SPECTRAL DEFORMATIONS OF JACOBI OPERATORS

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Abstract. We extend recent work concerning isospectral deformations for one-dimensional Schrödinger operators to the case of Jacobi operators. We provide a complete spectral characterization of a new method that constructs isospectral deformations of a given Jacobi operator $(Hu)(n) = a(n)u(n+1) + a(n-1)u(n-1) - b(n)u(n)$. Our technique is connected to Dirichlet data, that is, the spectrum of the operator $H_{n_0}^\infty$ on $\ell^2(-\infty, n_0) \oplus \ell^2(n_0, \infty)$ with a Dirichlet boundary condition at n_0. The transformation moves a single eigenvalue of $H_{n_0}^\infty$ and perhaps flips which side of n_0 the eigenvalue lives. On the remainder of the spectrum the transformation is realized by a unitary operator.

1. Introduction

Spectral deformations of Jacobi operators have proven useful in various applications such as inverse spectral theory and construction of solutions for the Toda and Kac van Moerbeke hierarchy [3], [7], [10], [14]. In [9] a powerful new spectral deformation method was introduced for Schrödinger operators. The aim of the present paper is to develop an analogous tool for Jacobi operators.

One approach to spectral deformations is to factor a given Jacobi operator

$$(Hu)(n) = a(n)u(n+1) + a(n-1)u(n-1) - b(n)u(n), \tag{1.1}$$

$u \in \mathfrak{D}(H) \subseteq \ell^2(\mathbb{Z})$, into a product of first order difference operators

$$H = A_\sigma^* A_\sigma + \lambda, \quad \sigma \in [-1, 1]. \tag{1.2}$$

Interchanging the order of A_σ^* and A_σ produces a second operator $H_\sigma = A_\sigma A_\sigma^* + \lambda$ whose spectral properties are closely related to those of H. In fact, depending on the parameter σ, one gets operators which are either isospectral to H or have the additional eigenvalue λ [4], [7] (see also [10]).

Clearly, the special form of (1.2) implies that $H - \lambda \geq 0$ and hence this single commutation method can only be applied to insert eigenvalues below the spectrum of H. However, ignoring this fact and performing two (suitable) commutation steps produces meaningful operators H_γ, $\gamma > 0$ (all intermediate operators are ill-defined unless λ is below the spectrum of H). The operators H_γ are isospectral to H except for the additional eigenvalue λ (for details see [7]).

The idea of our new method is to perform two single commutation steps as before (with possibly ill-defined intermediate operators), but now using different choices for the parameter λ in the first respectively second step. The investigation of the resulting transformed operator will be the task of this paper.

In order to further explain these ideas we need to introduce additional notation. The Dirichlet operator $H_{n_0}^\infty$ is obtained by restricting H to the subset of sequences

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$u \in \ell^2(\mathbb{Z})$ which satisfy $u(n_0) = 0$. It can be viewed as a rank one resolvent perturbation of H (cf. [6], [12]) implying that in each spectral gap (E_0, E_1) of H there can be at most one eigenvalue μ_0 of $H^\infty_{n_0}$. Since $H^\infty_{n_0}$ decomposes into a direct sum $H^\infty_{n_0} = H^\infty_{-,n_0} \oplus H^\infty_{+,n_0}$ (with respect to the decomposition $\ell^2(\mathbb{Z}) = \ell^2(-\infty, n_0) \oplus \ell^2(n_0, \infty)$) there is a sign σ_0 associated with each μ_0 such that $\mu_0 \in \sigma(H^\infty_{\sigma_0,n_0})$ ($\sigma(.)$ denoting the spectrum of an operator).

Let (E_0, E_1) be a spectral gap of H, $\mu_0 \in \sigma(H^\infty_{\sigma_0,n_0}) \cap (E_0, E_1)$ and pick $(\mu, \sigma) \in (E_0, E_1) \times \{\pm\}$. Then our transformation will send H to an operator $H_{(\mu,\sigma)}$ in such a way that $H, H_{(\mu,\sigma)}$ are unitarily equivalent and the Dirichlet datum (μ_0, σ_0) will be shifted to (μ, σ) whereas all other Dirichlet eigenvalues remain unchanged. We will hence refer to this transformation as the Dirichlet deformation method.

As anticipated, this transformation is realized by two single commutations; using $u_{\sigma_0}(\mu_0, .)$, $u_{-\sigma}(\mu, .)$ in the first, second factorization of H, respectively. Here $u_\pm(z, .)$, $z \in \mathbb{C}\backslash\sigma(H)$ denote weak solutions of $Hu = zu$ being square summable near $\pm\infty$. By [7] the operator $H_{(\mu,\sigma)}$ is associated with the sequences

$$a_{(\mu,\sigma)}(n) = a(n)\sqrt{\frac{W_{(\mu,\sigma)}(n-1)W_{(\mu,\sigma)}(n+1)}{W_{(\mu,\sigma)}(n)^2}}, \tag{1.3}$$

$$b_{(\mu,\sigma)}(n) = b(n) - \partial^* \frac{a(n)u_{\sigma_0}(\mu_0, n)u_{-\sigma}(\mu, n+1)}{W_{(\mu,\sigma)}(n)}, \tag{1.4}$$

where

$$W_{(\mu,\sigma)}(n) = \frac{W_n(u_{\sigma_0}(\mu_0)u_{-\sigma}(\mu))}{\mu - \mu_0}, \tag{1.5}$$

$W_n(.,..)$ denotes the (modified) Wronskian and $(\partial^* f)(n) = f(n) - f(n-1)$. Clearly, $H_{(\mu,\sigma)}$ is only well-defined if $W_{(\mu,\sigma)}(n+1)W_{(\mu,\sigma)}(n) > 0$; but this is ensured by [13], Theorem 4.6.

In the special case, where a, b are periodic (cf. [11]), these ideas have been used in [7] to give the discrete analogue of the FIT-formula derived in [5] for the isospectral torus of periodic Schrödinger operators.

