Non-integrability and continuation of fixed points of 2n-dimensional perturbed twist mappings

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Abstract. In this paper a simple criterion to prove non-integrability of symplectic, perturbed twist mappings in $2n$ dimensions is developed for sufficiently small perturbations. In addition an upper bound for the number of isolating integrals the system can possess is provided. A criterion for the analytic continuation of isolated periodic orbits in case of a small nonzero perturbation of twist maps is found. The evaluation of their linear stability character by obtaining a simplfied expression of the eigenvalues of the Jacobian matrix concludes the theoretical part. The theory is finally applied to a generalized standard map involving Jacobian elliptic functions.

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1. Introduction

In many applications of the theory of nonlinear dynamical systems in science, engineering and economy one deals with Hamiltonian systems. For a general introduction into the field see e.g. Moser (1968, 1973), Arnol'd (1980; *et al* 1988), Abraham and Marsden (1982), Lichtenberg and Liebermann (1983), MacKay *et al* (1984) and Wodnar (1994).

The study of continuous flows defined through differential equations corresponding to dynamical systems — as for instance Hamilton's equations — can be greatly simplified by considering a suitably selected Poincaré mapping, sampling the state of the system at significant discrete time steps. In practice, however, one can find only approximate expressions for the mapping corresponding to a specific continuous system, see e.g. Hadjidemetriou (1991; 1996). In any case, particular mapping constructions stated in terms of known functions serve as an important tool for the study of fundamental mechanisms which are mostly difficult to be observed in real physical systems corresponding to a flow. From the mapping approach there may also emerge conceptually interesting new proof techniques (e.g. Moser 1962, Mather 1984) as has been the case in this paper.

For the application of perturbation techniques one is led to consider *near integrable systems*. The class of mappings naturally corresponding to Hamiltonian systems are the *symplectic mappings* — which in the case of two dimensions reduce to *area preserving maps* — in particular *perturbed twist mappings*. In order to characterize the dynamics of a flow, it is essential to decide on integrability and to know its periodic orbits which correspond to fixed points of the Poincaré map.

In this paper a simple criterion to prove an obstruction to integrability of a perturbed twist mapping under certain conditions as well as a closely related theory on the fixed point structure are provided for small enough perturbations in systems of arbitrary even dimension. The essential steps of the procedure are the *implicit* formulation of the twist map in terms of a generating function of the second type and an *explicit* representation of the q-fold iterated map found perturbatively. In conjunction with previous work by Meletlidou & Ichtiaroglou (1994) and Ichtiaroglou & Meletlidou (1996) a non-integrability criterion is found which yields also an upper bound on the maximum number of analytic integrals that may exist for the system under consideration. Going further, we prove the analytic continuation of isolated periodic orbits for small perturbations by means of the implicit function theorem. In addition, in order to determine the linear stability of these orbits, we reduce the calculation of the eigenvalues of the Jacobian matrix to a linear space of half the dimension of the original system. The theory is well suited to prove the existence of a finite, but arbitrarily large number of *isolated* periodic orbits for generic systems, compare e.g. to Arnol'd *et al*

(1988 p 216) and Veerman & Holmes (1985). A similar theorem on the continuation of periodic orbits for Hamiltonian flows was proved by Poincaré (1890; 1892 secs 42, 79). By exploiting the symplectic structure of the Jacobian matrix of the mapping through generalizing the symmetry of the related characteristic polynomial to the matrix itself, we succeed in reducing the length of the proof considerably, compared to Poincaré's work. About the general stability theory of equilibria in Hamiltonian flows see e.g. MacKay (1986). We refer the reader to Birkhoff (1927 p 12 and p 140) for statements related to the general form of the implicit function theorem involving analytic functions as it will be heaviliy used in this paper and to Abraham and Marsden (1982) for a modern treatment of this fundamental theorem, however, involving just C^∞ functions.

The theory is demonstrated along with the example of a generalized Chirikov-Taylor standard map (Chirikov 1979), involving Jacobian elliptic funtions instead of the usual trigonometric ones.

The essential element for both non-integrability and fixed point structure of symplectic mappings is the occurance of *Poincaré resonances* which play a decisive rôle for the understanding of irreversibility on the macroscopic level, based on a generalized, but purely Hamiltonian dynamical approach to the microscopic world according to the fundamental work by Prigogine (e.g. 1962; 1980) and his coworkers (e.g. Petrosky 1984; 1985). See Prigogine (1995) for a review on the subject. This underlines the importance of the study of non-integrability and fixed points of symplectic mappings in many dimensions also for the solution of fundamental physical problems.

2. Perturbed Twist Maps and Hamiltonian Systems

Consider an $n+1$ degrees of freedom Hamiltonian system

$$H(\boldsymbol{w}^*, \boldsymbol{J}^*) = H_0(\boldsymbol{J}^*) + \varepsilon H_1(\boldsymbol{w}^*, \boldsymbol{J}^*) + o(\varepsilon^2) \tag{1}$$

represented in action-angle variables

$$\boldsymbol{w}^* = (w_0, \ldots, w_n)^T \quad \text{and} \quad \boldsymbol{J}^* = (J_0, \ldots, J_n)^T,$$

with H being analytic in perturbation parameter ε in an open interval $]-\varepsilon_1, \varepsilon_1[$, $\varepsilon_1 > 0$, and also analytic in $(\boldsymbol{w}^*, \boldsymbol{J}^*)$ in an open domain $\mathbb{D}_H^* := \mathbb{T}^{n+1} \times \mathbb{J}_H^*$, $\mathbb{J}_H^* \subseteq \mathbb{R}^{n+1}$, where $\ldots^T$ denotes matrix transposition. Define the *explicit Poincaré-map*

$$\boldsymbol{G}_\varepsilon : \left\{ \begin{array}{lcl} \boldsymbol{w}' & = & \bar{\boldsymbol{w}}_\varepsilon(\boldsymbol{w}, \boldsymbol{J}) \bmod 2\pi \quad \text{(componentwise)} \\ \boldsymbol{J}' & = & \bar{\boldsymbol{J}}_\varepsilon(\boldsymbol{w}, \boldsymbol{J}) \end{array} \right. \tag{2}$$

by the surfaces $w_0^{(k)} = 2k\pi$ — superscript (k) denoting the k-th iteration of the map — on the set of reduced action-angle variables $\boldsymbol{J} = (J_1, \ldots, J_n)^T$, $\boldsymbol{w} = (w_1, \ldots, w_n)^T$ in the corresponding projected open domain $\mathbb{D}_E := \mathbb{T}^n \times \mathbb{J}_H$, $\mathbb{J}_H \subseteq \mathbb{R}^n$ and for $\varepsilon \in]-\varepsilon_1, \varepsilon_1[$.

