \sum_{m=1}^N T_-^\Omega(P_- \theta f_m) \geq -c \int_\Omega V^{5/2} - C_2(k_1, k_2, k_3) w^{-2} \int_\Omega V^{3/2} \tag{4.48}$$

if $\delta = 1$ (strong cube). Here $C_{1,2}(k_1, k_2, k_3)$ are positive constants depending on k_1, k_2, and k_3. Note that the estimates are uniform in N.

Before the proof we state an important corollary.

Corollary 4.7 *We may choose k_1 and k_2 (recall that T_- depends on k_1 and k_2 through d and δ) such that for any orthonormal family of spinors $\{f_m\}_{m=1}^N$ we have, uniformly in N,*

$$\sum_{m=1}^N T_-(f_m) \tag{4.49}$$

$$\geq -c \int_{\mathbf{R}^3} V^{5/2} - c \left(\int_{\mathbf{R}^3} (B + d^{-2}) V^{3/2} + \int_{\mathbf{R}^3} (B + d^{-2}) d^{-1} V + \int_{\mathbf{R}^3} (B + d^{-2}) |\nabla V| \right)$$

Proof of Corollary 4.7.

We can apply Theorem 4.6 for each cube Ω_j separately. Since

$$\sum_{m=1}^N T_-(f_m) = \sum_{m=1}^N \sum_j \delta_j T_-^{\Omega_j}(P_-^{(j)} \theta_j f_m)$$

by (4.11), using (4.47), (4.48) and the universally bounded overlap of the cubes, we obtain (4.49). In the calculation we used

$$w^{-2} \leq c(k_3^{-2} B^{\#} + d^{-2}) \leq c(B(x) + d^{-2}) \tag{4.50}$$

for any $x \in \Omega$ by (4.46) and (4.29). $\square$

The rest of this Section contains the proof of Theorem (4.6). This will involve several steps. First we need a further decomposition of the cube Ω into *cylinders* (Section 4.3.1). Then in Section 4.3.2 the kinetic energy is considered within each cylinder, and we treat the contributions from the lowest and the higher Landau levels separately. The main result of that section is summarized in Proposition 4.11. In Section 4.3.3 we include the potential. Finally, in Sections 4.3.4 and 4.3.5 we treat the lowest and the higher level contributions separately.

4.3.1 Localizing into cylinders

Throughout Sections 4.3.2 and 4.3.3, g will denote a spinor with Dirichlet boundary conditions on Ω and with $P_- g = g$. Recall that P_- is the spin space projection corresponding to the direction of the magnetic field in the center of Ω. In applications, g will be $P_-\theta f$.

Corresponding to the cube Ω of side length $\lambda = 10d$ and the magnetic field $\mathbf{B}$ we can, as in Section 2.1, construct cylinders C_P of base radius w given by (4.46). These cylinders were defined by

$$C_P = \{\xi \,:\, |\xi_3| \le 10d, |\xi_\perp - P| \le w\}; \tag{4.51}$$

in terms of the coordinate system $\{\xi_i\}_{i=1}^3$ of Section 2.1 and of the points $P \in \mathcal{P} = \{\xi \mid \xi_3 = 0\}$.

Choose a regular square lattice Λ with spacing $w/2$ in the plane $\mathcal{P}$. We consider only the cylinders C_P corresponding to $P \in \Lambda$. We define also the *core* of the cylinder C_P as

$$C_P^{(0)} := \{\xi \,:\, |\xi_3| \le 10d, |\xi_\perp - P| \le w/2\}. \tag{4.52}$$

Recall that all cylinders (and cores) are perfectly aligned.

Consider the finite subset of $\Lambda' \subset \Lambda$, consisting of points P, such that the corresponding core $C_P^{(0)}$ intersects the cube Ω. From now on we shall consider only cylinders and cores corresponding to $P \in \Lambda'$.

It is straightforward from the construction above that the cores $\{C_P^{(0)}\}_{P\in\Lambda'}$ cover the whole cube Ω. It also follows from the geometry that the number of cylinders that overlap at a fixed point is bounded by a universal constant.

If we assume that k_1 is small enough we can apply Corollary 2.2 and conclude that we can find a constant magnetic field $\tilde{\mathbf{B}}_P$ such that the magnetic field $\mathbf{B}_C^*(\xi) := \mathbf{B}(\xi) - \tilde{\mathbf{B}}_P$ within the cylinder $C = C_P$ satisfies the bound

$$\|\mathbf{B}_C^*\|_{L^\infty(C)} \le cl^{-1}dB^\#(1 + d(l^{-1} + L^{-1})) + wB^\#(l^{-1} + L^{-1}).$$

If we use $w \le 10d$ we obtain

$$\|\mathbf{B}_C^*\|_{L^\infty(C)} \le ck_1 B^\# \tag{4.53}$$

and if we use $w \leq k_3(B^{\#})^{1/2}$ we find

$$\begin{aligned}\|\mathbf{B}_C^*\|_{L^\infty(C)} &\leq c(B^{\#})^{1/2}d^{-1}\left(d^2(B^{\#})^{1/2}l^{-1}(1+k_1)+k_3k_1\right) \\ &\leq ck_1(k_1(1+k_1)+k_3)(B^{\#})^{1/2}d^{-1}. \end{aligned} \tag{4.54}$$

