Fourier Duality as a Quantization Principle*

R. Aldrovandi† and L.A. Saeger‡

Instituto de Física Teórica - UNESP

Rua Pamplona 145

01405-900 - São Paulo, SP

Brazil

The Weyl-Wigner prescription for quantization on Euclidean phase spaces makes essential use of Fourier duality. The extension of this property to more general phase spaces requires the use of Kac algebras, which provide the necessary background for the implementation of Fourier duality on general locally compact groups. Kac algebras – and the duality they incorporate – are consequently examined as candidates for a general quantization framework extending the usual formalism. Using as a test case the simplest non-trivial phase space, the half-plane, it is shown how the structures present in the complete-plane case must be modified. Traces, for example, must be replaced by their noncommutative generalizations – weights – and the correspondence embodied in the Weyl-Wigner formalism is no more complete. Provided the underlying algebraic structure is suitably adapted to each case, Fourier duality is shown to be indeed a very powerful guide to the quantization of general physical systems.

Keywords: Weyl Quantization, Kac Algebras, Generalized Fourier Duality, Noncommutative Harmonic Analysis.

---

*Work supported by CAPES and CNPq, Brazil.

†e-mail: ra@axp.ift.unesp.br

‡e-mail: saeger@axp.ift.unesp.br

Contents

1 Introduction

The complete quantum description of a physical system presupposes the identification of the space of observables, the scene of its dynamical evolution. In the classical description, this space corresponds to a subalgebra of the algebra $C^\infty(M)$ of infinitely differentiable functions on the phase space. When the latter is the linear space $\mathbb{R}^{\nvDash}$ (or $\mathbb{R}^{\nvDash\ltimes}$), the relationship between the quantum and the classical cases is well known, given as it is by the Weyl correspondence prescription. The correspondence assumes the existence (and knowledge) of a possible classical version of the system and of a general prescription – "quantization" – to transform the classical into the quantum description. The ultimate goal is to uncover some "grand principle", a rule providing directly the quantal description: given a system, we should be able to identify the observables and their space without the mediation of classical quantities. It would then be possible to describe even purely quantum systems, for which there are no classical limits. Such an objective is still far ahead and for the present time we are condemned to proceed from classical systems, trying to work up a general procedure of quantization from particular examples. This trial-and-error approach blends rigorous assumptions with inferences from previous case-study experience.

The Weyl prescription, in its original form [**?**], makes use of a very particular kind of duality, the Pontryagin duality which holds only when the underlying group of linear symplectomorphisms is abelian. The Pontryagin dual of an abelian group G is the space of characters, which is also an abelian group, though not necessarily the same. Thus, the Euclidean spaces $\mathbb{R}^{\ltimes}$ and the cyclic groups $\mathbb{Z}_{\ltimes}$ are self-dual in this sense, but the circle and the group of integers are dual to each other. The link between these groups is provided by generalized two-way Fourier transforms mapping the L^1-space of one into the L^∞-space of the other, and for this reason we shall use the expression *Fourier duality* as a synonym to (eventually generalized) Pontryagin duality. In the Euclidean linear case, the abelian group involved is formed by the translations on phase space, which is isomorphic to its own Fourier dual: the Fourier transforms of functions on $\mathbb{R}^{\nvDash}$ are functions on $\mathbb{R}^{\nvDash}$. Given the Fourier transform $\tilde{f}$ of a classical dynamical function f, the Weyl prescription yields the corresponding operator (q-number function) as a Fourier transform of $\tilde{f}$ with a projective operator kernel. A formal inverse procedure (first considered by Wigner [**?**]), involving an integration on an operator space, gives then the corresponding c-number function [**?**]. These c-number functions and their Fourier transforms belong to twisted (noncommutative) algebras, different from the usual abelian algebras of convolution and pointwise product. The twisted convolution on $L^1(\mathbb{R}^{\nvDash})$ and the Moyal product on $L^\infty(\mathbb{R}^{\nvDash})$ arise naturally from the projective operator product through the Weyl correspondence. Because of this "quantum" origin, the deformation of the usual algebras of functions on phase space they represent is considered a quantization [**?**].