From fundamental theorems we know that such functions $\bar{\boldsymbol{w}}$ and $\bar{\boldsymbol{J}}$ exist and are analytic in ε and $(\boldsymbol{w}, \boldsymbol{J})$ in their open domains (see Birkhoff 1927 p 12). Conversly, a recent result by Treschev (1996) proved the existence of a Hamiltonian analytic in all of its variables within open domains corresponding to arbitrarily given functions $\bar{\boldsymbol{w}}$ and $\bar{\boldsymbol{J}}$ representing the Poincaré-map (2) with the appropriate analyticity properties. Furthermore — for sufficiently small ε — $\boldsymbol{G}_\varepsilon$ admits an implicit representation through a generating function S of the second type

$$S = S(\boldsymbol{w}, \boldsymbol{J}') = \underbrace{\boldsymbol{w}\boldsymbol{J}' + S_T(\boldsymbol{J}')}_{S_0 :=} + \varepsilon S_1(\boldsymbol{w}, \boldsymbol{J}') \; + \; o(\varepsilon^2) \tag{3}$$

in the following form

$$\boldsymbol{G}_\varepsilon : \left\{ \begin{array}{lllllllll} \boldsymbol{w}' & = & \dfrac{\partial S}{\partial \boldsymbol{J}'} & = & \boldsymbol{w} & + & \dfrac{\partial S_T}{\partial \boldsymbol{J}'} & + \; \varepsilon \dfrac{\partial S_1}{\partial \boldsymbol{J}'} & + \; o(\varepsilon^2) \\ \boldsymbol{J} & = & \dfrac{\partial S}{\partial \boldsymbol{w}} & = & \boldsymbol{J}' & & & + \; \varepsilon \dfrac{\partial S_1}{\partial \boldsymbol{w}} & + \; o(\varepsilon^2) \end{array} \right. \tag{4}$$

see e.g. Arnol'd (1980), Wodnar (1994). Again the righthand sides of (4) can be shown to be analytic in $\varepsilon \in]-\varepsilon_2, \varepsilon_2[,\; 0 < \varepsilon_2 \leq \varepsilon_1$ and $(\boldsymbol{w}, \boldsymbol{J}') \in \mathbb{D} := \mathbb{T}^n \times \mathbb{J},\; \mathbb{J} \subseteq \mathbb{J}_H,\; \mathbb{J}$ open. Subsequently we shall use the standard notation $2\pi\boldsymbol{\alpha}(\boldsymbol{J}') := \dfrac{\partial S_T}{\partial \boldsymbol{J}'}(\boldsymbol{J}')$. If the nondegeneracy condition

$$\mathrm{Det}\left(2\pi \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{J}'}\right) = \mathrm{Det}\, \frac{\partial^2 S_T}{\partial \boldsymbol{J}'^2} \neq 0 \tag{5}$$

holds in $\mathbb{D}$ we speak of a — *perturbed*, if $\varepsilon \neq 0$ — *(implicit) twist mapping*. For several theoretical considerations the definition of a twist map requires also the invariance of two tori bordering the domain of definition of $\boldsymbol{G}_\varepsilon$. This restriction, however, is of no relevance for the subsequent deductions. $\boldsymbol{G}_\varepsilon$ being canonical, its Jacobian matrix

$$\mathcal{D}\boldsymbol{G}_\varepsilon = \begin{pmatrix} \dfrac{\partial \bar{\boldsymbol{w}}}{\partial \boldsymbol{w}} & \dfrac{\partial \bar{\boldsymbol{w}}}{\partial \boldsymbol{J}} \\[2ex] \dfrac{\partial \bar{\boldsymbol{J}}}{\partial \boldsymbol{w}} & \dfrac{\partial \bar{\boldsymbol{J}}}{\partial \boldsymbol{J}} \end{pmatrix}$$

of the related explicit representation clearly satisfies the *local symplectic* or *canonical condition*

$$\mathcal{D}\boldsymbol{G}_\varepsilon^T \; \boldsymbol{\Omega} \; \mathcal{D}\boldsymbol{G}_\varepsilon = \boldsymbol{\Omega} \tag{6}$$

at any point of phase space, where

$$\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{O}_n & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{O}_n \end{pmatrix}$$

is the symplectic $2n \times 2n$ unit matrix.

We call a $2n$-dimensional symplectic mapping $\boldsymbol{G}_\varepsilon$ represented in any state variables in an open domain $\mathbb{D}$ completely (analytically) *integrable* if there exist n functions $\Phi_j^{(\varepsilon)}$ defined on $\mathbb{D}$ and on some open interval with respect to ε which are analytic in all of their variables and which are *integrals*, i.e. invariants of the mapping

$$\Phi_j^{(\varepsilon)} \circ \boldsymbol{G}_\varepsilon = \Phi_j^{(\varepsilon)}, \qquad j = 1, \ldots, n,$$

if those are *independent*,

$$\operatorname{rank}\left(\boldsymbol{\nabla}\Phi_1^{(\varepsilon)} \cdots \boldsymbol{\nabla}\Phi_n^{(\varepsilon)}\right) = n$$

in $\mathbb{D}$, and if they are in *involution* within $\mathbb{D}$, i.e.

$$(\boldsymbol{\nabla}\Phi_j^{(\varepsilon)})^T \boldsymbol{\Omega}\, \boldsymbol{\nabla}\Phi_k^{(\varepsilon)} = 0, \qquad j, k = 1, \ldots, n.$$

Otherwise such a mapping shall be termed *non-integrable*. The sufficiency of only n integrals for complete integrability rests on a reasoning analogous to the proof of the Liouville-Arnol'd theorem (see e.g. Arnol'd 1980 p 271) which is stated for canonical flows. In subsequent deductions no use of the involution property will be made.

If a mapping is integrable its generating function in action-angle variables attains the form of S_0 as defined in (3) in open domains. Thus according to (4) with $\varepsilon = 0$ the phase space is foliated into invariant tori each of which corresponds to a fixed value of $\boldsymbol{J}'$. The components of $\boldsymbol{J}'$ may therefore be regarded as n integrals of the map. We call such tori of $\boldsymbol{G}_0$ *resonant* if the components of $\boldsymbol{\alpha}$ are rational numbers.

3. Non-Integrability

Assume the existence of n independent integrals $\Phi_j(\boldsymbol{w}, \boldsymbol{J}) = \Phi_j^{(\varepsilon)}(\boldsymbol{w}, \boldsymbol{J})$, 2π-periodic in each component of $\boldsymbol{w}$, analytic in ε within $]-\varepsilon_0, \varepsilon_0[$, $0 < \varepsilon_0 \leq \varepsilon_2$, and analytic in $(\boldsymbol{w}, \boldsymbol{J})$ in the domain $\mathbb{D}$ of $\boldsymbol{G}_\varepsilon$. Thus for $j = 1, \ldots, n$

$$\Phi_j(\boldsymbol{w}, \boldsymbol{J}) = \Phi_{j0}(\boldsymbol{w}, \boldsymbol{J}) + \varepsilon\Phi_{j1}(\boldsymbol{w}, \boldsymbol{J}) + o(\varepsilon^2) = \Phi_j(\boldsymbol{w}', \boldsymbol{J}').$$