By (2.3) we get for the gradient

$$\|\nabla\mathbf{B}_C^*\|_{L^\infty(C)} = \|\nabla\mathbf{B}\|_{L^\infty(C)} \leq B^{\#}(l^{-1}+L^{-1}) \leq 2k_1B^{\#}d^{-1}. \tag{4.55}$$

Using Proposition 2.3 and the estimate (2.18) we find that there exists a vectorpotential $\mathbf{A}_C^*$ (depending on C), with $\nabla\times\mathbf{A}_C^* = \mathbf{B}_C^*$, such that

$$\|\mathbf{A}_C^*\|_{L^\infty(C)} \leq c\left[k_1(k_1(1+k_1)+k_3)(B^{\#})^{1/2}d^{-1}w + k_1B^{\#}d^{-1}w^2\right] \leq Kd^{-1} \tag{4.56}$$

using (4.46), where

$$K := ck_1k_3[k_1(k_1+1)+k_3]. \tag{4.57}$$

Define $\mathbf{A}_C^{(0)}(\xi) := \frac{1}{2}\tilde{\mathbf{B}}_P\times(\xi-P)$ the standard gauge depending on C, which generates the constant field $\tilde{\mathbf{B}}_P$ in the $\{\xi_i\}$ coordinate system (determined by Ω). Note that $\mathbf{A}_C^{(0)} = (A_{C,\xi_1}^{(0)}, A_{C,\xi_2}^{(0)}, 0)$, and $\mathbf{A}_C^{(0)}$ depends only on ξ_1 and ξ_2. The direction of $\tilde{\mathbf{B}}_P$ is independent of P since it has the same direction as $\mathbf{B}(Q)$. The notion of spin-up and spin-down is therefore independent of the cylinder.

Following Section 3.1, let us define the operator $T_C := (p_{\xi_1}+A_{C,\xi_1}^{(0)}) - i(p_{\xi_2}+A_{C,\xi_2}^{(0)})$ on C with Neumann boundary condition (more precisely, $D(T_C) := H^1(C)$). Let Π_C be the zero level spectral projection of $T_C^*T_C$. From (3.2) we get

$$(I-\Pi_C)T_C^*T_C(I-\Pi_C) \geq 2\tilde{B}_P(I-\Pi_C) \tag{4.58}$$

and from (3.1)

$$T_C^*T_C = \sum_{j=1}^{2}(p_{\xi_j}+A_{C,\xi_j}^{(0)})^2 - \tilde{B}_P, \tag{4.59}$$

on $D(T_C^*T_C)$ defined in Section 3.1. Note that as in (4.30) we get from (4.53) that

$$\tilde{c}B^{\#} \leq \tilde{B}_P \leq (\tilde{c})^{-1}B^{\#}. \tag{4.60}$$

We need the following lemma to estimate the kinetic energy within the cylinders.

We remark, that the operators T_C and Π_C are apriori defined on a subspace of $L^2(C)$ but they naturally act on the corresponding subspace of the spinor space $L^2(C,\mathbf{C}^2)$ as well, acting separately on both components. We shall not indicate this fact, i.e., in the notations, we do not distinguish between T_C and $T_C\otimes\mathrm{Id}_{\mathbf{C}^2}$.

Lemma 4.8 *For each cylinder $C = C_P$ with $P \in \Lambda'$ there exists a gauge function $\varphi_C : C \to \mathbf{R}$ such that the following lower bound holds for any g with $P_- g = g$*

$$\int_C |\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A})g|^2 \geq c \int_C \left(|T_C e^{i\varphi_C} g|^2 + |p_{\xi_3} e^{i\varphi_C} g|^2 \right) - cK^2 d^{-2} \int_C |g|^2. \tag{4.61}$$

Proof. The vectorpotential $\mathbf{A}_C^{(0)} + \mathbf{A}_C^*$ generates the original field $\mathbf{B}$ within C. Therefore there exists a real function φ_C (depending on C) on the cylinder C such that $\mathbf{A}_C^{(0)} + \mathbf{A}_C^* + \nabla\varphi_C = \mathbf{A}$, and

$$\int_C |\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A})g|^2 = \int_C |\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}_C^{(0)} + \mathbf{A}_C^*) e^{i\varphi_C} g|^2. \tag{4.62}$$

Now we can use a Cauchy-Schwarz inequality to estimate

$$\int_C |\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A})g|^2 \geq \frac{1}{2} \int_C |\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}_C^{(0)}) e^{i\varphi_C} g|^2 - 2\,(\sup_C |\mathbf{A}_C^*|)^2 \int_C |g|^2 \tag{4.63}$$

$$\geq \frac{1}{2} \int_C |\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}_C^{(0)}) e^{i\varphi_C} g|^2 - cK^2 d^{-2} \int_C |g|^2.$$