2. Preliminary definitions

Throughout this paper we denote by $\ell(I) = \ell(M, N)$, $I = \{n \in \mathbb{Z} | M < n < N\}$, $M, N \in \mathbb{Z} \cup \{\pm\infty\}$ the set of complex-valued sequences $\{u(n)\}_{n\in I}$ and by $\ell^p(I)$, $1 \le p \le \infty$ the sequences $u \in \ell(I)$ such that $|u|^p$ is summable over I. The scalar product in the Hilbert space $\ell^2(I)$ will be denoted by

$$\langle u, v\rangle = \sum_{n\in I} \overline{u(n)}v(n), \quad u, v \in \ell^2(I). \tag{2.1}$$

We will be concerned with operators on $\ell^2(\mathbb{Z})$ associated with the difference expression

$$(\tau f)(n) = a(n)f(n+1) + a(n-1)f(n-1) + b(n)f(n), \tag{2.2}$$

where $a, b \in \ell(\mathbb{Z})$ satisfy

Hypothesis H. 2.1. *Suppose*

$$a(n) \in \mathbb{R}\backslash\{0\}, \quad b(n) \in \mathbb{R}, \quad n \in \mathbb{Z}. \tag{2.3}$$

If τ is limit point ($l.p.$) at both $\pm\infty$ (cf., e.g., [1], [2]), then τ gives rise to a unique self-adjoint operator H when defined maximally. Otherwise, we need to fix a boundary condition at each endpoint where τ is limit circle ($l.c.$) (cf., e.g., [1], [2]). Throughout this paper we denote by $u_\pm(z,.)$, $z \in \mathbb{C}$, nontrivial solutions of $\tau u = zu$ which satisfy the boundary condition at $\pm\infty$ (if any) with $u_\pm(z,.) \in \ell^2_\pm(\mathbb{Z})$, respectively. Here $\ell^2_\pm(\mathbb{Z})$ denotes the sequences in $\ell(\mathbb{Z})$ being ℓ^2 near $\pm\infty$. The solution $u_\pm(z,.)$ might not exist for $z \in \mathbb{R}$ (cf. [13], Lemma A.1), but if it exists it is unique up to a constant multiple.

Picking a fixed $z_0 \in \mathbb{C}\backslash\mathbb{R}$ we can characterize H by

$$H: \begin{array}{ccc} \mathfrak{D}(H) & \to & \ell^2(\mathbb{Z}) \\ f & \mapsto & \tau f \end{array}, \tag{2.4}$$

where the domain of H is explicitly given by

$$\begin{aligned} \mathfrak{D}(H) = \{f \in \ell^2(\mathbb{Z}) |\ \tau f \in \ell^2(\mathbb{Z}), \lim_{n\to+\infty} W_n(u_+(z_0), f) = 0, \\ \lim_{n\to-\infty} W_n(u_-(z_0), f) = 0\} \end{aligned} \tag{2.5}$$

and

$$W_n(f,g) = a(n)\Big(f(n)g(n+1) - f(n+1)g(n)\Big) \tag{2.6}$$

denotes the (modified) Wronskian. The boundary condition at $\pm\infty$ imposes no additional restriction on f if τ is $l.p.$ at $\pm\infty$ and can hence be omitted in this case.

Next, denote by P_{n_0} the orthogonal projection onto the one-dimensional subspace spanned by δ_{n_0} in $\ell^2(\mathbb{Z})$, where $\delta_{n_0}(n)$ is 1 for $n = n_0$ and 0 else. The Dirichlet operator is now defined by

$$H^\infty_{n_0} = (\mathbb{1} - P_{n_0}) H (\mathbb{1} - P_{n_0}) \tag{2.7}$$

in the Hilbert space $(\mathbb{1} - P_{n_0})\ell^2(\mathbb{Z}) = \{f \in \ell^2(\mathbb{Z}) | \langle \delta_{n_0}, f\rangle = 0\}$. Clearly, $H^\infty_{n_0}$ decomposes into a direct sum $H^\infty_{n_0} = H^\infty_{-,n_0} \oplus H^\infty_{+,n_0}$ with respect to the decomposition $\ell^2(\mathbb{Z}) = \ell^2(-\infty, n_0) \oplus \ell^2(n_0,\infty)$ and we have $\sigma_{ess}(H) = \sigma_{ess}(H^\infty_{n_0}) = \sigma_{ess}(H^\infty_{-,n_0}) \cup \sigma_{ess}(H^\infty_{+,n_0})$.

Without restriction we will only consider the case $n_0 = 0$ and abbreviate $H^\infty_{\pm,0} = H_\pm$ to simplify notation. This enables us to formulate our basic hypothesis.

Hypothesis H.2.2. *(i). Let (E_0, E_1) be a spectral gap of H, that is, $(E_0, E_1) \cap \sigma(H) = \{E_0, E_1\}$.*
(ii). Suppose $\mu_0 \in \sigma_d(H_{\sigma_0}) \cap [E_0, E_1]$.
(iii). Let $(\mu, \sigma) \in [E_0, E_1] \times \{\pm\}$ and $\mu \in (E_0, E_1)$ or $\mu \in \sigma_d(H)$.

Here $\sigma_d(H)$ denotes the discrete spectrum (i.e., $\sigma_d(H) = \sigma_d(H)\backslash\sigma_{ess}(H)$).

Remark 2.3. *Clearly, if μ_0 is an eigenvalue of two of the operators H, H_-, H_+, then it is also one of the third. Hence if $\mu_0 \in \sigma_d(H_{\sigma_0})\backslash\sigma_d(H_{-\sigma_0})$ then $\mu_0 \in (E_0, E_1)$ and if $\mu_0 \in \sigma_d(H_{\sigma_0}) \cap \sigma_d(H_{-\sigma_0})$ then $\mu_0 \in \{E_0, E_1\}$. (The choice of σ_0 in the latter case is irrelevant). Condition (ii) thus says that $\mu_0 = E_{0,1}$ is only allowed if $E_{0,1}$ is a discrete eigenvalue of H. Similar in (iii) for μ.*

Our next objective is to define the operator $H_{(\mu,\sigma)}$ of the Introduction. Since $H_{(\mu_0,\sigma_0)} = H$, we will assume $(\mu, \sigma) \neq (\mu_0, \sigma_0)$ without restriction.