Now, since we have

$$\begin{aligned}
\Phi_j(\boldsymbol{w}, \boldsymbol{J}) \;&= \Phi_{j0}(\boldsymbol{w}, \boldsymbol{J}') \\
&\quad + \;\varepsilon\bigg(\frac{\partial \Phi_{j0}}{\partial \boldsymbol{J}'}(\boldsymbol{w}, \boldsymbol{J}')\,\frac{\partial S_1}{\partial \boldsymbol{w}}(\boldsymbol{w}, \boldsymbol{J}') \\
&\quad + \;\Phi_{j1}(\boldsymbol{w}, \boldsymbol{J}')\bigg) + o(\varepsilon^2), \\
\Phi_j(\boldsymbol{w}', \boldsymbol{J}') &= \Phi_{j0}(\boldsymbol{w} + 2\pi\alpha(\boldsymbol{J}'), \boldsymbol{J}') \\
&\quad + \;\varepsilon\bigg(\frac{\partial \Phi_{j0}}{\partial \boldsymbol{w}}(\boldsymbol{w} + 2\pi\boldsymbol{\alpha}(\boldsymbol{J}'), \boldsymbol{J}')\,\frac{\partial S_1}{\partial \boldsymbol{J}'}(\boldsymbol{w}, \boldsymbol{J}') \\
&\quad + \;\Phi_{j1}(\boldsymbol{w} + 2\pi\boldsymbol{\alpha}(\boldsymbol{J}'), \boldsymbol{J}')\bigg) + o(\varepsilon^2),
\end{aligned}$$

respectively, by collecting orders ε^0 and ε^1 we obtain the relations

$$\Phi_{j0}(\boldsymbol{w}, \boldsymbol{J}') = \Phi_{j0}(\boldsymbol{w} + 2\pi\boldsymbol{\alpha}(\boldsymbol{J}'), \boldsymbol{J}'), \tag{7}$$

$$\begin{aligned}\frac{\partial \Phi_{j0}}{\partial \boldsymbol{J}'}(\boldsymbol{w}, \boldsymbol{J}')\,\frac{\partial S_1}{\partial \boldsymbol{w}}(\boldsymbol{w}, \boldsymbol{J}') - \frac{\partial \Phi_{j0}}{\partial \boldsymbol{w}}(\boldsymbol{w} + 2\pi\boldsymbol{\alpha}(\boldsymbol{J}'), \boldsymbol{J}')\,\frac{\partial S_1}{\partial \boldsymbol{J}'}(\boldsymbol{w}, \boldsymbol{J}')&\\ = \Phi_{j1}(\boldsymbol{w} + 2\pi\boldsymbol{\alpha}(\boldsymbol{J}'), \boldsymbol{J}') - \Phi_{j1}(\boldsymbol{w}, \boldsymbol{J}').&\end{aligned} \tag{8}$$

From equation (7) Φ_{j0} are integrals of the mapping generated by S_0. Expanding Φ_{j0} into a Fourier series

$$\Phi_{j0}(\boldsymbol{w}, \boldsymbol{J}') = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} A_{\boldsymbol{k}j0}(\boldsymbol{J}')\exp(\mathrm{i}\boldsymbol{k}\boldsymbol{w})$$

and applying relation (7) yields

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^n} A_{\boldsymbol{k}j0}(\boldsymbol{J}')[\exp(2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{\alpha}(\boldsymbol{J}')) - 1]\exp(\mathrm{i}\boldsymbol{k}\boldsymbol{w}) \equiv 0$$

so that either

$$\boldsymbol{k}\boldsymbol{\alpha}(\boldsymbol{J}') \in \mathbb{Z} \qquad \text{or} \qquad A_{\boldsymbol{k}j0}(\boldsymbol{J}') \equiv 0$$

in the open domain of definition and for all $\boldsymbol{k} \neq \boldsymbol{O}$. But if the first alternative is true for at least one $\boldsymbol{k} \neq \boldsymbol{O}$ in any open subset of $\mathbb{R}^n$, then $\boldsymbol{\alpha}(\boldsymbol{J}') \equiv \boldsymbol{Const}$ and this is a contradiction to the nondegeneracy condition (5), therefore

$$\frac{\partial \Phi_{j0}}{\partial \boldsymbol{w}}(\boldsymbol{w}, \boldsymbol{J}') \equiv 0 \quad \Longrightarrow \quad \Phi_{j0} = \Phi_{j0}(\boldsymbol{J}'). \tag{9}$$

We point out that — following Poincaré (1892 sec 81) — one can show that the Φ_j can always be selected such that the Φ_{j0} are independent, see Meletlidou (1995 p 43) for a detailed proof.

Using (9) we simplify relation (8) to

$$\frac{\partial \Phi_{j0}}{\partial \boldsymbol{J}'}(\boldsymbol{J}')\frac{\partial S_1}{\partial \boldsymbol{w}}(\boldsymbol{w}, \boldsymbol{J}') = \Phi_{j1}(\boldsymbol{w} + 2\pi\boldsymbol{\alpha}(\boldsymbol{J}'), \boldsymbol{J}') - \Phi_{j1}(\boldsymbol{w}, \boldsymbol{J}'). \tag{10}$$

Applying (10) $q \in \mathbb{N}$ times iteratively, we obtain

$$\begin{aligned}&\Phi_{j1}(\boldsymbol{w} + 2\pi q\boldsymbol{\alpha}(\boldsymbol{J}'), \boldsymbol{J}') - \Phi_{j1}(\boldsymbol{w}, \boldsymbol{J}')\\ &= \sum_{k=0}^{q-1}\left(\frac{\partial \Phi_{j0}}{\partial \boldsymbol{J}'}\,\frac{\partial S_1}{\partial \boldsymbol{w}}\right)(\boldsymbol{w} + 2\pi k\boldsymbol{\alpha}(\boldsymbol{J}'), \boldsymbol{J}').\end{aligned} \tag{11}$$

Now let $\boldsymbol{\alpha}$ be of the form

$$\alpha_j(\boldsymbol{J}') = p_j/q, \qquad p_j \in \mathbb{Z}, \quad q \in \mathbb{N}, \quad j = 1, \ldots, n, \tag{12}$$

such that q is relative prime to the largest common divisor of the p_j. Then $(\boldsymbol{w} + 2\pi k\boldsymbol{\alpha}(\boldsymbol{J}'), \boldsymbol{J}')$, $k = 0, \ldots, q-1$, corresponds to a representative segment of a q-periodic orbit of the unperturbed part $\boldsymbol{G}_0$ of mapping $\boldsymbol{G}_\varepsilon$. This form of $\boldsymbol{\alpha}$ can be achieved on a dense set of actions $\boldsymbol{J}'$ because of the nondegeneracy condition (5). For such a choice of $\boldsymbol{\alpha}$, because of periodicity, the left hand side of (11) vanishes, so that, using again (9),

$$\frac{\partial \Phi_{j0}}{\partial \boldsymbol{J}'}(\boldsymbol{J}') \sum_{k=0}^{q-1} \frac{\partial S_1}{\partial \boldsymbol{w}}(\boldsymbol{w} + 2\pi k \boldsymbol{\alpha}(\boldsymbol{J}'), \boldsymbol{J}') \equiv 0, \qquad j = 1, \ldots, n,$$

identically on $(\boldsymbol{w}, \boldsymbol{J}') \in \mathbb{D}_{\mathbb{Q}} := \mathbb{T}^n \times \mathbb{J}_{\mathbb{Q}}$, $\mathbb{J}_{\mathbb{Q}} = \{\boldsymbol{J} \in \mathbb{J} | \boldsymbol{\alpha}(\boldsymbol{J}) \in \mathbb{Q}^n\}$. In other words

$$\mathbf{F}(\boldsymbol{J}') \, \frac{\partial \Sigma^{(q)}}{\partial \boldsymbol{w}}(\boldsymbol{w}, \boldsymbol{J}') \equiv 0 \tag{13}$$

holds on $\mathbb{D}_{\mathbb{Q}}$ where

$$\begin{aligned} (\mathbf{F}(\boldsymbol{J}'))_{j,l=1,\ldots,n} &:= \frac{\partial \Phi_{j0}}{\partial J'_l} \\ \Sigma^{(q)}(\boldsymbol{w}, \boldsymbol{J}') &:= \sum_{k=0}^{q-1} S_1(\boldsymbol{w} + 2\pi k \boldsymbol{\alpha}(\boldsymbol{J}'), \boldsymbol{J}'). \end{aligned}$$