Using $P_- g = g$, the kinetic energy term splits:

$$\int_C |\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}_C^{(0)}) e^{i\varphi_C} g|^2 \tag{4.64}$$

$$= \int_C \left(|[\sigma_{\xi_1}(p_{\xi_1} + A^{(0)}_{C,\xi_1}) + \sigma_{\xi_2}(p_{\xi_2} + A^{(0)}_{C,\xi_2})] e^{i\varphi_C} g|^2 + |p_{\xi_3} e^{i\varphi_C} g|^2 \right),$$

with $p_{\xi_i} = \sum_j e^j_{\xi_i} p_j$, and similarly for $\sigma_{\xi_i} = \sum_j e^j_{\xi_i} \sigma_j$; where $e_{\xi_i} = (e^1_{\xi_i}, e^2_{\xi_i}, e^3_{\xi_i})$ is the i^{th} unit coordinate vector of the $\{\xi_i\}$ system in the standard basis. The proof of (4.64) is obvious by

$$\int_C |\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}_C^{(0)}) e^{i\varphi_C} g|^2 = \int_C |\sum_{j=1}^3 \sigma_{\xi_j} (p_{\xi_j} + A^{(0)}_{C,\xi_j}) e^{i\varphi_C} g|^2 \tag{4.65}$$

and by $P_- = (1 - \sigma_{\xi_3})/2$, as an easy calculation shows that the cross terms, involving p_{ξ_3}, are all zero. For example, if v and w are vectors such that $P_- v = v$ and $P_- w = w$, i.e. $\sigma_{\xi_3} v = -v$ and $\sigma_{\xi_3} w = -w$, then

$$\overline{\sigma_{\xi_2} v} \cdot \sigma_{\xi_3} w = -\overline{\sigma_{\xi_2} v} \cdot w = \overline{\sigma_{\xi_2} \sigma_{\xi_3} v} \cdot w = -\overline{\sigma_{\xi_3} \sigma_{\xi_2} v} \cdot w = -\overline{\sigma_{\xi_2} v} \cdot \sigma_{\xi_3} w \tag{4.66}$$

shows that the cross terms vanish, where v or w can be any vector of the form $(p_{\xi_j} + A^{(0)}_{C,\xi_j}) e^{i\varphi_C} g$.

By $P_{-}g = g$, one sees

$$\int_C |[\sigma_{\xi_1}(p_{\xi_1} + A^{(0)}_{C,\xi_1}) + \sigma_{\xi_2}(p_{\xi_2} + A^{(0)}_{C,\xi_2})]e^{i\varphi_C} g|^2$$

$$= \int_C |[(p_{\xi_1} + A^{(0)}_{C,\xi_1}) - i(p_{\xi_2} + A^{(0)}_{C,\xi_2})]e^{i\varphi_C} g|^2 = \int_C |Te^{i\varphi_C} g|^2, \tag{4.67}$$

since for any two vectors v and w we have, by an easy calculation, that

$$|\sigma_{\xi_1} P_{-}v + \sigma_{\xi_2} P_{-}w|^2 = |P_{-}(v - iw)|^2. \tag{4.68}$$

Putting together (4.63), (4.64) and (4.67) we obtain (4.61). $\square$

4.3.2 Splitting into lower and upper Landau levels

In this section we split the kinetic energy within the cube Ω into contributions from lower and higher 'local' Landau levels. This means that first we have to localize the kinetic energy in each cylinder $C := C_P$ and consider the constant magnetic field $\tilde{\mathbf{B}}_P$ within the cylinder .

Proposition 4.9 *There exist gauge functions $\varphi_C : C \to \mathbf{R}$ for each cylinder C intersecting Ω such that for any spinor g satisfying Dirichlet boundary conditions on Ω ($g \in H^1_0(\Omega, \mathbf{C}^2)$), and $P_{-}g = g$ we have with K as defined in (4.57)*

$$\int_\Omega |\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A})g|^2 \geq c \sum_C \int_C |\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A})g|^2 \geq c \sum_C T^{\Omega}_{up}(g, C) \tag{4.69}$$

$$+ c \sum_C \int_C \left(|p_{\xi_3} \Pi_C e^{i\varphi_C} \chi_C g|^2 - cK^2 d^{-2} \chi_\Omega \chi_{C^{(0)}} |\Pi_C e^{i\varphi_C} \chi_C g|^2 \right)$$

with

$$T^{\Omega}_{up}(g, C) := \tag{4.70}$$

$$\int_C \left(|T_C (I - \Pi_C) e^{i\varphi_C} \chi_C g|^2 + |p_{\xi_3} (I - \Pi_C) e^{i\varphi_C} \chi_C g|^2 - cK^2 d^{-2} \chi_\Omega \chi_{C^{(0)}} |(I - \Pi_C) e^{i\varphi_C} \chi_C g|^2 \right).$$

Here χ_Ω and $\chi_{C^{(0)}}$ denote the characteristic functions of the cube and the core of the cylinder respectively. The sum $\sum_C$ represents a sum over the family $\{C = C_P\}_{P \in \Lambda'}$.

Remark. The importance of introducing the core of the cylinders is due to the singularity in the spectral density of Π_C near the boundary of C (see (3.3).

Proof of Proposition 4.9.