Due to our assumption (H.2.2) we can find solutions $u_{\sigma_0}(\mu_0,.)$, $u_{-\sigma}(\mu,.)$ (cf. [13], Lemma A.1) and define

$$W_{(\mu,\sigma)}(n) = \begin{cases} \frac{W_n(u_{\sigma_0}(\mu_0)u_{-\sigma}(\mu))}{\mu-\mu_0} , & \mu \neq \mu_0 \\ \sum\limits_{m=\sigma_0\infty}^{n} u_{\sigma_0}(\mu_0,m)^2, & (\mu,\sigma)=(\mu_0,-\sigma_0) \end{cases} , \tag{2.8}$$

where $\sum_{m=+\infty}^{n} = -\sum_{m=n+1}^{\infty}$. The motivation for the case $(\mu,\sigma) = (\mu_0,-\sigma_0)$ follows from (assuming $u_{-\sigma}(\mu,m)$ holomorphic w.r.t. μ)

$$\begin{aligned} \lim_{\mu\to\mu_0} \frac{W_n(u_{\sigma_0}(\mu_0),u_{\sigma_0}(\mu))}{\mu-\mu_0} &= \lim_{\mu\to\mu_0} \sum_{m=\sigma_0\infty}^{n} u_{\sigma_0}(\mu_0,m)u_{\sigma_0}(\mu,m) \\ &= \sum_{m=\sigma_0\infty}^{n} u_{\sigma_0}(\mu_0,m)^2. \end{aligned} \tag{2.9}$$

¿From the proof of [13], Theorem 4.6 we infer

Lemma 2.4. *Suppose (H.2.2), then*

$$W_{(\mu,\sigma)}(n+1)W_{(\mu,\sigma)}(n) > 0, \quad n \in \mathbb{Z}. \tag{2.10}$$

Thus the sequences

$$a_{(\mu,\sigma)}(n) = a(n)\sqrt{\frac{W_{(\mu,\sigma)}(n-1)W_{(\mu,\sigma)}(n+1)}{W_{(\mu,\sigma)}(n)^2}}, \tag{2.11}$$

$$b_{(\mu,\sigma)}(n) = b(n) - \partial^* \frac{a(n)u_{\sigma_0}(\mu_0,n)u_{-\sigma}(\mu,n+1)}{W_{(\mu,\sigma)}(n)}, \tag{2.12}$$

are both well-defined and we can consider the associated difference expression

$$(\tau_{(\mu,\sigma)}u)(n) = a_{(\mu,\sigma)}(n)u(n+1) + a_{(\mu,\sigma)}(n-1)u(n-1) + b_{(\mu,\sigma)}(n)u(n). \tag{2.13}$$

The next lemma collects some basic properties which follow either from [7], Section 3 (choosing $N=2$) or can be verified directly.

Lemma 2.5. *Let*

$$(A_{(\mu,\sigma)}u)(z,n) = \frac{W_{(\mu,\sigma)}(n)u(z,n) - \frac{1}{z-\mu_0}u_{-\sigma}(\mu,n)W_n(u_{\sigma_0}(\mu_0),u(z))}{\sqrt{W_{(\mu,\sigma)}(n-1)W_{(\mu,\sigma)}(n)}}, \tag{2.14}$$

where $u(z)$ solves $\tau u = zu$ for $z \in \mathbb{C}\backslash\{\mu_0\}$. Then we have

$$\tau_{(\mu,\sigma)}(A_{(\mu,\sigma)}u)(z,n) = z(A_{(\mu,\sigma)}u)(z,n) \tag{2.15}$$

and

$$|(A_{(\mu,\sigma)}u)(z,n)|^2 = |u(z,n)|^2 + \frac{1}{|z-\mu|^2}\frac{u_{-\sigma}(\mu,n)}{u_{\sigma_0}(\mu_0,n)}\partial^*\frac{|W_n(u_{\sigma_0}(\mu_0),u(z))|^2}{W_{(\mu,\sigma)}(n)}. \tag{2.16}$$

Moreover, the sequences

$$u_{\mu_0}(n) = \frac{u_{-\sigma}(\mu,n)}{\sqrt{W_{(\mu,\sigma)}(n-1)W_{(\mu,\sigma)}(n)}}, \tag{2.17}$$

$$u_{\mu}(n) = \frac{u_{\sigma_0}(\mu_0,n)}{\sqrt{W_{(\mu,\sigma)}(n-1)W_{(\mu,\sigma)}(n)}}, \tag{2.18}$$

satisfy $\tau_{(\mu,\sigma)}u = \mu_0 u$, $\tau_{(\mu,\sigma)}u = \mu u$ respectively. Note also

$$u_{\sigma_0}(\mu_0,0) = u_\mu(0) = 0 \tag{2.19}$$

and

$$u_{\mu_0}(n)u_\mu(n) = \partial^* \frac{1}{W_{(\mu,\sigma)}(n)}. \tag{2.20}$$

In addition, let $u(z), \hat{u}(z)$ satisfy $\tau u = zu$, then

$$W_{(\mu,\sigma),n}(u_{\mu_0}, A_{(\mu,\sigma)}u(z)) = \frac{z-\mu}{z-\mu_0}\frac{W_n(u_{-\sigma}(\mu), u(z))}{W_{(\mu,\sigma)}(n)}, \tag{2.21}$$

$$W_{(\mu,\sigma),n}(u_\mu, A_{(\mu,\sigma)}u(z)) = \frac{W_n(u_{\sigma_0}(\mu_0), u(z))}{W_{(\mu,\sigma)}(n)}, \tag{2.22}$$

$$\begin{aligned} W_{(\mu,\sigma),n}(A_{(\mu,\sigma)}u(z), A_{(\mu,\sigma)}\hat{u}(\hat{z})) &= \frac{z-\mu}{z-\mu_0} W_n(u(z), \hat{u}(\hat{z})) \\ &+ \frac{z-\hat{z}}{(z-\mu)(\hat{z}-\mu_0)} \frac{W_n(u_\sigma(\mu), u(z)) W_n(u_{-\sigma_0}(\mu_0), \hat{u}(\hat{z}))}{W_{(\mu,\sigma)}(n)}, \end{aligned} \tag{2.23}$$

where

$$W_{(\mu,\sigma),n}(u,v) = a_{(\mu,\sigma)}(n)\Big(u(n)v(n+1) - u(n+1)v(n)\Big). \tag{2.24}$$

Having these preliminaries out of the way, we will now define operators associated with $\tau_{(\mu,\sigma)}$ by introducing suitable boundary conditions (since $\tau_{(\mu,\sigma)}$ is not necessarily $l.p.$ at $\pm\infty$). We single out the following three situations, which are the only ones where the spectra of H and $H_{(\mu,\sigma)}$ are closely related.