$\Sigma^{(q)}(\boldsymbol{w}, \boldsymbol{J}')$ is called the *action function* along an orbital segment of length q. See e.g. MacKay *et al* (1984) for the definition of the action function in context with type one generating functions. In this paper we concentrate on length q segments of q-periodic orbits of the unperturbed system, therefore q is always determined by $\boldsymbol{J}'$. In analogy to the case of continuous systems we define the *ultra-subharmonic Poincaré-Melnikov-Arnol'd vector* of a mapping as

$$\frac{\partial \Sigma^{(q)}}{\partial \boldsymbol{w}}(\boldsymbol{w}, \boldsymbol{J}').$$

If we are now able to show that this vector does not vanish identically in $(\boldsymbol{w}, \boldsymbol{J}')$ on a dense or key set (see Arnol'd *et al* 1988 p 213) of $\boldsymbol{J}'$ values, we would have $\operatorname{Det} \mathbf{F} \equiv 0$ on this set which is a contradiction to the independence of integrals $\Phi_{i0}, \quad i = 0, \ldots, n$. Consequently the system determined by $\boldsymbol{G}_\varepsilon$ cannot have a complete set of independent analytic integrals. This establishes the first part of our non-integrability criterion.

Going further, we differentiate (13) with respect to $\boldsymbol{w}$ yielding

$$\mathbf{L}(\boldsymbol{w}, \boldsymbol{J}') \, \mathbf{F}(\boldsymbol{J}') \equiv \mathbf{O},$$

identically on $\mathbb{D}_{\mathbb{Q}}$, with

$$(\mathbf{L}(\boldsymbol{w}, \boldsymbol{J}'))_{j,l=1,\ldots,n} := \frac{\partial^2}{\partial w_j \partial w_l} \Sigma^{(q)}(\boldsymbol{w}, \boldsymbol{J}').$$

Now suppose the system represented by mapping $\boldsymbol{G}_\varepsilon$ has at least s independent integrals $\Phi_j, \quad j = 1, \ldots, s$ for some fixed s, $1 \leq s \leq n$, then $\operatorname{rank} \mathbf{F}(\boldsymbol{J}') \geq s$ in $\mathbb{D}$. Because of $\operatorname{rank} \mathbf{F} \leq \dim \ker \mathbf{L}$ and $\dim \ker \mathbf{L} + \operatorname{rank} \mathbf{L} = n$ it follows that

$$s \;\leq\; \operatorname{rank} \mathbf{F}(\boldsymbol{J}') \;\leq\; n - \operatorname{rank} \mathbf{L}(\boldsymbol{w}, \boldsymbol{J}').$$

Note that $\mathbf{F}$ and its rank depend just on the resonant torus selected, while $\mathbf{L}$ also depends on the particular periodic orbit considered. Thus passing to the maximum rank over all such orbits on the torus determined by $\boldsymbol{J}'$

$$r_{\boldsymbol{J}'} := \max\{\operatorname{rank}\mathbf{L}(\boldsymbol{w},\boldsymbol{J}')|\boldsymbol{w}\in\mathbb{T}^n\},$$

we know there exists at least one orbit with initial $\boldsymbol{w} = \boldsymbol{w}_{\boldsymbol{J}'}$, such that $r_{\boldsymbol{J}'} = \operatorname{rank}\mathbf{L}(\boldsymbol{w}_{\boldsymbol{J}'},\boldsymbol{J}')$, thus $s \le \operatorname{rank}\mathbf{F}(\boldsymbol{J}') \le n - r_{\boldsymbol{J}'}$. Note that the supremum taken here is really a maximum since a rank is an integer. Now let

$$r := \min\{r_{\boldsymbol{J}'}|\boldsymbol{J}'\in\mathbb{J}_1\},$$

where $\mathbb{J}_1 \subseteq \mathbb{J}_{\mathbb{Q}}$ is a dense or a key set of resonant $\boldsymbol{J}'$, then we have

$$s \;\le\; \operatorname{rank}\mathbf{F}(\boldsymbol{J}') \;\le\; n-r$$

on this set. If now $s > n-r$ we have arrived at a contradiction. Thus there cannot be more than $n-r$ independent integrals with the prescribed properties. Summarizing we have proved the following theorem.

Theorem 1. Let a one parameter family of discrete dynamical systems of dimension $2n$ be represented through the family of canonical, i.e. symplectic mappings

$$\boldsymbol{G}_\varepsilon : \mathbb{D}_E \to \mathbb{T}^n\times\mathbb{R}^n, (\boldsymbol{w},\boldsymbol{J}) \mapsto (\boldsymbol{w}',\boldsymbol{J}'), \qquad \mathbb{D}_E = \mathbb{T}^n\times\mathbb{J}_H,\ \mathbb{J}_H\subseteq\mathbb{R}^n,\ \mathbb{J}_H \text{ open}$$

analytic in its parameter $\varepsilon \in]-\varepsilon_1,\varepsilon_1[,\ \varepsilon_1 > 0$ and on $\mathbb{D}_E$ with respect to $(\boldsymbol{w},\boldsymbol{J})$. Moreover assume $\boldsymbol{G}_0$ to be of twist type, i.e. in canonical action-angle variables $(\boldsymbol{w},\boldsymbol{J})$

$$\boldsymbol{G}_0 : \left\{ \begin{array}{rcl} \boldsymbol{w}' & = & \boldsymbol{w} + 2\pi\boldsymbol{\alpha}(\boldsymbol{J}) \\ \boldsymbol{J}' & = & \boldsymbol{J}, \end{array} \right.$$

fulfilling the nondegeneracy condition

$$\operatorname{Det}\frac{\partial\boldsymbol{\alpha}}{\partial\boldsymbol{J}'}(\boldsymbol{J}') \neq 0.$$

Then for sufficiently small ε let $\boldsymbol{G}_\varepsilon$ be implicitly defined through its second type generating function

$$\begin{aligned} S(\boldsymbol{w},\boldsymbol{J}') \; &= S_0(\boldsymbol{w},\boldsymbol{J}') + \varepsilon S_1(\boldsymbol{w},\boldsymbol{J}') + o(\varepsilon^2), \\ S_0(\boldsymbol{w},\boldsymbol{J}') &= \boldsymbol{w}\boldsymbol{J}' + S_T(\boldsymbol{J}') \end{aligned}$$

as

$$\boldsymbol{G}_\varepsilon : \left\{ \begin{array}{rcl} \boldsymbol{w}' & = & \dfrac{\partial S}{\partial \boldsymbol{J}'}(\boldsymbol{w},\boldsymbol{J}') \qquad \text{and} \\ \boldsymbol{J} & = & \dfrac{\partial S}{\partial \boldsymbol{w}}(\boldsymbol{w},\boldsymbol{J}') \end{array} \right.$$

respectively, $\boldsymbol{G}_\varepsilon$ being analytic in an open domain $\mathbb{D} = \mathbb{T}^n \times \mathbb{J}$, $\mathbb{J} \subseteq \mathbb{J}_H$ in $(\boldsymbol{w}, \boldsymbol{J}')$ and in $\varepsilon \in]-\varepsilon_2, \varepsilon_2[$, $0 < \varepsilon_2 \leq \varepsilon_1$.