Since the cylinders C cover the cube Ω, with a universally bounded overlap, we have

$$\int_\Omega |\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})g|^2 \geq c\sum_C \int_C |\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})g|^2. \tag{4.71}$$

We estimate the right side of (4.71) using Lemma 4.8. Since the *cores* also cover the all of Ω, we can reallocate the error in (4.61) to get

$$\sum_C \int_C |\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})g|^2 \tag{4.72}$$

$$\geq \sum_C \left[c\int_C \Big(|[(p_{\xi_1}+A^{(0)}_{C,\xi_1}) - i(p_{\xi_2}+A^{(0)}_{C,\xi_2})]e^{i\varphi_C}g|^2 + |p_{\xi_3}e^{i\varphi_C}g|^2 \Big) - cK^2d^{-2}\int_C \chi_\Omega\chi_{C^{(0)}}|g|^2 \right],$$

where we have inserted χ_Ω, the characteristic function of the cube Ω, since g is supported in Ω. Note that as a consequence of the reallocation it is the characteristic function $\chi_{C^{(0)}}$ which appears in the error term.

Let χ_C be the characteristic function of the cylinder C; instead of g we shall use $\chi_C g$ in order to be in the domain of T_C and Π_C. From (4.72) we obtain (recalling that p_{ξ_3} and Π_C commute)

$$\sum_C \int_C |\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})g|^2 \tag{4.73}$$

$$\geq c\sum_C \int_C \Big(|T_C\Pi_C e^{i\varphi_C}\chi_C g|^2 + |p_{\xi_3}\Pi_C e^{i\varphi_C}\chi_C g|^2 - cK^2d^{-2}\chi_\Omega\chi_{C^{(0)}}|\Pi_C e^{i\varphi_C}\chi_C g|^2 \Big)$$

$$+c\sum_C \int_C \Big(|T_C(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2 + |p_{\xi_3}(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2 - cK^2d^{-2}\chi_\Omega\chi_{C^{(0)}}|(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2 \Big),$$

using the pointwise inequality on C

$$|g|^2 = |e^{i\varphi_C}\chi_C g|^2 \tag{4.74}$$

$$\leq 2|\Pi_C e^{i\varphi_C}\chi_C g|^2 + 2|(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2.$$

Notice that the first term on the right hand side of (4.73), $|T_C\Pi_C e^{i\varphi_C}\chi_C g|^2$, is simply zero as Π_C projects onto the zero level of $T_C^*T_C$. This completes the proof of (4.69). $\square$

We next rewrite the upper spectral part, $T^\Omega_{up}(g,C)$ in order to introduce Dirichlet boundary condition. This rewriting will give further contributions to the lower spectral part (i.e., to the error term containing Π_C in (4.69)).

We use a further localization function. For each cylinder $C = C_P$ we choose a function $\theta_C \in C^\infty(\mathbf{R}^3)$, $0 \le \theta_C \le 1$, with the properties

$$\theta_C \text{ depends only on } \xi_\perp,\ \theta_C \equiv 1 \text{ on } C_P^{(0)},\ |\nabla \theta_C| \le cw^{-1} \tag{4.75}$$

$$\theta_C \equiv 0 \text{ for } |\xi_\perp - P| \ge w. \tag{4.76}$$

Then, the following lower bound is valid.

Lemma 4.10 *There exists $c' > 0$ such that if $k_1 < c'$ and $k_3 > c'^{-1}$ then for any $g \in H_0^1(\Omega, \mathbf{C}^2)$, with $P_- g = g$*

$$T_{up}^\Omega(g, C) \ge c \int_C \Big(|(\mathbf{p} + \mathbf{A}_C^{(0)})\theta_C(I - \Pi_C)e^{i\varphi_C}\chi_C g|^2 - c(K^2+1)d^{-2}\chi_\Omega |g|^2 \Big), \tag{4.77}$$

where K is given in (4.57).

The kinetic energy term is just the magnetic spinless Laplacian, since θ_C ensures Dirichlet boundary condition (recall that g is zero outside of Ω, in particular it automatically satisfies Dirichlet boundary condition at the ends of the cylinder, since the cylinders are longer than the cube). The error term will later (in the next Section 4.3.3) be reallocated into the cores of the cylinders, where the localization functions are identically 1.

Proof of Lemma 4.10.

Using (4.58) and (4.59) one easily obtains the operator inequality

$$(I - \Pi_C)T_C^* T_C (I - \Pi_C) \ge \frac{1}{3}(1 - \Pi_C)(\sum_{j=1}^2 (p_{\xi_j} + A_{C,\xi_j}^{(0)})^2 + 3\tilde{B}_P)(I - \Pi_C) \tag{4.78}$$

on $D(T_C^* T_C)$. Using the lower bound in (4.60) we find

$$\int_C |T_C(I - \Pi_C)\phi|^2 \ge \frac{1}{3}\int_C \sum_{j=1}^2 |(p_{\xi_j} + A_{C,\xi_j}^{(0)})(I - \Pi_C)\phi|^2 + \tilde{c}B^\# \int_C |(I - \Pi_C)\phi|^2 \tag{4.79}$$

for any $\phi \in D(T_C)$ (note that $I - \Pi_C$ maps $D(T_C)$ into itself, since Π_C maps the whole $L^2(C)$ into $D(T_C)$ as $T_C \Pi_C \phi = 0$ for any $\phi \in L^2(C)$).