Let $\omega \in \{\pm\}$ and

$$BC_\omega(f) = \begin{cases} \lim_{n\to\omega\infty} W_n(u_\omega, f) = 0 & \text{if } \tau_{(\mu,\sigma)} \text{ is } l.c. \text{ at } \omega\infty \\ 0 & \text{if } \tau_{(\mu,\sigma)} \text{ is } l.p. \text{ at } \omega\infty \end{cases}, \tag{2.25}$$

where u_ω is chosen according to one of the following cases:

Case I: τ is $l.p.$ at $\omega\infty$. Choose $u_\omega = u_\mu$ or $u_\omega = u_{\mu_0}$.
Case II: τ is $l.c.$ at $\omega\infty$.

- (i). If $\omega = \sigma = \sigma_0$ choose $u_\omega = u_\mu$.
- (ii). If $-\omega = \sigma = \sigma_0$ choose $u_\omega = u_{\mu_0}$.
- (iii). If $\sigma = -\sigma_0$ and $\mu \in \sigma_d(H)$ choose $u_\omega = u_{\mu_0}$.
- (iv). If $\sigma = -\sigma_0$ and $\mu_0 \in \sigma_d(H)$ choose $u_\omega = u_\mu$.

(Note that in Case II (iii) and (iv) are the same if $(\mu,\sigma) = (\mu_0, -\sigma_0)$.)

Using this boundary conditions we define

$$H_{(\mu,\sigma)}: \begin{array}{ccc} \mathfrak{D}(H_{(\mu,\sigma)}) & \to & \ell^2(\mathbb{Z}) \\ f & \mapsto & \tau_{(\mu,\sigma)} f \end{array}, \tag{2.26}$$

where the domain of $H_{(\mu,\sigma)}$ is explicitly given by

$$\mathfrak{D}(H_{(\mu,\sigma)}) = \{f \in \ell^2(\mathbb{Z}) | \tau_{(\mu,\sigma)} f \in \ell^2(\mathbb{Z}), BC_-(f) = BC_+(f) = 0\}. \tag{2.27}$$

As always, there is no boundary condition at $\omega\infty$ in (2.27) if $\tau_{(\mu,\sigma)}$ is $l.p.$ at $\omega\infty$, $\omega \in \{\pm\}$. Furthermore, $H_{(\mu,\sigma),\pm}$ denote the corresponding Dirichlet half-line operators with respect to the base point $n_0 = 0$.

3. Half-line operators

In this section we will give a complete spectral characterization of the half-line operators $H_{(\mu,\sigma),\pm}$. In addition, this will provide all necessary results for the investigation of $H_{(\mu,\sigma)}$.

To begin with we compute the Weyl $\tilde{m}$-functions

$$\tilde{m}_\pm(z) = \mp \frac{u_\pm(z,1)}{a(0) u_\pm(z,0)}. \tag{3.1}$$

We recall that $\tilde{m}_\pm(z)$ are equivalently given by

$$\tilde{m}_\pm(z) = \frac{\pm 1}{a(0)} \lim_{n\to\pm\infty} \frac{W_n(c(z),f)}{W_n(s(z),f)}, \tag{3.2}$$

where $c(z,n)$, $s(z,n)$ form a fundamental system for τ (i.e., $(\tau - z)c(z) = (\tau - z)s(z) = 0$, $s(z,0) = c(z,1) = 0$, and $s(z,1) = c(z,0) = 1$) and f is arbitrary if τ is *l.p.* at $\pm\infty$ respectively solves $(\tau-\lambda)f = 0$, $\lambda \in \mathbb{R}$, and satisfies the boundary condition at $\pm\infty$ if τ is *l.c.* at $\pm\infty$ (cf. [7], Appendix B).

Theorem 3.1. *Let $\tilde{m}_\pm(z)$, $\tilde{m}_{(\mu,\sigma),\pm}(z)$ denote the Weyl $\tilde{m}$-functions of H, $H_{(\mu,\sigma)}$ respectively. Then we have*

$$\tilde{m}_{(\mu,\sigma),\pm}(z) = \frac{1}{1+\gamma_{(\mu,\sigma)}} \Big(\frac{z-\mu_0}{z-\mu} \tilde{m}_\pm(z) \mp \frac{\gamma_{(\mu,\sigma)}}{z-\mu} \Big), \tag{3.3}$$

where

$$\gamma_{(\mu,\sigma)} = \left\{ \begin{array}{ll} -\sigma(\mu-\mu_0)\tilde{m}_{-\sigma}(\mu), & \mu \neq \mu_0 \\ -\frac{\sigma_0 u_{\sigma_0}(\mu_0,\sigma_0 1)^2}{\sum\limits_{n\in\sigma_0\mathbb{N}} u_{\sigma_0}(\mu_0,n)^2}, & (\mu,\sigma) = (\mu_0,-\sigma_0) \end{array} \right. . \tag{3.4}$$

Proof. We first note that

$$\begin{aligned} c_{(\mu,\sigma)}(z,n) &= \frac{z-\mu_0}{z-\mu} (A_{(\mu,\sigma)} c)(z,n) - \frac{\gamma_{(\mu,\sigma)}}{z-\mu} a(0) (A_{(\mu,\sigma)} s)(z,n), && (3.5)\\ s_{(\mu,\sigma)}(z,n) &= \sqrt{1+\gamma_{(\mu,\sigma)}} (A_{(\mu,\sigma)} s)(z,n) && (3.6) \end{aligned}$$

constructed from the fundamental system $c(z,n)$, $s(z,n)$ for τ form a fundamental system for $\tau_{(\mu,\sigma)}$ corresponding to the same initial conditions. Furthermore, note

$$\frac{W_{(\mu,\sigma)}(1)}{W_{(\mu,\sigma)}(0)} = 1 + \gamma_{(\mu,\sigma)}, \quad \frac{W_{(\mu,\sigma)}(0)}{W_{(\mu,\sigma)}(-1)} = 1. \tag{3.7}$$

Now the result follows upon evaluating

$$\tilde{m}_{(\mu,\sigma),\pm}(z) = \frac{\pm 1}{a_{(\mu,\sigma)}(0)} \lim_{n\to\pm\infty} \frac{W_{(\mu,\sigma),n}(c_{(\mu,\sigma)}(z),u_\omega)}{W_{(\mu,\sigma),n}(s_{(\mu,\sigma)}(z),u_\omega)}. \tag{3.8}$$