Let furthermore the action function $\Sigma^{(q)}$ be defined as

$$\Sigma^{(q)}(\boldsymbol{w}, \boldsymbol{J}') = \sum_{k=0}^{q-1} S_1(\boldsymbol{w} + 2\pi\boldsymbol{\alpha}(\boldsymbol{J}'), \boldsymbol{J}')$$

such that $\boldsymbol{J}'$ corresponds to a resonant torus, i.e. to a q-periodic orbit of $\boldsymbol{G}_0$,

$$\boldsymbol{\alpha}(\boldsymbol{J}') = \frac{1}{q}\boldsymbol{p}, \qquad \boldsymbol{p} \in \mathbb{Z}^n, \quad q \in \mathbb{N},$$

with q being relative prime to the largest common divisor of the p_j. Then,

(i) if within $\mathbb{D}$ a dense or a key set of resonant tori of $\boldsymbol{G}_0$ is found such that for at least one orbit on each of them

$$\frac{\partial}{\partial \boldsymbol{w}}\Sigma^{(q)}(\boldsymbol{w}, \boldsymbol{J}') \not\equiv_{\boldsymbol{w}} 0,$$

then $\boldsymbol{G}_\varepsilon$ does not possess a complete set of analytic integrals for $\varepsilon \in]-\varepsilon_0, \varepsilon_0[\backslash\{0\}$, $0 < \varepsilon_0 \leq \varepsilon_2$; and

(ii) if, likewise, on some dense or key set of resonant tori there is at least one orbit on each of them such that

$$\operatorname{rank}\left(\frac{\partial^2\Sigma^{(q)}}{\partial \boldsymbol{w}^2}(\boldsymbol{w}, \boldsymbol{J}')\right) \geq r \in \mathbb{N},$$

then $\boldsymbol{G}_\varepsilon$ cannot possess more than $n - r$ independent integrals being analytic in the same set of ε values.

4. Continuation of Periodic Orbits and Linear Stability

Again we assume the implicit map $\boldsymbol{G}_\varepsilon$ to be analytic in $(\boldsymbol{w}, \boldsymbol{J}')$ and ε in open domains $\mathbb{D} = \mathbb{T} \times \mathbb{J}$, $\mathbb{J} \subseteq \mathbb{R}^n$ and $]-\varepsilon_2, \varepsilon_2[$, $\varepsilon_2 > 0$, respectively. Expanding with respect to ε we obtain the following explicit first order formulation of the q-fold iterated mapping

$$\boldsymbol{G}_\varepsilon^{(q)} : \begin{cases} \boldsymbol{w}^{(q)} & = \quad \boldsymbol{w} + 2\pi q\boldsymbol{\alpha}(\boldsymbol{J}) \\ & \qquad + \varepsilon\left(\dfrac{\partial\Sigma^{(q)}}{\partial \boldsymbol{J}}(\boldsymbol{w}, \boldsymbol{J}) - 2\pi\dfrac{\partial\boldsymbol{\alpha}}{\partial \boldsymbol{J}}(\boldsymbol{J})\,\dfrac{\partial\Sigma^{(q)}}{\partial \boldsymbol{w}}(\boldsymbol{w}, \boldsymbol{J})\right) \quad + o(\varepsilon^2) \\ \boldsymbol{J}^{(q)} & = \quad \boldsymbol{J} - \varepsilon\dfrac{\partial\Sigma^{(q)}}{\partial \boldsymbol{w}}(\boldsymbol{w}, \boldsymbol{J}) \qquad\qquad + o(\varepsilon^2). \end{cases}$$

The condition for a point $(\boldsymbol{w}, \boldsymbol{J})$ to be q-periodic, which will meet the requirements for a proper application of the implicit function theorem in order to show continuation of the periodicity of the specific point from $\varepsilon = 0$ to small $|\varepsilon| \neq 0$, takes the form

$$\boldsymbol{f}(\varepsilon, \boldsymbol{w}, \boldsymbol{J}) := \begin{pmatrix} \boldsymbol{w}^{(q)}(\varepsilon, \boldsymbol{w}, \boldsymbol{J}) - \boldsymbol{w} - 2\pi \boldsymbol{p} \\ \dfrac{\boldsymbol{J}^{(q)}(\varepsilon, \boldsymbol{w}, \boldsymbol{J}) - \boldsymbol{J}}{\varepsilon} \end{pmatrix} = \boldsymbol{O}, \qquad \boldsymbol{p} \in \mathbb{Z}^n, \quad q \in \mathbb{N}. \tag{14}$$

The division by ε involved in the $\boldsymbol{J}$-terms is consistent, because an orbit being continued for small enough $|\varepsilon| \neq 0$ must fulfill (14) in its original form without division, thus, since $\varepsilon \neq 0$, division is possible and the condition must be still valid in the limit of $\varepsilon \to 0$ because of continuity. The necessity of this procedure will become clear below. We have $\boldsymbol{f}(\varepsilon, \boldsymbol{w}, \boldsymbol{J}) = \boldsymbol{f}_0(\boldsymbol{w}, \boldsymbol{J}) + o(\varepsilon)$ with

$$\boldsymbol{f}_0(\boldsymbol{w}, \boldsymbol{J}) = \begin{pmatrix} 2\pi q \boldsymbol{\alpha}(\boldsymbol{J}) - 2\pi \boldsymbol{p} \\ \dfrac{\partial \Sigma^{(q)}}{\partial \boldsymbol{w}}(\boldsymbol{w}, \boldsymbol{J}) \end{pmatrix}.$$

The implicit function theorem now ensures that when

$$\boldsymbol{f}_0(\boldsymbol{w}_0, \boldsymbol{J}_0) = \boldsymbol{O}$$

holds for certain $(\boldsymbol{w}_0, \boldsymbol{J}_0)$ and if also

$$\mathrm{Det}\, \frac{\partial \boldsymbol{f}_0}{\partial(\boldsymbol{w}, \boldsymbol{J})}(\boldsymbol{w}_0, \boldsymbol{J}_0) = \begin{vmatrix} \mathbf{O}_n & \dfrac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{J}}(\boldsymbol{J}_0) \\ \dfrac{\partial^2 \Sigma^{(q)}}{\partial \boldsymbol{w}^2}(\boldsymbol{w}_0, \boldsymbol{J}_0) & \dfrac{\partial^2 \Sigma^{(q)}}{\partial \boldsymbol{w} \partial \boldsymbol{J}}(\boldsymbol{w}_0, \boldsymbol{J}_0) \end{vmatrix} \neq 0, \tag{15}$$

then there is an open interval of ε values around zero, such that the perturbed twist map $\boldsymbol{G}_\varepsilon$ has a q-periodic orbit with initial condition $(\boldsymbol{w}, \boldsymbol{J}) = (\boldsymbol{w}(\varepsilon), \boldsymbol{J}(\varepsilon)) = (\boldsymbol{w}_0 + \varepsilon \boldsymbol{w}_1 + o(\varepsilon^2), \boldsymbol{J}_0 + \varepsilon \boldsymbol{J}_1 + o(\varepsilon^2))$ analytically depending on ε which for $\varepsilon = 0$ coincides with $(\boldsymbol{w}_0, \boldsymbol{J}_0)$. We note that condition (15), because of nondegeneracy reduces to

$$\mathrm{Det}\, \frac{\partial^2 \Sigma^{(q)}}{\partial \boldsymbol{w}^2}(\boldsymbol{w}_0, \boldsymbol{J}_0) \neq 0$$

and also that without the above division by ε in the periodicity conditions for $\boldsymbol{J}$ this condition would have been void in any case.