Now we insert θ_C (which commutes with p_{ξ_3}) in (4.79), and obtain

$$\begin{aligned}
\int_C |T_C(I-\Pi_C)\phi|^2 &\geq \frac{1}{3}\int_C \sum_{j=1}^2 \theta_C^2 |(p_{\xi_j} + A^{(0)}_{C,\xi_j})(I-\Pi_C)\phi|^2 + \tilde{c}B^{\#}\int_C |(I-\Pi_C)\phi|^2 \\
&\geq \frac{1}{6}\int_C \sum_{j=1}^2 |(p_{\xi_j} + A^{(0)}_{C,\xi_j})\theta_C(I-\Pi_C)\phi|^2 - \frac{1}{3}\int_C |\nabla\theta_C|^2 |(I-\Pi_C)\phi|^2 + \tilde{c}B^{\#}\int_C |(I-\Pi_C)\phi|^2 \\
&\geq \frac{1}{6}\int_C \sum_{j=1}^2 |(p_{\xi_j} + A^{(0)}_{C,\xi_j})\theta_C(I-\Pi_C)\phi|^2 + \Big((\tilde{c} - ck_3^{-2})B^{\#} - cd^{-2}\Big)\int_C |(I-\Pi_C)\phi|^2 \\
&\geq \frac{1}{6}\int_C \sum_{j=1}^2 |(p_{\xi_j} + A^{(0)}_{C,\xi_j})\theta_C(I-\Pi_C)\phi|^2 - cd^{-2}\int_C |(I-\Pi_C)\phi|^2, \qquad (4.80)
\end{aligned}$$

using (4.75), (4.46), and assuming, in the last step, that the constant k_3 is sufficiently large.

We shall substitute (4.80) into the definition of $T^{\Omega}_{up}(g, C)$ (see (4.70)). First, using that θ_C commutes with p_{ξ_3}, we get (choose ϕ to be the components of the spinor $e^{i\varphi_C}\chi_C g$ in (4.80) and use that the operators Π_C, T_C etc. are diagonal in the spin space, i.e. they act separately on both components of a spinor)

$$\begin{aligned}
T^{\Omega}_{up}(g, C) &\geq c\int_C \Bigg(\sum_{j=1}^2 |(p_{\xi_j} + A^{(0)}_{C,\xi_j})\theta_C(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2 + |p_{\xi_3}\theta_C(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2 \\
&\qquad\qquad -c(K^2+1)d^{-2}|(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2\Bigg) \\
&\geq c\int_C \Big(|(\mathbf{p} + \mathbf{A}^{(0)}_C)\theta_C(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2 - c(K^2+1)d^{-2}|(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2\Big). \qquad (4.81)
\end{aligned}$$

In formula (4.81) we have neglected $\chi_{C^{(0)}}$ in the error term, as we shall have to reallocate that error term again anyway.

The last term in (4.81) is estimated trivially, as

$$\int_C |(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2 \leq \int_C |g|^2, \qquad (4.82)$$

using that $I - \Pi_C$ is a projection on $L^2(C)$; and we get (4.77) from (4.81) since we can insert χ_Ω for free, as $g\chi_\Omega = g$. □

The result of this section is summarized in the following kinetic energy bound. Its proof is straightforward from (4.69), (4.70) and (4.77).

Proposition 4.11 *There exist a universal constant $c' > 0$ and a constant K_1 depending on k_1 and k_3 such that if $k_1 < c'$ and $k_3 > c'^{-1}$ then for any spinor $g \in H_0^1(\Omega, \mathbf{C}^2)$, supported in Ω and satisfying $P_- g = g$, we have*

$$\int_\Omega |\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})g|^2 \geq \sum_C \int_C c|\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})g|^2 \tag{4.83}$$

$$\geq \sum_C \int_C \Big(c|p_{\xi_3}\Pi_C e^{i\varphi_C}\chi_C g|^2 - K_1 d^{-2}\chi_\Omega \chi_{C^{(0)}} |\Pi_C e^{i\varphi_C}\chi_C g|^2 \Big)$$

$$+\sum_C \int_C \Big(c|(\mathbf{p}+\mathbf{A}_C^{(0)})\theta_C(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2 - K_1 d^{-2}\chi_\Omega |g|^2\Big) . \ \square$$

4.3.3 Including the potential

Now we are ready to estimate $T_-^\Omega(g)$ (see (4.45)) completely in terms of the contributions from the lowest and higher Landau levels.

The result is

$$T_-^\Omega(g) \geq \sum_C \int_C \Big\{ c|p_{\xi_3}\Pi_C e^{i\varphi_C}\chi_C g|^2 - K_1 d^{-2}\chi_\Omega \chi_{C^{(0)}} |\Pi_C e^{i\varphi_C}\chi_C g|^2 \Big\} \tag{4.84}$$

$$+\sum_C \int_C \Big\{ c|(\mathbf{p}+\mathbf{A}_C^{(0)})\theta_C(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2 - K_1 d^{-2}\chi_\Omega |g|^2$$

$$-(V\delta^{-1} + cd^{-2})\chi_\Omega\chi_{C^{(0)}}|g|^2\Big\}$$

(in the last line we used that the cores of the cylinders, belonging to a cube, cover it, therefore the potential term in $T_-^\Omega(g)$ can be allocated into the cores, and we again inserted χ_Ω for free). After reallocating the $-K_1 d^{-2}\chi_\Omega|g|^2$ term into the cores and including it into the other error term, we have

$$T_-^\Omega(g) \geq \sum_C \int_C \Big\{ c|p_{\xi_3}\Pi_C e^{i\varphi_C}\chi_C g|^2 - K_1 d^{-2}\chi_\Omega \chi_{C^{(0)}} |\Pi_C e^{i\varphi_C}\chi_C g|^2 \Big\} \tag{4.85}$$

$$+\sum_C \int_C \Big\{ c|(\mathbf{p}+\mathbf{A}_C^{(0)})\theta_C(I-\Pi_C)e^{i\varphi_C}\chi_C g|^2 - (V\delta^{-1} + (K_1+c)d^{-2})\chi_\Omega\chi_{C^{(0)}}|g|^2\Big\} ,$$