Using (2.21) one obtains for $u_\omega = u_\mu(n), u_{\mu_0}(n)$, $v_\omega = u_{\sigma_0}(\mu_0,n), u_{\sigma_0}(\mu_0,n)$ respectively (according to Case I or II above)

$$\tilde{m}_{(\mu,\sigma),\pm}(z) = \frac{1}{1+\gamma_{(\mu,\sigma)}} \Big(\frac{z-\mu_0}{z-\mu} \frac{\pm 1}{a(0)} \lim_{n\to\pm\infty} \frac{W_n(c(z),v_\omega)}{W_n(s(z),v_\omega)} \mp \frac{\gamma_{(\mu,\sigma)}}{z-\mu} \Big) \tag{3.9}$$

the claim follows. □

Observe that even if there seems to be some freedom in the choice of the boundary condition BC_ω at first sight, Theorem 3.1 shows that different choices give rise to the same operator $H_{(\mu,\sigma)}$ (since $\tilde{m}_{(\mu,\sigma),\pm}(z)$ determine $H_{(\mu,\sigma)}$ uniquely). This fact will be used in the proof of Lemma 4.7. As a second consequence we note

Corollary 3.2. *The sequences*

$$u_{(\mu,\sigma),\pm}(z,n) = (A_{(\mu,\sigma)} u_\pm)(z,n), \quad z \in \mathbb{C}\backslash\{\mu,\mu_0\}, \tag{3.10}$$

are square summable near $\pm\infty$ and satisfy the boundary condition of $H_{(\mu,\sigma),\pm}$ at $\pm\infty$ (if any). Moreover, the same is true for

$$u_{(\mu,\sigma),-\sigma}(\mu,n) = W_{(\mu,\sigma)}(n) u_{\mu_0}(n) - u_\mu(n) \sum_{m=-\sigma\infty}^{n} u_{-\sigma}(\mu,m)^2, \tag{3.11}$$

$$u_{(\mu,\sigma),\sigma_0}(\mu_0,n) = W_{(\mu,\sigma)}(n) u_\mu(n) - u_{\mu_0}(n) \sum_{m=\sigma_0\infty}^{n} u_{\sigma_0}(\mu_0,m)^2, \tag{3.12}$$

and

$$u_{(\mu,\sigma),\sigma}(\mu,n) = u_\mu(n), \quad \mu \notin \sigma_d(H), \tag{3.13}$$

$$u_{(\mu,\sigma),-\sigma_0}(\mu_0,n) = u_{\mu_0}(n), \quad \mu_0 \notin \sigma_d(H). \tag{3.14}$$

If μ or $\mu_0 \in \sigma_d(H)$ one has to replace the last formulas by

$$u_{(\mu,\sigma),\sigma}(\mu,n) = W_{(\mu,\sigma)}(n) u_{\mu_0}(n) - u_\mu(n) \sum_{m=\sigma\infty}^{n} u_{-\sigma}(\mu,m)^2, \tag{3.15}$$

$$u_{(\mu,\sigma),-\sigma_0}(\mu_0,n) = W_{(\mu,\sigma)}(n) u_\mu(n) - u_{\mu_0}(n) \sum_{m=-\sigma_0\infty}^{n} u_{\sigma_0}(\mu_0,m)^2, \tag{3.16}$$

respectively.

Proof. Follows immediately form

$$u_{(\mu,\sigma),\pm}(z,n) = \frac{c_\pm(z)}{1+\gamma_{(\mu,\sigma)}} \frac{z-\mu_0}{z-\mu} \Big(\frac{c_{(\mu,\sigma)}(z,n)}{a_{(\mu,\sigma)}(0)} \mp \tilde{m}_{(\mu,\sigma),\pm}(z) s_{(\mu,\sigma)}(z,n) \Big) \tag{3.17}$$

if

$$u_\pm(z,n) = c_\pm(z) \Big(\frac{c(z,n)}{a(0)} \mp \tilde{m}_\pm(z) s(z,n) \Big). \tag{3.18}$$

If $z=\mu,\mu_0$ one can assume $u_\pm(z)$ holomorphic with respect to z near μ,μ_0 ([13], Theorem A.1) and consider limits (compare [13], Theorem A.3). □

Next we are interested in the pole structure of $\tilde{m}_{(\mu,\sigma),\pm}(z)$ near $z=\mu,\mu_0$. A straightforward investigation of (3.3) (invoking the Herglotz property of $\tilde{m}_{(\mu,\sigma),\pm}(z)$) shows

Corollary 3.3. *We have*

$$\tilde{m}_{(\mu,\sigma),\pm}(z) = \begin{cases} -\frac{\gamma_\mu}{z-\mu} + O(z-\mu)^0, & \omega = \sigma \\ O(z-\mu)^0, & \omega = -\sigma \end{cases}, \quad \omega \in \{\pm\}, \tag{3.19}$$

where

$$\gamma_\mu = \begin{cases} (\mu-\mu_0)(\tilde{m}_+(\mu) + \tilde{m}_-(\mu)), & \mu \neq \mu_0 \\ \frac{u_-(\mu,-1)^2}{\sum_{n=-\infty}^{-1} u_-(\mu,n)^2} + \frac{u_+(\mu,1)^2}{\sum_{n=1}^{+\infty} u_+(\mu,n)^2}, & (\mu,\sigma) = (\mu_0,-\sigma_0) \end{cases} \geq 0. \tag{3.20}$$

Moreover, $\gamma_\mu = 0$ if $\mu \in \sigma_d(H)\backslash\{\mu_0\}$ and $\gamma_\mu > 0$ else. If $\mu \neq \mu_0$, then

$$\tilde{m}_{(\mu,\sigma),\pm}(z) = O(z-\mu_0)^0 \tag{3.21}$$

and note $\gamma_\mu = a(0)^{-2}(\mu-\mu_0) G(\mu,0,0)^{-1}$.