The main difficulty comes from the fact that, for general systems, including those whose phase space is the Euclidean space, the group acting on phase space (a group of linear symplectomorphisms, here called *special canonical group*) is not abelian and/or compact. On compact groups the integration implied in the Fourier transform is defined in a simple way, as there exists a unique Haar measure, which is both left- and right-invariant. Amongst non-compact groups, the existence of Haar measures is assured only for those which are locally compact, though in general the left-invariant and the right-invariant measures differ (when they happen to be equal, the group is said to be *unimodular*). Group-to-group duality is, however, restricted to the commutative case: the space dual to a non-abelian group is no more a group, but an algebra. Duality must be understood no more as a relationship between groups, but as a relationship in a wider category. A fair formulation for the general locally compact case has been obtained in the early seventies, and led to the introduction of Kac algebras. These are Hopf-von Neumann algebras with peculiar generalized measures, called Haar weights. Actually, for non-unimodular locally compact groups, Fourier duality is only possible in the Kac algebra framework. We must abstract from groups to Kac algebras in order to have a Fourier duality. In this sense the usual Weyl correspondence is part of a highly nontrivial *projective duality* for the abelian group $\mathbb{R}^{\nvDash}$, where an algebra generated no more by linear, but by projective operators comes into play [**?**].

Our objective here is to give a step further in the question of quantization through the study of these analytic-algebraic aspects. The algebraic facet is better known: it is necessary to resort to Hopf-von Neumann algebras. These algebras are, however, rather involved operator algebras, on which many different topologies and measures can be defined. The analytic facet lies precisely in the choice of the correct topology and measure. Our guiding idea will be the assumption of Fourier duality, which stands at the heart both of the Weyl quantization approach and of the group duality alluded to above. Since Fourier duality in its more general form is implemented in the Kac algebra structural frame [**?**], we argue that they are also able to provide a generalized Weyl prescription for quantizing a phase space on which a separable locally compact type I group acts by symplectomorphisms.

We take as a test case the simplest non-trivial example of phase space: the half-plane. The fact that the configuration space $\mathbb{R}_+$ is abelian, that the manifold of the special canonical group coincides with the phase space manifold and that there is no need to consider central extensions of this canonical group accounts for the relative simplicity of the example. Its non-triviality comes from the non-trivial properties of the group which, besides being non-abelian and non-compact, is non-unimodular. These are important features, which bring to light the main difficulties of quantization on a general phase space. Specifically, this example also shows why the usual Weyl-Wigner quantization procedure does not generalize straightforwardly and does not always lead to a generalized Moyal bracket [**?**], or to a deformation of the algebra of functions on phase space. Although this example does not cover quantization on a general phase space, where the respective canonical group may have little to do with the space it acts [**?**], we believe the duality principle it illustrates can be generalized to quantization on any phase space to which an operator algebra can be associated, as it is done in [**?**].

We begin with an exposition of the classical picture on the half-plane in section 2. We use, in order to select a group on phase space, Isham's canonical approach which, though not quite general, is enough for the case in view. In the next section we give some details and classify the induced irreducible representations of the half-plane special canonical group, which is in fact a special parametrization of the affine group on the line (conversely, we show in the Appendix that the half-plane is the unique non-trivial homogeneous symplectic manifold of the affine group on the line). Since it seems that neither Hopf-von Neumann nor Kac algebras are structures quite familiar to the Physics community, we review them in a separate section. We emphasize those Kac algebras which are related to groups, in order to show how group duality is attained. At the end of that section we also show how to decompose the operator Kac algebra of a type I group according to its unitary dual. This is not found in the Kac algebra literature and will be essential to the interpretation of the Weyl formula in the duality framework. The half-plane case, used all along more as a gate into non-trivial aspects, is finally retaken for its own sake and given its finish in section **??**. The whole treatment leads to a reappraisal of the reach and limitations of the Weyl-Wigner formalism as a guide for quantization on general phase spaces.