Now using the pointwise inequality

$$|\psi|^2 = |e^{i\varphi_C}\chi_C\psi|^2 \leq 2|\Pi_C e^{i\varphi_C}\chi_C\psi|^2 + 2|(I-\Pi_C)e^{i\varphi_C}\chi_C\psi|^2 \tag{4.86}$$

for any $\psi \in H^1(C, \mathbf{C}^2)$, we get

$$\begin{aligned} T_-^\Omega(g) \;\geq\; & \sum_C \int_C \Big\{ c|p_{\xi_3} \Pi_C e^{i\varphi_C} \chi_C g|^2 - c \sum_C \int_C \Big(V\delta^{-1} + (K_1 + c) d^{-2} \Big) \chi_\Omega \chi_{C^{(0)}} |\Pi_C e^{i\varphi_C} \chi_C g|^2 \Big\} \\ & + \sum_C \int_C \Big\{ c|(\mathbf{p} + \mathbf{A}_C^{(0)}) \theta_C (I - \Pi_C) e^{i\varphi_C} \chi_C g|^2 \\ & - c(V\delta^{-1} + (K_1 + c) d^{-2}) \chi_\Omega \chi_{C^{(0)}} |(I - \Pi_C) e^{i\varphi_C} \chi_C g|^2 \Big\}. \end{aligned} \tag{4.87}$$

Finally, in the last line we can estimate $\chi_{C^{(0)}}$ by θ_C^2, since θ_C is 1 on the core.
We can summarize our result in the following Proposition.

Proposition 4.12 *There exists a universal constant $c' > 0$ such that if $k_1 < c'$ and $k_3 > c'^{-1}$ we obtain for any spinor $g \in H_0^1(\Omega, \mathbf{C}^2)$, with $P_- g = g$*

$$T_-^\Omega(g) \geq T_-^{\Omega,\,low}(g) + T_-^{\Omega,\,up}(g) \tag{4.88}$$

with

$$T_-^{\Omega,\,low}(g) := \sum_C \int_C \left\{ c|p_{\xi_3} \Pi_C e^{i\varphi_C} \chi_C g|^2 - U \chi_{C^{(0)}} |\Pi_C e^{i\varphi_C} \chi_C g|^2 \right\}, \tag{4.89}$$

$$T_-^{\Omega,\,up}(g) := \sum_C \int_C \left\{ c|(\mathbf{p} + \mathbf{A}_C^{(0)}) \theta_C (I - \Pi_C) e^{i\varphi_C} \chi_C g|^2 - U |\theta_C (I - \Pi_C) e^{i\varphi_C} \chi_C g|^2 \right\}, \tag{4.90}$$

where, for shortness, we denoted

$$U := c(V\delta^{-1} + K_2 d^{-2}) \chi_\Omega, \tag{4.91}$$

with K_2 depending on k_1 and k_3. Note that U depends only on the cube (via δ and χ_Ω) but not on the cylinders. $\square$

4.3.4 Finishing the lowest Landau level

The contribution from $T_-^{\Omega,\,low}(g)$ is estimated as follows.

Lemma 4.13 *There exist a universal constant $c' > 0$ and a constant K_3 depending on k_1, k_2 and k_3 such that if $k_1 < c'$ then for any orthonormal family of spinors $\{f_m\}_{m=1}^N$ we have $\theta \in C_0^\infty(\Omega)$, $0 \leq \theta \leq 1$. Then for any strong cube Ω we have*

$$\sum_{m=1}^N T_-^{\Omega,\,low}(P_- \theta f_m) \geq -K_3 w^{-2} \int_\Omega V^{3/2} \tag{4.92}$$

and for any weak cube Ω

$$\delta \sum_{m=1}^{N} T_{-}^{\Omega,\, low}(P_{-}\theta f_m) \geq -cK_3 w^{-2}\left(\int_{\Omega}|\nabla V| + d^{-1}\int_{\Omega} V\right). \tag{4.93}$$

Proof of Lemma 4.13

First we can write

$$\sum_{m=1}^{N} T_{-}^{\Omega,\, low}(P_{-}\theta f_m) \tag{4.94}$$

$$= \sum_{m=1}^{N}\sum_{C}\int_{C}\left(c|p_{\xi_3}\Pi_C e^{i\varphi_C}\chi_C P_{-}\theta f_m|^2 - U\chi_{C^{(0)}}|\Pi_C e^{i\varphi_C}\chi_C P_{-}\theta f_m|^2\right) = \sum_{C}\mathrm{Tr}(H_C^{low}\gamma_{C,N}^{low}),$$

where

$$H_C^{low} := \Pi_C(p_{\xi_3}^2 - U\chi_{C^{(0)}})\Pi_C \tag{4.95}$$

acting on $L^2(C)$ with Dirichlet boundary conditions on the third direction and Neumann boundary conditions along the transversal directions; and we have defined the operator

$$\gamma_{C,N}^{low} := e^{i\varphi_C}\chi_C P_{-}\theta\left(\sum_{m=1}^{N}|f_m\rangle\langle f_m|\right)\theta P_{-}\chi_C e^{-i\varphi_C}. \tag{4.96}$$