Using the previous corollary plus weak convergence of $\pi^{-1}\mathrm{Im}(\tilde{m}_\pm(\lambda+\mathrm{i}\varepsilon))d\lambda$ to the corresponding spectral measure $d\tilde{\rho}_\pm(\lambda)$ as $\varepsilon\downarrow 0$ implies

Lemma 3.4. *Let $d\tilde{\rho}_\pm(\lambda)$ and $d\tilde{\rho}_{(\mu,\sigma),\pm}(\lambda)$ be the respective spectral measures of $\tilde{m}_\pm(z)$ and $\tilde{m}_{(\mu,\sigma),\pm}(z)$. Then we have*

$$d\tilde{\rho}_{(\mu,\sigma),\pm}(\lambda) = \frac{1}{1+\gamma_{(\mu,\sigma)}}\Big(\frac{\lambda-\mu_0}{\lambda-\mu}d\tilde{\rho}_\pm(\lambda) + \left\{\begin{array}{ll} \gamma_\mu\, d\Theta(\lambda-\mu) & \sigma=\pm \\ 0 & \sigma=\mp \end{array}\right\}\Big), \tag{3.22}$$

where $\gamma_{(\mu,\sigma)},\gamma_\mu$ are defined in (3.4), (3.20) and $d\Theta$ is the unit point measure concentrated at 0.

Let $P_\pm(\mu_0)$, $P_{(\mu,\sigma),\pm}(\mu)$ denote the orthogonal projections onto the subspaces spanned by $u_{\sigma_0}(\mu_0,.)$, $u_\mu(.)$ in $\ell^2(\pm\mathbb{N})$ respectively. Then the above results clearly imply

Theorem 3.5. *The operators $(\mathbb{1}-P_\pm(\mu_0))H_\pm$ and $(\mathbb{1}-P_{(\mu,\sigma),\pm}(\mu))H_{(\mu,\sigma),\pm}$ are unitarily equivalent. Moreover, we have $\mu\notin\sigma(H_{(\mu,\sigma),-\sigma})$ and $\mu_0\notin\sigma(H_{(\mu,\sigma),\pm})\backslash\{\mu\}$.*

If $\mu\notin\sigma_d(H)$ or $(\mu,\sigma)=(\mu_0,-\sigma_0)$, then $\mu\in\sigma(H_{(\mu,\sigma),\sigma})$ and thus

$$\sigma(H_{(\mu,\sigma),\pm}) = \Big(\sigma(H_\pm)\backslash\{\mu_0\}\Big)\cup\left\{\begin{array}{ll} \{\mu\}, & \sigma=\pm \\ \emptyset, & \sigma=\mp \end{array}\right. . \tag{3.23}$$

Otherwise, that is, if $\mu\in\sigma_d(H)\backslash\{\mu_0\}$, then $\mu\notin\sigma(H_{(\mu,\sigma),\sigma})$ and thus

$$\sigma(H_{(\mu,\sigma),\pm}) = \sigma(H_\pm)\backslash\{\mu_0\}. \tag{3.24}$$

In essence, Theorem 3.5 says that, as long as $\mu\notin\sigma_d(H)\backslash\{\mu_0\}$, the Dirichlet datum (μ_0,σ_0) is rendered into (μ,σ), whereas everything else remains unchanged. If $\mu\in\sigma_d(H)\backslash\{\mu_0\}$, that is, if we are trying to move μ_0 to an eigenvalue, then μ_0 is removed. This latter case reflects the fact that we cannot move μ_0 to an eigenvalue E without moving the Dirichlet eigenvalue on the other side of E to E at the same time.

We end this section with a few additions.

Remark 3.6. *(i). For $f\in\ell(\mathbb{N})$ set*

$$\begin{aligned}(A_{(\mu,\sigma),+}f)(n) &= \sqrt{\frac{W_{(\mu,\sigma)}(n)}{W_{(\mu,\sigma)}(n-1)}}f(n) \\ &\quad - u_{-\sigma_0,(\mu,\sigma)}(\mu_0,n)\sum_{j=1}^n u_{\sigma_0}(\mu_0,j)f(j),\end{aligned} \tag{3.25}$$

$$\begin{aligned}(A^{-1}_{(\mu,\sigma),+}f)(n) &= \sqrt{\frac{W_{(\mu,\sigma)}(n-1)}{W_{(\mu,\sigma)}(n)}}f(n) \\ &\quad - u_{-\sigma}(\mu,n)\sum_{j=1}^n u_{\sigma,(\mu,\sigma)}(\mu,j)f(j).\end{aligned} \tag{3.26}$$

Then we have $A_{(\mu,\sigma),+}A^{-1}_{(\mu,\sigma),+} = A^{-1}_{(\mu,\sigma),+}A_{(\mu,\sigma),+} = \mathbb{1}_{\ell(\mathbb{N})}$ and

$$\tau_{(\mu,\sigma),+} = A_{(\mu,\sigma),+}\tau_+A^{-1}_{(\mu,\sigma),+}. \tag{3.27}$$

Similarly, for $f \in \ell(-\mathbb{N})$ set

$$(A_{(\mu,\sigma),-}f)(n) = \sqrt{\frac{W_{(\mu,\sigma)}(n)}{W_{(\mu,\sigma)}(n-1)}}f(n) - u_{-\sigma_0,(\mu,\sigma)}(\mu_0,n)\sum_{j=n+1}^{1} u_{\sigma_0}(\mu_0,j)f(j), \tag{3.28}$$

$$(A^{-1}_{(\mu,\sigma),-}f)(n) = \sqrt{\frac{W_{(\mu,\sigma)}(n-1)}{W_{(\mu,\sigma)}(n)}}f(n) - u_{-\sigma}(\mu,n)\sum_{j=n+1}^{1} u_{\sigma,(\mu,\sigma)}(\mu,j)f(j). \tag{3.29}$$

Then we have $A_{(\mu,\sigma),-}A^{-1}_{(\mu,\sigma),-} = A^{-1}_{(\mu,\sigma),-}A_{(\mu,\sigma),-} = \mathbb{1}_{\ell(-\mathbb{N})}$ and

$$\tau_{(\mu,\sigma),-} = A_{(\mu,\sigma),-}\tau_- A^{-1}_{(\mu,\sigma),-}. \tag{3.30}$$

(ii). Note that the case $(\mu,\sigma) = (\mu_0, -\sigma_0)$ corresponds to the double commutation method with $\gamma = \infty$ (cf. [7], Section 4). Furthermore, the operators $A_{(\mu,\sigma),\pm}$ are unitary when restricted to proper subspaces of $\ell^2(\pm\mathbb{N})$ in this case.
(iii). Due to the factor $\frac{z-\mu_0}{z-\mu}$ in front of $\tilde m_{(\mu,\sigma),\pm}(z)$, all norming constants (i.e., the negative residues at each pole of $\tilde m_{(\mu,\sigma),\pm}(z)$) are altered.
(iv). Clearly our transformation preserves reflectionless properties.