2 The Special Canonical Group

The phase space we want to quantize on is the half-plane $\mathbb{R}_+ \times \mathbb{R}$, the cotangent bundle of the configuration space given by the half-line $\mathbb{R}_+$. On this symplectic manifold we use the coordinates x and p, in terms of which the symplectic form, given as the derivative of the Liouville canonical 1-form, is

$$\omega = d\theta_o = dp \wedge dx, \qquad x \in \mathbb{R}_+, \ | \in \mathbb{R}.$$

The symplectic form implements, through the equation

$$i_{X_f}\omega = -df, \tag{1}$$

a homomorphism between the space of C^∞-functions (hamiltonians) and the space of symplectic hamiltonian vector fields, whose kernel are the constants. Since ω is non-degenerate, (1) can be solved for the vector fields and yields, in the above coordinates, $X_f = \partial_p f \, \partial_x - \partial_x f \, \partial_p$. The symplectic form also provides a Lie algebra structure on the C^∞-functions, as it defines the Poisson bracket by

$$\{f, g\} = -\omega(X_f, X_g), \tag{2}$$

which is isomorphic to the hamiltonian vector fields Lie algebra through

$$[X_f, X_g] = -X_{\{f,g\}}.$$

We shall follow Isham [**?**] in the first steps. To quantize a phase space we start by looking for a finite dimensional (for simplicity) group whose elements act as symplectomorphisms, that is, preserve the symplectic structure. The action must be transitive, so as to avoid any lack of globality in the quantum description, and also (quasi-)effective. It is thus necessary to find a finite dimensional group G_{hp} under whose action the half-plane is a symplectic homogeneous G-space. The task can be simplified by proceeding as follows. Consider a group whose manifold is the configuration space, $(\mathbb{R}_+, \cdot)$, and make it act on a linear space so as to get at least an almost-faithful representation (a representation whose kernel is discrete). Take the action of $\mathbb{R}_+$ on $\mathbb{R}$ given by

$$\lambda \in \mathbb{R}_+ \mapsto \mathbb{R}_\lambda(\Game) = \Game\lambda,\ \Game \in \mathbb{R}. \tag{3}$$

Construct the semi-direct product group $\mathbb{R}_+ \oslash \mathbb{R}$, with the product operation given by

$$(\lambda, a)(\overset{\circ}{;}b) = (\lambda\overset{\circ}{;}a + \phi_\lambda(b)),$$

where $\phi_\lambda(b) = R^*_{\lambda^{-1}}(b) = b/\lambda$ is the homomorphism on $\mathbb{R}$ given by the representation R^* contragradient to (3). The identity in $\mathbb{R}_+ \oslash \mathbb{R}$ is $(1, 0)$ and the inverse element of (λ, a) is given by $(\lambda^{-1}, \phi_\lambda^{-1}(-a))$, with $\phi_{\lambda^{-1}} = \phi_\lambda^{-1}$.

Considering the left action

$$l_{(\lambda,a)}(x, p) = (\lambda x, p/\lambda - a),$$

of this group on the space $\mathbb{R}_+ \times \mathbb{R}$, we see that $G_{hp} = \mathbb{R}_+ \oslash \mathbb{R}$ is formed by some special linear canonical transformations on the half-plane. Actually, we show in the Appendix, using Kirillov's orbits method [**?**], that the half-plane is the only non-trivial symplectic manifold canonically invariant by G_{hp}.