The linear stability of the fixed points is determined by the eigenvalues of the matrix $\mathbf{M}$ of the tangential map. Composing this matrix as

$$\mathbf{M} = \mathcal{D}\boldsymbol{G}_\varepsilon^{(q)} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

with four $n \times n$ matrices

$$\begin{aligned}
(\mathbf{A})_{ij} &= \delta_{ij} + \varepsilon \left(\frac{\partial^2}{\partial J_i \partial w_j} - 2\pi \sum_{l=1}^{n} \frac{\partial \alpha_i}{\partial J_l} \frac{\partial^2}{\partial w_l \partial w_j} \right) \Sigma^{(q)} && + o(\varepsilon^2) \\
(\mathbf{B})_{ij} &= 2\pi q \frac{\partial \alpha_i}{\partial J_j} + \varepsilon \left(\frac{\partial^2}{\partial J_i \partial J_j} - 2\pi \sum_{l=1}^{n} \frac{\partial \alpha_i}{\partial J_l} \frac{\partial^2}{\partial w_l \partial J_j} \right) \Sigma^{(q)} && + o(\varepsilon^2) \\
(\mathbf{C})_{ij} &= -\varepsilon \frac{\partial^2}{\partial w_i \partial w_j} \Sigma^{(q)} && + o(\varepsilon^2) \\
(\mathbf{D})_{ij} &= \delta_{ij} - \varepsilon \frac{\partial^2}{\partial w_i \partial J_j} \Sigma^{(q)} && + o(\varepsilon^2),
\end{aligned}$$

for $i, j = 1, \ldots, n,$ we have from (6), since $\mathbf{M}$ is a symplectic matrix,

$$\mathbf{M}^{-1} = -\boldsymbol{\Omega} \mathbf{M}^T \boldsymbol{\Omega} = \begin{pmatrix} \mathbf{D}^T & -\mathbf{B}^T \\ -\mathbf{C}^T & \mathbf{A}^T \end{pmatrix}.$$

If λ is an eigenvalue of $\mathbf{M}$ with eigenvector $\boldsymbol{x}$ then $1/\lambda$ is an eigenvalue of $\mathbf{M}^{-1}$ with the same eigenvector, consequently $\nu := \lambda + 1/\lambda$ is an eigenvalue to $\mathbf{N} := \mathbf{M} + \mathbf{M}^{-1}$ with eigenvector $\boldsymbol{x}$. Since $1/\lambda$ is also an eigenvalue of $\mathbf{M}$ with an eigenvector $\boldsymbol{y} \neq \boldsymbol{x}$, ν is necessarily at least a double eigenvalue of $\mathbf{N}$, which attains the form

$$\mathbf{N} = \begin{pmatrix} \mathbf{A} + \mathbf{D}^T & \mathbf{B} - \mathbf{B}^T \\ \mathbf{C} - \mathbf{C}^T & \mathbf{A}^T + \mathbf{D} \end{pmatrix} = \begin{pmatrix} 2\mathbf{I}_n + \varepsilon \boldsymbol{\Theta} + o(\varepsilon^2) & o(\varepsilon) \\ o(\varepsilon^2) & 2\mathbf{I}_n + \varepsilon \boldsymbol{\Theta}^T + o(\varepsilon^2) \end{pmatrix},$$

where

$$\boldsymbol{\Theta}(\boldsymbol{w}, \boldsymbol{J}) = -2\pi \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{J}}(\boldsymbol{J}) \, \frac{\partial^2 \Sigma^{(q)}}{\partial \boldsymbol{w}^2}(\boldsymbol{w}, \boldsymbol{J}).$$

Following the method of Yakubovich and Starzhinskii (1975 p 315) we may draw the following conclusions. For $\varepsilon = 0$ clearly $\nu = 2$, therefore — since $\mathrm{Det}\,(\mathbf{N} - \nu \mathbf{I}_{2n}) = 0$ defines ν in dependence of ε — we set $\varepsilon \beta_\varepsilon(\boldsymbol{w}, \boldsymbol{J}) = \nu_\varepsilon(\boldsymbol{w}, \boldsymbol{J}) - 2$. Dividing the above determinant by ε^{2n}, which is justified by the necessity of continuity with respect to ε at $\varepsilon = 0$, we obtain

$$\begin{vmatrix} \boldsymbol{\Theta} - \beta_\varepsilon \mathbf{I}_n & o(1) \\ o(\varepsilon) & \boldsymbol{\Theta}^T - \beta_\varepsilon \mathbf{I}_n + o(\varepsilon) \end{vmatrix} = 0.$$

If $\beta_0(\boldsymbol{w}_0, \boldsymbol{J}_0)$ is an eigenvalue of $\boldsymbol{\Theta}(\boldsymbol{w}_0, \boldsymbol{J}_0)$ we have

$$\beta_\varepsilon(\boldsymbol{w}, \boldsymbol{J}) = \beta_0(\boldsymbol{w}_0, \boldsymbol{J}_0) + o(\varepsilon^{1/l}), \qquad l \in \mathbb{N},$$

which depends analytically on some fractional power of ε. Thus

$$\nu_\varepsilon(\boldsymbol{w}, \boldsymbol{J}) = 2 + \varepsilon \beta_0(\boldsymbol{w}_0, \boldsymbol{J}_0) + o(\varepsilon^{1+1/l}),$$

and consequently

$$\lambda = \lambda_\varepsilon(\boldsymbol{w}, \boldsymbol{J}) = 1 \pm \sqrt{\varepsilon}\rho(\boldsymbol{w}_0, \boldsymbol{J}_0) + o(\varepsilon^{\min(1,1/2+1/l)}), \qquad l \in \mathbb{N}$$

where $\rho^2(\boldsymbol{w}_0, \boldsymbol{J}_0) = \beta_0(\boldsymbol{w}_0, \boldsymbol{J}_0)$. It follows that the linear stability of the periodic orbit continued is just determined by the $n \times n$ (Hessian) twist matrix $\frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{J}}(\boldsymbol{J}_0)$ and the $n \times n$ Hessian $\frac{\partial^2 \Sigma^{(q)}}{\partial \boldsymbol{w}^2}(\boldsymbol{w}_0, \boldsymbol{J}_0)$. Note, that because of (15), ρ is always different from zero so the continued periodic orbits are always isolated. The linear stability analysis is thus reduced to consideration of a linear space of half the dimension of the system. The following theorem results:

Theorem 2. Let $\boldsymbol{G}_\varepsilon$ be the implicit twist mapping defined through a generating function S as in theorem 1, so that the nondegeneracy condition holds. Assume $\boldsymbol{G}_\varepsilon$ to be analytic in $(\boldsymbol{w}, \boldsymbol{J}')$ and analytic in ε in open domains. Then for all resonant tori of the unperturbed mapping $\boldsymbol{G}_0$ determined by $\boldsymbol{J}_0$ such that

$$\boldsymbol{\alpha}(\boldsymbol{J}_0) = \frac{1}{q}\boldsymbol{p}, \qquad \boldsymbol{p} \in \mathbb{Z}^n, \quad q \in \mathbb{N}$$

a q-periodic orbit of $\boldsymbol{G}_0$ containing the point $(\boldsymbol{w}_0, \boldsymbol{J}_0)$ is analytically continued to $(\boldsymbol{w}, \boldsymbol{J}) = (\boldsymbol{w}_0, \boldsymbol{J}_0) + o(\varepsilon)$ for small enough $|\varepsilon| \neq 0$, if the vector $\partial\Sigma^{(q)}/\partial\boldsymbol{w}$ has a simple zero in $(\boldsymbol{w}_0, \boldsymbol{J}_0)$, i.e.

$$\frac{\partial \Sigma^{(q)}}{\partial \boldsymbol{w}}(\boldsymbol{w}_0, \boldsymbol{J}_0) = \boldsymbol{O}, \qquad \mathrm{Det}\, \frac{\partial^2 \Sigma^{(q)}}{\partial \boldsymbol{w}^2}(\boldsymbol{w}_0, \boldsymbol{J}_0) \neq 0,$$

$\Sigma^{(q)}$ being defined as in theorem 1. Moreover the eigenvalues of the Jacobian $\mathcal{D}\boldsymbol{G}_\varepsilon^{(q)}$ at such a point are of the form

$$\lambda_\varepsilon(\boldsymbol{w}, \boldsymbol{J}) = 1 \pm \sqrt{\varepsilon}\rho(\boldsymbol{w}_0, \boldsymbol{J}_0) + o(\varepsilon^{\min(1,1/2+1/l)}), \qquad l \in \mathbb{N},$$

where ρ^2 are the eigenvalues of

$$-2\pi \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{J}}(\boldsymbol{J}_0)\, \frac{\partial^2 \Sigma^{(q)}}{\partial \boldsymbol{w}^2}(\boldsymbol{w}_0, \boldsymbol{J}_0)$$

and λ_ε are analytic functions of a fractional power of ε.

In the following we consider the linear stability of mapping $\boldsymbol{G}_\varepsilon$ in dependence of $\beta_0 = \beta_0(\boldsymbol{w}_0, \boldsymbol{J}_0)$ for sufficiently small $|\varepsilon|$ in more detail:

If at least one β_0 is complex, instability of complex type results. If all β_0 are real and simple with at least one β_0 such that $\varepsilon\beta_0 > 0$, we have real instability. If all β_0 are real and simple with $\varepsilon\beta_0 < 0$ we encounter the only certain case of linear stability based on the framework of theorem 2. In the latter two cases, if some β_0 is multiple, complex instability may result, depending on the behaviour of the higher order terms in ε.

We note that in the case of $n = 1$ we have

$$\rho^2(w_0, J_0) = -2\pi \frac{\mathrm{d}\alpha}{\mathrm{d}J}(J_0) \frac{\partial^2 \Sigma^{(q)}}{\partial w^2}(w_0, J_0).$$

Since $\Sigma^{(q)}$ is 2π-periodic with respect to w, two consecutive simple zeroes of $\partial\Sigma^{(q)}/\partial w$ correspond to a maximum and a minimum of $\Sigma^{(q)}$, i.e. positive and negative ρ^2 respectively, assuming positive twist direction $\frac{\mathrm{d}\alpha}{\mathrm{d}J}(J_0) > 0$. Thus we obtain the known result of Poincaré-Birkhoff for small enough $\varepsilon > 0$, see Poincaré (1912) and Birkhoff (1913). Note further that, in view of theorem 1, if the perturbed mapping possesses at least one integral, then for a resonant torus in general position, $\mathrm{Det}\,\mathbf{L} = 0$ and theorem 2 is not applicable.

5. Application to a Generalized Standard Map

In the following we want to demonstrate the application of the theory developed so far along the simple example of a two dimensional mapping ($n = 1$). In the case of the well known Chirikov-Taylor standard map (Chirikov 1979, Lichtenberg and Liebermann 1983)

$$\boldsymbol{G}_\varepsilon : \begin{cases} w' = w + J' \\ J' = J - \varepsilon \sin w \end{cases}$$

generated by

$$S(w, J') = wJ' + \frac{1}{2}J'^2 - \varepsilon \cos w$$

our non-integrability criterion stated in theorem 1 fails due to the absence of higher harmonics in the perturbation of the J-part; a dense set or at least a key set of resonant tori with the properties stated in theorem 1 cannot be found. Note that in the case of the standard map, since the perturbation of the J-part depends only on w and the perturbation of the w-part vanishes — which characterizes a so-called *radial map* — the mapping can be made explicit. In general a standard map for several degrees of freedom is defined as being derivable from a 'potential' function $V = V(\boldsymbol{w})$, periodic in its components, in the form $\boldsymbol{w}' = \boldsymbol{w} + \boldsymbol{J}'$ and $\boldsymbol{J}' = \boldsymbol{J} + \boldsymbol{\nabla}_{\boldsymbol{w}} V$ respectively, see e.g. Lomelí (1996).

For the mentioned reason we considered a *generalized standard map* by replacing trigonometric functions by their respective analogues of properly scaled Jacobian elliptic functions. In particular let

$$S(w, J') = wJ' + \frac{1}{2}J'^2 + \varepsilon \frac{\pi}{2Kk} \ln\left(\mathrm{dn}\frac{2K}{\pi}w - k\,\mathrm{cn}\frac{2K}{\pi}w\right),$$

then

$$\boldsymbol{G}_\varepsilon : \begin{cases} w' = w + J' \\ J' = J - \varepsilon \,\mathrm{sn}\left(\dfrac{2K}{\pi} w, k\right). \end{cases} \tag{16}$$

Thus for $\alpha(J') = p/q$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$

$$\frac{\partial}{\partial w}\Sigma^{(q)}(w, J') = \sum_{m=0}^{q-1} \mathrm{sn}\left(\frac{2K}{\pi}\left(w + 2\pi m \frac{p}{q}\right), k\right),$$

wherein k is the modulus of the Jacobian elliptic functions and $K = K(k)$ the related quarter period, $K(k)$ representing the complete elliptic integral of the first kind in the sense of Legendre, see Abramowitz and Stegun (1984 p 235). Using the well known relation — see again Abramowitz and Stegun (1984 p 229) —

$$\mathrm{sn}(u, k) = \frac{2\pi}{kK} \sum_{j=0}^{\infty} \frac{Q^{j+1/2}}{1 - Q^{2j+1}} \sin\left[(2j+1)\frac{\pi}{2K} u\right],$$

with $Q = e^{-\dfrac{\pi K'}{K}}$ being the elliptic nome and $K' = K(k')$ being the complementary quarterperiod corresponding to the complementary modulus $k' = \sqrt{1 - k^2}$, we find

$$\frac{\partial}{\partial w}\Sigma^{(q)}(w, J') = q \frac{\tilde{k}\tilde{K}}{kK} \mathrm{sn}\left(\frac{2\tilde{K} q}{\pi} w, \tilde{k}\right).$$