4. Full-line operators

Having the results of the previous section at our disposal we can now easily deduce all spectral properties of the operator $H_{(\mu,\sigma)}$. We recall the Weyl M-matrix

$$M(z) = \Big(\langle\delta_j, (H-z)^{-1}\delta_k\rangle\Big)_{0\le j,k\le 1} = \frac{a(0)^{-2}}{\tilde m_+(z)+\tilde m_-(z)}\begin{pmatrix} -1 & a(0)\tilde m_+(z) \\ a(0)\tilde m_+(z) & a(0)^2\tilde m_+(z)\tilde m_-(z)\end{pmatrix}, \tag{4.1}$$

associated with H. Then Theorem 3.1 yields

Theorem 4.1. *Given $H, H_{(\mu,\sigma)}$ the respective Weyl M-matrices $M(z), M_{(\mu,\sigma)}(z)$ are related by*

$$M_{(\mu,\sigma),0,0}(z) = \frac{1}{(1+\gamma_{(\mu,\sigma)})^2}\frac{z-\mu}{z-\mu_0}M_{0,0}(z), \tag{4.2}$$

$$M_{(\mu,\sigma),0,1}(z) = \frac{1}{1+\gamma_{(\mu,\sigma)}}\Big(M_{0,1}(z) + \frac{\gamma_{(\mu,\sigma)}}{z-\mu_0}a(0)M_{0,0}(z)\Big), \tag{4.3}$$

$$M_{(\mu,\sigma),1,1}(z) = \frac{z-\mu_0}{z-\mu}M_{0,1}(z) + \frac{\gamma_{(\mu,\sigma)}}{z-\mu_0}\Big(1-2a(0)M_{0,1}(z)\Big) + \frac{\gamma^2_{(\mu,\sigma)}}{(z-\mu_0)(z-\mu)}a(0)^2M_{0,0}(z). \tag{4.4}$$

Moreover, $M_{(\mu,\sigma),j,k}(z,m,n)$, $j,k \in \{0,1\}$ are holomorphic near $z = \mu, \mu_0$.

Given the connection between $M(z)$ and $M_{(\mu,\sigma)}(z)$ we can compute the corresponding Herglotz matrix measure of $M_{(\mu,\sigma)}(z)$ as in Lemma 3.4.

Lemma 4.2. *The matrix measures $d\rho, d\rho_{(\mu,\sigma)}$ corresponding to $M(z), M_{(\mu,\sigma)}(z)$ are related by*

$$
\begin{aligned}
(4.5)\qquad d\rho_{(\mu,\sigma),0,0}(\lambda) &= \frac{1}{(1+\gamma_{(\mu,\sigma)})^2}\frac{\lambda-\mu}{\lambda-\mu_0} d\rho_{0,0}(\lambda),\\
(4.6)\qquad d\rho_{(\mu,\sigma),0,1}(\lambda) &= \frac{1}{1+\gamma_{(\mu,\sigma)}}\Big(d\rho_{0,1}(\lambda) + \frac{\gamma_{(\mu,\sigma)}}{\lambda-\mu_0} a(0) d\rho_{0,0}(\lambda)\Big),\\
d\rho_{(\mu,\sigma),1,1}(\lambda) &= \frac{\lambda-\mu_0}{\lambda-\mu} d\rho_{1,1}(\lambda) - 2\frac{\gamma_{(\mu,\sigma)}}{\lambda-\mu_0} a(0) d\rho_{0,1}(\lambda)\\
(4.7)\qquad &\quad + \frac{\gamma_{(\mu,\sigma)}^2}{(\lambda-\mu_0)(\lambda-\mu)} a(0)^2 d\rho_{0,0}(\lambda).
\end{aligned}
$$

Equivalently

$$
\begin{aligned}
d\rho_{(\mu,\sigma)}(\lambda) &= \frac{1}{(z-\mu)(z-\mu_0)} \times\\
(4.8)\qquad & \begin{pmatrix} \frac{z-\mu}{1+\gamma_{(\mu,\sigma)}} & 0 \\ a(0)\gamma_{(\mu,\sigma)} & z-\mu_0 \end{pmatrix} d\rho(\lambda) \begin{pmatrix} \frac{z-\mu}{1+\gamma_{(\mu,\sigma)}} & a(0)\gamma_{(\mu,\sigma)} \\ 0 & z-\mu_0 \end{pmatrix}.
\end{aligned}
$$

This finally leads to our main theorem

Theorem 4.3. *Let $H, H_{(\mu,\sigma)}$ be defined as in (2.4), (2.26) respectively. Denote by $P(\mu_0), P(\mu)$ the orthogonal projections corresponding to the spaces spanned by $u_{\sigma_0}(\mu_0,.), u_{-\sigma}(\mu,0)$ in $\ell^2(\mathbb{Z})$ respectively. Then $(\mathbb{1} - P(\mu_0) - P(\mu))H$ and $H_{(\mu,\sigma)}$ are unitarily equivalent. In particular, H and $H_{(\mu,\sigma)}$ are unitarily equivalent if $\mu, \mu_0 \notin \sigma_d(H)$.*

Remark 4.4. *By inspection, Dirichlet deformations produce the commuting diagram*

$$
\begin{tikzcd}
 & (\mu_1,\sigma_1) \arrow[dr] & \\
(\mu_0,\sigma_0) \arrow[ur] \arrow[rr] & & (\mu_2,\sigma_2)
\end{tikzcd}
$$

for (μ_j, σ_j), $0 \le j \le 1$ according to (H.2.2).

Remark 4.5. *We have seen in Theorem 3.5 that the Dirichlet deformation method cannot create a situation where a discrete eigenvalue E_0 of H is rendered into a Dirichlet eigenvalue (i.e., moving μ_0 to the eigenvalue E_0). However, one can use the following three-step procedure to generate a prescribed degeneracy at an eigenvalue E_0 of H:*

(i). Use the Dirichlet deformation method to move μ to a discrete eigenvalue E_0 of H. (This removes both the discrete eigenvalue E_0 of H and the (Dirichlet) eigenvalue μ of $H_- \oplus H_+$).