Note that the operator inequality $0 \leq \gamma_{C,N}^{low} \leq 1$ is satisfied, i.e., $\gamma_{C,N}^{low}$ is a density matrix. We therefore conclude from Lemma 3.2, with $\mathcal{U} = U\chi_{C^{(0)}}$, that

$$\mathrm{Tr}_{L^2(C)}(H_C^{low}\gamma_{C,N}^{low}) \geq -L_{1,1}\int_C \Pi_C^{(2)}(\xi_\perp,\xi_\perp)(U\chi_{C^{(0)}})^{3/2}(\xi_3,\xi_\perp)\mathrm{d}\xi_3\,\mathrm{d}\xi_\perp. \tag{4.97}$$

For any $|\xi_\perp - P| \leq w/2$ (for points in the core $C^{(0)}$), we find by (3.3),

$$\Pi_C^{(2)}(\xi_\perp;\xi_\perp) \leq e^{\tilde{B}_P w^2/4}\frac{w^2}{\pi\left(w^2 - \left(\frac{w}{2}\right)^2\right)^2} \leq cw^{-2}e^{ck_3^2}, \tag{4.98}$$

where we also used $w \leq k_3(B^{\#})^{-1/2}$ and (4.60).

Inserting these estimates into (4.94) we have, independently of N,

$$\sum_{m=1}^{N} T_{-}^{\Omega,\, low}(P_{-}\theta f_m) \geq -ce^{ck_3^2}w^{-2}\sum_C\int_{C^{(0)}} U^{3/2} \geq -ce^{ck_3^2}w^{-2}\int_{\Omega}(V\delta^{-1} + K_2 d^{-2})^{3/2} \tag{4.99}$$

$$\geq -ce^{ck_3^2}w^{-2}\delta^{-3/2}\int_{\Omega} V^{3/2} - ce^{ck_3^2}w^{-2}K_2^{3/2} \tag{4.100}$$

using the universally bounded overlap of the cylinders and inserting the definition of U (see (4.91)).

Using the definition of δ, for a *strong cube* Ω ($\delta = 1$) we obtain (4.92) immediately.

For a *weak cube* Ω we have

$$\begin{aligned}\sum_{m=1}^{N} T_-^{\Omega,\,low}(P_-\theta f_m) &\geq -ce^{ck_3^2} w^{-2}\delta^{-1}[k_2^{-1/2} + k_2 K_2^{3/2}] \left(\int_\Omega V^{3/2}\right)^{2/3} \\ &\geq -ce^{ck_3^2} w^{-2}\delta^{-1}[k_2^{-1/2} + k_2 K_2^{3/2}] \left(\int_\Omega |\nabla V| + d^{-1}\int_\Omega V\right) \quad (4.101)\end{aligned}$$

In the last step in (4.101) we again used Sobolev's inequality (4.42) to obtain a quantity which is additive in the cubes. □

4.3.5 Finishing the upper Landau levels

Here we consider the contribution from $T_-^{\Omega,\,up}(f)$ when summing up for an orthonormal set. The goal is

Lemma 4.14 *There exist a universal constant $c' > 0$ and a constant K_4 depending on k_1, k_2 and k_3 such that if $k_1 < c'$ and $k_2 > c'^{-1}$ then for any orthonormal family of spinors $\{f_m\}_{m=1}^{\infty}$ and $\theta \in C_0^\infty(\Omega)$, $0 \leq \theta \leq 1$ we obtain*

$$\sum_{m=1}^{N} T_-^{\Omega,\,up}(P_-\theta f_m) \geq -c\int_\Omega V^{5/2} - K_4 d^{-2}\int_\Omega V^{3/2}, \quad (4.102)$$

if Ω is a strong cube and

$$\delta\sum_{m=1}^{N} T_-^{\Omega,\,up}(P_-\theta f_m) \geq -K_4 d^{-2}\left(d^{-1}\int_\Omega V + \int_\Omega |\nabla V|\right). \quad (4.103)$$

if Ω is a weak cube.

Proof of Lemma 4.14.

As in the proof of Lemma 4.13, we can write

$$\sum_{m=1}^{N} T_-^{\Omega,\,up}(P_-\theta f_m) \quad (4.104)$$

$$= \sum_{m=1}^{N} \sum_{C} \int_C \left\{ c|(\mathbf{p} + \mathbf{A}_C^{(0)})\theta_C(I - \Pi_C)e^{i\varphi_C}\chi_C P_- \theta f_m|^2 - U|\theta_C(I - \Pi_C)e^{i\varphi_C}\chi_C P_- \theta f_m|^2 \right\}$$

$$= \sum_C \mathrm{Tr}(H_C^{up}\gamma_{C,N}^{up}),$$

where

$$H_C^{up} := (\mathbf{p} + \mathbf{A}_C^{(0)})^2 - cU\chi_C \tag{4.105}$$

with Dirichlet boundary conditions on C, and

$$\gamma_{C,N}^{up} := \theta_C(I - \Pi_C)e^{i\varphi_C}\chi_C P_- \theta \left(\sum_{m=1}^{N} |f_m\rangle\langle f_m| \right) \theta P_- \chi_C e^{-i\varphi_C}(I - \Pi_C)\theta_C \tag{4.106}$$

is an operator (density matrix) between 0 and 1.