In this relation $\tilde{k}$ is uniquely defined through the equations

$$\sigma(\tilde{k}) := \frac{\tilde{K}'}{\tilde{K}} = \frac{K'}{K} q, \qquad \tilde{K} = K(\tilde{k}), \qquad \tilde{K}' = K(\tilde{k}'), \qquad \tilde{k}' = \sqrt{1 - \tilde{k}^2},$$

because σ is continuous on $\tilde{k} \in]0, 1[$, $\sigma(0) = \infty$, $\sigma(1) = 0$ and $\sigma(\tilde{k})$ decreases monotonically with $\tilde{k}$.

We conclude first, that e.g. for the key set of tori $J' = 2\pi \dfrac{q+1}{q} \to 2\pi$ $(q \to \infty)$, $\dfrac{\partial}{\partial w}\Sigma^{(q)}(w, J') \not\equiv_w 0$ holds, therefore the dynamical system determined by the mapping (16) is *non-integrable* according to theorem 1.

Second, for every rational value p/q of $\alpha(J_0) = \frac{1}{2\pi} J_0$, the zeroes of $\dfrac{\partial}{\partial w}\Sigma^{(q)}(w_0, J_0)$ are $w_0 = \dfrac{\pi j}{q}$, $j = 0, \ldots, 2q-1$ for which $\dfrac{\partial^2}{\partial w^2}\Sigma^{(q)}(w_0, J_0)$ is positive — corresponding to a stable periodic orbit — if j is even, and negative — corresponding to an unstable periodic orbit — if j is odd.

In figure 1 the continued fixed points at some main resonances are shown for $\varepsilon = 0.1$ and $k = \sin 80^{\mathrm{o}}$.

6. Final Remarks

Comparing the results of this paper with the analogous ones of Meletlidou and Ichtiaroglou (1994) and Ichtiaroglou and Meletlidou (1996), there is a striking resemblence to the theory developed here visible if the expression $\langle H_1 \rangle$ (the average value of H_1 taken over one period) in the quoted papers is replaced by $\Sigma^{(q)}$. Indeed, the theory of symplectic integrators (see e.g. Yoshida 1993; Hadjidemetriou 1991, 1996) shows that the first order symplectic integrator implicitly defined by a generating function of the second type establishes a close connection between those two expressions. Particularly, if we deal with a Hamiltonian system (1) the first order symplectic integrator takes the form

$$\tilde{\boldsymbol{G}}_\varepsilon : \begin{cases} \boldsymbol{w}' &= \boldsymbol{w} \quad + \tau \dfrac{\partial H_0}{\partial \boldsymbol{J}'}(\boldsymbol{J}') \quad + \tau\varepsilon \dfrac{\partial H_1}{\partial \boldsymbol{J}'}(\boldsymbol{w}, \boldsymbol{J}') \\ \boldsymbol{J} &= \boldsymbol{J}' \qquad\qquad\qquad\quad + \tau\varepsilon \dfrac{\partial H_1}{\partial \boldsymbol{w}}(\boldsymbol{w}, \boldsymbol{J}'), \end{cases} \tag{17}$$

where τ is the integration time step assumed to be a small quantity. Taking now heuristically $\tau = 2\pi$ in (17) and comparing the result with (4) we recognize that in (17) $2\pi H_1$ takes the rôle of S_1 in (4). However, this is only a rough consideration, the development of a more rigorous theory describing the connection between symplectic maps and Hamiltonian flows in more detail will be the subject of a forthcoming paper.

In literature, extensive use has been made of the generating function approach and related methods in order to describe the dynamics of mappings, see e.g. Percival (1979), Aubry and Le Daeron (1983), Aubry (1983), MacKay and Meiss (1983), MacKay *et al* (1984), Mather (1984), Lomelí (1996) for the use of the first type generators mostly relying on stationarity of the related action functions with respect to certain orbital variations, as far as fixed points and their linear stability or invariant sets in general are concerned. In our present work we used the more convenient way of stationarity against variations of the initial conditions and preferred to take advantage of the natural occurence of type two generators in the context of perturbed twist maps as long as the perturbation is small enough. Generating functions of the second type and their critical points when expressed in non-mixed form play a decisive rôle in generalisations of the Poincaré-Birkhoff fixed point theorem to higher dimensions, see Arnol'd (1980) and Milani (1991) for supplementary remarks. The main difference of our approach to the cited papers lies in the fact that in Arnold's work the generating function has to be modified to yield a non-mixed representation by means of a function that is known to exist, but unknown in explicit terms which results in just an existence statement. In contrast, our result on fixed points is of constructive nature, well suited for practical calculations, however, it is a perturbative one, valid for small ε only. We note that in the method of Arnol'd the sufficient closeness of the mapping to identity or a pure twist also imposes strong restrictions on the applicability of the theory. The question of the actual necessity of such a confinement is not yet settled.

In this paper we defined the *ultra-subharmonic Poincaré-Melnikov-Arnol'd vector* for the case of a symplectic near twist map and used it for showing both non-integrability and continuation of fixed points. Related work on nonintegrability and the *homoclinic Poincaré-Melnikov-Arnol'd vector* has been done e.g. by Glasser *et al* (1989), Delshams and Ramírez-Ros (1996) for the case of planar maps and by Sun (1996) for high-dimensional maps of specific form. For canonical flows the definition and treatment of these vectors go back to Poincaré (1890), Melnikov (1963) and Arnol'd (1964). See Wiggins (1990 p 117) for a theorem on the rôle of the ultra-subharmonic Melnikov vector for the existence of periodic orbits and Wiggins (1990 p 483) for a treatment of the homoclinic Melnikov vector, both in the context of flows.

For a specific system of two degrees of freedom Veerman and Holmes (1985) proved a theorem which establishes the existence of an arbitrarily large but finite number of isolated fixed points in the neighbourhood of an elliptic fixed point for sufficiently small perturbation away from integrability. From our present work, using further estimates involved in the application of the implicit function theorem not published here, we conclude that under appropriate conditions such a result may be proved for a large class of symplectic maps. However, since the domain of validity in ε of this approach without further assumptions shrinks to zero as the period q of an orbit under consideration tends to infinity, the number of isolated fixed points that can be shown to exist with this method in some open set remains always bounded.

We note finally that the two theorems presented here are coordinate independent except for the determination of rank $\partial^2\Sigma^{(q)}/\partial \boldsymbol{w}^2$ unless the zeroes of $\partial\Sigma^{(q)}/\partial \boldsymbol{w}$ are considered.

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Figure 1. Some orbits around fixed points of resonances 2/1, 11/6, 9/5, 5/3, 8/5, 3/2, 7/5, 4/3 (from above) of the generalized standard map.