(ii) As a consequence of step (i), there is now another eigenvalue $\tilde{\mu}$ of $H_- \oplus H_+$ in the resulting larger spectral gap of H. Move $\tilde{\mu}$ to E_0 using the Dirichlet deformation method.

(iii) Use the double commutation method to insert an eigenvalue of H at E_0.

Finally, use the double commutation method at the beginning of this remark to change σ_0 into any allowed value.

Theorem 4.4 of [7] *then shows that the resulting operator is unitarily equivalent to the original operator H, and Theorem 5.4 of* [7] *then proves that the remaining Dirichlet eigenvalues remain invariant.*

Next, we provide two limit point results. The first, although trivial from a technical point of view, nevertheless will apply in a great variety of situations.

Lemma 4.6. *Given $H, H_{(\mu,\sigma)}$, let $\omega \in \{\pm\}$ and suppose that one of the following conditions (1)–(2) holds.*

- $\sigma_{ess}(H_\omega) \neq \emptyset$.
- $\sigma(H_\omega) = \sigma_d(H_\omega) = \{E_{\omega,n}\}_{n\in\mathbb{N}}$ *with* $\sum_{n\in\mathbb{N}}(1+E_{\omega,n}^2)^{-1} = \infty$.

Then, both τ and $\tau_{(\mu,\sigma)}$ are l.p. at $\omega\infty$.

Proof. A simple consequence of the fact that τ *l.c.* at $\omega\infty$ implies that the resolvent of H_ω is Hilbert–Schmidt. □

Our second limit point result is more tailored toward the Dirichlet deformation method .

Lemma 4.7. *Assume that $\mu_0, \mu \in (E_0, E_1)$ and $\sigma = \sigma_0$. Then $\tau_{(\mu,\sigma)}$ is in the limit point (resp., l.c.) case at $\omega\infty$ if and only if τ is l.p (resp., l.c.) at $\omega\infty$, $\omega \in \{\pm\}$.*

Proof. It suffices to consider $\mu \neq \tilde{\mu}$. Assume that τ is *l.p* at $\omega\infty$ and suppose the contrary for $\tau_{(\mu,\sigma)}$, that is, suppose $\tau_{(\mu,\sigma)}$ is *l.c.* at $\omega\infty$. The fact that both choices $u_\omega = u_\mu$ and $u_\omega = u_{\mu_0}$ in (2.25) yield equivalent boundary condition (since they yield the same operator) implies

$$\lim_{n\to\omega\infty} W_n(u_{\mu_0}, u_\mu) = 0. \tag{4.9}$$

In other words, u_{μ_0}, u_μ both satisfy the boundary condition $BC_\omega(u_{\mu_0}) = BC_\omega(u_\mu) = 0$ at $\omega\infty$. If $-\omega = \sigma_0$ we infer that $\mu \in \sigma(H_{(\mu,\sigma)}) = \sigma(H)$ since $BC_{-\omega}(u_\mu) = 0$ by Corollary 3.2. But this contradicts $\mu \in (E_0, E_1)$. Similarly, if $\omega = \sigma_0$ we infer that $\mu_0 \in \sigma(H_{(\mu,\sigma)}) = \sigma(H)$ since $BC_{-\omega}(u_{\mu_0}) = 0$ by Corollary 3.2. But this contradicts $\mu_0 \in (E_0, E_1)$. By symmetry in τ and $\tau_{(\mu,\sigma)}$, the proof is complete. □

Remark 4.8. *If $\sigma = -\sigma_0$ then the Dirichlet deformation method does not necessarily preserve the l.p. property. By Remark 4.4 it suffices to consider the case $\mu = \mu_0$. Take an operator H being l.c. at $\sigma_0\infty$ and such that $\sigma_{ess}(H_{-\sigma_0}) \neq \emptyset$. Then $\tau(\mu,\sigma)$ is l.p. at $\sigma\infty$ since $u_\mu \notin \ell^2(\sigma\mathbb{N})$ (by (2.20)) and l.p. at $-\sigma\infty$ since $\sigma_{ess}(H_{(\mu,\sigma),-\sigma_0}) \neq \emptyset$.*

Finally, we briefly comment on how to iterate Dirichlet deformation method (see [7], Section 3). Suppose

$$(E_{0,j}, E_{1,j}), \quad (\mu_{0,j}, \sigma_{0,j}), (\mu_j, \sigma_j) \in [E_{0,j}, E_{1,j}] \times \{\pm\} \tag{4.10}$$

satisfy (H.2.2) for each $j = 1, \dots, N$, $N \in \mathbb{N}$. Then the Dirichlet deformation result after N steps, denoted by $H_{(\mu_1,\sigma_1),\dots,(\mu_N,\sigma_N)}$, is associated with the sequences

$$\begin{aligned} a_{(\mu_1,\sigma_1),\dots,(\mu_N,\sigma_N)}(n) &= \sqrt{a(n-N)a(n+N)} \\ &\quad \times \frac{\sqrt{C_{n-N}(U_{1,\dots,N})C_{n-N+2}(U_{1,\dots,N})}}{C_{n-N+1}(U_{1,\dots,N})}, \\ b_{(\mu_1,\sigma_1),\dots,(\mu_N,\sigma_N)}(n) &= b(n) - \partial^*\, a(n)\frac{D_{n-N+1}(U_{1,\dots,N})}{C_{n-N+1}(U_{1,\dots,N})}, \end{aligned} \tag{4.11}$$

where $(U_{1,\dots,N}) = (u_{\sigma_{0,1}}(\mu_{0,1}), u_{\sigma_1}(\mu_1), \dots, u_{\sigma_{0,N}}(\mu_{0,N}), u_{\sigma_N}(\mu_N))$ and C_n, D_n are given by

$$C_n(u_1, \dots, u_N) = \det\{u_i(n+j-1)\}_{1\le i,j\le N}, \tag{4.12}$$

$$D_n(u_1, \dots, u_N) = \det \left\{ \begin{array}{ll} u_i(n+j-1), & j < N \\ u_i(n+N), & j = N \end{array} \right\}_{1\le i,j\le N}. \tag{4.13}$$

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