We have to consider the cases that Ω is strong or weak separately. If Ω is a *strong cube* (i.e. $\delta = 1$), using the variational principle, the standard magnetic Lieb-Thirring inequality, the fact that the cylinders have universally bounded overlap, and finally the definition of U, we get

$$\sum_{m=1}^{N} T_-^{\Omega, up}(P_- \theta f_m) = \sum_C \mathrm{Tr}(H_C^{up}\gamma_{C,N}^{up}) \geq -c \sum_C \int_C U^{5/2} \geq -c \int_\Omega U^{5/2} \tag{4.107}$$

$$\geq -c \int_\Omega \left(V^{5/2} + K_2^{5/2} d^{-5} \right) = -c \int_\Omega V^{5/2} - cK_2^{5/2} d^{-2}$$

$$\geq -c \int_\Omega V^{5/2} - cd^{-2} k_2^{3/2} K_2^{5/2} \int_\Omega V^{3/2},$$

where, in the last step, we used that for strong cubes $k_2^{3/2} \int_\Omega V^{3/2} \geq 1$. This proves (4.102).

For Ω being a *weak cube* (i.e. $\delta < 1$), the operator $H_C^{up} = (\mathbf{p} + \mathbf{A}_C^{(0)})^2 - cU\chi_C$ with Dirichlet b.c. on C is bounded from below by $-cK_2 d^{-2}$ since $(\mathbf{p} + \mathbf{A}_C^{(0)})^2 - cV\delta^{-1}\chi_\Omega \chi_C$, defined on C with Dirichlet boundary conditions, is nonnegative if k_2 is large enough. For, by the CLR bound (recall that the CLR bound is true for spinless magnetic operator as well),

$$L_{3,1} \int_C (cV\delta^{-1}\chi_\Omega)^{3/2} \leq cL_{3,1}\delta^{-3/2} \int_\Omega V^{3/2} = cL_{3,1} k_2^{-3/2} < 1 \tag{4.108}$$

if k_2 satisfies $cL_{3,1} k_2^{-3/2} < 1$.

The dimension of the spectral projection $\Pi_{\{H_C^{up}<0\}}$ onto the negative spectrum of H_C^{up} is estimated, by the CLR-bound, as

$$\operatorname{Tr}\Pi_{\{H_C^{up}<0\}} \le cL_{3,1}\int_C \chi_\Omega(V\delta^{-1}+K_2d^{-2})^{3/2} \le cK_2^{3/2}w^2d^{-2}+c\delta^{-3/2}\int_C \chi_\Omega V^{3/2}, \tag{4.109}$$

where we used that the volume of the cylinder is $|C| \le cdw^2$. We have a lower bound for each of these eigenvalues, therefore, for Ω weak,

$$\begin{aligned}
\sum_{m=1}^N T_-^{\Omega,\,up}(P_-\theta f_m) &= \sum_C \operatorname{Tr}\left(H_C^{up}\gamma_{C,N}^{up}\right) \ge -cK_2d^{-2}\sum_C \operatorname{Tr}\Pi_{\{H_C^{up}<0\}} && (4.110)\\
&\ge -c\sum_C K_2d^{-2}\left(cK_2^{3/2}w^2d^{-2}+c\delta^{-3/2}\int_C\chi_\Omega V^{3/2}\right)\\
&\ge -cK_3^{5/2}w^2d^{-4}\left(\frac{cd^2}{w^2}\right)-c\delta^{-3/2}K_2d^{-2}\int_\Omega \chi_\Omega V^{3/2}\\
&\ge -cK_4d^{-2}\delta^{-1}\left(\int_\Omega V^{3/2}\right)^{2/3}\\
&\ge -cK_4d^{-2}\delta^{-1}\left(d^{-1}\int_\Omega V+\int_\Omega|\nabla V|\right), && (4.111)
\end{aligned}$$

where, from the second to the third line, we used that the number of cylinders belonging to a given cube is bounded by cd^2w^{-2} by elementary geometry. We used, moreover, that the cylinders have a finite overlap. We also used the definition of δ twice, and, in the last line, we applied Sobolev's inequality (4.42). □

4.3.6 Completing the treatment of the spin-down part

To obtain Theorem 4.6, we use Proposition 4.12 and Lemmas 4.13 and 4.14. Using the estimate (4.88) one easily obtains (4.47) from (4.93) and (4.103) by a simple calculation and using $w \le 10d$. Similarly, (4.92) and (4.102) imply (4.48). This completes the proof of Theorem 4.6. □

4.4 Completing the proof of the Lieb-Thirring theorem

Using Lemma 4.1 we may assume that V satisfies (4.3).

Obviously Lemma 4.3 and Corollaries 4.5 and 4.7 then give our Lieb-Thirring inequality (1.10). □.

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