The Lie algebra $\mathcal{G}_{hp}$ of G_{hp} can be obtained from the group product with the help of the formula $e^{tA}e^{sB}e^{-tA}e^{-sB} = e^{ts[A,B] + \textit{ higher orders in } t,s}$ and is given on $\mathbb{R} \oplus \mathbb{R}$ by

$$[(l, a), (r, b)] = (0, ar - lb). \tag{4}$$

It is straightforward to realize this Lie algebra in terms of symplectic hamiltonian vector fields on the half-plane. By the exponential mapping $(l, a) \mapsto (e^l, a) \in \mathbb{R}_+ \oslash \mathbb{R}$ we introduce the 1-parameter subgroups $t \mapsto (e^{lt}, 0),\ s \mapsto (1, as)$, whose action on $\mathbb{R}_+ \times \mathbb{R}$,

$$l_{l,a}(x, p) = (e^{lt}x, e^{-lt}p - as)$$

is easily found to be generated by the symplectic hamiltonian (right-invariant) vector fields

$$X_{l,a}(x, p) = lx\partial_x - (lp + a)\partial_p,$$

corresponding to the hamiltonians $h_{l,a}(x, p) = ax + lxp$. On $C^\infty(\mathbb{R}_+ \times \mathbb{R})$ these hamiltonians define a Poisson subalgebra by

$$\{h_{l,a}, h_{r,b}\} = h_{0,ar-lb}, \tag{5}$$

whose structure is identical to that of (4). We can then say that there is a faithful momentum mapping $J : T^*\mathbb{R}_+ \to \mathcal{G}^*_{\approx\shortmid}$, allowing the association of the pair $(l, a) \in \mathcal{G}_{hp}$ to the hamiltonian function $h_{l,a}$ by the duality pairing $\langle J(x,p), (l,a)\rangle = h_{l,a}(x,p)$. By this Lie algebra isomorphism we privilege the functions $h_{l,a}$ as a preferred class of observables to be quantized. Also because of this isomorphism, there is no need to central-extend $\mathcal{G}_{hp}$ as it happens in the complete-plane case. In other words, the cohomology space $H^2(\mathcal{G}_{hp}, \mathbb{R}) \sim \mathbb{H}^{\not\in}(\mathbb{G}_{\approx\shortmid}, \mathbb{R})$ is trivial [?]. It is then possible to take the unitary irreducible linear representations of the special canonical group realized in terms of operators on a given Hilbert space and try to find an unbounded operator (representation generator) in correspondence with each preferred observable on the half-plane. In the next section we will provide the representations necessary to characterize such operators but, differently from Isham's approach, functions will be associated to bounded operators *a la* Weyl, which means that we shall work at the group representation level.

3 Induced Representations and the Unitary Dual

Irreducible unitary representations of semi-direct product groups are easily constructed via Mackey's induced representation theory (see [?, ?, ?, ?]). In this section we construct irreducible unitary representations of $G_{hp} = \mathbb{R}_+ \oslash \mathbb{R}$ by that method. We first note that a G_{hp} element $g = (\lambda, a)$ can be decomposed in its $\mathbb{R}_+$ and $\mathbb{R}$ parts according to

$$(\lambda, a) = (\lambda, 0)(1, \phi_\lambda^{-1}(a)) = g_{\mathbb{R}_+} g_{\mathbb{R}} \tag{6}$$

and

$$(\lambda, a) = (1, a)(\lambda, 0). \tag{7}$$

We begin by looking for unitary irreducible representations of the subgroup $\mathbb{R}$. This is immediate, since $\mathbb{R}$ is an abelian normal subgroup. Its character (one-dimensional) representations on $\mathbb{C}$ are given by $V_x(a) = e^{ixa}$, where x is a label contained in the unitary dual group $\hat{\mathbb{R}} \sim \mathbb{R}$. The Hilbert space where our induced representation of G_{hp} will be realized is constructed as the space of functions $f : G_{hp} \to \mathbb{C}$ which can be decomposed into wavefunctions ξ, $f_x(g) = V_x^{-1}(g_{\mathbb{R}})\xi(g_{\mathbb{R}_+})$, or, using (6) with $g = (\lambda, a)$,

$$f_x(\lambda, a) = e^{-ix\phi_\lambda^{-1}(a)}\xi(\lambda), \qquad \xi \in L^2(\mathbb{R}_+), \tag{8}$$